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Induced Gauge Fields in the Path Integral

Shogo TANIMURA
Department of Applied Mathematics and Physics
Faculty of Engineering, Kyoto University
*Kyoto 606-01, Japan**

Izumi TSUTSUI
Institute for Nuclear Study, University of Tokyo
Midori-cho, Tanashi-shi, Tokyo 188, Japan†

Abstract

The path integral on a homogeneous space G/H is constructed, based on the guiding principle 'first lift to G and then project to G/H'. It is then shown that this principle admits inequivalent quantizations inducing a gauge field (the canonical connection) on the homogeneous space, and thereby reproduces the result obtained earlier by algebraic approaches.

*E-mail: tanimura@kuamp.kyoto-u.ac.jp
†E-mail: tsutsui@ins.u-tokyo.ac.jp

1 Introduction

Geometric approaches to quantum mechanics have been studied by various groups ever since the foundation of quantum mechanics was laid down. The prime aim of such approaches is to render quantum mechanics applicable to more general settings, not just to Euclidean space as originally done. However, it is by now well recognized that quantization is generally difficult to carry out unless the setting is fairly simple. A system whose classical configuration space Q is a homogeneous space given by a coset G/H falls into this simple category. An important lesson learned when quantizing on homogeneous spaces is that there are actually (infinitely) many *inequivalent quantizations* allowed [1, 2, 3]. In other words, there exist unitarily inequivalent Hilbert spaces where physical properties, such as their energy spectra, may differ from each other. These inequivalent quantizations are classified according to the induced representation [1] which is used for the quantization.

Interest in the inequivalent quantizations has been renewed recently after Landsman and Linden examined the physical implications of the quantizations and found that a special type of gauge field is induced on homogeneous spaces [4, 5, 6, 7, 8]. The gauge fields are a (topological) solution of the Yang-Mills equation on the spaces, called the *canonical connection* (or *H-connection*). However, the previous arguments leading to the gauge fields are algebraic and abstract, and there is no intuitive account of this rather mysterious appearance of gauge fields. It would be therefore desirable to develop a path integral account, which normally admits a more intuitive understanding based on the geometry of the configuration space. In this note we wish to take a step in this direction — we shall show that, the path integral on a homogeneous space carries the canonical connection as a gauge field, if we adopt the guiding principle that the path integral be constructed first on the group manifold G and then projected down to the homogeneous space G/H. This 'first lift and then project' principle may be arguable, but it is certainly true that the case $Q = S^1$, where the path integral is known to reproduce the inequivalent quantizations correctly [9], relies on this principle. We shall not dwell on this issue until the end of the paper where a possible explanation is given. We here mention that similar induced gauge fields appear in various other contexts as well; *e.g.*, in the context of Berry's (geometric) phase in quantum mechanics [10, 11, 12, 13] or in the kinematics of molecules and deformable bodies [14, 15, 16, 17]. Moreover, induced gauge fields play an important role in high energy physics too; *e.g.*, in the so-called hidden local symmetry of nonlinear sigma models [18] and in the search for a possible origin of dynamical gauge bosons [19, 20]. We hope that the path integral account given in this paper may shed some light on the machinery for those phenomena in general.

The plan of this paper is as follows. First we review quantum mechanics on $Q = S^1$ to see how the gauge field is induced. Motivated by this simple example, we then generalize the construction of the path integral to the case of homogeneous spaces $Q = G/H$ following the above guiding principle. We shall find that the canonical connection does appear in the path integral in the form expected, once the induced representation is incorporated in the path integral scheme. Finally we will argue a possible generalization to inhomogeneous spaces, together with a restriction that may underlie the guiding principle we adopted.

2 Covering the path integral

Let us begin by reviewing the path integral on a circle S^1 [9]. (A further discussion can be found in refs.[5, 21].) We first regard S^1 as the coset $S^1 \cong \boldsymbol{R}/2\pi\boldsymbol{Z}$ by identifying the point x of $\boldsymbol{R}$ with other points $x + 2\pi n$ for $n \in \boldsymbol{Z}$. This identification defines a covering map $\pi : \boldsymbol{R} \to S^1$. Our idea is then to construct the path integral on S^1 from the path integral on $\boldsymbol{R}$ with the above identification in mind.

Let $K_R(x', x; t) = \langle x'|e^{-iHt}|x\rangle$ be a propagator on $\boldsymbol{R}$ which is invariant under the translation by 2π,

$$K_R(x' + 2\pi, x + 2\pi; t) = K_R(x', x; t). \tag{1}$$

On account of the identification of points $x' + 2\pi n$ with x', summation over n may lead to a propagator on S^1;

$$K_{S^1}(x', x; t) = \sum_{n=-\infty}^{\infty} K_R(x' + 2\pi n, x; t), \tag{2}$$

where we interpret the integer n as the winding number of a path connecting two points x and x' along the circle S^1. Clearly, this expression admits an immediate generalization. In fact, we do not have an *a priori* physical reason to add up propagators for different winding numbers with the same weight, as long as the weight is a phase factor. Based on this observation Schulman [9] proposed to insert a weight factor ω_n with $|\omega_n| = 1$ to obtain a more general propagator

$$K^{\omega}_{S^1}(x', x; t) = \sum_{n=-\infty}^{\infty} \omega_n K_R(x' + 2\pi n, x; t). \tag{3}$$

The composition law of the propagator

$$\int_0^{2\pi} dx'\, K^{\omega}_{S^1}(x'', x'; t')\, K^{\omega}_{S^1}(x', x; t) = K^{\omega}_{S^1}(x'', x; t + t') \tag{4}$$

is guaranteed if the weight satisfies[‡]

$$\omega_m \, \omega_n = \omega_{m+n} . \tag{5}$$

This implies that $\omega : \pi_1(S^1) \to U(1)$ is a unitary representation of the first homotopy group $\pi_1(S^1) \cong \boldsymbol{Z}$ and hence given by $\omega_n = e^{i\alpha n}$ with a real parameter $\alpha \in [0,\, 2\pi)$. For each value of α, the propagator (3) furnishes an inequivalent quantum theory on S^1. To see the physical meaning of α, we assume K_R to be of the standard form

$$K_R(x', x; t) = \int_x^{x'} [dx] \, \exp\left[i \int dt \left\{ \frac{1}{2}\left(\frac{dx}{dt}\right)^2 - V(x) \right\} \right], \tag{6}$$

where $V(x + 2\pi) = V(x)$ in order to satisfy (1). Putting $A = \alpha/(2\pi)$, we find that the propagator (3) can be rewritten as

$$K^{\omega}_{S^1}(x', x; t) = e^{-i\frac{\alpha}{2\pi}(x'-x)} \sum_{n=-\infty}^{\infty} \int_x^{x'+2\pi n} [dx] \, \exp\left[i \int dt \left\{ \frac{1}{2}\left(\frac{dx}{dt}\right)^2 - V(x) + A\, \frac{dx}{dt} \right\} \right]. \tag{7}$$

We therefore see that the insertion of the weight $\omega_n = e^{i\alpha n}$ just amounts to introduction of the minimal coupling with the vector potential A. Being constant, the vector potential has vanishing curvature on S^1 but the flux penetrating the circle is finite. Hence, its physical consequence is analogous to that of the Aharonov-Bohm effect.

3 Lifting the path integral

What we have considered above is a covering $\pi : \boldsymbol{R} \to S^1 \cong \boldsymbol{R}/2\pi\boldsymbol{Z}$. A point x' in S^1 is lifted to points $x' + 2\pi n$ in $\boldsymbol{R}$, which are translated by the action of the group $\boldsymbol{Z}$. For each lifted point a propagator in $\boldsymbol{R}$ is defined, then we add them up with a weight factor given by the representation $\omega : \boldsymbol{Z} \to U(1)$ to obtain a propagator in S^1. Thus a path in S^1 is lifted up to $\boldsymbol{R}$ once, and then it is projected down to S^1 with a nontrivial weight multiplied, resulting in inequivalent quantizations and inducing a $U(1)$ gauge field. In this section, we shall repeat the above construction of the path integral to a homogeneous space G/H, where G is a compact Lie group and H its closed subgroup. In order to set up a framework where a generalization of the covering $\boldsymbol{R} \to S^1 \cong \boldsymbol{R}/2\pi\boldsymbol{Z}$ can be realized for $Q \cong G/H$, we take the principal fiber bundle $\pi : G \to G/H$ in which H acts on G from the right and G acts on G/H from the left. The difference from the former case is that H can be a continuous group or a nonabelian group in general, and hence the summation over the winding numbers

[‡]The weight can actually be determined by requiring consistency against a shift of winding numbers [9] but here we use the composition law for our later generalization.

$\sum_n$ ($n \in \boldsymbol{Z}$) will be replaced by the integration over the group $\int_H dh$. For a nonabelian H its 1-dimensional representation is always trivial, but if we use higher dimensional nontrivial representations we will get inequivalent quantizations, as we shall see below.

According to our guiding principle, we first lift our system from Q to G, and consider a propagator in G which is a map $K_G : G \times G \times \boldsymbol{R}^+ \to \boldsymbol{C}$. The propagator we are interested in is one which is invariant under the H action (as in (1)),

$$K_G(g'h, gh; t) = K_G(g', g; t) \ , \tag{8}$$

for arbitrary g, $g' \in G$ and $h \in H$. As before, we take the standard form for the propagator on G,

$$K_G(g', g; t) = \int_g^{g'} [dg] \, \exp\left[i \int dt \left\{ \frac{1}{2} \left|\left| \frac{dg}{dt} \right|\right|^2 - V(g) \right\} \right] \ . \tag{9}$$

Since the condition (8) implies the invariance of the potential $V(gh) = V(g)$, which corresponds to the periodicity $V(x + 2\pi) = V(x)$ in (6), the potential V is actually a function of the homogeneous space $V : Q \to \boldsymbol{R}$. The norm $||\cdot||$ used in (9) is given by the invariant metric on G, that is, $||\dot{g}||^2 = \mathrm{Tr}(g^{-1}\dot{g})^2$ where 'Tr' is a matrix trace properly normalized in some irreducible representation. (The expression (9) is rather symbolic; for a concrete expression, see [5].) Now we define two unitary operators U_t and R_h acting on $\psi \in L_2(G)$ by

$$(U_t\psi)(g') = \int_G dg \, K_G(g', g; t) \, \psi(g) \ , \tag{10}$$

$$(R_h\psi)(g) = \psi(gh) \ , \tag{11}$$

for each $t > 0$ and $h \in H$, where dg in (10) is the normalized Haar measure of G. Then, the invariance (8) states that $U_t \, R_h = R_h \, U_t$, and hence there exists a conserved quantity associated with this invariance. Consequently, the Hilbert space $L_2(G)$ can be decomposed into the irreducible representations of H.

To implement the decomposition, let (V_χ, ρ_χ) be an irreducible unitary representation of H, where V_χ is a representation space labeled by χ. A function $f : G \to V_\chi$ is called χ-*equivariant* if it satisfies $f(gh) = \rho_\chi(h)^{-1} f(g)$. In other words, f is a section of the associated vector bundle $E_\chi = G \times_\rho V_\chi$. The space of χ-equivariant functions is denoted by Γ^χ, which is equipped with the inner product

$$\langle f_1, f_2 \rangle = \int_G dg \, \langle f_1(g), f_2(g) \rangle, \tag{12}$$

where in the right-hand side $\langle \cdot, \cdot \rangle$ denotes the inner product of the linear space V_χ. Consider then the operator $I^{(\chi,j)} : L_2(G) \to \Gamma^\chi$ defined by

$$(I^{(\chi,j)}\psi)^i(g) = \sqrt{d_\chi} \int_H dh \, \rho_\chi^{ij}(h) \, \psi(gh) \ . \tag{13}$$

Here $d_\chi = \dim V_\chi$, the indices $i, j = 1, \cdots, d_\chi$ run over the components of V_χ, $\rho_\chi^{ij}(h)$ is a matrix element of an unitary representation $\rho_\chi(h)$, and dh the normalized Haar measure of H. This operator $I^{(\chi,j)}$ provides a partial isometry in the sense that $I^{(\chi,j)}$ is isometric on $(\ker I^{(\chi,j)})^\perp$ (for more detail on $I^{(\chi,j)}$, see [5]). The adjoint operator $I^{(\chi,j)\dagger} : \Gamma^\chi \to L_2(G)$ is defined by the relation $\langle I^{(\chi,j)\dagger} f, \psi\rangle = \langle f, I^{(\chi,j)}\psi\rangle$, where the former bracket is the inner product of $L_2(G)$ while the latter is the one of Γ^χ. One can then show that $I^{(\chi,j)\dagger}$ picks up the j-th component of a χ-equivariant function:

$$(I^{(\chi,j)\dagger} f)(g) = \sqrt{d_\chi} f^j(g). \tag{14}$$

Next let us turn to the time evolution operator U_t. Observe first that, thanks to the invariance (8), the product $U_t^{(\chi,j)} = I^{(\chi,j)}\, U_t\, I^{(\chi,j)\dagger}$ may be used to define a unitary time evolution projected on Γ^χ. Explicitly, it is given by

$$(U_t^{(\chi,j)} f)^i(g') = \int_G dg \int_H dh \sum_{k=1}^{d_\chi} \rho_\chi(h)^{ik}\, K_G(g'h, g; t) f^k(g), \tag{15}$$

which shows that $U_t^{(\chi,j)}$ is in fact independent of j, and hence can be written simply as U_t^χ. From this expression we can deduce the projected propagator K_Q^χ acting on Γ^χ via $(U_t^\chi f)(g') = \int_G dg\, K_Q^\chi(g', g; t)\, f(g)$, that is,

$$K_Q^\chi(g', g; t) = \int_H dh\, \rho_\chi(h)\, K_G(g'h, g; t)\ . \tag{16}$$

The projected propagator K_Q^χ is a map $K_Q^\chi : G \times G \times \boldsymbol{R}^+ \to \mathrm{End}(V_\chi)$, which is an analogue of (3). Note that the summation $\sum_n (n \in \boldsymbol{Z})$ with respect to covering points is replaced by the integration $\int_H dh$ along the fiber as planned, whereas the phase factor ω_n is now replaced by the nonabelian weight $\rho_\chi(h)$. Note also that the composition law $U_{t+t'}^\chi = U_{t'}^\chi U_t^\chi$ is ensured by the homomorphism $\rho_\chi(h'h) = \rho_\chi(h')\rho_\chi(h)$ of the representation. The projected propagator K_Q^χ has the following properties,

$$K_Q^\chi(g'h, g; t) = \rho_\chi(h)^{-1}\, K_Q^\chi(g', g; t)\ , \tag{17}$$

$$K_Q^\chi(g', gh; t) = K_Q^\chi(g', g; t)\, \rho_\chi(h)\ . \tag{18}$$

Thus we see that our path integral on Q has successfully accommodated the inequivalent quantizations which are labeled by the irreducible representation χ of H.

4 Inducing the gauge field

Having found the path integral which reproduces the inequivalent quantizations obtained in algebraic approaches, we now move on to examine whether it carries the canonical connection as a gauge field in the form of the (nonabelian) minimal coupling, as we have seen in (7) for the case S^1. This requires to analyze the local structure of the propagator (17) by dividing a path in Q into small intervals, and for this we need some preparations.

Recall first that the Haar measure dg of G induces the G-invariant measure $dq = \pi_*(dg)$ on Q, whereby a function $\phi : Q \to \boldsymbol{C}$ can be integrated as

$$\int_Q dq\, \phi(q) = \int_G dg\, \phi(\pi(g)). \tag{19}$$

Let $\{D_\alpha\}$ be an open covering of $Q = \cup_\alpha D_\alpha$, $\{s_\alpha : D_\alpha \to G\}$ be a set of local sections of the fiber bundle $\pi : G \to Q$, and $\{w_\alpha : Q \to \boldsymbol{R}\}$ be a partition of unity associated with the covering $\{D_\alpha\}$. Then they give local expressions to various objects: for a χ-equivariant function f its pullback is $f_\alpha = s_\alpha^* f = f \circ s_\alpha : D_\alpha \to V_\chi$; the pullback of the projected propagator K_Q^χ is a map $K_{\alpha\beta}^\chi : D_\alpha \times D_\beta \times \boldsymbol{R}^+ \to \mathrm{End}(V_\chi)$ defined by $K_{\alpha\beta}^\chi(q', q; t) = K_Q^\chi(s_\alpha(q'), s_\beta(q); t)$; and if $q' \in D_\alpha \cap D_\gamma$ the local expressions are related by $K_{\gamma\beta}^\chi(q', q; t) = \rho_\chi(t_{\gamma\alpha}(q')) K_{\alpha\beta}^\chi(q', q; t)$ with a transition function $t_{\gamma\alpha}(q') = s_\gamma(q')^{-1} s_\alpha(q')$. In terms of these, the time evolution operator reads

$$(U_t^\chi f)_\alpha(q') = \sum_\beta \int_Q dq\, K_{\alpha\beta}^\chi(q', q; t)\, w_\beta(q) f_\beta(q). \tag{20}$$

Hence the composition law $U_{t+t'}^\chi = U_{t'}^\chi U_t^\chi$ implies

$$\begin{aligned} K_Q^\chi(g'', g; t+t') &= \int_G dg'\, K_Q^\chi(g'', g'; t')\, K_Q^\chi(g', g; t) \\ &= \sum_\alpha \int_{D_\alpha} dq'\, w_\alpha(q')\, K_Q^\chi(g'', s_\alpha(q'); t')\, K_Q^\chi(s_\alpha(q'), q; t). \end{aligned} \tag{21}$$

Inserting intermediate points repeatedly, we obtain

$$\begin{aligned} K_Q^\chi(g_n, g_0; t) = & \sum_{\alpha_1, \cdots, \alpha_{n-1}} \int_{D_{\alpha_{n-1}}} dq_{n-1} \cdots \int_{D_{\alpha_1}} dq_1\, w_{\alpha_{n-1}}(q_{n-1}) \cdots w_{\alpha_1}(q_1) \\ & \times K_Q^\chi(g_n, s_{\alpha_{n-1}}(q_{n-1}); \epsilon)\, K_Q^\chi(s_{\alpha_{n-1}}(q_{n-1}), s_{\alpha_{n-2}}(q_{n-2}); \epsilon) \cdots \\ & \times K_Q^\chi(s_{\alpha_2}(q_2), s_{\alpha_1}(q_1); \epsilon)\, K_Q^\chi(s_{\alpha_1}(q_1), g_0; \epsilon), \end{aligned} \tag{22}$$

where $\epsilon = t/n$. When two points $q_k = q(\tau)$ and $q_{k+1} = q(\tau + \epsilon)$ are close enough to be contained in a single patch D_α, one of the factorized propagator becomes

$$K_{\alpha\alpha}^\chi(q(\tau+\epsilon), q(\tau); \epsilon) = \int_H dh\, \rho_\chi(h(\epsilon))\, K_G(s_\alpha(q(\tau+\epsilon))h(\epsilon), s_\alpha(q(\tau)); \epsilon), \tag{23}$$

where we extend $h \in H$ to be a smooth function $h : (-\epsilon, \epsilon) \to H$ such that $h(0) = e$ (e is the identity element of H) and $h(\epsilon) = h$. Then eq.(9) tells that for a short time interval ϵ,

$$
\begin{aligned}
&K_G(s_\alpha(q(\tau+\epsilon))h(\epsilon), s_\alpha(q(\tau)); \epsilon) \\
&\qquad \approx \exp\left[i\epsilon\left\{ \frac{1}{2}\left|\left| \frac{d}{d\epsilon} s_\alpha(q(\tau+\epsilon))h(\epsilon)\Big|_{\epsilon=0} \right|\right|^2 - V(q(\tau)) \right\} \right], \qquad (24)
\end{aligned}
$$

with

$$
\begin{aligned}
&\left|\left| \frac{d}{d\epsilon} s_\alpha(q(\tau+\epsilon))h(\epsilon)\Big|_{\epsilon=0} \right|\right|^2 \\
= \; & \mathrm{Tr}\left[h(\epsilon)^{-1} s_\alpha(q(\tau))^{-1} \frac{ds_\alpha(q(\tau))}{d\tau} h(\epsilon)\Big|_{\epsilon=0} + h(\epsilon)^{-1}\frac{dh(\epsilon)}{d\epsilon}\Big|_{\epsilon=0} \right]^2 \\
= \; & \mathrm{Tr}\left[s_\alpha(q(\tau))^{-1} \frac{ds_\alpha(q(\tau))}{d\tau} + \frac{dh(\epsilon)}{d\epsilon} h(\epsilon)^{-1}\Big|_{\epsilon=0} \right]^2 \\
= \; & \mathrm{Tr}\left[P_{\mathcal{H}}\Big(s_\alpha(q(\tau))^{-1} \frac{ds_\alpha(q(\tau))}{d\tau} \Big) + \frac{dh(\epsilon)}{d\epsilon} h(\epsilon)^{-1}\Big|_{\epsilon=0} \right]^2 \\
& + \mathrm{Tr}\left[P_{\mathcal{H}}^{\perp}\Big(s_\alpha(q(\tau))^{-1} \frac{ds_\alpha(q(\tau))}{d\tau} \Big) \right]^2, \qquad (25)
\end{aligned}
$$

where $P_{\mathcal{H}}$ is a projector from the Lie algebra of G onto the Lie algebra of H, and $P_{\mathcal{H}}^{\perp}$ denotes its orthogonal complement.

Now, if we take the interval ϵ small enough, then the contribution from the stationary point of (25) with respect to the variation of h will dominate in the integration $\int_H dh$ in (23). Thus in the limit $\epsilon \to 0$ the integration may be replaced by the value at the stationary point

$$
\frac{dh(\epsilon)}{d\epsilon} h(\epsilon)^{-1}\Big|_{\epsilon=0} = -P_{\mathcal{H}}\Big(s_\alpha(q(\tau))^{-1} \frac{ds_\alpha(q(\tau))}{d\tau} \Big). \qquad (26)
$$

This result may be interpreted that for a small change of the parameter $q(\tau)$ in the base manifold Q, the lifted point in the fiber space G moves along the shortest path, *i.e.*, it acquires the smallest change. Now we notice that the right-hand side of (26) is nothing but (the pullback of) the canonical connection A, which is just the Maurer-Cartan 1-form $g^{-1}dg$ projected down to the subalgebra $\mathcal{H}$,

$$
A = P_{\mathcal{H}}(g^{-1}dg). \qquad (27)
$$

It is worth mentioning that this connection is invariant under the G-action over the homogeneous space Q and provides various topological solutions of the Yang-Mills equation, for instance, the Dirac monopole and the BPST instanton on $Q = S^2$ and S^4, respectively (see, for example [23, 7]).

Writing the pullback of A by the section s_α as $A_\alpha = P_{\mathcal{H}}(s_\alpha^{-1} ds_\alpha)$, we can write the solution of (26) as $h_{\alpha\alpha}[q_{k+1}] = h(\epsilon) = \mathcal{P}\ \exp[-\int_{q_k}^{q_{k+1}} A_\alpha]$, where the symbol $\mathcal{P}$ denotes the path-ordering. Using this and gathering scattered pieces, we finally obtain the propagator (22) in the desired form,

$$K^{\chi}_{\alpha_n\alpha_0}(q_n, q_0; t) = \int_{q_0}^{q_n} [dq]\, \rho_\chi(h_{\alpha_n\alpha_0}[q]) \exp\left[i \int dt \left\{ \frac{1}{2} \left\| \frac{dq}{dt} \right\|^2 - V(q) \right\} \right], \qquad (28)$$

where $h_{\alpha_n\alpha_0}$ is a nonabelian weight factor

$$\begin{aligned} h_{\alpha_n\alpha_0}[q] &= t_{\alpha_n\alpha_{n-1}}(q_n)\, \mathcal{P} \exp[-\int_{q_{n-1}}^{q_n} A_{\alpha_{n-1}}] \\ &\times t_{\alpha_{n-1}\alpha_{n-2}}(q_{n-1})\, \mathcal{P} \exp[-\int_{q_{n-2}}^{q_{n-1}} A_{\alpha_{n-2}}] \cdots t_{\alpha_1\alpha_0}(q_1)\, \mathcal{P} \exp[-\int_{q_0}^{q_1} A_{\alpha_0}] \end{aligned} \qquad (29)$$

with $q_k \in D_{k-1} \cap D_k$ $(k = 1, \cdots, n)$ being intermediate points. The factor $h_{\alpha_n\alpha_0}[q]$ is actually a holonomy associated with the path $q : [0, t] \to Q$, and the above expression (29) shows that the gauge field interacts minimally in the nonabelian sense [7]. We therefore reached the path integral (28) with (29) which precisely reproduces the result found earlier in algebraic approaches [4, 5, 6].

5 Concluding remarks

We considered the path integral on a homogeneous space $Q = G/H$, and showed that the propagator on Q can be reduced from the one on G by integration of redundant degrees of freedom in the fiber direction of H, with a non-trivial weight factor $\rho_\chi(h)$ multiplied. Being a unitary representation of H, the factor $\rho_\chi(h)$ preserves the composition law of the propagator, and a different ρ_χ leads to a different (inequivalent) quantization. The composition law then allows for a decomposition of the propagator into small intervals, and integration over intermediate points eventually results in the path integral expression. Examination of the propagator at short distance reveals that a gauge field is induced in the path integral in the form of the canonical connection. Thus we have shown that our guiding principle — 'first lift and then project' — yields the inequivalent quantizations and the induced gauge field correctly. The basic tool used here is essentially the one used in [5], where the same path integral expression has been derived from the self-adjoint Hamiltonian through the Trotter formula. In this paper we put an emphasis on the role of the geometry and adopted the guiding principle in order to reach the path integral expression, rather than defining the quantum theory algebraically first.

Several questions are still left open. One obvious question is how we construct a quantum theory on inhomogeneous spaces. Inhomogeneous spaces often arise in physics, with the one most frequently discussed being a Riemann surface with higher genus. Actually our formulation is not restricted to homogeneous spaces. A more general situation which allows our principle to be employed is the following[§]. Let P be a Riemannian manifold with a metric $\tilde{g}$ and let a Lie group H act on P freely and isometrically. Then the manifold $M = P/H$ admits an induced metric g, with which the projection $\pi : P \to M$ defines a principal bundle and a Riemannian submersion. Assume that a propagator in P is H-invariant, $K_P(p'h, ph; t) = K_P(p', p; t)$. Then our formulation of the path integral can be applied straightforwardly. Indeed, the propagator on M can be defined by

$$K_M^\chi(p', p; t) = \int_H dh\, \rho_\chi(h) K_P(p'h, p; t), \tag{30}$$

which acts on a χ-equivariant function $f : P \to V_\chi$; $f(ph) = \rho_\chi(h)^{-1} f(p)$. When the base space (M, g) is fixed, inequivalent quantizations are classified by choice of the principal bundle (P, M, π, H), the lifted metric $(P, \tilde{g})$ and the representation (H, V_χ, ρ_χ). However, this scheme may be too general; we do not have any criterion to choose a specific quantization. In fact, in this scheme the choice of $(P, M, \pi, H, \tilde{g})$ is equivalent to introduction of an arbitrary gauge field by hand and, as a result, we have no longer a natural explanation of inducing gauge fields.

In contrast, there exists such a criterion when the base space M is a homogeneous space $Q = G/H$. In fact, the invariance under the G-action determines both g and $\tilde{g}$ uniquely, and hence the induced gauge field, too. The only remaining arbitrariness is the choice of the representation ρ_χ, and accordingly there are (infinitely) many inequivalent quantizations. We may therefore conclude that the existence of inequivalent quantizations is the norm when quantizing on a general Riemannian manifold (M, g). If M admits a transitive action of some isometry group G, then the request of invariance will severely restrict possible quantizations. If M does not admit such an action, even a self-adjoint momentum operator cannot be defined globally as a generator of the transitive action, and hence in that case we are forced to give up the concept of momentum.

This last point may be important in realizing the significance of our guiding principle. Indeed, given a homogeneous space $Q = G/H$ there appears no compelling reason, at a glance, to lift it to G and consider quantization there. But this way we can guarantee that there exists a self-adjoint Hamiltonian given by the quadratic Casimir, which in turn ensures unitary time evolution of the system. The existence of such a Hamiltonian is by no means guaranteed for a system whose configuration space is nontrivial. Unfortunately, this is not derived on the sole ground of geometry, and finding such a

[§]Such a situation has already been considered by Montgomery [17] in investigating geometric properties of induced gauge fields of deformable bodies.

derivation will be crucial in developing a path integral based on a purely geometric and intuitive principle.

As a final remark, we add that we have begun a preliminary investigation in two dimensions into the meaning of nontrivial topology in field theories which admit inequivalent quantizations (see, for example [24]). We have however left untouched the higher dimensional cases, not to mention the path integral approach.

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