THE PROBLEM OF CONSTRUCTING THE CURRENT OPERATORS IN QUANTUM FIELD THEORY

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Abstract

Lorentz invariance of the current operators implies that they satisfy the well-known commutation relations with the representation operators of the Lorentz group. It is shown that if the standard construction of the current operators in quantum field theory is used then the commutation relations are broken by the Schwinger terms.

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In any relativistic quantum theory the system under consideration is described by some unitary representation of the Poincare group. The electromagnetic or weak current operator $\hat{J}^{\mu}(x)$ for this system (where $\mu = 0, 1, 2, 3$ and x is a point in Minkowski space) should satisfy the following necessary conditions.

Let $\hat{U}(a) = exp(\imath \hat{P}_{\mu} a^{\mu})$ be the representation operator corresponding to the displacement of the origin in spacetime translation of Minkowski space by the four-vector a. Here $\hat{P} = (\hat{P}^0, \hat{\mathbf{P}})$ is the operator of the four-momentum, $\hat{P}^0$ is the Hamiltonian, and $\hat{\mathbf{P}}$ is the operator of ordinary momentum. Let also $\hat{U}(l)$ be the representation operator corresponding to $l \in SL(2, C)$. Then

$$\hat{U}(a)^{-1} \hat{J}^{\mu}(x) \hat{U}(a) = \hat{J}^{\mu}(x - a) \tag{1}$$

$$\hat{U}(l)^{-1} \hat{J}^{\mu}(x) \hat{U}(l) = L(l)^{\mu}_{\nu} \hat{J}^{\nu}(L(l)^{-1} x) \tag{2}$$

where $L(l)$ is the element of the Lorentz group corresponding to l and a sum over repeated indices $\mu, \nu = 0, 1, 2, 3$ is assumed.

As follows from Eq. (1)

$$\hat{J}^\mu(x) = exp(\imath \hat{P}x)\hat{J}^\mu(0)exp(-\imath \hat{P}x) \tag{3}$$

Therefore, if the operators $\hat{P}$ are known, the problem of constructing $\hat{J}^\mu(x)$ can be reduced to that of constructing the operator $\hat{J}^\mu(0)$ with the correct properties.

Let $\hat{M}^{\mu\nu}$ ($\hat{M}^{\mu\nu} = -\hat{M}^{\nu\mu}$) be the representation generators of the Lorentz group. Then, as follows from Eq. (2), Lorentz invariance of the current operator implies

$$[\hat{M}^{\mu\nu}, \hat{J}^\rho(0)] = -\imath(g^{\mu\rho}\hat{J}^\nu(0) - g^{\nu\rho}\hat{J}^\mu(0)) \tag{4}$$

where $g^{\mu\nu}$ is the metric tensor in Minkowski space.

When the S-matrix is calculated in the framework of perturbation theory, Eq. (3) is used explicitly while Eq. (4) is not. It is shown instead (see, for example, Refs. [1, 2]) that Lorentz invariance can be ensured by using the appropriate definition of the T-product (taking into account the Schwinger terms [3]). However when we consider electromagnetic or weak properties of strongly interacting systems, perturbation theory does not apply and, for consistency, Eqs. (1) and (2) must be satisfied. One usually believes that, although the actual construction of the operator $\hat{J}^\mu(x)$ in QCD is very difficult technical problem, in principle QCD makes it possible to construct this operator in such a way that Eqs. (1) and (2) will be satisfied. The purpose of the present paper is to pay attention to the fact that the standard procedure of constructing the current operators in quantum field theory leads to the operators which do not satisfy Eq. (2).

In the standard formulation of quantum field theory the operators $\hat{P}_\mu$, $\hat{M}_{\mu\nu}$ are given by

$$\hat{P}_\mu = \int \hat{T}^\nu_\mu d\sigma_\nu(x), \quad \hat{M}_{\mu\nu} = \int \hat{M}^\rho_{\mu\nu} d\sigma_\rho(x) \tag{5}$$

where $\hat{T}^\nu_\mu$ and $\hat{M}^\rho_{\mu\nu}$ are the energy-momentum and angular momentum tensors and $d\sigma_\mu(x) = \lambda_\mu \delta(\lambda x - \tau)d^4x$ is the volume element of the space-like hypersurface defined by the time-like vector λ $(\lambda^2 = 1)$ and the evolution parameter τ. The initial commutation relations for the field operators are given on the hypersurface $\sigma_\mu(x)$.

For simplicity we will consider the well-known case of QED where the interaction Lagrangian is given by $L_{int}(x) = -e\hat{J}^\mu(x)\hat{A}_\mu(x)$, e is the electron charge and $\hat{A}_\mu(x)$ is the operator of the photon field. Then it is easy to show (see, for example, Refs. [4, 5]) that

$$\hat{P}^\mu = P^\mu - \lambda^\mu \int L_{int}(x)\delta(\lambda x - \tau)d^4x,$$
$$\hat{M}^{\mu\nu} = M^{\mu\nu} - \int L_{int}(x)(x^\nu\lambda^\mu - x^\mu\lambda^\nu)\delta(\lambda x - \tau)d^4x \qquad (6)$$

where we use P^μ and $M^{\mu\nu}$ to denote the four-momentum operator and the generators of the Lorentz group in the case when all interactions are turned off.

It is well-known (see, for example, Refs. [4, 6]) that the operator $\hat{J}^\mu(x)$ in QED is given by

$$\hat{J}^\mu(x) = \frac{1}{2}[\hat{\bar{\psi}}(x), \gamma^\mu\hat{\psi}(x)] \qquad (7)$$

where $\hat{\psi}(x)$ is the Heisenberg operator of the electron-positron field. If $x = 0$ this operator coincides with the free operator $\psi(0)$ in the interaction representation and therefore, as follows from Eq. (7), $\hat{J}^\mu(0) = J^\mu(0)$ where $J^\mu(x)$ is the current operator in the free theory.

The most often considered case is $\tau = 0$, $\lambda = (1, 0, 0, 0)$. Then $\delta(\lambda x - \tau)d^4x = d^3\mathbf{x}$ and the integration in Eq. (6) is taken over the hyperplane $x^0 = 0$. Therefore, as follows from Eq. (6), $\hat{\mathbf{P}} = \mathbf{P}$, and, as follows from Eq. (3), the operator $\hat{J}^\mu(0, \mathbf{x}) \equiv \hat{J}^\mu(\mathbf{x})$ is free, i.e. $\hat{J}^\mu(\mathbf{x}) = J^\mu(\mathbf{x})$. It is also obvious that $\hat{A}^\mu(\mathbf{x}) = A^\mu(\mathbf{x})$.

Since $J^\mu(0)$ satisfies Eq. (4) if $\hat{M}^{\mu\nu} = M^{\mu\nu}$, it follows from the second expression in Eq. (6) that $\hat{J}^\nu(0)$ will satisfy Eq. (4) if

$$\int x^i A_\mu(\mathbf{x})[J^\mu(\mathbf{x}), J^\nu(0)]d^3\mathbf{x} = 0 \quad (i = 1, 2, 3) \qquad (8)$$

for all ν.

It is well-known that if the standard equal-time commutation relations are used naively then the commutator in Eq. (8) vanishes for all μ, ν and therefore this equation is satisfied. However the famous Schwinger result [3] is

$$[J^i(\mathbf{x}), J^0(0)] = C\frac{\partial}{\partial x^i}\delta(\mathbf{x}) \qquad (9)$$

where C is some (infinite) constant. Therefore Eq. (8) is not satisfied and the current operator $\hat{J}^\mu(x)$ does not satisfy Lorentz invariance.

Let us now consider the following question. While the arguments given in Ref. [3] prove that the commutator in Eq. (9) cannot vanish, one might doubt whether the singularity of the commutator is indeed given by the right hand side of this expression. However it is easy to show that only this form of the commutator is compatible with the continuity equation $\partial \hat{J}^{\mu}(x)/\partial x^{\mu} = 0$. Indeed, as follows from this equation and Eq. (3), $[\hat{J}^{\mu}(0), \hat{P}_{\mu}] = 0$. Since $\hat{J}^{\mu}(0) = J^{\mu}(0)$ and $J^{\mu}(0)$ satisfies the condition $[J^{\mu}(0), P_{\mu}] = 0$, it follows from Eq. (6) that the continuity equation is satisfied only if

$$\int A_{\mu}(\mathbf{x})[J^{\mu}(\mathbf{x}), J^{0}(0)]d^{3}\mathbf{x} = 0 \tag{10}$$

In turn this relation is satisfied only if $[J^{0}(\mathbf{x}), J^{0}(0)] = 0$ and Eq. (9) is valid, since $div(\mathbf{A}(\mathbf{x})) = 0$.

As pointed out by Dirac [7], any physical system can be described in different forms of relativistic dynamics. By definition, the description in the point form implies that the operators $\hat{U}(l)$ are the same as for noninteracting particles, i.e. $\hat{U}(l) = U(l)$ and $\hat{M}^{\mu\nu} = M^{\mu\nu}$, and thus interaction terms can be present only in the four-momentum operators $\hat{P}$ (i.e. in the general case $\hat{P}^{\mu} \neq P^{\mu}$ for all μ). The description in the instant form implies that the operators of ordinary momentum and angular momentum do not depend on interactions, i.e. $\hat{\mathbf{P}} = \mathbf{P}$, $\hat{\mathbf{M}} = \mathbf{M}$ ($\hat{\mathbf{M}} = (\hat{M}^{23}, \hat{M}^{31}, \hat{M}^{12})$), and therefore interaction terms may be present only in $\hat{P}^{0}$ and the generators of the Lorentz boosts $\hat{\mathbf{N}} = (\hat{M}^{01}, \hat{M}^{02}, \hat{M}^{03})$. In the front form with the marked z axis we introduce the + and - components of the four-vectors as $x^{+} = (x^{0} + x^{z})/\sqrt{2}$, $x^{-} = (x^{0} - x^{z})/\sqrt{2}$. Then we require that the operators $\hat{P}^{+}, \hat{P}^{j}, \hat{M}^{12}, \hat{M}^{+-}, \hat{M}^{+j}$ ($j = 1, 2$) are the same as the corresponding free operators, and therefore interaction terms may be present only in the operators $\hat{M}^{-j}$ and $\hat{P}^{-}$.

In quantum field theory the form of dynamics depends on the choice of the hypersurface $\sigma_{\mu}(x)$. In particular, it is clear from the above consideration that the choice $\tau = 0$, $\lambda = (1, 0, 0, 0)$ leads to the instant form [7]. The front form can be formally obtained from Eq. (6) as follows. Consider the vector λ with the components

$$\lambda^{0} = \frac{1}{(1 - v^{2})^{1/2}}, \quad \lambda^{j} = 0, \quad \lambda^{3} = -\frac{v}{(1 - v^{2})^{1/2}} \quad (j = 1, 2) \tag{11}$$

Then taking the limit $v \to 1$ in Eq. (6) we get

$$\hat{P}^{\mu} = P^{\mu} - \omega^{\mu} \int L_{int}(x)\delta(x^{+})d^{4}x,$$

$$\hat{M}^{\mu\nu} = M^{\mu\nu} - \int L_{int}(x)(x^\nu\omega^\mu - x^\mu\omega^\nu)\delta(x^+)d^4x \tag{12}$$

where the vector ω has the components $\omega^- = 1$, $\omega^+ = \omega^j = 0$. It is obvious that the generators (12) are given in the front form and that's why Dirac [7] related this form to the choice of the light front $x^+ = 0$.

By analogy with the above consideration it is easy to show that the standard light front quantization also leads to the current operator which does not satisfy Eq. (2). Let us also note that if the theory should be invariant under the space reflection or time reversal, the corresponding representation operators in the front form $\hat{U}_P$ and $\hat{U}_T$ are necessarily interaction dependent (this is clear, for example, from the relations $\hat{U}_P P^+ \hat{U}_P^{-1} = \hat{U}_T P^+ \hat{U}_T^{-1} = \hat{P}^-$) and the operator $\hat{J}^\mu(0)$ should satisfy the conditions

$$\hat{U}_P(\hat{J}^0(0), \hat{\mathbf{J}}(0))\hat{U}_P^{-1} = \hat{U}_T(\hat{J}^0(0), \hat{\mathbf{J}}(0))\hat{U}_T^{-1} = (\hat{J}^0(0), -\hat{\mathbf{J}}(0)) \tag{13}$$

Therefore it is not clear whether these conditions are compatible with the relation $\hat{J}^\mu(0) = J^\mu(0)$. However this difficulty is a consequence of the difficulty with Eq. (2) since, as noted by Coester [8], the interaction dependence of the operators $\hat{U}_P$ and $\hat{U}_T$ in the front form does not mean that there are discrete dynamical symmetries in addition to the rotations about transverse axes. Indeed, the discrete transformation P_2 such that $P_2\,x := \{x^0, x_1, -x_2, x_3\}$ leaves the light front $x^+ = 0$ invariant. The full space reflection P is the product of P_2 and a rotation about the 2-axis by π. Thus it is not an independent dynamical transformation in addition to the rotations about transverse axes. Similarly the transformation TP leaves $x^+ = 0$ invariant and $T = (TP)P_2R_2(\pi)$.

The fact that the choice $\hat{J}^\mu(0) = J^\mu(0)$ is incompatible with Lorentz invariance was pointed out by several authors investigating relativistic effects in nuclear physics (see, for example, Refs. [9, 10]). In our opinion this problem is in fact algebraic. Indeed, in quantum field theory the standard quantization procedure implies that $\hat{J}^\mu(0) = J^\mu(0)$ while some of the operators $\hat{M}^{\mu\nu}$ contain interaction terms. Since there is no reason to believe that they commute with $J^\mu(0)$, it is reasonable to conclude that Eq. (4) is not satisfied and therefore Eq. (2) is not satisfied too.

At the same time, if $\hat{J}^\mu(0) = J^\mu(0)$ then Eq. (4) is obviously satisfied in the point form. In Ref. [7] the point form was related to the hypersurface $t^2 - \mathbf{x}^2 > 0$, $t > 0$, but as argued by Sokolov [11] the point form should be related to the hyperplane orthogonal to the four-velocity of the system under consideration. In view of the above discussion this problem deserves investigation.

Let us summarize our discussion. Lorentz invariance of the operator $\hat{J}^\mu(x)$ implies that the commutation relations (2) are satisfied. However if the standard construction of $\hat{J}^\mu(x)$ in quantum field theory is used then the inevitable presence of the Schwinger terms leads to the conclusion that these relations are not satisfied.

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References

[1] J.D.Bjorken, Phys. Rev. **147**, 1467 (1966).

[2] C.Itzykson and J.-B.Zuber. Quantum Field Theory (McGraw-Hill Book Company, New York 1982).

[3] J.Schwinger, Phys. Rev. Lett. **3**, 296 (1959).

[4] A.I.Akhiezer and V.B.Berestetsky. Quantum Electrodynamics (Nauka, Moscow 1969).

[5] N.N.Bogolubov and D.V.Shirkov. Introduction to the Theory of Quantized Fields (Nauka, Moscow 1976).

[6] J.D.Bjorken and S.D.Drell. Relativistic Quantum Fields (McGraw-Hill Book Company, New York 1976).

[7] P.A.M.Dirac, Rev. Mod. Phys. **21**, 392 (1949).

[8] F.Coester. Private Communication of May 23, 1995.

[9] I.L.Grach and L.A.Kondratyuk, Yad. Fiz. **39**, 316 (1984).

[10] P.L.Chung, F.Coester, B.D.Keister and W.N.Polyzou, Phys. Rev. **C37**, 2000 (1988).

[11] S.N.Sokolov, in *Problems in High Energy Physics and Quantum Field Theory* (IHEP, Protvino, 1984) p. 85.