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N=2 Supersymmetric QCD and Four Manifolds;

(I) the Donaldson and the Seiberg-Witten Invariants

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We study the path integral of a twisted $N = 2$ supersymmetric Yang-Mills theory coupled with hypermultiplet having the bare mass. We explicitly compute the topological correlation functions for the $SU(2)$ theory on a compact oriented simply connected simple type Riemann manifold with $b_2^+ \geq 3$. As the corollaries, we determine the topological correlation functions of the theory without the bare mass and those of the theory without coupling to the hypermultiplet. This includes a concrete field theoretic proof of the relation between the Donaldson and the Seiberg-Witten invariants.

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1. Introduction

Both the Donaldson and the Seiberg-Witten invariants of smooth four-manifolds are closely related to the $N = 2$ supersymmetric Yang-Mills (SYM) theory [1].

The cohomological description of the Donaldson theory [2] (for review, see [3]) is represented by the twisted version of $N = 2$ SYM theory [4], called the topological Yang-Mills (TYM) theory (for reviews and references, see [5][6]). The semi-classical analysis of the TYM theory, based on the ultraviolet weak coupling limit of the underlying physical theory, was used to reformulate the Donaldson theory in a concrete way. Those cohomological descriptions, however, turned out to be surprisingly difficult to obtain explicit results .

On the other hand, the physical interpretation of Witten opened the door to an entirely different formulation of the Donaldson theory. This is due to the asymptotic freedom of the underlying physical theory. In the infrared or the large scaling limit, the physical theory is strongly coupled and the semi-classical description is not valid. Since the TYM theory is metric independent, the Donaldson theory can be reformulated in terms of the new degrees of freedom that may appear in the strong coupling vacua. In a seminal paper [7], Seiberg and Witten determined the exact infrared behavior of the $N = 2$ SYM theory. The celebrated Seiberg-Witten invariant originates from the resulting low-energy effective theory [1], which is a simple and powerful new tool for the study of differential-topology of four-manifold [8][9].

For an oriented simply connected compact Riemann four-manifold X of simple type [10] with $b_2^+ \geq 3$, Witten conjectured a precise formula relating the $SU(2)$ Donaldson invariants with the Seiberg-Witten invariants[1]

$$\begin{aligned}\langle \exp(\hat{v} + \tau\hat{u})\rangle =& 2^{1+\frac{1}{4}(7\chi+11\sigma)}\biggl(\exp\left(\frac{v\cdot v}{2} + 2\tau\right)\sum_x n_x e^{v\cdot x} \\ &+ i^{(\chi+\sigma)/4}\exp\left(-\frac{v\cdot v}{2} - 2\tau\right)\sum_x n_x e^{-iv\cdot x}\biggr),\end{aligned} \tag{1.1}$$

where x is the Seiberg-Witten basic class, n_x is the algebraic sum of the number of the solutions of the Seiberg-Witten equation and $v \in H_2(X;\mathbb{Z})$. The above formula agrees with the structure formula of Kronheimer-Mrowka for the simple type manifold as well as with the results of the paper [11] where the Donaldson invariants on Kähler surface was determined almost completely using the known vacuum structure of $N = 1$ SYM theory. Some progress in a mathematical proof has been announced by Pidstrigach and Tyurin [12]. More recently, Witten explained the appearance of the $spin^c$ in the low energy effective theory [13].

In our previous paper [14], we showed that the $N = 2$ SYM theory coupled with hypermultiplet (SQCD) can be twisted after picking a *spin*^{c} structure $\mathfrak{c}$ to define a global supersymmetric theory called topological QCD (TQCD) on an arbitrary oriented Riemann four-manifold X. TQCD is a generalization of the TYM theory. TQCD shares many of the properties of the TYM theory as a cohomological field theory in which a suitable path integral defines differential-topological invariants of smooth four-manifold. The topological amplitudes of TQCD are the intersection pairings, analogous to the Donaldson invariants, in the moduli space $\mathcal{M}(k, \mathfrak{c})$ of the non-abelian version of the Seiberg-Witten monopoles.

We note that the twisting of the general $N = 2$ hypermultiplets including gravity was previously studied by Anselmi and Fré using the σ-model interpretation [15]. On $K3$ surface, their theory is equivalent to TQCD with the choice of trivial *spin*^{c} strucuture. Based on the Mathai-Quillen formalism, Labastida and Mariño constructed a topological theory on spin manifold [16] which is equivalent to TQCD with the choice of the trivial *spin*^{c} structure.

At first sight, TQCD looks like just a clone of the TYM theory after replacing the moduli space $\mathcal{M}(k)$ of anti-self-dual (ASD) connections with the moduli space $\mathcal{M}(k, \mathfrak{c})$. The cohomological interpretation of the invariants is also based on the weak coupling limit of the underlying physical theory. The theory also shares the notorious difficulty for explicit computations with the TYM theory. Furthermore, those invariants defined by TQCD appear not to contain any information beyond the Donaldson and the Seiberg-Witten invariants.

However, TQCD has an interesting property. By introducing the $N = 2$ supersymmetric bare mass term to the hypermultiplet, we will show that the resulting TQCD (massive TQCD) interpolates the Donaldson and the Seiberg-Witten theories. Our picture rather contrasts with the approach of Seiberg and Witten in that the genuine quantum scaling behavior of the underlying $N = 2$ SYM theory interpolates the above two limits [7]. On the other hand, we consider a weak coupling limit of a different asymptotically free theory after turning on the bare mass for the additional matter-multiplet. Our computation does not use any known structures of the vacua or the electro-magnetic duality of the underlying physical $N = 2$ supersymmetric theories.

In this paper, we study the $SU(2)$ theory with the hypermultiplet carrying the fundamental representation. We determine the topological correlation functions of massive TQCD on a simple type manifold. As the corollaries, we derive the topological correlation functions of the theory without the bare mass as well as those of the theory without hypermultiplet (TYM theory). The last result is a simple and perfectly concrete path integral proof of the formula (1.1).

Introducing the bare mass term to the hypermultiplet, we show that the path integral is localized to two different branches; in branch (i) the dominant contribution comes from the moduli space of ASD connections, while in branch (ii) the dominant contribution comes from the moduli space of the (abelian) Seiberg-Witten monopoles. To put it differently, branch (i) is governed by the TYM theory and branch (ii) is governed by a topological QED coupled with massless hypermultiplet. We show the above property by using Witten's fixed point theorem for the global supersymmetry [17] as well as by the semi-classical analysis combined with an additional stationary phase method after adding a BRST trivial deformation related to the bare mass term. It turns out that a key simplication occurs in the large scale limit of the metric. Our computation is rather elementary using the simple Gaussian integrals. In the subsequent paper, we will give the the explicit computations of topological invariants defined by $SU(N_c)$ TYM and TQCD with $N_f = 1$ hypermultiplets carrying the fundamental representation and clarify their relations with the underlying physical theories [18].

This paper is organized as follows. In section 2, we briefly review TQCD without the bare mass [14]. The purpose of this section is to establish our notations and to make this paper reasonably self-contained. We add some important remarks as well. In section 3, we study the twisted theory after introducing the bare mass term to the hypermultiplet. Adopting various arguments we show that the path integral of the resulting theory is localized to two types of the branches. Then, we show that the key step is to take the large scale limit of the Riemann metric. In section 4, the path integrals of TQCD having the bare mass is computed in the large scale limit. As the corollaries, we prove the formula (1.1) and obtain the precise formula for the invariants defined by TQCD with an arbitrary $spin^c$ structure. In section 5, we briefly discuss some relations with the physical theory.

The appendix is devoted to a demonstration of the path integral method which is a slightly simpler version of the technique we used in the actual computations.

2. The Topological QCD

In this section, we briefly review the $N = 2$ global (rigid) supersymmetric Yang-Mills theory coupled with hypermultiplet (TQCD) on the general oriented compact Riemann 4-manifolds [14].

2.1. Twisting

In the flat 4-manifold $\mathbb{R}^4$, the $N=2$ rigid supersymmetric theories are well-defined. The theory contains an $N=2$ vector multiplet in the adjoint representation. In addition, one can couple $N=2$ matter multiplets known as the hypermultiplets carrying a representation R and its conjugate representation $\tilde{R}$ of the gauge group. Those multiplets contain various spinor fields which are well defined.

Now we consider a compact oriented simply connected Riemann four manifold X. We want to define the supersymmetric theory on X without destroying the global (rigid) supersymmetry or, equivalently, without introducing the dynamical gravity. Then there are two obstructions;

(a) The non-existence of the spinor fields on the manifold X with the non-vanishing second Stiefel-Whitney $w_2(X)\neq 0$ class.

(b) The non-existence of the nowhere vanishing constant supersymmetry charge.

The obstruction (b) is generic since the supercharges transform as the spinor and the rigid supersymmetry requires the existence of nowhere vanishing constant spinor. Thus, the rigid supersymmetric theory can be defined only on the parallelizable (topologically trivial) manifolds. On a topologically non-trivial manifold, the only quantity that can be defined as a nowhere vanishing constant is the one which transforms as the scalar. The twisting procedure introduced by Witten is the recipe to make the supercharge transform as the scalar [4][11].

For the $N=2$ vector multiplet, the twisting resolves obstructions (a) and (b) simultaneously. After twisting, there are no fields which transform as the spinor. And, the supercharges have components which transform as the vector, the self-dual tensor and the scalar. If we take the component which transforms as the scalar and use it as the new global supercharge, the obstruction (b) is removed as well. One important remark is that the resulting action functional is independent of the spin structure on X and the Riemann curvature of the manifold.

Twisting of the hypermultiplet is slightly more subtle. Since the twisted supercharge transforms as a scalar, the obstruction (b) does not exist. After twisting, every component field of hypermultiplet transforms as spinor. If the manifold has spin structure, i.e., $w_2(X)=0$, the resulting theory will be well-defined. Unlike the theory without hypermultiplet, the twisted theory depends on the spin structure.[1] If the manifold X does not

[1] A simply connected spin manifold has unique spin structure.

admit a spin structure the twisted theory may not exist. A practical way of resolving the obstruction (a) is to regard the twisted hypermultiplet as *spin*^{c} spinor which exists in any oriented Riemann 4-manifold. Roughly speaking, the *spin*^{c} spinor transforms as the spinor with certain background $U(1)$ charge whose connection couples with the Dirac operator. The additional $U(1)$ connection compensates the obstruction to define the spinor. A *spin*^{c} structure $\mathfrak{c} \in H^2(X;\mathbb{Z})$ is an integral lift of the Stifel-Whitney class $w_2(X) \in H^2(X;\mathbb{Z}/2)$, i.e. $\mathfrak{c} \equiv w_2(X)$ mod 2 [19]. The price we should pay is that the twisted theory depends on a particular *spin*^{c} structure we choose. The space $H^2(X;\mathbb{R})$ of harmonic two-forms on X is an b_2-dimensional flat space with signature $(b_2^+ - b_2^-)$. The space $H^2(X;\mathbb{Z})$ is the integral lattice in $H^2(X;\mathbb{R})$. Then, the set $H^2_s(X;\mathbb{Z})$ of all *spin*^{c} structure is an affine sublattice of $H^2(X;\mathbb{Z})$. Obviously, there are b_2 independent generators $\mathfrak{s}$ of the transitive action on $H^2_s(X;\mathbb{Z})$. Thus we have a family of the TQCD parametrized by the space $H^2_s(X;\mathbb{Z})$ of the *spin*^{c} structures on X. It would be worthy to remark that even in a spin manifold one can consider the theory defined by an arbitrary *spin*^{c} structure. In our viewpoint, the only difference between the spin and non-spin manifold is that the spin manifold has the *spin*^{c} bundle with the trivial determinant line bundle (trivial background $U(1)$ connections).

Let P be a principal G-bundle over a simply connected compact oriented Riemann manifold X. Pick a representation R of G such that $c_2(adj) - T(R) \geq 0$ and consider the associated vector bundle E whose fiber is the vector space V, $R: G \to V$. We pick a *spin*^{c} structure $\mathfrak{c}$ on X and consider the associated *spin*^{c} bundle $W^{\pm}_{\mathfrak{c}}$.[2] Let $\mathcal{A}$ be the space of all connections on P and $\Gamma(W^+_{\mathfrak{c}} \otimes E)$ the space of the sections of the *spin*^{c} bundle twisted by the vector bundle E. After twisting, the complex boson (squark) in the hypermultiplet become a section of $W^+_{\mathfrak{c}} \otimes E$;

$$q \in \Gamma(W^+_{\mathfrak{c}} \otimes E), \qquad q^\dagger \in \Gamma(\overline{W}^+_{\mathfrak{c}} \otimes \tilde{E}), \tag{2.1}$$

where $\tilde{E}$ denotes the vector bundle conjugate to E. The *spin*^{c} Dirac operator

$$\sigma^\mu D_\mu : \Gamma(W^+_{\mathfrak{c}} \otimes E) \to \Gamma(W^-_{\mathfrak{c}} \otimes E), \tag{2.2}$$

is the Dirac operator for the *spin*^{c} bundle twisted by E. We will sometimes denote $\sigma^\mu D_\mu$ by $\not{D}$ or by $\not{D}^E_{\mathfrak{c}}$. One effect of twisting the hypermultiplet is that the Dirac operator is

[2] We will occasionally confuse with the determinant line bundle $det(W^{\pm}_{\mathfrak{c}}) = L_{\mathfrak{c}}$ and its first Chern class $c_1(det(W^{\pm}_{\mathfrak{c}}))$. Both will be denoted by $\mathfrak{c}$. To avoid possible confusion we will always denote the intersection number explicitly using '$\cdot$'. For example, $\mathfrak{c}\cdot\mathfrak{c} = L_{\mathfrak{c}} \cdot L_{\mathfrak{c}} = c_1(L_{\mathfrak{c}}) \cdot c_1(L_{\mathfrak{c}}) = \int_X c_1(L_{\mathfrak{c}}) \wedge c_1(L_{\mathfrak{c}})$. The expression $\mathfrak{c}^2$ can mean $L^2_{\mathfrak{c}}$ or $2\mathfrak{c}$. Note also that $\mathfrak{c} + 2\zeta \equiv L_{\mathfrak{c}} \otimes L^2_{\zeta}$.

coupled with the background $U(1)$ connection of the determinant line bundle $det(W_{\mathfrak{c}}^{+})$. This can be summarized by the Weitzenböck formula;

$$(\not{D})^2 = -g^{\mu\nu}D_\mu D_\nu - F_A^+ - p^+ + \frac{1}{4}R, \tag{2.3}$$

where F_A^+ is the self-dual part of the gauge field strength, p^+ denotes the self-dual part of the curvature 2-form on $det(W_{\mathfrak{c}}^{+})$ and R denotes the scalar curvature of the metric.

Throughout this paper, we restrict our attention to the case that the gauge group is $SU(2)$ and the theory is coupled with one hypermultiplet carrying the fundamental representation (the 2-dimensional representation).

2.2. The Action Functional

The topological action of the twisted $N = 2$ super-Yang-Mills theory coupled with the hypermultiplet is given by

$$S = -i\{Q, V_T\}, \tag{2.4}$$

where

$$V_T = \frac{1}{h^2}\int d^4x\sqrt{g}\Big[\chi_a^{\mu\nu}\left(H_{\mu\nu}^a - i(F_{\mu\nu}^{+a} + q^\dagger\sigma_{\mu\nu}T^a q)\right) - \frac{1}{2}g^{\mu\nu}(D_\mu\bar{\phi})_a\lambda_\nu^a + \frac{1}{8}[\phi,\bar{\phi}]_a\eta^a \\ + \left(X_{\tilde{q}}^{\alpha}\psi_{q\alpha} + \psi_{\tilde{q}}^{\alpha}X_{q\alpha}\right) + i\left(q_{\dot{\alpha}}^\dagger\bar{\phi}_a T^a\bar{\psi}_{\tilde{q}}^{\dot{\alpha}} + \bar{\psi}_{q\dot{\alpha}}\bar{\phi}_a T^a q^{\dot{\alpha}}\right)\Big]. \tag{2.5}$$

The supersymmetry transformation laws for the fields in the adjoint representation are

$$\begin{aligned}
&\hat{\delta}A_\mu = i\varrho\lambda_\mu, && && \\
&\hat{\delta}\lambda_\mu = -\varrho D_\mu\phi, && \hat{\delta}\chi_{\mu\nu} = \varrho H_{\mu\nu}, && \hat{\delta}H_{\mu\nu} = i\varrho[\phi,\chi_{\mu\nu}], \\
&\hat{\delta}\phi = 0, && \hat{\delta}\bar{\phi} = i\varrho\eta, && \hat{\delta}\eta = \varrho[\phi,\bar{\phi}],
\end{aligned} \tag{2.6}$$

where $\hat{\delta}(field) = -i\varrho\{Q, field\}$. The fields carrying the representation R and $\tilde{R}$ transform as

$$\begin{aligned}
\hat{\delta}q^{\dot{\alpha}} &= -\,\varrho\bar{\psi}_{\tilde{q}}^{\dot{\alpha}}, & \hat{\delta}\bar{\psi}_{\tilde{q}}^{\dot{\alpha}} &= -\,i\varrho\phi^a T_a q^{\dot{\alpha}}, \\
\hat{\delta}q_{\dot{\alpha}}^\dagger &= -\,\varrho\bar{\psi}_{q\dot{\alpha}}, & \hat{\delta}\bar{\psi}_{q\dot{\alpha}} &= i\varrho q_{\dot{\alpha}}^\dagger\phi^a T_a,
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
\hat{\delta}\psi_{q\alpha} &= -i\varrho\sigma^\mu{}_{\alpha\dot{\alpha}}D_\mu q^{\dot{\alpha}} + \varrho X_{q\alpha}, \\
\hat{\delta}X_{q\alpha} &= i\varrho\phi^a T_a\psi_{q\alpha} - i\varrho\sigma^\mu{}_{\alpha\dot{\alpha}}D_\mu\bar{\psi}_{\tilde{q}}^{\dot{\alpha}} + \varrho\sigma^\mu{}_{\alpha\dot{\alpha}}\lambda_\mu^a T_a q^{\dot{\alpha}}, \\
\hat{\delta}\psi_{\tilde{q}}^{\alpha} &= i\varrho D_\mu q_{\dot{\alpha}}^\dagger\bar{\sigma}^{\mu\dot{\alpha}\alpha} - \varrho X_{\tilde{q}}^{\alpha}, \\
\hat{\delta}X_{\tilde{q}}^{\alpha} &= i\varrho\psi_{\tilde{q}}^{\alpha}\phi^a T_a - i\varrho D_\mu\bar{\psi}_{q\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha} + \varrho q_{\dot{\alpha}}^\dagger\bar{\sigma}^{\mu\dot{\alpha}\alpha}\lambda_\mu^a T_a.
\end{aligned} \tag{2.8}$$

The above transformation laws satisfy

$$(\hat{\delta}_{\varrho}\hat{\delta}_{\varrho'} - \hat{\delta}_{\varrho'}\hat{\delta}_{\varrho})(field) = T_{\varepsilon}(field), \tag{2.9}$$

where $T_{\varepsilon}(field)$ denotes the variation of a field under a gauge transformation generated by an infinitesimal parameter $\varepsilon = -2i\varrho\varrho' \cdot \phi$. We introduce an additive quantum U-number called the ghost number. The global supercharge Q carries $U = 1$ and V_T is designed to have $U = -1$ such that the action has the zero-ghost number. So the theory has the U-number symmetry at the classical level. The U-number for the various fields are given by

$$\begin{array}{ccccccccccccccc} A_\mu & \lambda_\mu & \phi & \bar{\phi} & \eta & \chi_{\mu\nu} & H_{\mu\nu} & q^{\dot{\alpha}} & q^{\dagger}_{\alpha} & \bar{\psi}^{\dot{\alpha}}_{\tilde{q}} & \bar{\psi}_{q\dot{\alpha}} & \psi_{q\alpha} & \psi^{\alpha}_{\tilde{q}} & X_{q\alpha} & X^{\alpha}_{\tilde{q}} \\ 0 & 1 & 2 & -2 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 \end{array} . \tag{2.10}$$

The topological action is given by[3]

$$\begin{aligned} S = \frac{1}{h^2}\int d^4x\sqrt{g}\Big[&\left(H^{\mu\nu}_a - \frac{i}{2}(F^{+\mu\nu}_a + q^\dagger\bar{\sigma}^{\mu\nu}T_a q)\right)\left(H^a_{\mu\nu} - \frac{i}{2}(F^{+a}_{\mu\nu} + q^\dagger\bar{\sigma}_{\mu\nu}T^a q)\right) \\ &+ \frac{1}{4}\left(F^{+\mu\nu}_a + q^\dagger\bar{\sigma}^{\mu\nu}T_a q\right)\left(F^{+a}_{\mu\nu} + q^\dagger\bar{\sigma}_{\mu\nu}T^a q\right) - 2X^{\alpha}_{\tilde{q}}X_{q\alpha} \\ &+ iX^{\alpha}_{\tilde{q}}\sigma^{\mu}{}_{\alpha\dot{\alpha}}D_\mu q^{\dot{\alpha}} + iD_\mu q^\dagger_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}X_{q\alpha} - \frac{1}{2}g^{\mu\nu}(D_\mu\bar{\phi})_a(D_\nu\phi)^a + \frac{1}{8}[\phi,\bar{\phi}]_a[\phi,\bar{\phi}]^a \\ &- i\chi^{\mu\nu}_a[\phi,\chi_{\mu\nu}]^a + \chi^{\mu\nu}_a(d_A\lambda)^{+a}_{\mu\nu} + \frac{i}{2}g^{\mu\nu}(D_\mu\eta)_a\lambda^a_\nu - \frac{i}{2}g^{\mu\nu}[\lambda_\mu,\bar{\phi}]_a\lambda^a_\nu + \frac{i}{8}[\phi,\eta]_a\eta^a \\ &- i\chi^{\mu\nu}_a\bar{\psi}_q\bar{\sigma}_{\mu\nu}T^a q + i\chi^{\mu\nu}_a q^\dagger\bar{\sigma}_{\mu\nu}T^a\bar{\psi}_{\tilde{q}} - iD_\mu\bar{\psi}_{q\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\psi_{q\alpha} - i\psi^{\alpha}_{\tilde{q}}\sigma^{\mu}{}_{\alpha\dot{\alpha}}D_\mu\bar{\psi}^{\dot{\alpha}}_{\tilde{q}} \\ &+ 2i\psi^{\alpha}_{\tilde{q}}\phi_a T^a\psi_{q\alpha} - 2i\bar{\psi}_{q\dot{\alpha}}\bar{\phi}_a T^a\bar{\psi}^{\dot{\alpha}}_{\tilde{q}} - q^\dagger_{\dot{\alpha}}\lambda_{\mu a}T^a\bar{\sigma}^{\mu\dot{\alpha}\alpha}\psi_{q\alpha} - \psi^{\alpha}_{\tilde{q}}\sigma^{\mu}{}_{\alpha\dot{\alpha}}\lambda_{\mu a}T^a q^{\dot{\alpha}}. \\ &+ q^\dagger_{\dot{\alpha}}\eta_a T^a\bar{\psi}^{\dot{\alpha}}_{\tilde{q}} - \bar{\psi}_{q\dot{\alpha}}\eta_a T^a q^{\dot{\alpha}} - q^\dagger_{\dot{\alpha}}T^a T^b\left(\phi_a\bar{\phi}_b + \phi_b\bar{\phi}_a\right)q^{\dot{\alpha}}\Big]. \end{aligned} \tag{2.11}$$

[3] We replaced the complex scalar fields B and $\bar{B}$ of the physical theory with $\frac{i}{2\sqrt{2}}\phi$ and $i\sqrt{2}\bar{\phi}$. This does not necessarily mean that ϕ and $-\bar{\phi}$ should be complex conjugate after twisting.

After integrating out the auxiliary fields $H_{\mu\nu}$, $X^{\alpha}_{\tilde{q}}$ and $X_{q\alpha}$, we have

$$
\begin{aligned}
S = \frac{1}{h^2}\int d^4x\sqrt{g}\Big[&\frac{1}{4}F^{+\mu\nu}_{a}F^{+a}_{\mu\nu} - \frac{1}{2}p^{+}_{\mu\nu}q^{\dagger}\bar{\sigma}^{\mu\nu}q + \frac{1}{2}g^{\mu\nu}D_{\mu}q^{\dagger}_{\dot{\alpha}}D_{\nu}q^{\dot{\alpha}} \\
&+ \frac{1}{4}(q^{\dagger}\bar{\sigma}^{\mu\nu}T_a q)(q^{\dagger}\bar{\sigma}_{\mu\nu}T^a q) + \frac{1}{8}R(q^{\dagger}_{\dot{\alpha}}q^{\dot{\alpha}}) - \frac{1}{2}g^{\mu\nu}(D_{\mu}\bar{\phi})_a(D_{\nu}\phi)^a + \frac{1}{8}[\phi,\bar{\phi}]_a[\phi,\bar{\phi}]^a \\
&- i\chi^{\mu\nu}_{a}[\phi,\chi_{\mu\nu}]^a + \chi^{\mu\nu}_{a}(d_A\lambda)^{+a}_{\mu\nu} + \frac{i}{2}g^{\mu\nu}(D_{\mu}\eta)_a\lambda^a_{\nu} - \frac{i}{2}g^{\mu\nu}[\lambda_{\mu},\bar{\phi}]_a\lambda^a_{\nu} + \frac{i}{8}[\phi,\eta]_a\eta^a \\
&- i\chi^{\mu\nu}_{a}\bar{\psi}_{q}\bar{\sigma}_{\mu\nu}T^a q + i\chi^{\mu\nu}_{a}q^{\dagger}\bar{\sigma}_{\mu\nu}T^a\bar{\psi}_{\tilde{q}} - iD_{\mu}\bar{\psi}_{q\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\psi_{q\alpha} - i\psi^{\alpha}_{\tilde{q}}\sigma^{\mu}{}_{\alpha\dot{\alpha}}D_{\mu}\bar{\psi}^{\dot{\alpha}}_{\tilde{q}} \\
&+ 2i\psi^{\alpha}_{\tilde{q}}\phi_a T^a\psi_{q\alpha} - 2i\bar{\psi}_{q\dot{\alpha}}\bar{\phi}_a T^a\bar{\psi}^{\dot{\alpha}}_{\tilde{q}} + q^{\dagger}_{\dot{\alpha}}\lambda_{\mu a}T^a\bar{\sigma}^{\mu\dot{\alpha}\alpha}\psi_{q\alpha} + \psi^{\alpha}_{\tilde{q}}\sigma^{\mu}{}_{\alpha\dot{\alpha}}\lambda_{\mu a}T^a q^{\dot{\alpha}}. \\
&+ q^{\dagger}_{\dot{\alpha}}\eta_a T^a\bar{\psi}^{\dot{\alpha}}_{\tilde{q}} - \bar{\psi}_{q\dot{\alpha}}\eta_a T^a q^{\dot{\alpha}} - q^{\dagger}_{\dot{\alpha}}T^a T^b\left(\phi_a\bar{\phi}_b + \phi_b\bar{\phi}_a\right)q^{\dot{\alpha}}\Big].
\end{aligned}
\tag{2.12}
$$

The above action is Q invariant after changing the transformation law as

$$
\begin{aligned}
\hat{\delta}\chi^{a}_{\mu\nu} &= \frac{i}{2}\varrho(F^{+a}_{\mu\nu} + q^{\dagger}\bar{\sigma}_{\mu\nu}T^a q), \\
\hat{\delta}\psi_{q\alpha} &= -\frac{i}{2}\varrho\sigma^{\mu}{}_{\alpha\dot{\alpha}}D_{\mu}q^{\dot{\alpha}}, \\
\hat{\delta}\psi^{\alpha}_{\tilde{q}} &= \frac{i}{2}\varrho D_{\mu}q^{\dagger}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}.
\end{aligned}
\tag{2.13}
$$

By the fixed point theorem of Witten the path integral is localized to the locus of the fixed point of the global supersymmetry, modulo gauge symmetry. The important fixed points are $\delta\chi_{\mu\nu} = \delta\psi_{q\alpha} = 0$;

$$
F^{+a}_{\mu\nu} + q^{\dagger}\bar{\sigma}_{\mu\nu}T^a q = 0, \qquad \sigma^{\mu}D_{\mu}q = 0, \tag{2.14}
$$

which is the non-abelian version of the Seiberg-Witten monopoles. Note that the curvature $F_{\mu\nu}$ is the curvature of the bundle E (or P), while $q \in \Gamma(W^{+}_{\mathfrak{c}} \otimes E)$ is the section of $W^{+}_{\mathfrak{c}} \otimes E$. The Dirac operator $\sigma^{\mu}D_{\mu} : \Gamma(W^{+}_{\mathfrak{c}} \otimes E) \to \Gamma(W^{-}_{\mathfrak{c}} \otimes E)$ is the Dirac operator for the *spin*^{c} bundle twisted by E.[4] Note also that the curvature F is a $ad(P)$-valued 2-form

[4] In particular, if we consider the $U(1)$ gauge theory, $G = U(1)$ such that E is a line bundle, the equation (2.14) is not identical to the Seiberg-Witten monopole equation. The only special property for the $U(1)$ theory is that if the first Chern class $c_1(E)$ is integral the tensor product bundle $W^{+}_{\mathfrak{c}} \otimes E$ becomes another *spin*^{c} bundle $W^{+}_{\mathfrak{c}'}$ defined by the new *spin*^{c} structure $\mathfrak{c}' = \mathfrak{c} + 2c_1(E)$. Thus the equation that can be obtained by twisting the $N = 2$ super-Maxwell theory coupled with hypermultiplet is a perturbed Seiberg-Witten equation rather than the original equation.

(or trace-free endomorphism $End(E) = E \otimes \tilde{E}$ valued 2-form). Since $q \in \Gamma(W^+_{\mathfrak{c}} \otimes E)$ and $q^\dagger \in \Gamma(\overline{W}^+_{\mathfrak{c}} \otimes \tilde{E})$, the product $q \otimes q^\dagger$ lies in

$$W^+ \otimes_{\mathfrak{c}} E \otimes \overline{W}^+_{\mathfrak{c}} \otimes \tilde{E} \sim \Omega^0(End(E)) \oplus \Omega^2_+(End(E)),$$

where $\Omega^0(End(E))$ and $\Omega^2_+(End(E))$ denote the spaces of $End(E)$-valued zero-forms and $End(E)$-valued self-dual-two-forms respectively. An equivalent description can be obtained by examining the semi-classical limit $h^2 \to 0$, which is exact. The relevant bosonic part of the action can be written as

$$\begin{aligned}
&\frac{1}{h^2}\int d^4x\sqrt{g}\left(\frac{1}{4}|s|^2 + \frac{1}{2}|k|^2\right)\\
&\qquad = \frac{1}{h^2}\int d^4x\sqrt{g}\bigg(\frac{1}{4}F^{+\mu\nu}_a F^{+a}_{\mu\nu} - \frac{1}{2}p^+_{\mu\nu}q^\dagger\bar{\sigma}^{\mu\nu}q + \frac{1}{2}g^{\mu\nu}D_\mu q^\dagger_{\dot{\alpha}} D_\nu q^{\dot{\alpha}}\\
&\qquad\qquad + \frac{1}{4}(q^\dagger\bar{\sigma}^{\mu\nu}T_a q)(q^\dagger\bar{\sigma}_{\mu\nu}T^a q) + \frac{1}{8}R(q^\dagger_{\dot{\alpha}}q^{\dot{\alpha}})\bigg),
\end{aligned} \tag{2.15}$$

where $s = F^{+a}_{\mu\nu} + q^\dagger\bar{\sigma}_{\mu\nu}T^a q$ and $k = \sigma^\mu D_\mu q$. Thus, the path integral has the dominant contributions from the solutions of (2.14).

We also have another fixed point equation $\delta\lambda_\mu = \delta\eta = 0$;

$$D_\mu\phi = 0, \qquad [\phi, \bar{\phi}] = 0. \tag{2.16}$$

A connection is reducible if there exists non-zero solution ϕ of $D_\mu\phi = 0$.

The virtual (or formal) dimension of the moduli space $\mathcal{M}(\mathfrak{c}, k)$, the space of solutions of (2.14) in $\mathcal{A} \times \Gamma(W^+_{\mathfrak{c}} \otimes E)$ modulo the gauge symmetry, is

$$\begin{aligned}
dim\ \mathcal{M}(k, \mathfrak{c}) &= index\left(d^+_A \oplus d^*_A\right) + 2index\left(\not{D}^E_{\mathfrak{c}}\right) = dim\ \mathcal{M}(k) + 2index\left(\not{D}^E_{\mathfrak{c}}\right)\\
&= 8k - \frac{3}{2}(\chi + \sigma) - 2k + \frac{1}{2}(\mathfrak{c}\cdot\mathfrak{c} - \sigma),
\end{aligned} \tag{2.17}$$

where $\mathcal{M}(k)$ denotes the moduli space of $SU(2)$ ASD connection with the instanton number k and $'\cdot'$ denotes the intersection pairing. We will use the following notations;

$$2d(\mathfrak{c}, k) = \dim\mathcal{M}(k, \mathfrak{c}), \quad 2d(k) = \dim\mathcal{M}(k), \quad d_0(\mathfrak{c}, k) = d(\mathfrak{c}, k) - d(k) = index\ \not{D}^E_{\mathfrak{c}}. \tag{2.18}$$

With the choice of $b^+_2 = 1 + 2a$ for a positive integer a, both dimensions are even. We would like to remind the reader that the net U-number violation $\triangle U$ in the path integral measure due to the fermionic zero-modes equals the virtual dimension of the moduli space $\mathcal{M}(k, \mathfrak{c})$ [18]. The virtual dimension of $\mathcal{M}(\mathfrak{c}, k)$ becomes the real dimension if there are no fermionic zero-modes except those of λ and $(\bar{\psi}_{\tilde{q}\dot{\alpha}}, \bar{\psi}^{\dot{\alpha}}_{\tilde{q}})$ which span the tangent space over $\mathcal{M}(\mathfrak{c}, k)$. However, such an ideal situation will hardly be the case.

The actions (2.11) and (2.12) are also invariant under the global scaling of the metric if the scaling dimensions of the various fields are assigned as

$$\begin{array}{ccccccccccccccc} A_\mu & \lambda_\mu & \phi & \bar{\phi} & \eta & \chi_{\mu\nu} & H_{\mu\nu} & q^{\dot{\alpha}} & q^\dagger_{\dot{\alpha}} & \bar{\psi}^{\dot{\alpha}}_{\tilde{q}} & \bar{\psi}_{q\dot{\alpha}} & \psi_{q\alpha} & \psi^{\alpha}_{\tilde{q}} & X_{q\alpha} & X^{\alpha}_{\tilde{q}} \\ 1 & 1 & 0 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \end{array}. \tag{2.19}$$

If an operator O has the scaling dimension n, the integral $\int d^4x\sqrt{g}O$ scales as t^{4-n} under $g \to tg$. The transformation laws (2.6)(2.7) and (2.8) also preserve the scaling dimensions.

The crucial property of any cohomological field theory is that the energy-momentum tensor should be a Q-commutator,

$$\delta S = \frac{1}{2}\int_X \sqrt{g}\delta g^{\mu\nu}T_{\mu\nu}, \qquad T_{\mu\nu} = \{Q, \lambda_{\mu\nu}\}. \tag{2.20}$$

This immediately follows from the relation (2.4) if the variation operator $\delta/\delta g^{\mu\nu}$ commutes with Q in off shell. The only subtlety comes from the fields which are subject to the self-duality condition that the variation of the metric should be accompanied by the variations of the fields to preserve the self-duality. Since the algebra (2.6) is closed in off shell it can be guaranteed. Most of the important properties of the topological theory can be derived from the property (2.20). In particular, the topological invariance of the suitable correlation function is based on the property. The only obstruction is the possible metric dependency of the path integral measure. For a more detailed discussion on those properties, we refer the reader to [4].

2.3. The $\mathcal{G}$-Equivariant Cohomology

In this subsection, we briefly describe the relation between the twisted supersymmetry and the equivariant cohomology [20][6].

Consider the space $\mathcal{A}\times\Gamma(W^+_{\mathfrak{c}}\otimes E)$. The group $\mathcal{G} = Map(X, G)$ of the gauge transformation acts on $\mathcal{A}$ in the usual way and on $\Gamma(W^+_{\mathfrak{c}}\otimes E)$ according to the representation R and $\tilde{R}$ of G. Let $Lie(\mathcal{G})$ be the Lie algebra of $\mathcal{G}$. The $\mathcal{G}$ action on $\mathcal{A}\times\Gamma(W^+_{\mathfrak{c}}\otimes E)$ is generated by vector fields V_{a}, where we pick an orthonormal basis $\mathcal{T}_{\mathrm{a}}$ of $Lie(\mathcal{G})$. Let $\mathrm{Fun}(Lie(\mathcal{G}))$ be the algebra of polynomial functions, generated by ϕ^{a} with degree 2, on $Lie(\mathcal{G})$.

Now, one can formally define the (infinite dimensional) $\mathcal{G}$-equivariant de Rham cohomology. Let $\Omega^*(\mathcal{A}\times\Gamma(W^+_{\mathfrak{c}}\otimes E))$ be the de Rham complex on $\mathcal{A}\times\Gamma(W^+_{\mathfrak{c}}\otimes E)$. The equivariant de Rham complex is defined by

$$\Omega^*_{\mathcal{G}}(\mathcal{A}\times\Gamma(W^+_{\mathfrak{c}}\otimes E) = \left(\Omega^*(\mathcal{A}\times\Gamma(W^+_{\mathfrak{c}}\otimes E))\otimes Fun(Lie(\mathcal{G}))\right)^{\mathcal{G}}. \tag{2.21}$$

The associated differential operator δ can be formally represented as

$$\delta = -\int d^4x\sqrt{g}\Psi^I(x)\frac{\delta}{\delta A^I(x)} + i\int d^4x\sqrt{g}V(\phi(x))\frac{\delta}{\delta\Psi^I(x)}, \tag{2.22}$$

where A^I denote collectively the local coordinates on $\mathcal{A}\times\Gamma(W^+_{\mathfrak{c}}\otimes E)$ and Ψ^I denote the basis of the cotangent space. We have

$$\delta^2 = -i\int d^4x\sqrt{g}\phi^{\mathrm{a}}(x)\mathcal{L}_{\mathrm{a}}(x), \tag{2.23}$$

where $\mathcal{L}_{\mathrm{a}}$ is the Lie derivative with respect to V_{a}. Thus, $\delta^2=0$ on the $\mathcal{G}$-invariant subspace $\Omega^*_{\mathcal{G}}(\mathcal{A}\times\Gamma(W^+_{\mathfrak{c}}\otimes E))$ of $\Omega^*(\mathcal{A}\times\Gamma(W^+_{\mathfrak{c}}\otimes E))\otimes\mathrm{Fun}(Lie(\mathcal{G}))$. The $\mathcal{G}$-equivariant de Rham cohomology $H^*_{\mathcal{G}}(\mathcal{A}\times\Gamma(W^+_{\mathfrak{c}}\otimes E))$ is defined as the pairs $(\Omega^*_{\mathcal{G}}(\mathcal{A}\times\Gamma(W^+_{\mathfrak{c}}\otimes E)),\ \delta)$.

The basic supersymmetry algebra

$$\begin{aligned}
&\hat{\delta}A_\mu = +\,i\varrho\lambda_\mu, && \hat{\delta}\lambda_\mu = -\,\varrho D_\mu\phi, && \\
&\hat{\delta}q^{\dot\alpha} = -\,\varrho\bar\psi^{\dot\alpha}_{\tilde q}, && \hat{\delta}\bar\psi^{\dot\alpha}_{\tilde q} = -\,i\varrho\phi^a T_a q^{\dot\alpha}, && \hat{\delta}\phi = 0,\\
&\hat{\delta}q^\dagger_{\dot\alpha} = -\,\varrho\bar\psi_{q\dot\alpha}, && \hat{\delta}\bar\psi_{q\dot\alpha} = +\,i\varrho q^\dagger_{\dot\alpha}\phi^a T_a, &&
\end{aligned} \tag{2.24}$$

suggests that the twisted supercharge of the theory without the mass term can be interpreted as the generator of the $\mathcal{G}$-equivariant de Rham cohomology $H^*_{\mathcal{G}}(\mathcal{A}\times\Gamma(W^+_{\mathfrak{c}}\otimes E))$. The relation

$$(\hat\delta_\varrho\hat\delta_{\varrho'} - \hat\delta_{\varrho'}\hat\delta_\varrho)(field) = T_\varepsilon(field), \quad \text{where} \quad \varepsilon = -2i\varrho\varrho'\cdot\phi, \tag{2.25}$$

corresponds to the property (2.23).

Similarly to the TYM theory, one can interpret the twisted supercharge Q of TQCD as the $\mathcal{G}$-equivariant cohomology operator. Then, the topological action can also be interpreted as a certain Mathai-Quillen representative of a universal Thom class [21] based on the non-abelian version of Seiberg-Witten equations. For such a construction of the similar topological theory, we refer the reader to the paper [16].

2.4. The Observable and the Correlation Function

In the simply connected Riemann manifold, the second cohomology class determines the essential cohomological data. Picking a 2-dimensional homology class $\Sigma \in H_2(X;\mathbb{Z})$ which is Poincarè dual to $v \in H^2(X;\mathbb{Z})$, one defines the associated topological observable

$$\hat{v} = \frac{1}{4\pi^2}\int_\Sigma \mathrm{Tr}\,(i\phi F + \frac{1}{2}\lambda\wedge\lambda), \tag{2.26}$$

which carries the U-number 2. This observable $\hat{v}$ defines a 2-dimensional $\mathcal{G}$-equivariant cohomology class, i.e., $\hat{v} \in H^2_{\mathcal{G}}(\mathcal{A}\times\Gamma(W^+_{\mathfrak{c}}\otimes E))$. The Q-cohomology class of $\hat{v}$ depends only on the homology class of Σ. One also has the topological observable

$$\hat{u} = -\frac{1}{8\pi^2}\mathrm{Tr}\,\phi^2, \tag{2.27}$$

carrying the U-number 4 and depending only on $H_0(X;\mathbb{Z})$. The observable $\hat{u}$ defines a 4-dimensional class $H^4_{\mathcal{G}}$. TQCD has no additional non-trivial topological observables beyond those of the TYM theory. This may be an indication that the theory would have no new differential-topological information.

Now we consider the topological correlation function

$$\langle\hat{v}^r\hat{u}^s\rangle_{TQCD} = \frac{1}{\mathrm{vol}(\mathcal{G})}\int \mathcal{D}Y e^{-S}\cdot\hat{v}^r\hat{u}^s. \tag{2.28}$$

Due to the ghost number anomaly in the path integral measure the above topological amplitude vanishes unless $2r+4s$ is identical to the *formal dimension* $2d(\mathfrak{c},k)$ of the moduli space $\mathcal{M}(\mathfrak{c},k)$. If we consider, presumably, the favorable condition that the formal dimension of the moduli space is the actual dimension, the path integral reduces to an integration of the wedge product of differential forms on the moduli space $\mathcal{M}(\mathfrak{c},k)$. This can be seen by both the Q-fixed point theorem and the semi-classical analysis. The differential form is given by the restriction and the reduction $\hat{v}_0$ of $\hat{v}$ to the moduli space $\mathcal{M}(\mathfrak{c},k)$. That is, $\hat{v}_0$ defines an element of the de Rham cohomology class on $\mathcal{M}(\mathfrak{c},k)$.[5] The standard recipe similar to the TYM theory leads that $\hat{v}_0$ can be obtained by replacing λ by its zero-modes, F by the Q-fixed point value and ϕ by $<\phi>$.

[5] Of course, the above cohomological definition can be mathematically meaningful after some suitable compactification of the moduli space is understood. Here the above argument is just a formal cohomological interpretation of the path integral. *Whatever properties the moduli space has, the path integral, as we shall see, gives a perfectly concrete formula for those invariants.* See [4].

The integration over $\bar{\phi}$ in (2.12) gives a delta-function constraint

$$\frac{1}{2}g^{\mu\nu}D_\mu D_\nu\phi^a + \frac{i}{2}g^{\mu\nu}[\lambda_\mu,\lambda_\nu]^a + q^\dagger_{\dot\alpha}(T^aT^b + T^bT^a)q^{\dot\alpha}\phi^b - 2i\bar\psi_{q\dot\alpha}T^a\bar\psi^{\dot\alpha}_{\tilde q} = 0. \tag{2.29}$$

The η equation of motion of (2.12) gives

$$\frac{1}{2}g^{\mu\nu}D_\mu\lambda_\nu + iq^\dagger_{\dot\alpha}T^a\bar\psi^{\dot\alpha}_{\tilde q} + i\bar\psi_{q\dot\alpha}T^a q^{\dot\alpha} = 0, \tag{2.30}$$

which expresses the fact that the zero-modes of $(\lambda, \bar\psi^{\dot\alpha}_{\tilde q}, i\bar\psi_{q\dot\alpha})$ are orthogonal to the gauge variation. The $\bar\phi$ equation of motion (2.29) is just the supersymmetry transformation of (2.30). We can write (2.29) as

$$\left(D^\mu D_\mu \delta^a_b + 2q^\dagger_{\dot\alpha}(T^aT^b + T^bT^a)q^{\dot\alpha}\right)\phi^b = -i[\lambda^\mu,\lambda_\mu]^a + 4i\bar\psi_{q\dot\alpha}T^a\bar\psi^{\dot\alpha}_{\tilde q}. \tag{2.31}$$

On the other hand, the theory without coupling to the matter leads to

$$(D^\mu D_\mu)\,\phi = -i[\lambda^\mu,\lambda_\mu]. \tag{2.32}$$

Furthermore, we can substitute the vacuum expectation value $<\phi>$ of ϕ by

$$<\phi^a> = -i\int_X d^4y\sqrt{g}\,G^{ab}(x,y)\left([\lambda^\mu(y),\lambda_\mu(y)]^b - 4\bar\psi_{q\dot\alpha}(y)T^b\bar\psi^{\dot\alpha}_{\tilde q}(y)\right), \tag{2.33}$$

where

$$\left(D^\mu D_\mu\delta_{ab} + 2q^\dagger_{\dot\alpha}(T_aT_b + T_bT_a)q^{\dot\alpha}\right)G^{ab}(x,y) = \delta^{ab}\delta^4(x-y), \tag{2.34}$$

provided that we replace $q^{\dot\alpha}$, $q^\dagger_{\dot\alpha}$ by the non-abelian Seiberg-Witten monopole (2.14) and λ_μ, $\bar\psi_{q\dot\alpha}$ and $\bar\psi^{\dot\alpha}_{\tilde q}$ by their zero-modes which represent the tangent vectors of the moduli space $\mathcal{M}(k,\mathfrak{c})$. As the standard recipe of the TYM theory, we can replace ϕ with (2.33) whenever it appears in the topological observables.[6] This explains one of the mysteries of TQCD that no new and non-trivial observables are introduced due to the hypermultiplet.

3. The Massive TQCD

In the previous sections we only considered the theory with massless hyper-multiplet and its topological twisting. If we twist the $N = 2$ supersymmetric Yang-Mills theory coupled with massive hypermultiplet, a remarkable thing happens. We will show that the correlation function can be expressed by the sum of the contributions due to the Donaldson invariants and the Seiberg-Witten invariants.

[6] In TYM theory, the replacement of ϕ with $\langle\phi\rangle$ is the procedure for recovering the universal bundle construction of the cohomology class on $\mathcal{A}/\mathcal{G}$ [3][5][6]. The formula (2.33) implies that such a replacement in (2.26) and (2.27) will recover the analogous universal bundle construction for the extended space.

3.1. The Massive Hypermultiplet

In the $N=2$ supersymmetric QCD, the hypermultiplets can have the bare mass term which is invariant under the $N=2$ supersymmetry. We will always consider the theory with one hypermultiplet. In the on-shell action, the mass term can be written as[7]

$$S_{mass} = \int d^4x \Big[-m^2 q_i^\dagger q^i - \sqrt{2} m q_i^\dagger B^a T_a q^i + \sqrt{2} m q_i^\dagger \bar{B}^a T_a q^i - m\bar{\psi}_{\tilde{q}}^{\dot{\alpha}} \bar{\psi}_{q\dot{\alpha}} - m\psi_{\tilde{q}}^{\alpha}\psi_{q\alpha} \Big]. \quad (3.1)$$

Adding the mass term leads to the following on-shell supersymmetry transformation of the hypermultiplet

$$\begin{aligned} \delta q^i =& -\sqrt{2}\xi^{\alpha i}\psi_{q\alpha} + \sqrt{2}\bar{\xi}_{\dot{\alpha}}{}^i \bar{\psi}_{\tilde{q}}^{\dot{\alpha}}, \\ \delta\bar{\psi}_{\tilde{q}}^{\dot{\alpha}} =& -\sqrt{2} i \bar{\sigma}^{m\dot{\alpha}\alpha} D_m q^i \xi_{\alpha i} + 2T_a q^i B^a \bar{\xi}^{\dot{\alpha}}{}_i + \sqrt{2} m q^i \bar{\xi}^{\dot{\alpha}}{}_i, \\ \delta\psi_{q\alpha} =& -\sqrt{2} i \sigma^m_{\alpha\dot{\alpha}} D_m q^i \bar{\xi}^{\dot{\alpha}}{}_i - 2T_a q^i \bar{B}^a \xi_{\alpha i} + \sqrt{2} m q^i \xi_{\alpha i}, \end{aligned} \quad (3.2)$$

and of the conjugate fields

$$\begin{aligned} \delta q_i^\dagger =& -\sqrt{2}\bar{\psi}_{q\dot{\alpha}}\bar{\xi}^{\dot{\alpha}}{}_i - \sqrt{2}\psi_{\tilde{q}}^{\alpha}\xi_{\alpha i}, \\ \delta\bar{\psi}_{q\dot{\alpha}} =& \sqrt{2} i \xi^{\alpha i} D_m q_i^\dagger \sigma^m_{\alpha\dot{\alpha}} - 2\bar{\xi}_{\dot{\alpha}}{}^i q_i^\dagger B^a T_a - \sqrt{2} m \bar{\xi}_{\dot{\alpha}}{}^i q_i^\dagger, \\ \delta\psi_{\tilde{q}}^{\alpha} =& -\sqrt{2} i \bar{\xi}^i_{\dot{\alpha}} D_m q_i^\dagger \bar{\sigma}^{m\dot{\alpha}\alpha} - 2\xi^{\alpha i}\bar{B}^a q_i^\dagger T_a + \sqrt{2} m \xi^{\alpha i} q_i^\dagger. \end{aligned} \quad (3.3)$$

If we twist the above supersymmetry, there appear several problems concerning the mass term of the hypermultiplet. We first twist the supersymmetry transformation laws of the hypermultiplet following the recipe of our previous paper [14]. The twisted transformation laws of the hypermultiplet are given by

$$\begin{aligned} \hat{\delta}_m q^{\dot{\alpha}} =& -\varrho\bar{\psi}_{\tilde{q}}^{\dot{\alpha}}, \qquad & \hat{\delta}_m \bar{\psi}_{\tilde{q}}^{\dot{\alpha}} =& -i\varrho\phi^a T_a q^{\dot{\alpha}} - \varrho m q^{\dot{\alpha}}, \\ \hat{\delta}_m q_{\dot{\alpha}}^\dagger =& -\varrho\bar{\psi}_{q\dot{\alpha}}, \qquad & \hat{\delta}_m \bar{\psi}_{q\dot{\alpha}} =& i\varrho q_{\dot{\alpha}}^\dagger \phi^a T_a + \varrho m q_{\dot{\alpha}}^\dagger, \end{aligned} \quad (3.4)$$

and

$$\begin{aligned} \hat{\delta}_m \psi_{\tilde{q}}^{\alpha} =& i\varrho D_\mu q_{\dot{\alpha}}^\dagger \bar{\sigma}^{\mu\dot{\alpha}\alpha} - \varrho X_{\tilde{q}}^{\alpha}, \\ \hat{\delta}_m X_{\tilde{q}}^{\alpha} =& i\varrho\psi_{\tilde{q}}^{\alpha}\phi^a T_a - i\varrho D_\mu \bar{\psi}_{q\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha} + \varrho q_{\dot{\alpha}}^\dagger \bar{\sigma}^{\mu\dot{\alpha}\alpha}\lambda_\mu^a T_a + \varrho m \psi_{\tilde{q}}^{\alpha}, \\ \hat{\delta}_m \psi_{q\alpha} =& -i\varrho\sigma^\mu{}_{\alpha\dot{\alpha}} D_\mu q^{\dot{\alpha}} + \varrho X_{q\alpha}, \\ \hat{\delta}_m X_{q\alpha} =& i\varrho\phi^a T_a \psi_{q\alpha} - i\varrho\sigma^\mu{}_{\alpha\dot{\alpha}} D_\mu \bar{\psi}_{\tilde{q}}^{\dot{\alpha}} + \varrho\sigma^\mu{}_{\alpha\dot{\alpha}}\lambda_\mu^a T_a q^{\dot{\alpha}} + \varrho m \psi_{q\alpha}, \end{aligned} \quad (3.5)$$

[7] In terms of $N=1$ superspace notation, the action functional for hypermultiplet is given by $W = \sqrt{2}\tilde{Q}_h \Phi Q_h + m\tilde{Q}_h Q_h$, where Q_h and $\tilde{Q}_h$ are chiral superfields carrying a representation R and its conjugate $\tilde{R}$, respectively, Φ is the $N=1$ chiral multiplet carrying the adjoint representation and m is the bare mass for hypermultiplet. The untwisted action functional we are using is just the expansion of W in terms of the component fields. We follow the conventions in our previous paper [14].

while the transformation laws for the $N=2$ vector multiplet remains unchanged. Note that the twisted algebra (3.4) and (3.5) closed in off shell. Due to the new terms proportional to the mass m, the commutator of the supersymmetry is no longer the gauge transformation generated by ϕ. Furthermore, the above twisted supersymmetry does not preserve the U-number. Note, however, the global scaling dimensions (2.19) are preserved if the scaling dimension of m is 0.

Due to the new transformation laws, the action functional has the following additional terms;

$$S' = -i\{Q_m, V_T\} = S + \frac{1}{h^2}\int_X d^4x\sqrt{g}\left(2imq^\dagger_{\dot\alpha}\bar\phi^a T_a q^{\dot\alpha} + m\psi^\alpha_{\tilde q}\psi_{q\alpha}\right). \tag{3.6}$$

Note that the additional terms carry the U-number -2 while S' maintains the global scale invariance. The property that the energy-momentum tensor is a Q_m commutator remains unchanged.

$$T'_{\mu\nu} = \{Q_m, \lambda_{\mu\nu}\}. \tag{3.7}$$

Now we have two problems;

(1) The modified transformation laws (3.4) and (3.5) for the hypermultiplet break the basic commutation relation (2.25) due to the mass term;

$$(\hat\delta_{m\varrho}\hat\delta_{m\varrho'} - \hat\delta_{\varrho'}\hat\delta_{\varrho})(hypermultiplet) = T_\varepsilon(hypermultiplet) - 2\varrho\varrho'\cdot m, \tag{3.8}$$

where $\varepsilon = -2i\varrho\varrho'\cdot\phi$ as before. Furthermore, they do not preserve the U-number. Note, however, that if we assign the U-number 2 to m the U-number is preserved.

(2) The modified action (3.6) does not contain the full mass terms of the hypermultiplet.

We will temporarily ignore the problem (1) which will be resolved in Sect. 3.5.

Another very important effect is that the $\bar\phi$ equation of motion is changed from (2.29) to

$$\frac{1}{2}g^{\mu\nu}D_\mu D_\nu\phi^a + \frac{i}{2}g^{\mu\nu}[\lambda_\mu,\lambda_\nu]^a + q^\dagger_{\dot\alpha}(T^aT^b+T^bT^a)q^{\dot\alpha}\phi^b - 2i\bar\psi_{q\dot\alpha}T^a\bar\psi^{\dot\alpha}_{\tilde q} - 2imq^\dagger_{\dot\alpha}T^a q^{\dot\alpha} = 0.$$

$$\left(D^\mu D_\mu\delta^a_b + 2q^\dagger_{\dot\alpha}(T^aT^b+T^bT^a)q^{\dot\alpha}\right)\phi^b = -i[\lambda^\mu,\lambda_\mu]^a + 4i\bar\psi_{q\dot\alpha}T^a\bar\psi^{\dot\alpha}_{\tilde q} + 4imq^\dagger_{\dot\alpha}T^a q^{\dot\alpha}, \tag{3.9}$$

and the equation (2.33)is changed accordingly.

We have the same form of topological observable $\hat v$ in (2.26) which carries the U-number 2 and is invariant under the global scaling of the metric. The Q or Q_m-cohomology class of $\hat{v}$ depends only on the homology class of Σ. The same is true for $\hat{u}$ in (2.27). However, they are effectively different from the massless case due to the difference between (2.31) and (3.9). We shall see that the key simplification occurs due to the extra term in (3.9). The topological correlation function of the massive theory

$$\langle \hat{v}^r \hat{u}^s \rangle_{TQCD,m} = \frac{1}{\mathrm{vol}(\mathcal{G})} \int \mathcal{D}Y e^{-S'} \cdot \hat{v}^r \hat{u}^s, \tag{3.10}$$

also has the same U-number anomaly cancellation laws, $r + 2s = d(\mathfrak{c}, k)$.

3.2. The Q_m-Fixed Points

The effect of introducing the bare mass to the hypermultiplet can be most easily seen by checking the fixed point equations for the new global supercharge Q_m. In addition to the fixed point equations (2.14) and (2.16) of the original supersymmetry, we have new fixed point equations

$$\begin{cases} \hat{\delta}_m \bar{\psi}^{\dot{\alpha}}_{\tilde{q}} = 0, \\ \hat{\delta}_m \bar{\psi}_{q\dot{\alpha}} = 0, \end{cases} \Longrightarrow \begin{cases} m q^{\dot{\alpha}} + i\phi_a T^a q^{\dot{\alpha}} = 0, \\ m q^{\dagger}_{\dot{\alpha}} + i q^{\dagger}_{\dot{\alpha}} \phi^a T_a = 0, \end{cases} \tag{3.11}$$

If we collect the other Q_m-fixed point equations, $\hat{\delta}_m \chi_{\mu\nu} = \hat{\delta}_m \psi_{q\alpha} = \hat{\delta}_m \lambda_\mu = \hat{\delta}_m \eta = 0$, we have

$$\begin{cases} F^{+a}_{\mu\nu} + q^{\dagger} \bar{\sigma}_{\mu\nu} T^a q = 0, \\ \sigma^\mu D_\mu q = 0, \end{cases} \qquad \begin{cases} D_\mu \phi = 0, \\ [\phi, \bar{\phi}] = 0, \end{cases} \tag{3.12}$$

The first pair of the equations say that the fixed point locus is the moduli space $\mathcal{M}(\mathfrak{c}, k)$. The second pair of the equations means that ϕ is zero at the fixed point locus if the connection is irreducible (the gauge symmetry is unbroken) and ϕ is non-zero if the connection is reducible (the gauge symmetry is broken down to $U(1)$).

This is the judicious moment to study the new fixed point equation (3.11). One obvious fixed point is $q^{\dagger}_{\dot{\alpha}} = q^{\dot{\alpha}} = 0$, which will be called branch (i). In this branch, the fixed point equations reduce to those of the TYM theory;

$$F^{+a}_{\mu\nu} = 0, \qquad D_\mu \phi = 0, \qquad [\phi, \bar{\phi}] = 0. \tag{3.13}$$

Thus the fixed point locus is the moduli space of irreducible ASD connections for $\phi = 0$.[8] Note also that the fixed point equation (3.11) reduces to $q^{\dagger}_{\dot{\alpha}} = q^{\dot{\alpha}} = 0$ if $\phi = 0$. *So,*

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[8] For the manifold with $b_2^+ > 1$, there are no reducible ASD connections for a generic choice of the metric as well as for a smooth path joining two generic metrics. We will assume that the moduli space $\mathcal{M}(k)$ is connected.

whenever gauge symmetry is unbroken, the fixed point locus of Q_m is the moduli space $\mathcal{M}(k)$ of ASD connections.

Another type of fixed points with $q \neq 0$, which will be called branch (ii), are in the abelian Coulomb phase ($D_\mu\phi = 0$ and $\phi \neq 0$). In this branch the gauge symmetry is broken down to $U(1)$ and the vector bundle E reduces to the sum of line bundles $E = \zeta \oplus \zeta^{-1}$ where

$$\zeta \cdot \zeta = -k, \qquad c_1(\zeta) = -c_1(\zeta^{-1}) \in H^2(X;\mathbb{Z}). \tag{3.14}$$

The curvature two-form F of E reduces to

$$F \longrightarrow \frac{1}{2i}\begin{pmatrix} F_3 & 0 \\ 0 & -F_3 \end{pmatrix} \in \mathfrak{su}(2), \tag{3.15}$$

where $\frac{1}{2}F_3$ is the curvature of the line bundle ζ. Now equation (3.11) becomes

$$mq^{\dot\alpha} + i\phi_3 T_3 q^{\dot\alpha} = 0, \quad \textit{where} \quad \phi_a T^a \longrightarrow \phi_3 T_3 = \frac{1}{2i}\begin{pmatrix} \phi_3 & 0 \\ 0 & -\phi_3 \end{pmatrix}, \tag{3.16}$$

which can be written as

$$\begin{aligned} mq^{\dot\alpha}_{(1)} + \frac{1}{2}\phi_3 q^{\dot\alpha}_{(1)} &= 0, \\ mq^{\dot\alpha}_{(2)} - \frac{1}{2}\phi_3 q^{\dot\alpha}_{(2)} &= 0, \end{aligned} \tag{3.17}$$

where

$$q^{\dot\alpha} = \begin{pmatrix} q^{\dot\alpha}_{(1)} \\ q^{\dot\alpha}_{(2)} \end{pmatrix}, \tag{3.18}$$

and (1), (2) denote the color index. Thus the only nontrivial solutions for $q^{\dot\alpha}$ are either

$$q^{\dot\alpha} = \begin{pmatrix} q^{\dot\alpha}_{(1)} \\ 0 \end{pmatrix} \quad \textit{and} \quad 2m + \phi_3 = 0, \tag{3.19}$$

or

$$q^{\dot\alpha} = \begin{pmatrix} 0 \\ q^{\dot\alpha}_{(2)} \end{pmatrix} \quad \textit{and} \quad 2m - \phi_3 = 0. \tag{3.20}$$

Now the Q_m fixed point equation (3.12) (or the monopole equation (2.14)) becomes

$$\begin{aligned} F^{+3}_{\mu\nu} + \frac{1}{2i}\left(q^\dagger_{(1)}\bar\sigma_{\mu\nu}q_{(1)} - q^\dagger_{(2)}\bar\sigma_{\mu\nu}q_{(2)}\right) &= 0, \\ \begin{pmatrix} \sigma^\mu\mathfrak{D}_{\mu+} & 0 \\ 0 & \sigma^\mu\mathfrak{D}_{\mu-} \end{pmatrix}\begin{pmatrix} q_{(1)} \\ q_{(2)} \end{pmatrix} &= 0, \end{aligned} \tag{3.21}$$

where we have either $q_{(1)} = 0$ or $q_{(2)} = 0$. Here, $\sigma^\mu\mathfrak{D}_{\mu\pm}$ denotes the abelian *spin*^{c} the Dirac operator acting on $\Gamma(W^+_{\mathfrak{c}} \otimes \zeta^{\pm 1})$ satisfying the following Weitzenbök formula

$$(\sigma^\mu\mathfrak{D}_{\mu\pm})^2 = -g^{\mu\nu}\mathfrak{D}_{\mu\pm}\mathfrak{D}_{\nu\pm} \mp F^+_{3\mu\nu} - p^+_{\mu\nu} + \frac{1}{4}R, \tag{3.22}$$

and p^+ is the self-dual part of the curvature of $det(W_{\mathfrak{c}}^+) = L_{\mathfrak{c}}$. If we exchange $A^3 \to -A^3$ and $q_{(1)} \to q_{(2)}$ the equation is symmetric. Since we always have a pair of line bundles $\pm\zeta$ for each bundle reduction, we can always fix $q_{(2)} = 0$ and $q_{(1)} = M \neq 0$, and regard the two solutions as the same equation for the line bundles satisfying $\zeta \cdot \zeta = -k$. We also set $\sigma^\mu \mathfrak{D}_{\mu+}$ to $\sigma^\mu \mathfrak{D}_\mu$.

Now we have the celebrated Seiberg-Witten monopole equations

$$
\begin{aligned}
F^+_{3\mu\nu} + \frac{1}{2i} M^\dagger \bar{\sigma}_{\mu\nu} M = 0, \\
\sigma^\mu \mathfrak{D}_\mu M = 0.
\end{aligned}
\tag{3.23}
$$

Note that curvature $F^+_{3\mu\nu}$ is the curvature of the line bundle ζ^2, while M is the section of $W^+_{\mathfrak{c}} \otimes \zeta$. Since ζ is an integral class, one can regard $W^+_{\mathfrak{c}} \otimes \zeta = W^+_{\mathfrak{c}'}$ as a different $spin^c$ bundle for the different $spin^c$ structure $\mathfrak{c}' = \mathfrak{c} + 2\zeta$, i.e., $det(W^+_{\mathfrak{c}} \otimes \zeta) = L_{\mathfrak{c}} \otimes \zeta^2$. The Weitzenböck formula (3.22), which can be written as

$$
(\sigma^\mu \mathfrak{D}_\mu)^2 = -g^{\mu\nu} \mathfrak{D}_\mu \mathfrak{D}_\nu - (F^+_{3\mu\nu} + p^+_{\mu\nu}) + \frac{1}{4} R,
\tag{3.24}
$$

also shows that $\sigma^\mu \mathfrak{D}_\mu$ is the $spin^c$ Dirac operator acting on $W^+_{\mathfrak{c}'}$. We will frequently use the notation $\not{\mathfrak{D}}^{\mathfrak{c}'}$ for $\sigma^\mu \mathfrak{D}_\mu$. The original Seiberg-Witten equation consists of the curvature of $det(W^+_{\mathfrak{c}'})$ and the section of $W^+_{\mathfrak{c}'}$ [1]. The equation (3.23) should be viewed as a perturbed Seiberg-Witten monopole equation for the $spin^c$ structure $\mathfrak{c}'$;

$$
\begin{aligned}
F^{\mathfrak{c}'+}_{\mu\nu} + \frac{1}{2i} M^\dagger \bar{\sigma}_{\mu\nu} M = p^+_{\mu\nu}, \\
\not{\mathfrak{D}}^{\mathfrak{c}'} M = 0,
\end{aligned}
\tag{3.25}
$$

where $F^{\mathfrak{c}'} = F^+_{3\mu\nu} + p^+_{\mu\nu}$ denotes the curvature of $det(W^+_{\mathfrak{c}'})$.

All this can also be seen from the action (2.11) or (2.12). The relevant part is the Cartan subalgebra part of Eq. (2.15). Note that

$$
\begin{aligned}
\frac{1}{h^2} \int d^4x \sqrt{g} \biggl[\frac{1}{4} F_3^{+\mu\nu} F^+_{3\mu\nu} - \frac{1}{2} p^+_{\mu\nu} M^\dagger \sigma^{\mu\nu} M + \frac{1}{2} g^{\mu\nu} \mathfrak{D}_\mu M^\dagger_{\dot\alpha} \mathfrak{D}_\nu M^{\dot\alpha} \\
- \frac{1}{16} (M^\dagger \bar{\sigma}^{\mu\nu} M)(M^\dagger \bar{\sigma}_{\mu\nu} M) + \frac{1}{8} R (M^\dagger_{\dot\alpha s} M^{\dot\alpha}) \biggr],
\end{aligned}
\tag{3.26}
$$

which can be rewritten as $\frac{1}{h^2} \int \sqrt{g} d^4x \left(\frac{1}{4} |s_3|^2 + \frac{1}{2} |k_3|^2 \right)$, where $s_3 = F^+_{3\mu\nu} + \frac{1}{2i} M^\dagger \bar{\sigma}^{\mu\nu} M$ and $k_3 = \sigma^\mu \mathfrak{D}_\mu M$. The perturbation can be removed by replacing $F^+_{3\mu\nu}$ with $F^+_{3\mu\nu} + p^+_{\mu\nu}$ in s_3.

Then, we have

$$\frac{1}{h^2}\int\sqrt{g}d^4x\left(\frac{1}{4}|s_3|^2+\frac{1}{2}|k_3|^2\right)$$
$$=\frac{1}{h^2}\int d^4x\sqrt{g}\bigg[\frac{1}{4}F_3^{\mathfrak{c}'+\mu\nu}F_{\mu\nu}^{\mathfrak{c}'+}-\frac{1}{16}(M^\dagger\bar{\sigma}^{\mu\nu}M)(M^\dagger\bar{\sigma}_{\mu\nu}M)$$
$$+\frac{1}{2}g^{\mu\nu}\mathfrak{D}_\mu M^\dagger_{\dot\alpha}\mathfrak{D}_\nu M^{\dot\alpha}+\frac{1}{8}R(M^\dagger_{\dot\alpha}M^{\dot\alpha})\bigg]. \tag{3.27}$$

That is, one can rewrite (3.26) in terms of the the curvature of $det(W^+_{\mathfrak{c}'})$, which is equivalent to absorbing the term $-\frac{1}{2}p^+_{\mu\nu}M^\dagger\sigma^{\mu\nu}M$ by a field redefinition.

Thus we have the localization to the moduli space $\mathcal{M}(\mathfrak{c}')$ of Seiberg-Witten monopole with the *spin*^{c} structure $\mathfrak{c}'$, i.e., the space of solutions of (3.25) in $\mathcal{A}_{det(W^+_{\mathfrak{c}'})}\times\Gamma(W^+_{\mathfrak{c}'})$ modulo the gauge symmetry S^1. The formal dimension of the moduli space $\mathcal{M}(\mathfrak{c}')$ is given by

$$dim\mathcal{M}(\mathfrak{c}')=2index\left(\not{\mathfrak{D}}^{\mathfrak{c}'}\right)-\frac{\chi+\sigma}{2}=\frac{\mathfrak{c}'\cdot\mathfrak{c}'}{4}-\frac{2\chi+3\sigma}{4}. \tag{3.28}$$

We will denote a *spin*^{c} structure $\mathfrak{c}'$ by x if $dim\mathcal{M}(\mathfrak{c}')=0$, i.e.,[9]

$$x\cdot x=\frac{2\chi+3\sigma}{4}. \tag{3.29}$$

Then the moduli space $\mathcal{M}(x)$ consists of a finite collection of points. The Seiberg-Witten invariant n_x is the algebraic sum of the number of points counted with sign.

Applying the fixed point theorem of Witten for the global supersymmetry (see Sect. 3.1 of [17]), the path integral can be written as the sum of contributions of the branch (i) and the branch (ii). So the path integrals can be written as a certain sum of the Donaldson and the Seiberg-Witten invariants.

Note that we should have chosen a *spin*^{c} structure $\mathfrak{c}$ to define the twisting of the hypermultiplet. And the TQCD depends on the choice of the *spin*^{c} structure. Now, in branch (ii), one can view the choice of different twisting as the choice of different perturbation of the Seiberg-Witten equations.[10] It is shown that the Seiberg-Witten invariant n_x is independent of the (generic) perturbation [1]. *Consequently, the family of TQCD parametrized*

---

[9] This condition can be written by $\mathfrak{c}'=\mathfrak{c}+2\zeta$ with

$$\zeta\cdot\zeta<0,\qquad \mathfrak{c}\cdot\zeta=-index\left(\not{D}^E_{\mathfrak{c}}\right)+2\Delta,$$

where $D=(\chi+\sigma)/4$.

[10] Note that the perturbed term $p^+_{\mu\nu}$ in (3.25) is the self-dual part of the curvature of $det(W^+_{\mathfrak{c}})$. See [1] for the applications of the perturbation.

by the different choice of the spin^c structure is governed by the the same Seiberg-Witten invariants in branch (ii). This is a crucial property since we will use the path integral of TQCD, which depends on the choice of the *spin*^{c} structure, to obtain the path integral of the TYM theory.

Before moving to the next topic, we review the orientation of both moduli spaces $\mathcal{M}(k)$ and $\mathcal{M}(x)$. The proof of the orientability of a moduli space amounts to showing the triviality of a determinant line of elliptic operator arising from the linearization of the moduli space [3]. For the moduli space $\mathcal{M}(k)$ of ASD connection, the elliptic operator is $(d_A^+ \oplus d_A^*)$. Donaldson showed that an orientation of the space $H^1(X;\mathbb{R}) \oplus H^+(X;\mathbb{R})$ induces orientations of $\mathcal{M}(k)$. For the moduli space $\mathcal{M}(x)$ of the Seiberg-Witten monopoles the elliptic operator is $((d+d^*) \oplus \not{\mathfrak{D}}^x)$ and the triviality of its determinant line was shown [1]. The orientation of determinant line of $(d+d^*)$ is fixed once and for all by picking an orientation of $H^1(X;\mathbb{R}) \oplus H^+(X;\mathbb{R})$. Since the determinant line bundle of Dirac operator $\not{\mathfrak{D}}^x$ is naturally trivial, one can define an orientation of $\mathcal{M}(x)$. If we replace x with $-x$ which corresponds to different trivialization of the determinant line, we have

$$n_{-x} = (-1)^{\Delta} n_x, \qquad \Delta = \frac{\chi+\sigma}{4} = index\ \not{\mathfrak{D}}^x. \tag{3.30}$$

Since we will compare the contribution from the moduli space $\mathcal{M}(k)$ with those from the moduli spaces $\mathcal{M}(x)$, the relative orientations are important. Since the orientations of $det\ ind(d_A^+ \oplus d_A^*)$ and $det\ ind(d+d^*)$ are governed by the same data $H^1(X;\mathbb{R}) \oplus H^+(X;\mathbb{R})$, the ambiguity in the comparison can only come from the determinant line of the Dirac operator $\not{\mathfrak{D}}^x$. We will fix the orientation of $det\ ind(d_A^+ \oplus d_A^*)$ to the opposite of $det\ ind(d+d^*)$.

3.3. The Stationary Phases

In this subsection, we will address the problem (2) mentioned in Sect. 3.1. One can find the following combination is Q_m invariant

$$\frac{1}{2\pi}\int_X d^4x\sqrt{g}\left(m q^{\dagger}_{\dot{\alpha}} q^{\dot{\alpha}} + i\phi^a q^{\dagger}_{\dot{\alpha}} T_a q^{\dot{\alpha}} + \bar{\psi}^{\dot{\alpha}}_{\tilde{q}} \bar{\psi}_{q\dot{\alpha}}\right), \tag{3.31}$$

which is Q_m exact;

$$\begin{aligned} -i\left\{Q_m, -\frac{1}{4\pi}\int_X d^4x\sqrt{g}\left(q^{\dagger}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}_{\tilde{q}} + \bar{\psi}_{q\dot{\alpha}} q^{\dot{\alpha}}\right)\right\} \\ = \frac{1}{2\pi}\int_X d^4x\sqrt{g}\left(m q^{\dagger}_{\dot{\alpha}} q^{\dot{\alpha}} + i\phi^a q^{\dagger}_{\dot{\alpha}} T_a q^{\dot{\alpha}} + \bar{\psi}^{\dot{\alpha}}_{\tilde{q}} \bar{\psi}_{q\dot{\alpha}}\right). \end{aligned} \tag{3.32}$$

If we remove the term proportional to m in the supersymmetry transformation laws, the above term is no-longer invariant under the original supercharge Q. Instead, the Q invariant combination is

$$\frac{1}{2\pi}\int_X d^4x\sqrt{g}\left(i\phi^a q^\dagger_{\dot\alpha} T_a q^{\dot\alpha} + \bar\psi^{\dot\alpha}_{\tilde q}\bar\psi_{q\dot\alpha}\right). \tag{3.33}$$

We add the Q_m-exact term (3.31) to the topological action S' to get a one-parameter family of the topological theory

$$S'(t) = S' + \frac{t}{2\pi}\int_X d^4x\sqrt{g}\left(mq^\dagger_{\dot\alpha} q^{\dot\alpha} + i\phi^a q^\dagger_{\dot\alpha} T_a q^{\dot\alpha} + \bar\psi^{\dot\alpha}_{\tilde q}\bar\psi_{q\dot\alpha}\right). \tag{3.34}$$

If, in particular, we choose $t = 2\pi m/h^2$ we superficially recover the full mass terms of the hypermultiplet as the physical theory. Since the t dependent term is Q_m-exact, the theory does not depend on t as long as $t \neq 0$ by the standard argument of the cohomological theory.

Now we consider the topological correlation function

$$\langle \hat v^r \hat u^s\rangle_{TQCD(m,t)} = \frac{1}{\mathrm{vol}(\mathcal{G})}\int \mathcal{D}Y e^{-S' - \frac{t}{2\pi}\int_X d^4x\sqrt{g}\left(mq^\dagger_{\dot\alpha} q^{\dot\alpha} + i\phi^a q^\dagger_{\dot\alpha} T_a q^{\dot\alpha} + \bar\psi^{\dot\alpha}_{\tilde q}\bar\psi_{q\dot\alpha}\right)} \cdot \hat v^r \hat u^s. \tag{3.35}$$

In the $t = 0$ limit, the above formula is identical to (3.10). Thus, the path integral (3.35) can be evaluated in a suitable limit. One can consider t as purely imaginary and take the limit $Im(t) = \infty$, one may use the method of the stationary phases. In the $Im(t) \to \infty$ limit, the dominant contribution to the path integral comes from the stationary phases (the critical points). Such an approximation is exact, provided that we sum over the contributions of the all critical points.

The equation for the stationary phases in the t-dependent terms in (3.35) is

$$mq^{\dot\alpha} + i\phi_a T^a q^{\dot\alpha} = 0, \qquad mq^\dagger_{\dot\alpha} + iq^\dagger_{\dot\alpha}\phi^a T_a = 0. \tag{3.36}$$

Thus the stationary phase equation is identical to the Q_m-fixed point equation, $\delta_m \bar\psi^{\dot\alpha}_{\tilde q} = \delta_m \bar\psi_{q\dot\alpha} = 0$. This is not surprising as can be seen from relation (3.32).

Thus, the stationary phases have the same two branches as discussed in the previous subsection. Combining with the exact semi-classical limit $h^2 \to 0$, the exactness of the stationary phase approximation recovers the same localization of the path integral as predicted by the fixed point argument of the global supersymmetry Q_m.

Consequently, one can use either the Q_m-fixed point theorem or the combination of the semi-classical and the stationary phase approximation. Both methods say that one should evaluate the path integral exactly at the two branches and calculate Gaussian integral over the quadratic terms due to the transverse degrees of freedom.

Before moving to the next topic we should remark on three subtle points related to the stationary phase. There are some criteria for the independence on a BRST trivial deformation [20].

First of all, there should be no new fixed points flowing from infinity. In our case, the t-dependent term does not change any fixed points of the global supercharge Q_m. It is a rather natural deformation for the theory having the bare mass term.

Secondly, the additional BRST trivial term should preserve the U-number symmetry of the original theory. Note that the expression (3.31) contains the term with U-number 0 as well as the term with U-number 2. This means that the theory with $t \neq 0$ and $t = 0$ can be actually different.

Thirdly, the deformation term should not change the property that the energy-momentum tensor is a Q_m commutator. It can easily be seen that the particular deformation does not alter the property. However, the Q_m-exact term (3.31) does not preserve the global scaling invariance of the theory. Showing the global scaling invariance amounts to proving that the trace of energy-momentum tensor is a total divergence. The failure of the property can be seen by counting the net scaling dimension of the fields in (3.31) which is 2 rather than 4. We will return to this important issue later on.

3.4. The global S^1 symmetry

In this subsection, we study the global $U(1)$ symmetry on the hypermultiplet. We will show that there are two different types of the S^1 fixed points which are identical to the two branches of the stationary phases.

The theory has a global $U(1)$ symmetry acting on the hypermultiplet;

$$q \to e^{i\theta} q, \tag{3.37}$$

which leaves the action[11] as well as the fixed point equation of the global supersymmetry invariant

$$\begin{aligned} F^{+a}_{\mu\nu} + q^\dagger \bar{\sigma}_{\mu\nu} T^a q = 0, \\ \sigma^\mu D_\mu q = 0. \end{aligned} \tag{3.38}$$

[11] This S^1 symmetry should be read in general as $Q_h \to e^{i\theta} Q_h$ and $\tilde{Q}_h \to \tilde{Q}_h e^{-i\theta}$ which obviously leaves the action invariant.

This $U(1)$ action has two branches of the fixed points.

The obvious fixed point is when $q = 0$ and all the other fields belonging to the hypermultiplet vanish as well. Then the monopole equation (3.38) becomes the standard anti-self-duality equation. We will call this fixed point branch (i). There is another type of the fixed point. Note that the path integral is defined over the space of fields modulo the local gauge symmetry. Thus the S^1 action can have another fixed point if there are gauge transformations such that

$$g(\theta)q = e^{i\theta}q, \qquad g(\theta)^{-1}d_A g(\theta) = d_A. \tag{3.39}$$

The situation is very similar to the self-duality equations of Hitchin [22].[12] The first equation implies that if $q \neq 0$, then $g(\theta)$ is not an identity for $\theta \neq 2n\pi$ where n is an integer. And, then, the second equation implies that the connection A is reducible and that the $SU(2)$ bundle reduces to the direct sum of line bundles, i.e., $E = \zeta \oplus \zeta^{-1}$. Then, $g(\theta)$ becomes diagonalized. Since q belongs to the fundamental representation, q must be either $q = (q_{(1)}, 0)^T$ or $q = (0, q_{(2)})^T$ to satisfy the first equation.

Thus, two branches of the fixed point of S^1 action are identical to two branches of the stationary phases.

3.5. The $\mathcal{G} \times S^1$-equivariant Cohomology

The relation between the Q_m-fixed points and the fixed points of the S^1 action suggests that Q_m has a close relation with the S^1-equivariant cohomology. The commutation relation (3.8) implies that one can identify $-im$ as the generator of the $Fun(Lie(S^1))$ associated with the global S^1 action on the hypermultiplet. If we consider the space $\mathcal{A} \times \Gamma(W^+_{\mathfrak{c}} \otimes E)$, the global S^1-symmetry acts on $(A, q, q^\dagger)$ by $(A, e^{i\theta}q, q^\dagger e^{-i\theta})$. The $\mathcal{G}$ action on $\mathcal{A} \times \Gamma(W^+_{\mathfrak{c}} \otimes E)$ is generated by vector fields $V(\phi)$. We can consider the algebra

[12] In ref. [23], the similar symmetry was considered for the twisted $N = 4$ super-Yang-Mills theory which is a close cousin of TQCD as well as of the Hitchin equations. Vafa and Witten showed and computed that the path integral of the theory is the Euler character of the instanton moduli space provided that certain vanishing theorems hold. The vanishing theorem amounts to the absence of the branch (ii) contributions. In our case, we will express the invariant due to the branch (i) in terms of the contributions of the branch (ii). It will be interesting to see if the method we develop in this paper can be applied for computing the Euler character and testing the S-duality in a general case. An obvious starting point will be to study the model by adding the mass term which breaks $N = 4$ down to the $N = 2$ supersymmetry.

of polynomial functions Fun($Lie(S^1)$) generated by $-im$. Now, one can define the (infinite dimensional) $\mathcal{G} \times S^1$-equivariant de Rham cohomology. The equivariant de Rham complex is defined by

$$\begin{aligned} \Omega^*_{\mathcal{G}\times S^1}(\mathcal{A}\times&\Gamma(W^+_{\mathfrak{c}} \otimes E) \\ &= \left(\Omega^*(\mathcal{A}\times\Gamma(W^+_{\mathfrak{c}} \otimes E)) \otimes Fun(Lie(\mathcal{G})) \otimes Fun(Lie(S^1))\right)^{\mathcal{G}\times S^1}. \end{aligned} \tag{3.40}$$

The associated differential operator is δ which can be formally represented as

$$\delta = -\int d^4x\sqrt{g}\Psi^I(x)\frac{\delta}{\delta A^I(x)} + i\int d^4x\sqrt{g}V(\phi(x))^I\frac{\delta}{\delta\Psi^I(x)} + \int d^4x\sqrt{g}V(m)^J\frac{\delta}{\delta\Psi^J(x)}. \tag{3.41}$$

We have

$$\delta^2 = -\int d^4x\sqrt{g}(\mathcal{L}_{V(\phi(x))} + \mathcal{L}_{V(m)}). \tag{3.42}$$

Thus, $\delta^2 = 0$ on the $\mathcal{G} \times S^1$-invariant subspace $\Omega^*_{\mathcal{G}\times S^1}(\mathcal{A}\times\Gamma(W^+_{\mathfrak{c}} \otimes E)$ of $\Omega^*(\mathcal{A}\times\Gamma(W^+_{\mathfrak{c}} \otimes E))\otimes\text{Fun}(Lie(\mathcal{G}))\otimes\text{Fun}(Lie(S^1))$. The $\mathcal{G} \times S^1$-equivariant de Rham cohomology $H^*_{\mathcal{G}\times S^1}(\mathcal{A}\times\Gamma(W^+_{\mathfrak{c}} \otimes E))$ is defined as the pairs

$$(\Omega^*_{\mathcal{G}\times S^1}(\mathcal{A}\times\Gamma(W^+_{\mathfrak{c}} \otimes E)),\ \delta).$$

The basic supersymmetry algebra

$$\begin{aligned} &\hat{\delta}_m A_\mu = +i\varrho\lambda_\mu, &&\hat{\delta}_m\lambda_\mu = -\varrho D_\mu\phi, && \\ &\hat{\delta}_m q^{\dot{\alpha}} = -\varrho\bar{\psi}^{\dot{\alpha}}_{\tilde{q}}, &&\hat{\delta}_m\bar{\psi}^{\dot{\alpha}}_{\tilde{q}} = -i\varrho\phi^a T_a q^{\dot{\alpha}} - \varrho m q^{\dot{\alpha}}, &&\hat{\delta}_m\phi = 0, \\ &\hat{\delta}_m q^\dagger_{\dot{\alpha}} = -\varrho\bar{\psi}_{q\dot{\alpha}}, &&\hat{\delta}_m\bar{\psi}_{q\dot{\alpha}} = +i\varrho q^\dagger_{\dot{\alpha}}\phi^a T_a + \varrho m q^\dagger_{\dot{\alpha}}, &&\hat{\delta}_m m = 0, \end{aligned} \tag{3.43}$$

shows that the twisted supercharge of the theory with the massive hypermultiplet can be interpreted as the generator of the $\mathcal{G} \times S^1$-equivariant de Rham cohomology $H^*_{\mathcal{G}\times S^1}(\mathcal{A}\times\Gamma(W^+_{\mathfrak{c}} \otimes E))$.

Now, the problem of the U-number can be resolved. We define the degree of the $\mathcal{G} \times S^1$-equivariant complex by the formula

$$\deg(\alpha\otimes\beta\otimes\gamma) = \deg(\alpha) + 2\deg(\beta) + 2\deg(\gamma), \tag{3.44}$$

for $\alpha \in \Omega^*(\mathcal{A}\times\Gamma(W^+_{\mathfrak{c}} \otimes E))$, $\beta \in \text{Fun}(Lie(\mathcal{G}))$ and $\gamma \in \text{Fun}(Lie(S^1))$. That is, we assign $U = 2$ to the mass m,[13] regarding m as an operator or a constant field. Then Q_m increases

[13] The interpretation of the S^1-equivariant cohomology generator as a parameter or vise versa is not new [24].

the degree by one, as expected. Note that the action S' of (3.6) now has the U-number zero. The expression (3.31) also has the correct U-number 2. The familiar topological observable $\hat{v}$ (2.26) can be viewed as $\mathcal{G} \times S^1$-equivariant extension of a differential two-form on $\mathcal{A} \times \Gamma(W_{\mathfrak{c}}^+ \otimes E)$. Although $\hat{v}$ has the same form in the TYM, the massless TQCD and the massive TQCD, due to the differences between (2.32), (2.31) and (3.9), it is effectively different in each theory.

It is interesting to note that the last term in (3.31) is a closed form. The first and the second terms in (3.31) give the S^1 and $\mathcal{G}$-equivariant extensions of the last term, respectively. By comparing (3.31) with (3.32), one can view the changing of the supersymmetry from Q to Q_m as the recipe to introduce the term

$$\frac{1}{2\pi} \int_X d^4x \sqrt{g} \left(m q^\dagger_{\dot{\alpha}} q^{\dot{\alpha}} \right), \tag{3.45}$$

in the supersymmetric way. The TQCD with one hypermultiplet depends on the choice of the *spin*^{c} structure $\mathfrak{c}$. Thus, we have a family of TQCD parametrized by the space of *spin*^{c} structures on X. Since the TQCD depends on a choice of the *spin*^{c} structure $\mathfrak{c}$, it may be possible to embed the moduli space $\mathcal{M}(k)$ of anti-self-dual connections to the moduli space $\mathcal{M}(k, \mathfrak{c})$ as a connected component of the fixed points locus of global S^1 action on $\mathcal{M}(k, \mathfrak{c})$ by varying the *spin*^{c} structure. The S^1 action has also another type of the fixed points whose locus is the moduli space of the (abelian) Seiberg-Witten monopoles. Thus, in the formal level, the path integral evaluation can be quite similar to the Duistermaat-Heckmann (DH)[14] integral formula [25] or the equivariant integration formula of Berline and Vergne [26][24]. However, such an interpretation is clearly problematic unless the moduli space $\mathcal{M}(\mathfrak{c}, k)$ has highly favorable properties. We shall see that our path integral computation gives a perfectly concrete formula in any case.

An alternative formal viewpoint is to consider the equivariant S^1 localization from the beginning without referring to the moduli space $\mathcal{M}(\mathfrak{c}, k)$.

3.6. A Synthesis

The massless TQCD has almost the same properties and problems as the TYM theory. The cohomological interpretation of those two theories leads us to some integrals of differential forms over the moduli spaces which are rarely compact. Even though we assume

[14] If we consider the Kähler case, the term (3.45) without m can be identified with the momentum map (Hamiltonian) of the S^1 action on $\mathcal{A} \times \Gamma(W_{\mathfrak{c}}^+ \otimes E)$ Thus, the analogy between the DH integration formula and our path integral computation becomes much closer on Kähler manifolds. This is one way to see why the massive deformation dramatically enhances the computability of the path integral.

the case when the moduli space is actually compact, it is rarely possible to compute such an integral explicitly.

Now the role of introducing the bare mass term to the hypermultiplet becomes clear. The massive deformation we introduced further localizes the path integral to the moduli space $\mathcal{M}(k)$ of ASD connections and the moduli spaces $\mathcal{M}(x)$ of the (abelian) Seiberg-Witten monopoles. The latter spaces are compact and, if we assume the simple type condition, they are zero-dimensional. Although the path integral contributed from the moduli space $\mathcal{M}(k)$ would be almost impossible to compute, we can certainly get explicit results for the contributions of $\mathcal{M}(x)$.

We have seen the above localization by the various arguments which are closely related with each other. It is quite amusing to see that the massive deformation of TQCD leads to a beautiful synthesis of the various aspects of the cohomological field theory and the equivariant cohomology. However, some subtle points remain to be resolved.

The localization due to the global supersymmetry is based on a very general assumption. For example, even if we regard the action S' (3.6) as the complete action functional, the supersymmetry transformation law (3.4) predicts that the path integral should be localized according to equation (3.11). However, such a localization can be seen only after adding the Q_m exact term (3.31) to the action. In TYM theory and the massless TQCD, one can recover the same localization by the semi-classical limit as predicted by the Q-fixed point arguments. In fact, there is a drawback in the semi-classical analysis [4]. The kinetic energy term for the scalar fields ϕ and $\bar{\phi}$ is not positive definite. After twisting it seems to be more natural to regard those fields as independent fields. One can regard that ϕ is real and $\bar{\phi}$ is purely imaginary. Then, the localization $D_\mu\phi = 0$ can be seen by the stationary phase for $h^2 \to 0$. If we maintain the complex conjugation relation $\phi \sim -\bar{\phi}^*$ for ϕ and $\bar{\phi}$ as the physical theory, the kinetic energy is positive definite and the localization can be seen by the usual semi-classical limit.

The power of Witten's fixed point theorem is that it does not refer to such complications. However, this does not necessarily mean that we don't need to add the remaining mass term (3.31) to the action. To ensure the correct localization of the path integral, we should include all terms that produce all the relevant fixed point equations of the global supersymmetry by the equations of the motion. For example, even the problematic kinetic term for the scalar fields may not be necessary. However, the fixed point theorem says that one should replace a field by its fixed point value. In the TYM theory or in the branch (i) of our theory, the fixed point value for ϕ should be zero to avoid reducible instantons. Thus, the above replacement is obviously incorrect. The resolution is to try to integrate out ϕ which results in the replacement of (2.32). The kinetic term for the scalar fields is crucial.

In our viewpoint, one should maintain all the terms coming from the physical action. In fact, we can readily justify this. In the physical theory, the supercharge comes from the conserved supercurrent which can be calculated by the various terms in the action. The supersymmetry algebra then naturally follows. That is, the particular supersymmetry responsible for the crucial fixed point equation (3.11) originates from the full mass term for the hypermultiplet. The twisting procedure just couples one of the spinor indices to the internal global symmetry of the theory. Thus, we can not discard any term coming from the physical theory. Actually, our computation of the path integral will confirm the above assertion.

Now, the remaining problem is that the crucial term (3.31) in the exponent of (3.34) violates the global symmetries of the theory. We argue that the mass term should carry the U-number 2. This means that the theories with $t = 0$ and $t \neq 0$ are different, although there are no new fixed points flowing from infinity, since the positive ghost number of (3.31) will effect the ghost number anomaly cancellation due to the observables.[15] By choosing $t = 1/m$, we can make the t-dependent term in (3.34) to carry the U-number 0,

$$\frac{1}{2\pi m}\int_X d^4x\sqrt{g}\left(mq^\dagger_{\dot\alpha}q^{\dot\alpha} + i\phi^a q^\dagger_{\dot\alpha}T_a q^{\dot\alpha} + \bar\psi^{\dot\alpha}_{\tilde q}\bar\psi_{q\dot\alpha}\right), \tag{3.46}$$

such that the deformed theory does not depend on the additional Q_m-exact term. Now we can exponentiate the observable $\hat v$

$$\langle \exp(\hat v + \tau\hat u)\rangle_{m,\mathfrak{c},k} = \frac{1}{\mathrm{vol}(\mathcal{G})}\int \mathcal{D}Y e^{-S' - \frac{1}{2\pi}\int_X d^4x\sqrt{g}\left(q^\dagger_{\dot\alpha}q^{\dot\alpha} + i\phi^a q^\dagger_{\dot\alpha}T_a q^{\dot\alpha}/m + \bar\psi^{\dot\alpha}_{\tilde q}\bar\psi_{q\dot\alpha}/m\right) + \hat v + \tau\hat u}, \tag{3.47}$$

such that

$$\langle \exp(\hat v + \tau\hat u)\rangle_{m,\mathfrak{c},k} = \sum_{r+2s=d(\mathfrak{c},k)} \frac{(2\tau)^s}{r!s!}\langle \hat v^r \hat u^s\rangle_{m,\mathfrak{c},k}\,. \tag{3.48}$$

The final problem is that such a deformation violates the global scaling invariance. If we scale the metric by a constant $g \to tg$ both S' and $\hat v'$ remain invariant. On the other hand the deformation term (3.46) is scaled as t^2. Fortunately, this property does not change the independence of the correlation function on the metric.[16] Thus, *we can*

[15] One may still add the original term (3.31) to the action and control the theory if one can.

[16] The scaling independence can be seen by showing that the trace of energy-momentum tensor is a total divergence, which is independent of the Q_m-exactness of the energy-momentum tensor. Eventually, we will set $m = 0$. This limit is smooth since the additional term can be written as $\frac{1}{m}\{Q_m, O\}$ and there is no obstruction for going to $m = 0$. Such an argument is not valid in general, see the footnotes (10) and (11) of [4].

identify the stationary phase limit as the large scaling limit of the metric. Actually, *only the large scaling limit of the metric is the true stationary phase limit.* One may introduce an extra parameter and take the infinite limit. However, its effect can be absorbed by a rescaling of the metric. Since we are dealing with the metric invariant theory, the large scaling limit of the metric is the true stationary phase limit. It is quite amusing to see that the localization of the path integral, which is the key step of this paper, is achieved by the large scale limit of the metric.

In the physical theory, the analogous step would be taking the infinite mass limit such that the hypermultiplet can be integrated out. Then, the theory will reduce to the theory without the matter field. Though this is the most natural step to localize the theory to branch (i), it is not clear how the path integral localizes to branch (ii) as well. On the other hand, the physical theory has the asymptotic freedom so that it is scale-dependent. The $N = 2$ SYM theory interpolates the two branches by the genuine quantum scaling behaviours. The seminal work of Seiberg and Witten shows that the key simplification responsible for the Seiberg-Witten invariants occurs in the strong coupling vacuum which is equivalent to the weakly coupled vacua of massless $U(1)$ hypermultiplet.

In our case, the twisted theory with additional matter multiplet having the bare mass in the large scale limit of the metric localizes to the two branches corresponding to two different limits of the physical $N = 2$ SYM theory. Furthermore, condition (3.19) says that the $U(1)$ hypermultiplet is massless in branch (ii). *This is an amazing property. How is it that the essentially classical treatment of a theory understands the genuine quantum property of a different theory?* The only answer to this question seems to be the self-duality of the critical theory [27]. All those properties of the asymptotically free theories and their twisted versions can be some remnants of the critical theory through the massive deformations.

In the remaining sections, we will concretely realize the above picture. As all the three different viewpoints of the localization suggest, the path integral amounts to evaluating exactly at the locus of the each branch and computes the Gaussian integrals of quadratic terms due to the transverse degree. The favorable interpretation is to take the large scaling limit first. We define the action S_m by

$$
\begin{aligned}
S_m =& S' + \frac{1}{2\pi}\int_X d^4x\sqrt{g}\left(q^\dagger_{\dot\alpha} q^{\dot\alpha} + i\phi^a q^\dagger_{\dot\alpha} T_a q^{\dot\alpha}/m + \bar\psi^{\dot\alpha}_{\tilde q}\bar\psi_{q\dot\alpha}/m\right)\\
=& S + \frac{1}{h^2}\int_X d^4x\sqrt{g}\left(2im\bar\phi^a q^\dagger_{\dot\alpha} T_a q^{\dot\alpha} + m\psi^{\alpha}_{\tilde q}\psi_{q\alpha}\right)\\
&+ \frac{1}{2\pi}\int_X d^4x\sqrt{g}\left(q^\dagger_{\dot\alpha} q^{\dot\alpha} + i\phi^a q^\dagger_{\dot\alpha} T_a q^{\dot\alpha}/m + \bar\psi^{\dot\alpha}_{\tilde q}\bar\psi_{q\dot\alpha}/m\right).
\end{aligned}
\tag{3.49}
$$

where we use the action S in the form of Eq. (2.11).

Picking a Riemann metric g, we rescale $g \to tg$ and take $t \to \infty$ limit. In branch (i), the gauge symmetry is unbroken and the matter fields decouple as the transverse degrees of freedom. The dominant contribution to the path integral comes from the path integral of the TYM theory. In branch (ii), the gauge symmetry breaks down to $U(1)$ and the hypermultiplet reduces to one of its color. The suppressed color degrees of freedom for hypermultiplet and the components of the $N = 2$ vector multiplet which do not belong to the Cartan subalgebra part become the transverse degrees of freedom. In the infinite scaling limit, it is sufficient to keep only quadratic terms for the transverse degrees and compute the one-loop approximations which are arbitrarily good.

On the other hand, the path integrals for the non-transverse degrees should be computed exactly. These path integrals correspond to the path integral of TYM theory in branch (i) and the path integral of topological QED (Seiberg-Witten theory) in branch (ii).

We will use the notations $\langle O\rangle_{m,\mathfrak{c},k}$, $\langle O\rangle_{\mathfrak{c},k}$, $\langle O\rangle_{k}$, and $\langle O\rangle$ for the correlation functions evaluated in the massive TQCD for given *spin*^{c} structure and instanton number, in the massless TQCD for given *spin*^{c} structure and instanton number, in the TYM theory for given instanton number and in the TYM theory with summation over all instanton numbers, respectively.

4. The Computation of the Path Integral

In this section, we compute the path integral in the large scaling limit of the metric.

4.1. Branch (i) and the Donaldson-Witten theory

In this branch, the degree of freedom for the hypermultiplet becomes transverse. One can decompose the action S_m into two parts

$$S_m \approx S_m(i) + \delta^{(2)} S_m(i). \tag{4.1}$$

where $\delta^{(2)} S_m(i)$ denotes the quadratic action due to the transverse degrees (the matter fields Q_h and $\tilde{Q}_h$). The action $S_m(i)$ in branch (i) locus reduces to the familiar action of the TYM theory [4]

$$\begin{aligned} S_m(i) = \frac{1}{h^2}\int \sqrt{g} d^4x \Big[&\frac{1}{4} F_a^{+\mu\nu} F_{\mu\nu}^{+a} - \frac{1}{2} g^{\mu\nu} (D_\mu \bar{\phi})_a (D_\nu \phi)^a + \frac{1}{8}[\phi,\bar{\phi}]_a [\phi,\bar{\phi}]^a - i\chi_a^{\mu\nu}[\phi,\chi_{\mu\nu}]^a \\ &+ \chi_a^{\mu\nu}(d_A\lambda)_{\mu\nu}^{+a} + \frac{i}{2} g^{\mu\nu}(D_\mu\eta)_a\lambda_\nu^a - \frac{i}{2} g^{\mu\nu}[\lambda_\mu,\bar{\phi}]_a \lambda_\nu^a + \frac{i}{8}[\phi,\eta]_a\eta^a, \end{aligned} \tag{4.2}$$

which is the standard action for the TYM theory. Or, equivalently

$$
\begin{aligned}
S_m(i) =& -i\{Q_m, V_T(i)\} \\
=& -i\left\{Q, \frac{1}{h^2}\int \sqrt{g} d^4x \left[\chi_a^{\mu\nu}\left(H_{\mu\nu}^a - iF_{\mu\nu}^{+a}\right) - \frac{1}{2}g^{\mu\nu}(D_\mu\bar{\phi})_a\lambda_\nu^a + \frac{1}{8}[\phi,\bar{\phi}]_a\eta^a\right]\right\},
\end{aligned}
\tag{4.3}
$$

and integrate $H_{\mu\nu}$ out. Note that $Q = Q_m$.

The contribution of the branch (i) to the correlation function $\left\langle e^{\hat{v}}\right\rangle_{\mathfrak{c},m,k}$ can be written as

$$
\left\langle e^{\hat{v}}\right\rangle_{\mathfrak{c},m,k}(i) = \frac{1}{\mathrm{vol}(\mathcal{G})}\int \mathcal{D}W e^{-S(i)+\hat{v}(i)} \times \int \mathcal{D}\tilde{Q}_h\mathcal{D}Q_h e^{-\delta^{(2)}S_m(i)+\delta^{(2)}\hat{v}(i)}, \tag{4.4}
$$

where $\hat{v}(i)$ denotes the usual observable for TYM theory, $(\mathcal{D}W)$ denotes the path integral measure for TYM theory and $\delta^{(2)}\hat{v}(i)$ denotes the quadratic term of $\hat{v}$ due to the transverse degrees. Note that $\delta^{(2)}\hat{v}(i) = 0$.

4.2. Transverse Path Integral for the Branch (i)

The quadratic action due to the matter fields is

$$
\begin{aligned}
\delta^{(2)}S_m(i) = \frac{1}{h^2}\int & d^4x\sqrt{g}\Big[-2X_{\tilde{q}}^{\alpha}X_{q\alpha} + iX_{\tilde{q}}^{\alpha}\sigma^{\mu}{}_{\alpha\dot{\alpha}}D_\mu q^{\dot{\alpha}} + iD_\mu q_{\dot{\alpha}}^{\dagger}\bar{\sigma}^{\mu\dot{\alpha}\alpha}X_{q\alpha} + \frac{h^2}{2\pi}q_{\dot{\alpha}}^{\dagger}q^{\dot{\alpha}} \\
& - iD_\mu\bar{\psi}_{q\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\psi_{q\alpha} - i\psi_{\tilde{q}}^{\alpha}\sigma^{\mu}{}_{\alpha\dot{\alpha}}D_\mu\bar{\psi}_{\tilde{q}}^{\dot{\alpha}} + 2m\psi_{\tilde{q}}^{\alpha}\psi_{q\alpha} + \frac{h^2}{2\pi m}\bar{\psi}_{\tilde{q}}^{\dot{\alpha}}\bar{\psi}_{q\dot{\alpha}}\Big].
\end{aligned}
\tag{4.5}
$$

The Gaussian integrals over auxiliary fields $\psi_{\tilde{q}}^{\alpha}, \psi_{q\alpha}, X_{\tilde{q}}^{\alpha}$ and $X_{q\alpha}$ give

$$
\begin{aligned}
\delta^{(2)}S_m(i) = \frac{1}{h^2}\int & d^4x\sqrt{g}\Big[\frac{1}{2}D_\mu q_{\dot{\alpha}}^{\dagger}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\sigma^{\nu}{}_{\alpha\dot{\alpha}}D_\nu q^{\dot{\alpha}} + \frac{h^2}{2\pi}q_{\dot{\alpha}}^{\dagger}q^{\dot{\alpha}} \\
& - \frac{1}{2m}D_\mu\bar{\psi}_{q\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\sigma^{\nu}{}_{\alpha\dot{\alpha}}D_\nu\bar{\psi}_{\tilde{q}}^{\dot{\alpha}} - \frac{h^2}{2\pi m}\bar{\psi}_{q\dot{\alpha}}\bar{\psi}_{\tilde{q}}^{\dot{\alpha}}\Big],
\end{aligned}
\tag{4.6}
$$

with the following determinant;

$$
\frac{\Big[det\,(2m)\Big]_{(\psi_{\tilde{q}}^{\alpha},\psi_{\tilde{q}\alpha})}}{\Big[det\,(-\frac{1}{\pi})\Big]_{(X_{\tilde{q}}^{\alpha},X_{q\alpha})}} = \Big[det\,(-2\pi m)\Big]_{\Gamma(W_{\mathfrak{c}}^{-}\otimes E)}. \tag{4.7}
$$

Usually the infinite dimensional determinants are not well defined and need regularization. However, the above determinant ratio is perfectly well defined due to the global supersymmetry. Now we are left with

$$\int \mathcal{D}q^\dagger \mathcal{D}q \mathcal{D}\bar{\psi}_{\tilde{q}} \mathcal{D}\bar{\psi}_q e^{-\frac{1}{h^2}\int d^4x\sqrt{g}\left(\frac{1}{2}q^\dagger\left[\not{D}^2+\frac{1}{\pi}h^2\right]q+\frac{1}{2m}\bar{\psi}_q\left[\not{D}^2+\frac{1}{2}h^2\right]\bar{\psi}_{\tilde{q}}\right)}. \tag{4.8}$$

If we perform the Gaussian integral over $(q^\dagger_{\dot{\alpha}}, q^{\dot{\alpha}})$ and $(\bar{\psi}_{q\dot{\alpha}}, \bar{\psi}^{\dot{\alpha}}_{\tilde{q}})$ by ignoring $\delta^{(2)}\hat{v}(i)$ term, we have the following determinant[17]

$$\frac{\left[det\left(-\frac{\not{D}^2+\frac{1}{2\pi}h^2}{m}\right)\right]_{(\bar{\psi}_{q\dot{\alpha}},\bar{\psi}^{\dot{\alpha}}_{\tilde{q}})}}{\left[det\left(\frac{1}{2\pi}(\not{D}^2+\frac{1}{2\pi}h^2)\right)\right]_{(q^\dagger_{\dot{\alpha}},q^{\dot{\alpha}})}} = \left[det\left(-\frac{2\pi}{m}\right)\right]_{\Gamma(W^+_{\mathfrak{c}}\otimes E)}. \tag{4.9}$$

Combining (4.7) and (4.9), we have

$$\int \mathcal{D}Q_h \mathcal{D}\tilde{Q}_h e^{-\delta^{(2)}S_m(i)} = \left[det\left(-\frac{2\pi}{m}\right)\right]_{\Gamma(W^+_{\mathfrak{c}}\otimes E)\ominus\Gamma(W^-_{\mathfrak{c}}\otimes E)} = \left(-\frac{2\pi}{m}\right)^{index(\not{D}^E_{\mathfrak{c}})}. \tag{4.10}$$

Thus, we have

$$\left\langle e^{\hat{v}}\right\rangle_{\mathfrak{c},m,k}(i) = \left(-\frac{2\pi}{m}\right)^{index(\not{D}^E_{\mathfrak{c}})} \times \left\langle e^{\hat{v}}\right\rangle_k. \tag{4.11}$$

4.3. Branch (ii) and the Seiberg-Witten Theory

In this branch, the gauge symmetry breaks down to $U(1)$ (the maximal torus). The components of any field which do not belong to the Cartan subalgebra part becomes the transverse variable. One can decompose the action S_m into two parts

$$S_m \approx S_m(ii) + \delta^{(2)}S_m(ii). \tag{4.12}$$

[17] If we apply the Weitzenbóck formula, $\not{D}^2$ contains the usual connection Laplacian, the scalar curvature of metric and the curvature for connection. If we scale $g \to tg$, all the terms scale out as t^{-2}. The following formula clearly shows that the determinant ratio is independent of such a scaling.

To begin with, we regard that action $S_m(ii)$ consists of the Cartan subalgebra part of all the adjoint fields

$$
\begin{aligned}
S_m(ii) = \frac{1}{h^2}\int d^4x\sqrt{g}\Big[&\frac{1}{4}F_3^{+\mu\nu}F_{3\mu\nu}^{+} + \frac{i}{4}p_{\mu\nu}^{+}(M^{\dot\alpha\dagger}\bar\sigma^{\mu\nu}M^{\dot\alpha}) + \frac{1}{2}g^{\mu\nu}\mathfrak{D}_\mu M_{\dot\alpha}^{\dagger}\mathfrak{D}_\nu M^{\dot\alpha} \\
&- \frac{1}{16}(M^{\dagger}\bar\sigma^{\mu\nu}M)(M^{\dagger}\bar\sigma_{\mu\nu}M) + \frac{1}{8}R(M_{\dot\alpha}^{\dagger}M^{\dot\alpha}) - \frac{1}{2}g^{\mu\nu}\partial_\mu\bar\phi_3\partial_\nu\phi_3 \\
&+ \chi_3^{\mu\nu}(d\lambda_3)_{\mu\nu}^{+} - \frac{1}{2}\chi_3^{\mu\nu}\bar\psi_M\bar\sigma_{\mu\nu}M + \frac{1}{2}\chi_3^{\mu\nu}M^{\dagger}\bar\sigma_{\mu\nu}\bar\psi_{\tilde M} + \frac{i}{2}g^{\mu\nu}\partial_\mu\eta_3\lambda_{3\nu} \\
&+ \frac{1}{2i}M_{\dot\alpha}^{\dagger}\eta_3\bar\psi_{\tilde M}^{\dot\alpha} - \frac{1}{2i}\bar\psi_{M\dot\alpha}\eta_3M^{\dot\alpha} + \frac{1}{2i}M_{\dot\alpha}^{\dagger}\lambda_\mu\bar\sigma^{\mu\dot\alpha\alpha}\psi_{M\alpha} + \frac{1}{2i}\psi_{\tilde M}^{\alpha}\sigma^{\mu}{}_{\alpha\dot\alpha}\lambda_{3\mu}M^{\dot\alpha} \\
&- i\mathfrak{D}_\mu\bar\psi_{M\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\psi_{M\alpha} - i\psi_{\tilde M}^{\alpha}\sigma^{\mu}{}_{\alpha\dot\alpha}\mathfrak{D}_\mu\bar\psi_{\tilde M}^{\dot\alpha} - \bar\psi_{M\dot\alpha}\bar\phi_3\bar\psi_{\tilde M}^{\dot\alpha} + \psi_{\tilde M}^{\alpha}(\phi_3+2m)\psi_{M\alpha} \\
&+ \frac{1}{2}M_{\dot\alpha}^{\dagger}M^{\dot\alpha}\bar\phi_3(\phi_3+2m) + \frac{h^2}{2\pi}(1+\frac{\phi_3}{2m})M_{\dot\alpha}^{\dagger}M^{\dot\alpha} + \frac{h^2}{2\pi m}\bar\psi_{M\dot\alpha}\bar\psi_{\tilde M}^{\dot\alpha}\Big].
\end{aligned}
\tag{4.13}
$$

The above action has the following supersymmetry

$$
\begin{aligned}
&\hat\delta_m A_{3\mu} = i\varrho\lambda_{3\mu}, && \hat\delta_m\chi_{3\mu\nu} = \frac{i}{2}\varrho(F_{3\mu\nu}^{+} + \frac{1}{2i}M^{\dagger}\bar\sigma_{\mu\nu}M), && \hat\delta_m H_{3\mu\nu} = 0, \\
&\hat\delta_m\lambda_{3\mu} = -\varrho\partial_\mu\phi_3, && \hat\delta_m\bar\phi_3 = i\varrho\eta_3, && \hat\delta_m\eta_3 = 0, \\
&\hat\delta_m\phi_3 = 0, && &&
\end{aligned}
\tag{4.14}
$$

and

$$
\begin{aligned}
\hat\delta_m M^{\dot\alpha} &= -\varrho\bar\psi_{\tilde M}^{\dot\alpha}, & \hat\delta_m\bar\psi_{\tilde M}^{\dot\alpha} &= -\frac{1}{2}\varrho(\phi_3+2m)M^{\dot\alpha}, \\
\hat\delta_m M_{\dot\alpha}^{\dagger} &= -\varrho\bar\psi_{M\dot\alpha}, & \hat\delta_m\bar\psi_{M\dot\alpha} &= +\frac{1}{2}M_{\dot\alpha}^{\dagger}\varrho(\phi_3+2m), \\
\hat\delta_m\psi_{M\alpha} &= -\frac{i}{2}\varrho\sigma^{\mu}{}_{\alpha\dot\alpha}\mathfrak{D}_\mu M^{\dot\alpha}, && \\
\hat\delta_m\psi_{\tilde M}^{\alpha} &= +\frac{i}{2}\varrho\mathfrak{D}_\mu M_{\dot\alpha}^{\dagger}\bar\sigma^{\mu\dot\alpha\alpha}. &&
\end{aligned}
\tag{4.15}
$$

One can view the reduced action $S_m(ii)$ as a purely abelian theory which is the twisted version of the $N=2$ super-Maxwell theory coupled with hypermultiplet having the bare mass. The fixed point equations $\hat\delta_m\chi_{\mu\nu} = \hat\delta_m\psi_{M\alpha} = 0$ show that this theory describes the Seiberg-Witten invariants. Another important fixed point equation $\hat\delta_m\bar\psi_{\tilde M}^{\dot\alpha} = 0$ shows that one can replace ϕ with $-2m$,

$$
\phi \to \phi_3T_3 = \frac{1}{2i}\begin{pmatrix}\phi_3 & 0\\ 0 & -\phi_3\end{pmatrix} = \frac{1}{i}\begin{pmatrix}-m & 0\\ 0 & m\end{pmatrix}.
\tag{4.16}
$$

Of course, the above theory should be viewed as a subsector of the bigger theory. More precisely, we should interpret the theory as an effective theory in one of the two types of branches in the large scaling limit. As an embedded theory, the fixed point equation $(\phi+2m)=0$ implies that the $U(1)$ hypermultiplet is massless. Then our previous discussion clearly shows that branch (ii) of massive QCD in the large scale limit is described by the $U(1)$ massless hypermultiplet. The condition $(\phi+2m)=0$ also implies that the $U(1)$ hypermultiplet becomes massless at the S^1-fixed points. Up to now, we have regarded the bare mass m as a constant field carrying the U-number 2. This interpretation was based on the identification of m with the S^1-equivariant cohomology generator. Since this branch is in the fixed point of S^1 action we can treat m as a number. *This is because branch (ii) are S^1-fixed points where we undo the S^1 action by the local $U(1)$ gauge symmetry.* This can also be seen by writing $S_m(ii)$ honestly in the large scaling limit. Then the terms involving m in (4.13) decouple from the path integral. Of course, m should be treated as carrying the U-number 2 in the transverse integrations.

The contribution of branch (ii) to the correlation function $\left\langle e^{\hat{v}}\right\rangle_{\mathfrak{c},m,k}$ can be written as

$$\begin{aligned}\left\langle e^{\hat{v}}\right\rangle_{\mathfrak{c},m,k}(ii) =&\frac{1}{\mathrm{vol}(\mathcal{G})}\int \mathcal{D}\tilde{Q}_{h(1)}\mathcal{D}Q_{h(1)}\mathcal{D}W_3\int \mathcal{D}W e^{-S(ii)+\hat{v}(ii)}\\ &\times\int \mathcal{D}\tilde{Q}_{h(2)}\mathcal{D}Q_{h(2)}\mathcal{D}W_+\mathcal{D}W_- e^{-\delta^{(2)}S_m(ii)+\delta^{(2)}\hat{v}(ii)}.\end{aligned} \tag{4.17}$$

The contribution from

$$\frac{1}{\mathrm{vol}(\mathcal{G}_3)}\int \mathcal{D}\tilde{Q}_{h(1)}\mathcal{D}Q_{h(1)}\mathcal{D}W_3 e^{-S(ii)+\hat{v}(ii)}, \tag{4.18}$$

can be easily determined. First of all, in a simple type manifold we only need to consider the zero-dimensional moduli space of the Seiberg-Witten monopoles. Then there are no fermionic zero-modes. So one can simply replace $\hat{v}(ii)$ with its fixed point values

$$\hat{v}(ii) = \frac{m}{2\pi}(v\cdot x). \tag{4.19}$$

One can expand the action around the fixed points (a point in the zero dimensional moduli space $\mathcal{M}(x)$) up to the quadratic term. The action has the following general form:

$$\frac{1}{h^2}\int d^4x\sqrt{g}\Big[\Phi\Delta_B\Phi + i\Psi D_F\Psi\Big], \tag{4.20}$$

where Φ and Ψ denote the bosonic and fermionic fields, respectively, and Δ_B and D_F are their corresponding operators. The similar situation is discussed, in detail, in [4]. The Gaussian integral gives

$$\frac{\mathrm{Pfaff}D_F}{\sqrt{det\Delta_B}} = \pm 1. \tag{4.21}$$

The sign depends on a choice of orientation. To make the assignment of the sign meaningful, one should prove that the determinant line bundle of D_F (called the Pfaffian line bundle) can be trivialized. Note that D_F is precisely the linearization of the abelian Seiberg-Witten equations. Thus the orientatibility of Pfaffian line bundle follows from the orientability of the moduli space $\mathcal{M}(x)$. Thus we can read the result immediately as

$$\langle 1\rangle_x = \mathcal{N} n_x, \qquad \text{where} \qquad n_x = \sum_{s\in\mathcal{M}(x)} (-1)^{\varepsilon_s}, \tag{4.22}$$

where $\mathcal{N}$ denotes the standard renormalization due to the local operators constructed from metric depending only on χ and σ [4][11][23]. Consequently, we have

$$\int \mathcal{D}\tilde{Q}_{h(1)}\mathcal{D}Q_{h(1)}\mathcal{D}W_3 \int \mathcal{D}X e^{-S(ii)+\hat{v}(ii)} = \mathcal{N}\sum_x n_x e^{\frac{m}{2\pi}(v\cdot x)}. \tag{4.23}$$

Before leaving this subsection, we should address the problem of the compatibility between the replacement $\phi \to -2m$ and $\phi \to \langle\phi\rangle$. In the TYM theory, there should be no zero-modes of ϕ (the non-zero solution of ϕ for $D_\mu\phi = 0$). Since all the topological observables contain ϕ, the replacement of ϕ with its zero modes gives the vanishing results. The correct field theoretical treatment is to replace ϕ with its expectation value $\langle\phi\rangle$ which amounts to integrating it out [11]. In the massive TQCD, ϕ has the zero-modes in branch (ii) and the preferred value is $-2m$. This can be seen from the fixed point equation (4.15) as well as the large scaling limit, which we are considering, for the deformation term

$$\frac{1}{2\pi}\int_X d^4x\sqrt{g}\left(q^\dagger_{\dot\alpha} q^{\dot\alpha} + i q^\dagger_{\dot\alpha}\frac{\phi^a T^a}{m} q^{\dot\alpha} + \frac{\bar\psi^{\dot\alpha}_{\tilde q}\bar\psi_{q\dot\alpha}}{m}\right). \tag{4.24}$$

At branch (ii) the above expression is zero by definition. Of course, this is equivalent to the substitution (4.16). This means that the expectation value $\langle\phi\rangle$ reduces to $\phi(ii)$ in (4.16) for the effective $U(1)$ theory. The question is how this can be consistent with the substitution due to (3.9).

The relevant part is the ϕ_3 component in (3.9), we have

$$\begin{aligned} -\left(q^\dagger_{\dot\alpha} q^{\dot\alpha}\right)\langle\phi_3\rangle(ii) = &-\left(D_\mu D^\mu\right)\langle\phi_3\rangle(ii) - i[\lambda^\mu,\lambda^\nu]_3 \\ &+ 2\bar\psi_{q\dot\alpha}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\bar\psi^{\dot\alpha}_{\tilde q} + 2m q^\dagger_{\dot\alpha}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} q^{\dot\alpha}. \end{aligned} \tag{4.25}$$

In the branch (ii) stationary phase or the Q_m-fixed point, ϕ_3 is a non-zero covariant constant and there are no-zero modes of $\bar\psi_{q\dot\alpha}$, $\bar\psi^{\dot\alpha}_{\tilde q}$ and λ_μ. Furthermore, we have $\bar\psi_{q\dot\alpha(2)} = \bar\psi^{\dot\alpha(2)}_{\tilde q} = q^\dagger_{\dot\alpha(2)} = q^{\dot\alpha(2)} = 0$. Thus, the above formula reduces to

$$\left(q^\dagger_{\dot\alpha(1)} q^{\dot\alpha(1)}\right)\langle\phi_3\rangle(ii) = -2m q^\dagger_{\dot\alpha(1)} q^{\dot\alpha(1)}. \tag{4.26}$$

The solution for $\langle\phi_3\rangle\,(ii)$ is $-2m$ as desired.

The formula (4.25) is useful to extract the precise form of the quadratic terms for $iq^\dagger_{\dot\alpha}\phi^a T^a q^{\dot\alpha}$ due to the transverse variables $\bar\psi_{q\dot\alpha(2)}, \bar\psi_{\tilde q}^{\dot\alpha(2)}, q^\dagger_{\dot\alpha(2)}, q^{\dot\alpha(2)}$. From (4.25), we have the relation

$$-\frac{1}{2}\left(q^\dagger_{\dot\alpha(2)}q^{\dot\alpha(2)}\right)\langle\phi_3\rangle\,(ii) = \frac{i}{2}\langle\phi_3\rangle\,(ii)[A_\mu, A^\mu]_3 - \frac{i}{2}[\lambda^\mu,\lambda_\mu]_3 - \bar\psi_{q\dot\alpha(2)}\bar\psi^{\dot\alpha}_{\tilde q(2)} - mq^\dagger_{\dot\alpha(2)}q^{\dot\alpha(2)}. \tag{4.27}$$

Thus, the quadratic term for the deformation term (4.24) is

$$\begin{aligned}\delta^{(2)}\left(\frac{1}{2\pi}\int_X d^4x\sqrt{g}\left(q^\dagger_{\dot\alpha}q^{\dot\alpha} + iq^\dagger_{\dot\alpha}\frac{\phi^a T^a}{m}q^{\dot\alpha} + \frac{\bar\psi^{\dot\alpha}_{\tilde q}\bar\psi_{q\dot\alpha}}{m}\right)\right)(ii)\\ = \int_X d^4x\sqrt{g}\left(\frac{1}{\pi}A^+_\mu A^{\mu-} + \frac{\lambda^+_\mu\lambda^{\mu-}}{2\pi m}\right).\end{aligned} \tag{4.28}$$

Note that this quadratic expansion is compatible with the obvious choice

$$\frac{1}{2\pi}\int_X d^4x\sqrt{g}\left(q^\dagger_{\dot\alpha(2)}q^{\dot\alpha(2)} + \frac{\bar\psi^{\dot\alpha}_{\tilde q(2)}\bar\psi_{q\dot\alpha(2)}}{m}\right), \tag{4.29}$$

due to the Q_m-fixed point equation[18], $\hat\delta_m\chi_{\mu\nu} = F^{+a}_{\mu\nu} + q^\dagger\bar\sigma_{\mu\nu}T^a q = 0$.

4.4. Transverse Path Integral for Branch (ii)

In this branch the gauge symmetry is broken down to $U(1)$ and the $\pm$ components of the adjoint fields[19] and the components of the hypermultiplet with the suppressed color index become the transverse variable. Basically, we will integrate out all the transverse degrees. The relevant quadratic action is given by

$$\begin{aligned}\delta^{(2)}S_m(ii) = \frac{1}{h^2}\int d^4x\sqrt{g}\Big[&4H^{\mu\nu}_+H_{\mu\nu-} - 2iH^{\mu\nu}_+(\mathfrak{D}A_-)^+_{\mu\nu} - 2iH^{\mu\nu}_-(\mathfrak{D}A_+)^+_{\mu\nu} + 4\varphi\chi^{\mu\nu}_+\chi_{\mu\nu-}\\ &+ 2\chi^{\mu\nu}_+(\mathfrak{D}\lambda_-)^+_{\mu\nu} + 2\chi^{\mu\nu}_-(\mathfrak{D}\lambda^+)_{\mu\nu} - 2X^{\alpha(2)}_{\tilde q}X^{(2)}_{q\alpha} + iX^{\alpha(2)}_{\tilde q}\sigma^\mu{}_{\alpha\dot\alpha}\mathfrak{D}_\mu q^{\dot\alpha(2)}\\ &+ i\mathfrak{D}_\mu q^{\dagger(2)}_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}X^{(2)}_{q\alpha} - \varphi\psi^{\alpha(2)}_{\tilde q}\psi^{(2)}_{q\alpha} - i\psi^{\alpha(2)}_{\tilde q}\sigma^\mu{}_{\alpha\dot\alpha}\mathfrak{D}_\mu\bar\psi^{\dot\alpha(2)}_{\tilde q}\\ &- i\mathfrak{D}_\mu\bar\psi^{\dot\alpha(2)}_q\bar\sigma^{\mu\dot\alpha\alpha}\psi^{(2)}_{q\alpha} + \frac{1}{4}\varphi^2\bar\phi_+\bar\phi_- - i\varphi\bar\phi_+(\mathfrak{D}_\mu A^\mu_-) + i\varphi\bar\phi_-(\mathfrak{D}_\mu A^\mu_+)\\ &+ \frac{1}{4}\varphi\eta_+\eta_- + i\eta_+(\mathfrak{D}_\mu\lambda^\mu_-) + i\eta_-(\mathfrak{D}_\mu\lambda^\mu_+) + \frac{h^2}{\pi}A_{\mu+}A^\mu_- + \frac{h^2}{2\pi m}\lambda_{\mu+}\lambda^\mu_-\Big],\end{aligned} \tag{4.30}$$

[18] Obviously, the quadratic expansion in the neighborhood of the fixed point locus should be taken in the Q_m invariant way.

[19] Note that W decomposes as $W = W_3T_3 + W_+T_+ + W_-T_-$.

where $\pm$ in the subscript denotes the $T_\pm$ components while $+$ in the superscript denotes the self-dual part, and the substitution $\varphi = -2m$ is understood

Now we evaluate the transverse integral

$$\frac{1}{\mathrm{vol}(\mathcal{G}_\pm)} \int \mathcal{D}\tilde{Q}_{h(2)} \mathcal{D}Q_{h(2)} \mathcal{D}W_+ \mathcal{D}W_- e^{-\delta^{(2)} S_m(ii) + \delta^{(2)} \hat{v}(ii)}. \tag{4.31}$$

Note the ordering of the path integral measure should be used consistently. We choose unitary gauge in which

$$\phi_\pm = 0, \tag{4.32}$$

where

$$\phi = \varphi T_3 + \phi_+ T_+ + \phi_- T_-. \tag{4.33}$$

In this gauge ϕ has values on the maximal torus (Cartan sub-algebra). By following the standard Faddev-Povov gauge fixing, we introduce a new nilpotent BRST operator δ with the algebra

$$\begin{aligned} \delta\phi_\pm = \pm i c_\pm \varphi, \qquad \delta c_\pm = 0, \\ \delta\phi_3 = 0, \qquad \delta\bar{c}_\pm = b_\pm, \qquad \delta b_\pm = 0, \end{aligned} \tag{4.34}$$

where $c_\pm$ and $\bar{c}_\pm$ are anti-commuting ghosts and anti-ghosts, respectively, and $b_\pm$ are commuting auxiliary fields. The action for gauge fixing terms reads

$$\begin{aligned} S_{m,gauge}(ii) =& \delta\left[\frac{1}{h^2}\int_X i\big(\bar{c}_- * \varphi_+ + \bar{c}_+ * \varphi_-\big)\right] \\ =& \frac{1}{h^2}\int_X \Big[i\big(b_- * \phi_+ + b_+ * \phi_-\big) - \bar{c}_- * (\varphi) c_+ + \bar{c}_+ (\varphi) c_-\Big]. \end{aligned} \tag{4.35}$$

The integrations over the auxiliary fields $b_\pm$ lead to the gauge fixing condition (4.32). The Gaussian integrations over $\bar{c}$ and c give

$$\Big[det\,(\varphi)\Big]^{1/2}_{(c_+,\bar{c}_-)} \Big[det\,(\varphi)\Big]^{1/2}_{(\bar{c}_+,c_-)}. \tag{4.36}$$

Now consider the transverse part involving $\bar{\phi}_\pm$ and $\eta_\pm$. The quadratic action relevant to this sector is given by

$$\begin{aligned} \delta^{(2)} S_m(ii) = \frac{1}{h^2}\int d^4x\sqrt{g}\Big[&\frac{1}{4}\varphi^2\bar{\phi}_+\bar{\phi}_- - i\varphi\bar{\phi}_+(\mathfrak{D}_\mu A^\mu_-) + i\varphi\bar{\phi}_-(\mathfrak{D}_\mu A^\mu_+) \\ &+ \frac{1}{4}\varphi\eta_+\eta_- + i\eta_+(\mathfrak{D}_\mu\lambda^\mu_-) + i\eta_-(\mathfrak{D}_\mu\lambda^\mu_+)\Big]. \end{aligned} \tag{4.37}$$

The integrations over $(\bar{\phi}_+, \bar{\phi}_-)$ and over (η_+, η_-) combined with (4.36) give

$$\begin{aligned}\left[det\left(\frac{\varphi^2}{4\pi}\right)\right]^{-1/2}_{(\phi_+,\phi_-)}\left[det\left(\frac{\varphi}{4}\right)\right]^{1/2}_{(\eta_+,\eta_-)}\left[det\left(\varphi\right)\right]^{1/2}_{(c_+,\bar{c}_-)}\left[det\left(\varphi\right)\right]^{1/2}_{(\bar{c}_+,c_-)}& \\ \equiv\left[det(-2\pi m)\right]^{1/2}_{\Omega^0(\pm)}&.\end{aligned} \tag{4.38}$$

The transverse part involving $H^{\mu\nu}_{\pm}$ and $\chi^{\mu\nu}_{\pm}$ is given by

$$\begin{aligned}\delta^{(2)}S(ii) = \frac{1}{h^2}\int d^4x\sqrt{g}\Big[&4H^{\mu\nu}_+ H_{\mu\nu -} - 2iH^{\mu\nu}_+(\mathfrak{D}A_-)^+_{\mu\nu} - 2iH^{\mu\nu}_-(\mathfrak{D}A_+)^+_{\mu\nu} \\ &+ 4\varphi\chi^{\mu\nu}_+\chi_{\mu\nu -} + 2\chi^{\mu\nu}_+(\mathfrak{D}\lambda_-)^+_{\mu\nu} + 2\chi^{\mu\nu}_-(\mathfrak{D}\lambda_+)^+_{\mu\nu}\Big].\end{aligned} \tag{4.39}$$

The integrations over (H_+, H_-) and over (χ_+, χ_-) give

$$\left[det(4\varphi)\right]^{1/2}_{(\chi_+,\chi_-)} \times \left[det\left(\frac{4}{\pi}\right)\right]^{-1/2}_{(H_+,H_-)} \equiv \left[det(-2\pi m)\right]^{1/2}_{\Omega^{2+}(\pm)}. \tag{4.40}$$

The transverse part involving $X^{\alpha(2)}_{\tilde{q}}, X^{(2)}_{q\alpha}, \psi^{(2)}_{q\alpha}$ and $\psi^{\alpha(2)}_{\tilde{q}}$ is

$$\begin{aligned}\delta^{(2)}S(ii) = \frac{1}{h^2}\int d^4x\sqrt{g}\Big[&-2X^{\alpha(2)}_{\tilde{q}}X^{(2)}_{q\alpha} - \varphi\psi^{\alpha(2)}_{\tilde{q}}\psi^{(2)}_{q\alpha} \\ &+ iX^{\alpha(2)}_{\tilde{q}}\sigma^{\mu}{}_{\alpha\dot{\alpha}}\mathfrak{D}_\mu q^{\dot{\alpha}(2)} + i\mathfrak{D}_\mu q^{\dagger(2)}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}X^{(2)}_{q\alpha} \\ &- i\psi^{\alpha(2)}_{\tilde{q}}\sigma^{\mu}{}_{\alpha\dot{\alpha}}\mathfrak{D}_\mu\bar{\psi}^{\dot{\alpha}(2)}_{\tilde{q}} - i\mathfrak{D}_\mu\bar{\psi}^{\dot{\alpha}(2)}_{q}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\psi^{(2)}_{q\alpha}\Big].\end{aligned} \tag{4.41}$$

The integrations over $(X^{\alpha(2)}_{\tilde{q}}, X^{(2)}_{q\alpha})$ and over $(\psi^{(2)}_{q\alpha}, \psi^{\alpha(2)}_{\tilde{q}})$ give

$$\left[det\left(-\frac{1}{\pi}\right)\right]^{-1}_{(X^{\alpha(2)}_{\tilde{q}},X^{(2)}_{q\alpha})} \times \left[det(2m)\right]^{1/2}_{(\psi^{(2)}_{q\alpha},\psi^{\alpha(2)}_{\tilde{q}})}. \tag{4.42}$$

Now we collect all the remaining terms which came from the various Gaussian integrations

$$\begin{aligned}\delta^{(2)}S_m(ii) = \frac{1}{h^2}\int d^4x\sqrt{g}\Big[&(\mathfrak{D}_\mu A^\mu_-)(\mathfrak{D}_\nu A^\nu_+) - \frac{1}{4}(\mathfrak{D}A_-)^{+\mu\nu}(\mathfrak{D}A_-)^+_{\mu\nu} + \frac{h^2}{\pi}A_{\mu -}A^\mu_+ \\ &+\frac{1}{2m}\left((\mathfrak{D}_\mu\lambda^\mu_-)(\mathfrak{D}_\nu\lambda^\nu_+) - \frac{1}{4}(\mathfrak{D}\lambda_-)^{+\mu\nu}(\mathfrak{D}\lambda_+)^+_{\mu\nu} + \frac{h^2}{\pi}\lambda_{\mu +}\lambda^\mu_-\right) \\ &+\frac{1}{2}\not{\mathfrak{D}}q^{\dagger(2)}_{\dot{\alpha}}\not{\mathfrak{D}}q^{\dot{\alpha}(2)} - \frac{1}{2m}\not{\mathfrak{D}}\bar{\psi}^{(2)}_{q\dot{\alpha}}\not{\mathfrak{D}}\bar{\psi}^{\dot{\alpha}(2)}_{\tilde{q}}\Big].\end{aligned} \tag{4.43}$$

The Gaussian integrals over $(q^{\dagger(2)}_{\dot\alpha}, q^{\dot\alpha(2)})$ and over $(\bar\psi^{(2)}_{q\dot\alpha}, \bar\psi^{\dot\alpha(2)}_{\tilde q})$ give

$$\left[det\left(\frac{\not{\mathfrak{D}}^2}{4\pi}\right)\right]^{-1}_{(q^{\dagger(2)}_{\dot\alpha}, q^{\dot\alpha(2)})}\left[det\left(-\frac{\not{\mathfrak{D}}^2}{2m}\right)\right]_{(\bar\psi^{(2)}_{q\dot\alpha}, \bar\psi^{\dot\alpha(2)}_{\tilde q})}. \tag{4.44}$$

Now we are only left with $A_\pm$ and $\lambda_\pm$ whose quadratic action can be rewritten in a compact form

$$\delta^{(2)}S(ii) = \frac{1}{h^2}\int_X\left(A_+\wedge *(\nabla+\frac{h^2}{\pi})A_- - \frac{1}{2m}\lambda_+\wedge *(\nabla+\frac{h^2}{\pi})\lambda_-\right), \tag{4.45}$$

where $\nabla \equiv \mathfrak{D}\mathfrak{D}^* - \frac{1}{4}\mathfrak{D}^{+*}\mathfrak{D}^+$.

If we integrate over (A_+, A_-) and over (λ_+,λ_-), we get

$$\left[det\left(\frac{\nabla+h^2/\pi}{2\pi}\right)\right]^{-1/2}_{(A_+,A_-)}\times\left[det\left(-\frac{\nabla+h^2/\pi}{m}\right)\right]^{1/2}_{(\lambda_+,\lambda_-)} \equiv \left[det(-2\pi m)\right]^{-1/2}_{\Omega^1(\pm)}. \tag{4.46}$$

Collecting all the determinant ratios (4.38), (4.40), (4.42), (4.44) and (4.46), we have

$$\begin{aligned}
&\left[det(-2\pi m)\right]^{1/2}_{\Omega^0(\pm)\ominus\Omega^1(\pm)\oplus\Omega^{2+}(\pm)}\left[det(-2\pi m)\right]^{-1}_{[\bar\psi^{\dot\alpha(2)}_{\tilde q}]\ominus[\psi^{(2)}_{q\alpha}]}\\
&=(-2\pi m)^{-\frac{1}{2}\times index_\pm(\mathfrak{D}^++\mathfrak{D}^*)}(-2\pi m)^{-(index(\not{D}^E_{\mathfrak{c}})-\frac{1}{8}(x\cdot x-\sigma))}\\
&=(-2\pi m)^{-\frac{1}{2}\times index(d^+_A+d^*_A)-\Delta}\cdot(-2\pi m)^{-index(\not{D}^E_{\mathfrak{c}})+\Delta}\\
&=(-2\pi m)^{-(4k-3\Delta)}\cdot(-2\pi m)^{-index(\not{D}^E_{\mathfrak{c}})}\\
&=(-2\pi m)^{-d(k)-index(\not{D}^E_{\mathfrak{c}})}\\
&=(-2\pi m)^{-\frac{1}{2}\times dim\mathcal{M}(\mathfrak{c},k)} = (-2\pi m)^{-d(\mathfrak{c},k)}.
\end{aligned} \tag{4.47}$$

In the above, we used

$$\begin{aligned}
\frac{1}{2}\times index\left(d^+_A+d^*_A\right)\equiv d(k) &= \frac{1}{2}\left(dimH^1_A - dimH^0_A - dimH^2_A\right)\\
&= 4k - dim(G)\Delta,
\end{aligned} \tag{4.48}$$

where H^i_A denotes the three cohomology groups of the instanton complex (See for example [3]).[20] We also used

$$index\not{D}^E_{\mathfrak{c}} = -k + \frac{rank(E)}{8}(\mathfrak{c}\cdot\mathfrak{c}-\sigma), \tag{4.49}$$

[20] Note that $dim(SU(2)) = 3$. Since we already fixed gauge, we have $index(\mathfrak{D}^++\mathfrak{D}^*) = index(d^+_A+d^*_A)$. The action of $\mathfrak{D}^++\mathfrak{D}^*$ to the gauge singlet (the Cartan subalgebra part) is identical to that of (d^++d^*) which contributes -2Δ to the $index(\mathfrak{D}^++\mathfrak{D}^*)$. So we have

$$index_\pm(\mathfrak{D}^++\mathfrak{D}^*) = index(d^+_A+d^*_A)+2\Delta.$$

We used a similar procedure for the Dirac index.

and

$$\Delta = \frac{1}{8}(x \cdot x - \sigma), \tag{4.50}$$

which follows from the zero-dimensionality of the abelian Seiberg-Witten monopole moduli space $\mathcal{M}(x)$. As a check, we consider the case when $dim\mathcal{M}(x) = 2n$ such that

$$n = \frac{1}{8}(x \cdot x - \sigma) - \Delta. \tag{4.51}$$

The formula (4.47) becomes

$$(-2\pi m)^{-d(k) - index \not{D}^E_{\mathfrak{c}} + n} = (-2\pi m)^{-(dim\mathcal{M}(\mathfrak{c},k) - dim\mathcal{M}(x))/2}, \tag{4.52}$$

which is consistent.

Finally we evaluate the path integral

$$\begin{aligned} \int \mathcal{D}A^+\mathcal{D}A^-\mathcal{D}\lambda^+\mathcal{D}\lambda^- e^{-\frac{1}{h^2}\int_X \left(A_+\wedge *(\nabla+\frac{h^2}{\pi})A_- + \frac{1}{2m}\lambda_+\wedge *(\nabla+\frac{h^2}{\pi})\lambda_-\right)} \\ \times \exp\left(\frac{1}{4\pi^2}\int_\Sigma \left(mA_+\wedge A_- + \frac{1}{2}\lambda_+\wedge\lambda_-\right)\right). \end{aligned} \tag{4.53}$$

The path integral (4.53) can easily be done using the elementary techniques of quantum field theory. (See Appendix A.) The first step is to determine the Green's functions. For A_+ and A_-, we have

$$\Delta_F(x_1 - x_2) = \int \frac{d^4p}{(2\pi)^4}\sqrt{g}e^{-ip\cdot(x_1-x_2)}\frac{h^2}{p^2 + \frac{h^2}{\pi}}, \tag{4.54}$$

where p denotes the Fourier transformed variable or the four-momentum such that

$$\left(\nabla + \frac{h^2}{\pi}\right)\Delta_F(x_1 - x_2) = h^2\delta^{(4)}(x_1 - x_2). \tag{4.55}$$

The integral (4.54) is obviously divergent in the ultraviolet (the large momentum). However, the infinite scaling limit of the metric in a compact manifold is identical to the infinitesimally small limit of the momentum.[21] Thus, the above integral is simply a delta function;

$$\Delta_F(x_1 - x_2) = \pi\delta^{(4)}(x_1 - x_2). \tag{4.56}$$

[21] It may be tempting to think that a similar thing would happen if we set $h^2 \to 0$, which is not the case. The key simplification of the theory comes from the large scaling limit of the metric rather than the semi-classical limit.

The similar analysis for λ_+ and λ_- shows their propagator is given by

$$\Delta_F(x_1 - x_2) = 2\pi m \delta^{(4)}(x_1, x_2). \tag{4.57}$$

Now we can perform the Gaussian integral which gives

$$\exp\left(\frac{m^2}{4\pi^2}\left(\frac{v\cdot v}{2}\right)\right), \tag{4.58}$$

together with the determinant (4.46). Thus, we find

$$\begin{aligned}\frac{1}{\mathrm{vol}(\mathcal{G}_\pm)} \int \mathcal{D}\tilde{Q}_{h(2)}\mathcal{D}Q_{h(2)}\mathcal{D}W_+\mathcal{D}W_- e^{-\delta^{(2)}S_m(ii)+\delta^{(2)}\hat{v}(ii)} \\ = \left(-\frac{2\pi}{m}\right)^{d(k)+index(\not{D}^E_{\mathfrak{c}})} e^{\frac{m^2}{4\pi^2}\left(\frac{v\cdot v}{2}\right)}.\end{aligned} \tag{4.59}$$

Combining (4.59) with (4.23), we have the total contribution of branch (ii),

$$\left\langle e^{\hat{v}}\right\rangle_{m,\mathfrak{c},k}(ii) = \mathcal{N}\left(-\frac{2\pi}{m}\right)^{d(k)+index(\not{D}^E_{\mathfrak{c}})} \sum_x n_x e^{\frac{m^2}{4\pi^2}\left(\frac{v\cdot v}{2}\right)+\frac{m}{2\pi}(v\cdot x)}. \tag{4.60}$$

4.5. The Results

Collecting everything in the previous subsections, we get

$$\begin{aligned}&\left\langle e^{\hat{v}}\right\rangle_{m,\mathfrak{c},k}\\ &= -\left(-\frac{2\pi}{m}\right)^{d_0(\mathfrak{c},k)}\left\langle e^{\hat{v}}\right\rangle_k + \mathcal{N}\left(-\frac{2\pi}{m}\right)^{d(\mathfrak{c},k)}\sum_x n_x e^{\frac{m}{2\pi}(v\cdot x)+\frac{m^2}{4\pi^2}\left(\frac{v\cdot v}{2}\right)},\\ &= -(-1)^{d_0(\mathfrak{c},k)}\left(\frac{2\pi}{m}\right)^{d_0(\mathfrak{c},k)}\left[\left\langle e^{\hat{v}}\right\rangle_k - (-1)^{\Delta}\mathcal{N}\left(\frac{2\pi}{m}\right)^{d(k)}\sum_x n_x e^{\frac{m}{2\pi}(v\cdot x)+\frac{m^2}{4\pi^2}\left(\frac{v\cdot v}{2}\right)}\right],\end{aligned} \tag{4.61}$$

where $d_0(\mathfrak{c},k) = index\left(\not{D}^E_{\mathfrak{c}}\right)$, and $d(\mathfrak{c},k) = d_0(\mathfrak{c},k) + d(k)$. We multiplied the factor -1 to $\left\langle e^{\hat{v}}\right\rangle_k$ since we introduced the opposite orientations for $det(d_A^+ \oplus d_A^*)$ relative to $det(d+d^*)$. Note that we have an additional relative sign $(-1)^{\Delta}$ between the contributions of the two branches. If we replace m with $-m$, we have

$$\begin{aligned}&\left\langle e^{\hat{v}}\right\rangle_{-m,\mathfrak{c},k}\\ &= -\left(\frac{2\pi}{m}\right)^{d_0(\mathfrak{c},k)}\left[\left\langle e^{\hat{v}}\right\rangle_k - \mathcal{N}\left(\frac{2\pi}{m}\right)^{d(k)}\sum_x n_x e^{-\frac{m}{2\pi}(v\cdot x)+\frac{m^2}{4\pi^2}\left(\frac{v\cdot v}{2}\right)}\right]\\ &= -\left(\frac{2\pi}{m}\right)^{d_0(\mathfrak{c},k)}\left[\left\langle e^{\hat{v}}\right\rangle_k - (-1)^{\Delta}\mathcal{N}\left(\frac{2\pi}{m}\right)^{d(k)}\sum_x n_x e^{\frac{m}{2\pi}(v\cdot x)+\frac{m^2}{4\pi^2}\left(\frac{v\cdot v}{2}\right)}\right],\end{aligned} \tag{4.62}$$

where we have used

$$\begin{aligned}\sum_x n_x e^{-\frac{m}{2\pi}(v\cdot x)+\frac{m^2}{4\pi^2}\left(\frac{v\cdot v}{2}\right)} &= (-1)^\Delta \sum_x n_{-x} e^{\frac{m}{2\pi}(v\cdot(-x))+\frac{m^2}{4\pi^2}\left(\frac{v\cdot v}{2}\right)} \\ &= (-1)^\Delta \sum_x n_x e^{\frac{m}{2\pi}(v\cdot x)+\frac{m^2}{4\pi^2}\left(\frac{v\cdot v}{2}\right)}.\end{aligned} \tag{4.63}$$

Thus the relative sign $-(-1)^\Delta$ between the two branches remains unchanged. This is a crucial test since the theory without the bare mass or without hypermultiplet should be independent of m. Note also that the relative sign $-(-1)^\Delta$ does not depend on the *spin*^{c} structure chosen to define TQCD. This is also an important test of consistency since the result for the theory without hypermultiplet (TYM) should be independent of whatever TQCD we are using. We would like to emphasize again, as explained in Sect. 3.2, that the Seiberg-Witten invariants in (4.61) or in (4.62) are independent of the *spin*^{c} structure which defines a particular TQCD.

This is a judicious moment to determine all the invariants. Since the left-hand side of (4.61) or (4.62) is regular for $m=0$, the non-*zero*th powers in m of the right-hand side should vanish order by order. In particular

$$\langle \exp(\hat{v})\rangle_k = (-1)^\Delta \mathcal{N} \sum_{r=0}^{[d(k)/2]} \frac{(v\cdot v)^r}{2^r(d(k)-2r)!r!}\sum_x n_x (v\cdot x)^{d(k)-2r}. \tag{4.64}$$

This is a universal relation independent of the family of TQCD parametrized by the space of the *spin*^{c} structure. At this stage, we can determine the normalization term $\mathcal{N}$ by comparing with the known results, which turns out to be

$$\mathcal{N} = (-1)^\Delta 2^{2+\frac{1}{4}(7\chi+11\sigma)}, \tag{4.65}$$

where the last power of 2 appears in similar fashion with [1][11]. The extra $(-1)^\Delta$ originate from the ambiguity due to trivialization of *det ind*$(\not{\mathcal{D}}^x)$ in the relative orientation between the instanton moduli space and the (abelian) Seiberg-Witten moduli spaces.

It is easy to include the observable $\hat{u}$ and compute the general correlation function

$$\langle \exp(\hat{v}+\tau\hat{u})\rangle_{m,\mathfrak{c},k}\,. \tag{4.66}$$

We replace $\hat{u} = -\frac{1}{8\pi^2}\mathrm{Tr}\,\phi^2$ with its value in branch (ii), $\hat{u}(ii) = m^2/4\pi^2$. A similar manipulation leads to

$$\begin{aligned}\langle \exp(\hat{v}+\tau\hat{u})\rangle_{m,\mathfrak{c},k} = -\left(-\frac{2\pi}{m}\right)^{d_0(\mathfrak{c},k)} &\Bigg(\left\langle e^{\hat{v}+\tau\hat{u}}\right\rangle_k \\ &- 2^{2+\frac{1}{4}(7\chi+11\sigma)}\left(\frac{2\pi}{m}\right)^{d(k)} \sum_x n_x e^{\frac{m}{2\pi}(v\cdot x)+\frac{m^2}{4\pi^2}\left(\frac{v\cdot v}{2}\right)+\tau\frac{m^2}{2\pi^2}}\Bigg).\end{aligned} \tag{4.67}$$

We get

$$\langle\exp(\hat{v}+\tau\hat{u})\rangle_k = 2^{2+\frac{1}{4}(7\chi+11\sigma)}\sum_{r+s=0}^{[d(k)/2]}\frac{\left(\frac{v\cdot v}{2}\right)^r(2\tau)^s}{(d(k)-2r-2s))!r!s!}\sum_x n_x(v\cdot x)^{d(k)-2r-2s}. \quad (4.68)$$

In other words,

$$\left\langle\hat{v}^{d(k)-2s}\hat{u}^s\right\rangle_k = 2^{2+\frac{1}{4}(7\chi+11\sigma)}\sum_{r=0}^{[d(k)/2-s]}\frac{(d-2s)!(2\tau)^s}{(d(k)-2r-2s)!r!}\left(\frac{v\cdot v}{2}\right)^r\sum_x n_x(v\cdot x)^{d(k)-2r-2s}. \quad (4.69)$$

Note that one can write $\langle\exp(\hat{v}+\tau\hat{u})\rangle_k$ as

$$\langle\exp(\hat{v}+\tau\hat{u})\rangle_k = 2^{2+\frac{1}{4}(7\chi+11\sigma)}\left(\frac{2\pi}{m}\right)^{d(k)}\sum_x n_x e^{\frac{m}{2\pi}(v\cdot x)+\frac{m^2}{4\pi^2}\left(\frac{v\cdot v}{2}\right)+\tau\frac{m^2}{2\pi^2}}, \quad (4.70)$$

provided we take the *zero*th order of m only in the formal expansion of RHS. Recall that m was assigned to the U-number 2, so the U-number anomaly cancellation of the path integral of TYM is beautifully summarized in the above formula.

The correlation function $\langle\exp(\hat{v}+\tau\hat{u})\rangle_{\mathfrak{c},k}$ of the massless TQCD can also be obtained by collecting the *zero*th order of m in the formal expansion of (4.67). We have

$$\langle\exp(\hat{v}+\tau\hat{u})\rangle_{\mathfrak{c},k} = 2^{2+\frac{1}{4}(7\chi+11\sigma)}(-1)^{d_0(\mathfrak{c},k)}\left(\frac{2\pi}{m}\right)^{d(\mathfrak{c},k)}\sum_x n_x e^{\frac{m}{2\pi}(v\cdot x)+\frac{m^2}{4\pi^2}\left(\frac{v\cdot v}{2}\right)+\tau\frac{m^2}{2\pi^2}}, \quad (4.71)$$

provided we take the *zero*th order of m only. The above formula summarizes the U-number anomaly cancellation of the path integral of TQCD. There is a subtlety due to the additional sign factor $(-1)^{d_0(\mathfrak{c},k)}$. If we replace m with $-m$, we have

$$\langle\exp(\hat{v}+\tau\hat{u})\rangle_{\mathfrak{c},k} = 2^{2+\frac{1}{4}(7\chi+11\sigma)}\left(\frac{2\pi}{m}\right)^{d(\mathfrak{c},k)}\sum_x n_x e^{\frac{m}{2\pi}(v\cdot x)+\frac{m^2}{4\pi^2}\left(\frac{v\cdot v}{2}\right)+\tau\frac{m^2}{2\pi^2}}. \quad (4.72)$$

This shows that the polynomials vanish unless

$$d_0(\mathfrak{c},k)\equiv -k+\frac{1}{4}(\mathfrak{c}\cdot\mathfrak{c}-\sigma)=0 \text{ mod } 2. \quad (4.73)$$

Since $\mathfrak{c}\cdot\mathfrak{c}=\sigma$ mod 8, the polynomial identically vanishes if the instanton number k is odd. So the total degree of the polynomial $\langle\exp(\hat{v}+\tau\hat{u})\rangle_{\mathfrak{c},k}$ increases as $12\mathbb{Z}$ rather than

$6\mathbb{Z}$. Note that the degree of Donaldson's polynomial increases as $8\mathbb{Z}$.[22] Thus, for an even instanton number k, we have

$$\langle \exp(\hat{v}+\tau\hat{u})\rangle_{\mathfrak{c},k} = 2^{2+\frac{1}{4}(7\chi+11\sigma)} \sum_{r+s=0}^{[d(\mathfrak{c},k)/2]} \frac{\left(\frac{v\cdot v}{2}\right)^r (2\tau)^s}{(d(\mathfrak{c},k)-2r-2s))!r!s!} \sum_x n_x (v\cdot x)^{d(\mathfrak{c},k)-2r-2s}, \tag{4.74}$$

while for an odd instanton number k, we have

$$\langle \exp(\hat{v}+\tau\hat{u})\rangle_{\mathfrak{c},k} = 0. \tag{4.75}$$

By substituting (4.68) to (4.67) we also get

$$\begin{aligned}\left\langle e^{\hat{v}+\tau\hat{u}}\right\rangle_{m,\mathfrak{c},k} =& 2^{2+\frac{1}{4}(7\chi+11\sigma)} \left(-\frac{2\pi}{m}\right)^{d_0(\mathfrak{c},k)} \left[\left(\frac{2\pi}{m}\right)^{d(k)} \sum_x n_x e^{\frac{m}{2\pi}(v\cdot x)+\frac{m^2}{4\pi^2}\left(\frac{v\cdot v}{2}\right)+\tau\frac{m^2}{2\pi^2}}\right.\\ &\left. - \sum_{r+s=0}^{[d(k)/2]} \frac{\left(\frac{v\cdot v}{2}\right)^r (2\tau)^s}{(d(k)-2r-2s))!r!s!} \sum_x n_x (v\cdot x)^{d(k)-2r-2s}\right].\end{aligned} \tag{4.76}$$

The final step of our computation is to construct the generating functional

$$\langle \exp\left(\hat{v}+\lambda\hat{u}\right)\rangle = \sum_k \langle \exp(\hat{v}+\lambda\hat{u})\rangle_k \,. \tag{4.77}$$

Note that

$$d(k) = 4k - 3\Delta, \qquad n_{-x} = (-1)^\Delta n_x. \tag{4.78}$$

The result is

$$\begin{aligned}\langle \exp\left(\hat{v}+\tau\hat{u}\right)\rangle =& 2^{1+\frac{1}{4}(7\chi+11\sigma)} \left[\exp\left(\frac{v\cdot v}{2}+2\tau\right)\sum_x n_x e^{v\cdot x}\right.\\ &\left. + i^\Delta \exp\left(-\frac{v\cdot v}{2}-2\tau\right)\sum_x n_x e^{-iv\cdot x}\right],\end{aligned} \tag{4.79}$$

[22] We count the degrees of $\hat{v}$ and $\hat{u}$ by 2 and 4, repectively. The dimensions of the moduli space $\mathcal{M}(\mathfrak{c},k)$ is proportional to the instanton number k by the factor $2(4-1)=6$. The dimension of the moduli space $\mathcal{M}(k)$ is proportional to the instanton number k by the factor $2\cdot 4 = 8$. These factors are closely related to the anomaly free discrete subgroup of the global $U(1)_{\mathcal{R}}$ symmetry of the underlying physical theories. For the $SU(2)$ and $N_f=0$ theory, the discrete subgroup is Z_8. On the other hand, for $N_f=1$ the anomaly free discrete subgroup is Z_{12} rather than Z_6 [27]. After twisting the $U(1)_{\mathcal{R}}$ charge becomes the U-number. The property that $\langle\exp(\hat{v}+\tau\hat{u})\rangle_{\mathfrak{c},k}$ vanishes for odd k implies the fact that the anomaly free discrete subgroup is Z_{12} rather than Z_6 and vice versa.

which is the formula (1.1) of Witten. Note that

$$\left(\frac{d^2}{d\tau^2} - 2^2\right) \langle \exp\left(\hat{v} + \tau \hat{u}\right)\rangle = 0, \tag{4.80}$$

or equivalently

$$\left\langle \left(\hat{u}^2 - 2^2\right) \hat{z} \right\rangle = 0, \qquad \text{for any } \hat{z}. \tag{4.81}$$

The above two equivalent conditions are called the simple type conditions. One can define the Donaldson series $\mathbb{D}$

$$\begin{aligned} \mathbb{D}(v) &\equiv \frac{1}{2}\left(1 + \frac{1}{2}\frac{d}{d\tau}\right) \langle \exp(\hat{v} + \tau\hat{u})\rangle \bigg|_{\tau=0} \equiv \frac{1}{2}\left\langle (1 + \frac{\hat{u}}{2}) \exp(\hat{v}) \right\rangle \\ &= 2^{1+\frac{1}{4}(7\chi + 11\sigma)} \exp\left(\frac{v\cdot v}{2}\right) \sum_x n_x e^{v\cdot x}. \end{aligned} \tag{4.82}$$

Kronheimer and Mrowka proved that[10]

$$\mathbb{D}(v) = \exp\left(\frac{v\cdot v}{2}\right) \sum_x a_x e^{v\cdot x}, \tag{4.83}$$

where a_x is a (non-zero) rational number and x is a basic class which is an integral lift of the second Stifel-Whitney class $w_2(X)$ on X. The main predictions of the formula (1.1) of Witten are that the basic class of Kronheimer-Mrowka is the Seiberg-Witten class and that

$$a_x = 2^{1+\frac{1}{4}(7\chi+11\sigma)} n_x. \tag{4.84}$$

The formula (4.82) confirms those predictions.[23]

The generating functional for TQCD is defined by

$$\langle \exp\left(\hat{v} + \lambda\hat{u}\right)\rangle_{\mathfrak{c}} = \sum_k \langle \exp(\hat{v} + \tau\hat{u})\rangle_{\mathfrak{c},k}\,. \tag{4.85}$$

Note that

$$d(\mathfrak{c}, k) = 3k - 3\Delta + \frac{1}{4}(\mathfrak{c}\cdot\mathfrak{c} - \sigma), \qquad n_{-x} = (-1)^{\Delta} n_x, \tag{4.86}$$

[23] To complete the proof, the factor $2^{\frac{1}{4}(7\chi+11\sigma)}$ should be derived without referring to the known mathematical results. A step has been made by Witten in [13]. Our results suggest that the factor is universal for the family of TQCD with $N_f = 0, \ldots, 4$. Then, the factor may be derived by imposing the exact self-duality of the critical ($N_f = 4$) theory [27][23].

and the summation over k in (4.85) is only for even k. We have

$$\left\langle \exp\left(\hat{v}+\tau\hat{u}\right)\right\rangle_{\mathfrak{c}} = \frac{2}{3}\cdot 2^{1+\frac{1}{4}(7\chi+11\sigma)}\Bigg[\exp\left(\frac{v\cdot v}{2}+2\tau\right)\sum_x n_x e^{(v\cdot x)} + (-1)^{\Delta} e^{\frac{\pi i}{12}(\mathfrak{c}\cdot\mathfrak{c}-\sigma)}\exp\left(e^{-\frac{2\pi i}{3}}\left(\frac{v\cdot v}{2}+2\tau\right)\right)\sum_x n_x e^{e^{-\frac{\pi i}{3}}(v\cdot x)} + e^{\frac{\pi i}{6}(\mathfrak{c}\cdot\mathfrak{c}-\sigma)}\exp\left(e^{-\frac{4\pi i}{3}}\left(\frac{v\cdot v}{2}+2\tau\right)\right)\sum_x n_x e^{e^{-\frac{2\pi i}{3}}(v\cdot x)}\Bigg]. \tag{4.87}$$

Note that

$$\left(\frac{d^3}{d\tau^3}-2^3\right)\langle\exp\left(\hat{v}+\tau\hat{u}\right)\rangle_{\mathfrak{c}} = 0, \qquad \left\langle\left(\hat{u}^3-2^3\right)\hat{z}\right\rangle_{\mathfrak{c}} = 0, \qquad \text{for any } \hat{z}. \tag{4.88}$$

This is a generalized simple type condition for the polynomial invariants (4.87). Following Kronheimer and Mrowka, one can define the Donaldson series $\mathbb{D}(v)_{\mathfrak{c}}$ as

$$\mathbb{D}(v)_{\mathfrak{c}} \equiv \frac{1}{2}\left(1+\frac{1}{2}\frac{d}{d\tau}+\frac{1}{2^2}\frac{d^2}{d\tau^2}\right)\langle\exp(\hat{v}+\tau\hat{u})\rangle_{\mathfrak{c}}\Bigg|_{\tau=0} \equiv \frac{1}{2}\left\langle\left(1+\frac{\hat{u}}{2}+\frac{\hat{u}^2}{2^2}\right)\exp(\hat{v})\right\rangle_{\mathfrak{c}}. \tag{4.89}$$

We have

$$\mathbb{D}(v)_{\mathfrak{c}} = 2^{1+\frac{1}{4}(7\chi+11\sigma)}\exp\left(\frac{v\cdot v}{2}\right)\sum_x n_x e^{v\cdot x}. \tag{4.90}$$

Thus we get the same Donaldson series.

5. Relations with the Physical Theory

In paper [7], Seiberg and Witten studied the exact low energy effective theory of the $N=2$ supersymmetric $SU(2)$ Yang-Mills theory on the flat 4-manifolds. It turns out that the exact low energy effective theory can be determined by an analytic pre-potential which can be expressed in terms of an auxiliary elliptic curve varying over the quantum moduli space which parametrizes the different vacua. The elliptic curve is given by

$$y^2 = (x^2-\Lambda^4)(x-\hat{u}), \tag{5.1}$$

where Λ is the dynamically generated scale of the theory. In the finite region of the quantum moduli space, there are two singularities at $\hat{u}=\pm\Lambda^2$. At these two singular points, a new massless particle appears, which forms $N=2$ supersymmetric $U(1)$ hypermultiplets. Furthermore, the theory is weakly coupled near the singularities.

The above two singular points also correspond to the two vacua of the $N = 1$ theory. For a Kähler manifold with $b_2^+ > 1$, one can perturb the theory by adding $N = 2$ breaking, but $N = 1$ preserving mass term related to the holomorphic two-form [11]. In the large scale limit of the Kähler metric, the dominant contributions to the path integral only come from these two points. Witten argues that a similar thing is happening in a simple type manifold and that the contributions only come from a neighborhood of singular points [1]. In computing the path integrals of the TYM theory, one expands all the operators in terms of operators in the low energy effective theory. The simple type condition (4.81) arises when one replaces the operator $\hat{u}$ in terms of the c-number $\pm\Lambda^2$.

In this paper, we computed the topological correlation functions of the twisted $N = 2$ supersymmetric Yang-Mills theory coupled with hypermultiplet having the bare mass. Our purely semi-classical computation shows that the path integral of the TYM theory (Donaldson invariants) is expressed in terms of the branch (ii) contributions due to the $U(1)$ massless hypermultiplet. Note that such a dramatic localization of the path integral appears in the large scale limit of the metric. In that limit, the metric on the Riemann manifolds X becomes everywhere nearly flat. Thus, the topologically equivalent description of the twisted $N = 2$ SYM theory can also be the physically equivalent description of the untwisted theory in the low energy. This is reversing the logic of Witten. An interesting comparison is that we replace $\hat{u}$ with the bare mass of the hypermultiplet instead of the dynamically generated scaling parameter.

Our concrete and direct computation, then, clearly predicts the vacuum structure of the underlying physical theories. The simple type conditions (4.81) and (4.88) can be viewed as the predictions of the singularities in the quantum moduli spaces. The vacuum structure of the underlying physical theory of our model was also determined by Seiberg and Witten [27]. The simple type condition (4.88) is identical to the locus of the 3 singularities in the quantum moduli space of the $N = 2$ $SU(2)$ SYM theory coupled with one hypermultiplet. Our results can be viewed as a non-trivial check of their solutions. Our result also suggests that there should be some intriguing structure hidden in the solutions of Seiberg-Witten. Note also that the TYM theory and TQCD are governed by the same data and define the same Donaldson series. They belong to the same universality class. We believe those strange interrelations are originated from the critical theory $N_f = 4$.

The results in this paper can be generalized to the general gauge group. In the second paper of the series [18], we will completely determine the topological correlation functions of $SU(N_c)$ TQCD on a simple type manifold coupled with hypermultiplets in the fundamental representation having the bare mass. Similarly to this paper, we can also obtain the invariants of the massless theory and the $SU(N_c)$ Donaldson-Witten invariants as well. The relation between the vacuum structures of underlying physical theories and the generalized version of the simple type conditions for those invariants, as well as the universality of the Donaldson series all remain unchanged.

We conclude this paper with an interesting question. Fintushel and Stern determined the blowup formula for the $SU(2)$ Donaldson invariants [28]. Surprisingly, their formula is stated in terms of the same elliptic curve (5.1) which parametrizes the quantum moduli space. They also pointed out the relation between the simple type condition (4.81) and the discriminant locus of the curve. An analogous blowup formula might be constructed for the invariants (4.87). It will be interesting to see if such a formula recovers the elliptic curve for $N_f = 1$ theory [27].

After this work was completed, the announced paper [29] of Pidstrigach and Tyurin and a related work [30] of Labastida and Mariño with this paper appeared. In [30], Labastida and Mariño calculated the topological correlation functions of the $SU(2)$ theory with $N_f = 1$ using the physical method on a spin manifold with the canonical $spin_c$ structure. They also informed us that the twisting of N=2 hypermultiplets were considered in [31] and further elaborated in [32] .

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Appendix A.

In this appendix, we briefly explain the relation between the $2m$-point Green's function $G(x_1, \ldots, x_{2m})$ for a free field theory and a multilinear form $Q^{(m)}$ on $H_2(X)$. We will calculate the path integral of the simplest possible quantum field theory. The calculations in this paper are just some slightly elaborated variations of the following model. We will not consider the supersymmetric case.

Let X be a compact oriented Riemann four-manifold. The intersection form Q is a bilinear form

$$Q : H_2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z}) \to \mathbb{Z}. \tag{A.1}$$

Let $\Sigma_1, \Sigma_2 \in H_2(X;\mathbb{Z})$ be closed surfaces representing two dimensional homology cycles. The intersection number is defined by the algebraic sum of the number of transverse intersection points counted with a sign $\pm$ depending on the orientation near an intersection point p

$$TX_p = T_p\Sigma_1 \oplus T_p\Sigma_2. \tag{A.2}$$

We used notation $Q(\Sigma_1, \Sigma_2) = v_1 \cdot v_2$. This definition also makes sense for the self-intersection number $v \cdot v$; one can perturb two closed surfaces within the same homology class such that they intersect transversely at points. The multi-linear form $Q^{(m)}$ on $H_2(X)$ defined by [3]

$$Q^{(m)}(\Sigma_1, ..., \Sigma_{2m}) = \frac{1}{2^m m!}\sum_{\sigma\in S_{2m}} Q(\Sigma_{\sigma(1)}, \Sigma_{\sigma(2)}) \times \ldots \times Q(\Sigma_{\sigma(2m-1)}, \Sigma_{\sigma(2m)}), \tag{A.3}$$

where $Q(.,.)$ denotes the intersection form. For example

$$Q^{(2)}(\Sigma_1, \ldots, \Sigma_4) = (\Sigma_1\cdot\Sigma_2)(\Sigma_3\cdot\Sigma_4) + (\Sigma_1\cdot\Sigma_3)(\Sigma_2\cdot\Sigma_4) + (\Sigma_1\cdot\Sigma_4)(\Sigma_2\cdot\Sigma_3). \tag{A.4}$$

Now we consider a simple Gaussian integral

$$Z[J] = \int \mathcal{D}\phi e^{-\frac{1}{2}\int(\phi(x), A\phi(x)) + \int(J(x),\phi(x))}. \tag{A.5}$$

The Gaussian integral over ϕ gives

$$Z[J] = [det(A)]^{-1/2} e^{\frac{1}{2}\int(J, A^{-1}J)} = [det(A)]^{-1/2} e^{\frac{1}{2}\int d^4x d^4y J(x)\Delta_F(x-y)J(y)}, \tag{A.6}$$

where $\Delta_F(x-y)$ is called the Feynman propagator, defined by

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x-y)}}{(\widetilde{A^{-1}})}, \tag{A.7}$$

where p is the Fourier transformation variable (four-momentum) for x and $\tilde{A}$ is the Fourier transform of A. One usually defines

$$\langle \phi(x_1)\dots\phi(x_n)\rangle = \frac{\delta^n}{\delta J(x_1)\cdots\delta J(x_n)} Z(J)|_{J=0} = G(x_1,\dots,x_n). \tag{A.8}$$

Now we expand $Z[J]$

$$Z[J] = [det(A)]^{-1/2} \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \cdots d^4x_n G_n(x_1,\dots,x_n) J(x_1)\dots J(x_n). \tag{A.9}$$

By inspection, we find the odd order Green's functions vanish and

$$G_2(x_1,x_2) = \Delta_F(x_1 - x_2). \tag{A.10}$$

One can also find that

$$G_{2n}(x_1,...,x_{2m}) = \frac{1}{2^m m!} \sum_{\sigma\in S_{2m}} G_2(x_{\sigma(1)},x_{\sigma(2)}) \times \ldots \times G_2(x_{\sigma(2m-1)},x_{\sigma(2m)}). \tag{A.11}$$

Now we assume that the operator A is the identity such that $G_2(x,y) = \delta^{(4)}(x,y)$. Now we assume that $J(x)$ is a de Rham current supported on a closed surface Σ representing a homology cycle such that $\int_X (J,\phi) = \int_\Sigma \hat{j}(\phi)$. We can write

$$Z[J] = \left\langle e^{\int_\Sigma \hat{\phi}(x)} \right\rangle = e^{\frac{1}{2}\int d^4x d^4y J(x)\delta^{(4)}(x-y)J(y)}. \tag{A.12}$$

Near an intersection point p, we can write the exponent

$$\int d^4x d^4y J(x)\delta^{(4)}(x,y)J(y) = \int_{\Sigma_1\times\Sigma_2} d^2x d^2y \delta^{(4)}(x,y) = \pm 1, \tag{A.13}$$

where ± 1 is determined by the orientation (A.2). Thus we have

$$Z[J] = e^{\frac{v\cdot v}{2}}. \tag{A.14}$$

We can also use the version (A.9), we have

$$Z[J] = \sum_{n=0}^{\infty} \frac{1}{(2n)!} Q^{(n)}(\Sigma_1,...,\Sigma_{2n}) = \sum_{n=0}^{\infty} \left(\frac{1}{2^n n!} q^n \right), \tag{A.15}$$

where $q = v\cdot v$. When the surfaces Σ_i belong to the different homology classes, the notation $Q^{(n)}(\Sigma_1,\dots,\Sigma_{2n})$ will be more appropriate.

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