Applying black hole perturbation theory to numerically generated spacetimes

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Abstract

Nonspherical perturbation theory has been necessary to understand the meaning of radiation in spacetimes generated through fully nonlinear numerical relativity. Recently, perturbation techniques have been found to be successful for the time evolution of initial data found by nonlinear methods. Anticipating that such an approach will prove useful in a variety of problems, we give here both the practical steps, and a discussion of the underlying theory, for taking numerically generated data on an initial hypersurface as initial value data and extracting data that can be considered to be nonspherical perturbations.

I. INTRODUCTION

The formation of a black hole is, in principle, one of the most efficient mechanisms for generation of gravitational waves. Such sources tie together two major research initiatives. Laser interferometric gravity wave detectors [1] hold out a promise of the detection of gravitational waves from astrophysical events. To interpret the results of the gravitational wave signals, and to help find signals in the detector noise, a broad and detailed knowledge will be needed of astrophysical gravitational waveforms. This is one of the underlying motivations for the "grand challenge" [2] in high performance computing, aimed at computing the coalescence of black hole binaries.

Evolving numerical spacetimes and extracting outgoing radiation waveforms is indeed a challenge. In a straightforward numerical approach, a good estimate of the asymptotic waveform requires long numerical evolutions so that the emitted waves can be propagated far from the source. The necessary long evolutions are difficult for a number of reasons. General difficulties include throat stretching when black holes form, numerical instabilities

associated with curvilinear coordinate systems, and the effects of outer boundary conditions which are approximate. [3]

We suggest here that at least part of the cure for this problem may lie in the use of the theory and techniques of nonspherical perturbations of the Schwarzschild spacetime ("NPS"). By this we mean the techniques for treating spacetimes as deviations, first order in some smallness parameter, from the Schwarzschild spacetime. These techniques differ from "linearized theory" which treats perturbations of the spacetime from Minkowski spacetime and which cannot describe black holes. The basic ideas and methods were set down by many authors and lead to "wave equations" for the even parity [4] and odd parity [5] perturbations.

NPS has been used to compute outgoing radiation waveforms from a wide variety of black hole processes, including the scattering of waves [6], particles falling into a hole [7], and stellar collapse to form a hole [8]. The general scheme of NPS also underlies the techniques for extraction of radiation from numerically evolved spacetimes [9]. NPS computations have recently been used in conjunction with fully numerical evolution, as a code test [10] and as a strong-field radiation extraction procedure [3].

Here we are interested in another sort of application of NPS theory. To understand such applications we consider an example: Two very relativistic neutron stars falling into each other, coalescing and forming a horizon, as depicted in Fig. 1. The curve "hypersurface," in Fig. 1, indicates a spacelike "initial" surface. The spacetime can be divided into three regions by this initial surface and the horizon. The early evolution, in region I, below the initial hypersurface, is highly dynamical and nonspherical. Spherical perturbation theory is clearly inapplicable. Above the initial surface the spacetime remains highly nonspherical in region II inside the event horizon, but outside the event horizon, in region III, it may be justified to consider the spacetime to be a perturbation of a Schwarzschild spacetime. This is essentially guaranteed if the initial hypersurface is chosen late enough, in some sense, after the formation of the horizon. The evolution in region III, then, is determined by cauchy data on the initial hypersurface exterior to the horizon. It is important to note that this is made possible by the fact that the horizon is a causal boundary which shields the outer region from the dynamics of the highly nonspherical central region.

The scheme inherent in this division of spacetime has the potential greatly to increase the efficiency of the computation of the radiation generated when strong field sources form black holes. If one starts from the cauchy data on the initial hypersurface, one can evolve forward in time with the linear equations of perturbation theory. Many of the long-time evolution problems of numerical relativity are avoided and the interpretation of the computed fields in terms of radiation is immediate.

The approach suggested would then seem to be: Use numerical relativity up to the initial hypersurface; use the techniques of nonspherical perturbations in the future of the initial hypersurface. In fact, the efficiency that can be achieved may be even greater. In the early, highly nonspherical, pre-initial hypersurface phase of the development of the spacetime, there may be relatively little generation of gravitational radiation. By using a computational technique which suppresses the radiative degrees of freedom one may be able to compute the early stages of evolution relatively easily. There are two very recent examples of just such applications of this viewpoint. Price and Pullin [11] used as initial data the Misner's [12] solution to the initial value equations for two momentarily stationary black holes. Abrahams and Cook [13] considered two holes moving towards each other, and used numerical

2

values of the initial value equations. In neither case was there *any* use of fully nonlinear numerical evolution. The rather remarkable success of both computations suggests that there is something robust about the underlying idea of separating horizon-forming astrophysical scenarios into an early phase with no radiation and a late phase with small deviations from sphericity outside the horizon. It is plausible that the bulk of the radiation in most processes is generated only in the very strong-field interactions around the time of horizon formation and that radiation generation in the early dynamics can be ignored. One would, however, think that strong radiation would be emitted during the stages at which the early horizon is very nonspherical and at which time nonspherical perturbation theory would seem to be inapplicable. There should be a tendency for this "early" radiation, produced very close to the horizon, to go inward into the developing black hole, so that the application of nonspherical perturbation theory to the exterior really requires that on the initial spacetime the perturbation are small only well outside the horizon. It would seem that something of this sort would have to be happening to explain the accuracy of the Price-Pullin and Abrahams-Cook results.

Whether or not many problems can be treated with *no* use of fully numerical evolution, it appears clear to us that these perturbation methods will be applied to a variety of problems in which data on the initial hypersurface is available numerically. The primary purpose of this paper is to provide justification and background for earlier work on this subject and a clear recipe for future applications. In the next section we discuss the meaning, and limitations, of extracting a "perturbation" from this numerical data and computing radiated energies. The explicit process of extracting the perturbations from the numerical data is given in Sec. III. In Sec. IV we demonstrate the use of this procedure via application to a specific example, the Misner initial data.

II. INITIAL DATA AS SCHWARZSCHILD PERTURBATIONS

We outline here the formalism for perturbation theory based on work by Regge and Wheeler [5] and by Zerilli [4], but we will draw heavily on the gauge invariant reformulation of those earlier works by Moncrief [14]. Our starting point is an initial hypersurface which can be taken as a surface of constant Schwarzschild time. We assume that the coordinates x^i on that surface are almost Schwarzschild coordinates r, θ, ϕ and we assume that the values are known, on this hypersurface and in these coordinates, for the 3-metric γ_{ij} and the extrinsic curvature K_{ij}. The conditions for finding such a hypersurface and such coordinates will be made explicit in Sec. III.

Underlying perturbation theory is the idea of a family of metric functions $g_{\mu\nu}(x^\alpha; \epsilon)$, depending on the parameter ϵ, which satisfy the Einstein equations for all ϵ, and which, in the limit $\epsilon \to 0$, become the Schwarzschild metric functions, such as $g_{rr} = S^{-1}$. (Here $S \equiv 1 - 2M/r$ and M is the mass of the Schwarzschild spacetime; we use units throughout in which $c = G = 1$.) NPS theory amounts to the approximation

$$g_{\mu\nu}(x^\alpha; \epsilon) \approx g_{\mu\nu}(x^\alpha; \epsilon)|_{\epsilon=0} + \epsilon \frac{\partial}{\partial \epsilon} g_{\mu\nu}(x^\alpha; \epsilon)|_{\epsilon=0} \; . \tag{1}$$

A. Choice of expansion parameter

It is of some practical importance to realize that the choice of the expansion parameter can have a considerable effect on the range over which perturbation theory gives a good approximation. Let us imagine that we introduce a new parameter ϵ' which is a function of ϵ such that $d\epsilon'/d\epsilon$ approaches unity as $\epsilon \to 0$. If we take ϵ' to be the basis of our perturbation approach, the approximation becomes

$$g(x^\alpha; \epsilon) = g(x^\alpha; \epsilon(\epsilon')) = g_{\mu\nu}(x^\alpha; \epsilon(\epsilon'))|_{\epsilon'=0} + \epsilon' \frac{\partial}{\partial \epsilon'} g_{\mu\nu}(x^\alpha; \epsilon(\epsilon'))|_{\epsilon'=0} + \mathcal{O}(\epsilon'^2)$$

$$= g_{\mu\nu}(x^\alpha; \epsilon(\epsilon'))|_{\epsilon=0} + \left[\epsilon \frac{\partial}{\partial \epsilon'} g_{\mu\nu}(x^\alpha; \epsilon(\epsilon'))|_{\epsilon=0}\right] \left\{\frac{\epsilon'}{\epsilon}\right\} + \mathcal{O}(\epsilon'^2) . \tag{2}$$

At $\epsilon = 0$ the derivative of $g_{\mu\nu}$ with respect to ϵ and with respect to ϵ' have the same values, so for a given spacetime — that is, for a given value of ϵ — the nonspherical perturbation in (2) differs from that in (1) by the factor $\{\epsilon'/\epsilon\}$. Computed energies (which are quadratic in the nonspherical perturbations) will differ by the square of this ratio. Different choices of parameterization will change this factor and affect the accuracy of the linearized approximation.

To show the effects of this parameterization dependence, we take as an example Misner data [11] [12] for two holes. The initial separation of the holes, in units of the mass of the spacetime, is described by Misner's parameter μ_0. The metric perturbations, however, are not analytic in μ_0 as $\mu_0 \to 0$, so μ_0 cannot be used as the expansion parameter in (1). The actual expansion parameter used by Price and Pullin, was a function of μ_0 denoted κ_2. We consider here what would be the results of perturbation theory done with the expansion parameter

$$\epsilon = \frac{\kappa_2}{1 - k\kappa_2} . \tag{3}$$

Figure 2 shows the results, along with the energies computed by numerical relativity applied to full nonlinear evolution [15]. For all choices of k the agreement between perturbation theory and numerical relativity is good at sufficiently small initial separation (sufficiently small μ_0), but as μ_0 grows larger, the agreement increasingly depends on the which parameterization is used. The $k = 0$ parameterization, the parameter of the Price-Pullin paper, is a reasonably good approximation even up to separations ($\mu_0 > 1.36$) for which the initial apparent horizon consists of two disjoint parts. For positive values of k the agreement is less impressive, while for $k = -4$, it appears that perturbation theory is giving excellent answers for initial data that are very nonspherical. Clearly the $k = -4$ parameterization is "better," at least for the purpose of computing radiated energy. There exist yet better choices; in principle a parameterization could be found for which the energy computed by linearized theory is perfect for any initial separation. The crucial point is that we have no *a priori* way of choosing what is and what is not a good parameterization. The choice of expansion parameter κ_2 was made in the Price-Pullin analysis, because it occurred naturally in the mathematical expressions for the initial geometry. There was no *a priori* reason for believing it to be a particularly good, or particularly bad parameterization. This point will be discussed again, in connection with numerical results presented in Sec. IV.

The fact, demonstrated in Fig. 2, that the results of linear perturbation theory are arbitrary may seem to suggest that perturbation answers, from a formal expansion or numerical

4

initial data, are of little value. It should be realized, however, that the arbitrariness exhibited in Fig. 2 is simply a demonstration of the fact that linearized perturbation results are uncertain to second order in the expansion parameter. The fact that the results for different parameterizations start to differ from each other around $\mu_0 \approx 1.5$ simply signals that κ_2 is around unity. (In fact, $\kappa_2 \approx 0.24$ for $\mu_0 = 1.2$.) Higher order uncertainty is an unavoidable feature in the range where the expansion parameter is of order unity. But there is a potential misunderstanding about the meaning of "expansion parameter around unity." To see this consider a change to a new expansion parameter $\epsilon = 10^{-4} * \kappa_2$. The new expansion parameter ϵ is of order unity for $\mu_0 \approx 7$, yet we know that perturbation fails dramatically for such a large value of μ_0. The issue here is that we need some way of ascribing an appropriate "normalization" to the expansion parameter. A sign that the normalization is good is that physically-based measures of distortion start getting large for ϵ around unity. If we had reliable measures of this type then we could have some confidence about the range of the the expansion parameter for which we could neglect second order uncertainty, whether due to parameter arbitrariness or the omission of higher order terms in the calculation. One can formulate interesting measures for the normalization of the expansion parameter, such as the extent to which the linearized initial conditions violates the exact Hamiltonian constraint [16]. Most such measures are useful only for finding a very rough normalization for κ_2 (equivalently, for roughly finding the range in which linearized perturbation theory is reliable). The only reliable procedure for this is to carry out computations of radiated waveforms and energy to second order in the expansion parameter. The ratio of second order corrections to first order results gives the only direct measure of the reliability of perturbation results. If one computes an energy for which the second order correction to the first order result is 10%, then one knows that the third order correction (due to a change in parameterization or an inclusion of third order terms in the computation) will be on the order of 1%.

B. Treating nonlinear initial data as a perturbation expansion

We turn now to the central question of this paper: How does one apply perturbation theory to numerically generated initial data? To do this we consider our numerical initial data to be initial data for a solution in a parameterized family $g_{\mu\nu}(x^\alpha; \epsilon)$ corresponding to $\epsilon = \epsilon_{\text{num}}$. The application of perturbation theory is equivalent to replacing $g_{\mu\nu}(x^\alpha; \epsilon_{\text{num}})$ by

$$g_{\mu\nu}(x^\alpha; \epsilon)|_{\epsilon=0} + \frac{\partial}{\partial \epsilon} g_{\mu\nu}(x^\alpha; \epsilon)|_{\epsilon=0} \epsilon_{\text{num}}. \tag{4}$$

An added familiar complication is that we can introduce a family of coordinate transformations $x^\alpha = x^\alpha(x^{\mu'}; \epsilon)$ which reduces to $x^\alpha = x^{\alpha'}$ for $\epsilon \to 0$. Such a transformation takes the original family to a new family $g'_{\mu\nu}(x^{\alpha'}; \epsilon)$, which satisfies the same requirements as the original family. We follow Moncrief [14] in constructing, from the 3-metric γ_{ij} on constant-t surfaces, quantities q_i, which are invariant to first order in ϵ ("gauge invariant"), for coordinate transformations. The construction of these Moncrief q_i is done in two steps. First, the multipole moments of the metric are extracted. In practice this is done by multiplying the metric functions by certain angular factors and integrating over angles. Since we are only interested in quadrupole and higher order for radiation, this step also eliminates the

spherically symmetric background parts of the metric function. The second step is to form linear combinations of these multipoles and of their derivatives with respect to radius. We symbolically represent the process of forming these quantities as

$$q_i = Q_i(\gamma_{ij}, \partial_r \gamma_{ij}) \ . \tag{5}$$

Here the symbol "Q_i" represents the process of multiplying by angular functions and integrating, then multiplying by certain functions of r and taking linear combinations of the results. (Our notation here disagrees with that of Moncrief [14] in a potentially confusing way. Moncrief's perturbation quantities are independent of the size of ϵ. In order to have definitions that can be applied to numerical data we use quantities that – to first order – are proportional to ϵ.)

The Moncrief gauge invariants play two different roles. For even parity one of the gauge invariants, q_2, is a constraint; it vanishes in linearized theory as a result of the initial value equations. In linearized theory, the remaining Moncrief quantities, denoted q_1 here, satisfy wave equations $L(q_1) = 0$, the Regge-Wheeler equation in odd parity and Zerilli equation in even parity.

From our numerical data we construct the quantities q_i precisely according to (5). Our numerically constructed "perturbation" quantities will not be invariant under coordinate transformations, but rather will transform as $q_i' = q_i + \mathcal{O}(\epsilon_{\text{num}}^2)$. Similarly, the linearized constraint, q_2 will not vanish, but will be of order ϵ_{num}^2. The numerically constructed wavefunctions q_1 will satisfy $L(q_1) = \mathcal{O}((\epsilon_{\text{num}})^2)$, where L is the Regge-Wheeler or Zerilli wave operators.

The use of NPS methods is equivalent to ignoring the second order terms in the wave equations. The wavefunction q_i can then be propagated from the initial hypersurface forward and the radiation waveforms extracted from it. To evolve q_1 off the initial hypersurface, however, requires the initial time derivative $\partial q_1/\partial t$. This can be computed from the initial extrinsic curvature, but some care is needed. Indeed, the possible ambiguities that arise here are the justification for the somewhat protracted discussion in this section.

If $\mathbf{n}$ is the future-directed unit normal to the initial hypersurface then the rate at which the 3-metric is changing is given by

$$K_{ij} = -\frac{1}{2}\mathcal{L}_{\mathbf{n}}\gamma_{ij} \ , \tag{6}$$

where K_{ij} is the extrinsic curvature and $\mathcal{L}_{\mathbf{n}}$ is the Lie derivative along the unit normal. The unit normal is related to the derivative with respect to Schwarzschild time by $\partial/\partial t = S^{1/2}\,\mathbf{n}$. The time derivative of the Moncrief function then can be written

$$\partial q_1/\partial t = S^{1/2}\mathcal{L}_{\mathbf{n}}q_1$$

$$= S^{1/2}\mathcal{L}_{\mathbf{n}}Q_1(\gamma_{ij}, \partial\gamma_{ij}/\partial r) \ . \tag{7}$$

To evaluate the right hand side we need to know how Q_1 changes when it is Lie dragged by $\mathbf{n}$. Since Q_1 depends only on γ_{ij} it might appear that one need only Lie drag γ_{ij} to find the change in Q_1, and that $\mathcal{L}_{\mathbf{n}}Q_1 = Q_1(\mathcal{L}_{\mathbf{n}}\gamma_{ij}, \partial\mathcal{L}_{\mathbf{n}}\gamma_{ij}/\partial r)$. From this it would follow that $\partial q_1/\partial t = -2S^{1/2}Q_1(K_{ij}, \partial K_{ij}/\partial r)$. It is important to note that this is *not* the correct

relationship between K_{ij} and the cauchy data for the wave equation. The fallacy in this procedure lies in the fact that q_1 must be computed from the 3-metric on a slice for which Schwarzschild time is constant (to first order in ϵ_{num}). Lie dragging by $\mathbf{n}$ moves the 3-metric to a surface that is not (to first order) a constant time surface. The cure is clearly to compare quantities on surfaces of constant t by using $\mathcal{L}_t \equiv S^{1/2} \mathcal{L}_{\mathbf{n}}$. It is the Schwarzschild time derivative that commutes with the Schwarzschild radial derivative $\mathcal{L}_t(\partial/\partial r)^a = 0$. The correct prescription then follows from

$$\partial q_1/\partial t = S^{1/2} \mathcal{L}_{\mathbf{n}} q_1$$

$$= Q_1(S^{1/2} \mathcal{L}_{\mathbf{n}} \gamma_{ij}, \partial(S^{1/2} \mathcal{L}_{\mathbf{n}} \gamma_{ij})/\partial r)$$

$$= -2Q_1(S^{1/2} K_{ij}, \partial(S^{1/2} K_{ij})/\partial r) . \tag{8}$$

We note that the perturbed Schwarzschild metric does have a shift vector β_i of order ϵ, and in principle the shift vector influences the time development of γ_{ij} according to $\partial_t \gamma_{ij} = \partial_{t'} \gamma_{ij} + 2\nabla_{(i} \beta_{j)}$, where t' is a time coordinate in which the shift vector vanishes. But the shift vector can be considered to be "pure gauge." It is necessary if one wants a complete specification of the coordinates and the metric components, but its value is a matter of choice, and is not necessary for a complete specification of the physics. The initial value, and evolution, of the gauge invariant quantity q_1 is invariant with respect to the choice of β_i, and q_1 carries all the (physically meaningful) information about gravitational waves.

The evaluation of q_1 from (5) and $\partial q_1/\partial t$ from (8) completes the extraction, from the numerical data for γ_{ij}, K_{ij} of the cauchy data for the Regge-Wheeler or Zerilli wave equation. An alternative procedure arises if one uses the scalar wave-equations derived from the perturbative reduction of the nonlinear wave-equation for the extrinsic curvature which arises in a new explicitly hyperbolic form of the Einstein equations [17]. In this system, the scalar wave equations are one order lower in time derivative from the usual Regge-Wheeler and Zerilli equations, so the Cauchy data consists of the extrinsic curvature and its time-derivative (which involves the 3-dimensional Ricci curvature).

From the above it is clear that linearized evolution should give good accuracy when applied to numerically generated initial data with sufficiently small deviations from sphericity. For initial data which are known in analytic form one can, of course, apply linearized theory even to cases in which initial deviations from sphericity are only marginally small. The results in Fig. 2, for example, show that the results of such application of perturbation theory give reasonable accuracy for values of μ_0 at which an initial horizon is highly distorted. It is worrisome to apply linearized evolution to marginally nonspherical initial data, which do not, for example, satisfy the constraint $q_2 = 0$ with reasonable accuracy. Such a procedure — linear evolution of nonlinear initial data — has, among other disadvantages, no clear theoretical framework.

C. Calculating radiated energy by "forced linearization"

We wish to point out here that NPS methods can be used more broadly, and a procedure we call "forced linearization" can be applied to numerically generated initial data in a way

that amounts to extracting the linearized part of the data and evolving linearly. This procedure circumvents the difficulty of performing formal linearization to data which is only known numerically. We imagine that we start with an initial value problem in which there is some adjustable parameter, call it μ, such that $\mu = 0$ corresponds to the Schwarzschild initial data. There is no requirement that the family of solutions $g_{\mu\nu}(x^\alpha; \mu)$ be analytic in μ at $\mu \to 0$. There may be additional parameters, call them p_i, such as the parameters governing the initial momenta of holes. To apply forced linearization we fix the values of the p_i and make a choice of μ such that the computed initial data $\gamma_{ij}^{\mathrm{vns}}, K_{ij}^{\mathrm{vns}}$ are "very nearly spherical." One criterion for this would be that q_2 is very small. We then interpret this initial data as being essentially linearized data, to which the approximation in (4) applies. We extract multipoles, form a gauge invariant wave function q_1, and evolve it with the Zerilli or Regge-Wheeler equation, all as described above. The result of this will be a late-time waveform $q_1^{\mathrm{vns}}(r, t)$ and the energy E^{vns} that it carries. The next step is to characterize the results with a well behaved gauge invariant parameter. To do this we choose some fiducial radius r_{fid}, and evaluate $\epsilon^{\mathrm{vns}} \equiv q_1(r_{\mathrm{fid}}, t = 0)$ the gauge invariant wave function of the initial hypersurface at this radius.

Next, we leave the p_i unchanged, but choose a larger value of μ for which the numerically generated initial data set $\gamma_{ij}^{\mathrm{mrgnl}}, K_{ij}^{\mathrm{mrgnl}}$ is "marginal" in that it corresponds to deviations from sphericity large enough so that it differs significantly form linearized initial conditions; one sign of this would be that the condition $q_2 = 0$ is significantly violated. For this data set we go through the same procedure as above in characterizing the data set by a parameter $\epsilon^{\mathrm{mrgnl}} \equiv q_1(r_{\mathrm{fid}}, t = 0)$. For this marginally spherical initial data we take the solution for the wavefunction and energy to be

$$q_1^{\mathrm{mrgnl}}(r, t) = \left(\frac{\epsilon^{\mathrm{mrgnl}}}{\epsilon^{\mathrm{vns}}} \right) q_1^{\mathrm{vns}}(r, t) \qquad E^{\mathrm{mrgnl}} = \left(\frac{\epsilon^{\mathrm{mrgnl}}}{\epsilon^{\mathrm{vns}}} \right)^2 E^{\mathrm{vns}} . \tag{9}$$

The idea underlying this method is that the very nearly spherical data give us the solution for for $\partial g_{\mu\nu}/\partial\epsilon|_{\epsilon=0}$. For the marginal initial data set we then need only multiply this initial data by the appropriate factor telling us how much larger is the linear part of the nonsphericity than that of the very nearly spherical initial data. The success of forced linearization requires then that ϵ evaluated at r_{fid} be a well behaved parameterization of the linearized part of the nonsphericity in the numerical data. Since our expansion parameter ϵ is the magnitude of the perturbation, it will be a good expansion parameter as long as it is evaluated in a region where the nonlinear deviations from sphericity are small, i.e., where (4) is a good approximation. For this reason it is important that r_{fid} be chosen fairly large. For processes of the type pictured in Fig. 1, the deviations from sphericity fall off quickly in radius, so that at large enough r one can be certain that the initial data are an excellent approximation to linearized data. Evidence for this is that the violations of the $q_2 = 0$ constraint are always confined to small radii. One easily implemented check on the forced linearization procedure is to look at the factor $\epsilon^{\mathrm{mrgnl}}/\epsilon^{\mathrm{vns}}$ and confirm that it is independent of r for $r > r_{\mathrm{fid}}$. In Section III we show that this test is easily passed by a numerical example, and that the results of forced linearization are essentially the same as those of formal linearized theory.

III. EXTRACTION OF PERTURBATIONS FROM NUMERICAL DATA

Here we assume that the reader has numerical solutions for the 3-metric on an approximately t=const surface. The first step in applying NPS to numerical results is to transform to coordinates which are "almost Schwarzschild" coordinates. It is assumed that the numerical γ_{ij} and K_{ij} are expressed in a coordinate system R, θ, ϕ in which the approximate spherical symmetry is manifest. This means that K_{ij} and ratios like

$$\frac{\gamma_{R\theta}}{\sqrt{\gamma_{\theta\theta}}} \qquad \frac{\gamma_{R\phi}}{\sqrt{\gamma_{\theta\theta}}} \qquad \frac{\gamma_{\theta\phi}}{\gamma_{\theta\theta}} \tag{10}$$

must be small. They all are, in fact, formally of order ϵ_{num}, so if they are not all reasonably small compared to unity there is little reason to think that NPS will work. A Schwarzchild-like areal radial coordinate r needs to be introduced. This can be defined as a function of R by

$$r \equiv \left(\int \gamma_{\theta\theta} \, \gamma_{\phi\phi} \, d\Omega \ \right)^{1/4} / 4\pi. \tag{11}$$

where the integral is taken on a surface of constant R. The metric component γ_{rr}, in terms of this quantity, gives us another test of how close the geometry is to that of a constant time Schwarzschild slice. The quantity

$$r \left(1 - 1/\gamma_{rr} \right)$$

should be nearly equal to the constant $2M$, where M is the mass of the spacetime. The variability of this quantity in r, θ, and ϕ, is formally of order ϵ_{num}.

There are, of course, other ways of specifying the Schwarzschild-like coordinates. We could, for example, have defined $r^2 \equiv \gamma_{\theta\theta}$ All these coordinate choices, however, should agree to order ϵ_{num} and are therefore equivalent within a linearized gauge transformation.

To compute the gauge invariant perturbation functions, we first assume that an ℓm multipole of the 3-metric may be expanded as

$$\gamma_{ij} = c_1(\hat{e}_1)_{ij} + c_2(\hat{e}_2)_{ij} + h_1(\hat{f}_1)_{ij} + \frac{H_2}{S}(\hat{f}_2)_{ij} + r^2 K(\hat{f}_3)_{ij} + + r^2 G(\hat{f}_4)_{ij} \tag{12}$$

where, for clarity, we have suppressed multipole indices and have replaced Moncrief's h_1 and h_2 odd parity perturbation functions with c_1, c_2. The multipole moments c_1, c_2, h_1, H_2, K, and G are computed by projection onto the relevant spherical harmonics which can be found in Moncrief [14]. Explicit formulas for the important special case of even parity, axisymmetric perturbations may be found in Ref. [10].

For odd parity perturbations, one function can be constructed from the amplitudes c_1 and c_2 which is gauge invariant and satisfies the Regge-Wheeler equation (below),

$$Q^\times_{\ell m} = \sqrt{2\frac{(\ell+2)!}{(\ell-2)!}} \left[c_1 + \frac{1}{2} \left(\frac{\partial c_2}{\partial r} - \frac{2}{r}c_2 \right) \right] \frac{S}{r}. \tag{13}$$

The situation for even parity perturbations is more complicated. Two gauge invariant functions may be formed out of the multipole moments:

9

$$k_1 = K + \frac{S}{r}(r^2 \partial_r G - 2h_1) \tag{14}$$

$$k_2 = \frac{1}{2S}\left[H_2 - r\partial_r k_1 - \left(1 - \frac{M}{rS}\right)k_1 + S^{1/2}\partial_r(r^2 S^{1/2}\partial_r G - 2S^{1/2}h_1)\right] \tag{15}$$

From k_1 and k_2 it is possible to form two new functions, one which is radiative and one which is equivalent to the perturbed hamiltonian constraint

$$q_1 = 4rS^2 k_2 + \ell(\ell+1)rk_1 \tag{16}$$

$$q_2 = \partial_r[4rS^2 k_2 + \ell(\ell+1)rk_1] + \ell(\ell+1)[2Sk_2 + (1 - M/\{rS\})k_1]. \tag{17}$$

The scaled function

$$Q_{\ell m}^+ = \frac{q_1}{\Lambda}\sqrt{\frac{2(\ell-1)(\ell+2)}{\ell(\ell+1)}} \;, \tag{18}$$

with

$$\Lambda \equiv (\ell-1)(\ell+2) + 6M/r \;,$$

satisfies the Zerilli equation (below).

The time derivatives of the radiative gauge invariant functions $Q_{\ell m}^\times$ and $Q_{\ell m}^+$ are found by substituting $\sqrt{1 - 2M/r}K_{ij}$ for γ_{ij} in the multipole moment computation and forming the same combinations of moments.

The wavefunctions $Q_{\ell m}^\times$ and $Q_{\ell m}^+$ obey the Regge-Wheeler and Zerilli wave equations respectively:

$$\mathbf{L}Q_{\ell m}^\times + V_\ell^\times Q_{\ell m}^\times = 0 \tag{19}$$

$$\mathbf{L}Q_{\ell m}^+ + V_\ell^+ Q_{\ell m}^+ = 0 \tag{20}$$

where the wave operator appropriate to Schwarzschild spacetime is

$$\mathbf{L} = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} \tag{21}$$

in terms of the "tortoise coordinate" $r_* = r + 2M\ln(r/2M - 1)$, and where the potentials are given by

$$V_\ell^\times = (1 - 2M/r)\left[\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right] \tag{22}$$

and,

$$V_\ell^+(r) = (1 - 2M/r)\left[\frac{1}{\Lambda^2}\left(\frac{72M^2}{r^5} - \frac{12M}{r^3}(\ell-1)(\ell+2)(1 - 3M/r)\right) + \frac{\ell(\ell-1)(\ell+1)(\ell+2)}{r^2\Lambda}\right]. \tag{23}$$

Once the Zerilli and Regge-Wheeler equations are integrated for all the desired ℓ and m modes, the total radiated energy can be calculated from the asymptotic timeseries for $Q_{\ell m}^+$ and $Q_{\ell m}^\times$:

$$\frac{dE}{dt} = \frac{1}{32\pi}\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{dQ_{\ell m}^+}{dt}^2 + \frac{dQ_{\ell m}^\times}{dt}^2\right). \tag{24}$$

10

IV. EXAMPLE OF PERTURBATION EXTRACTION

In this section we demonstrate the extraction of a perturbation from a numerical solution to the nonlinear constraint equations – the Misner data representing two black holes at a moment of time symmetry. The Misner 3-geometry may be written [11] as

$$dl^2 = \Phi(r, \theta, \mu_0)^4 (S^{-1} dr^2 + r^2 d\Omega^2). \tag{25}$$

The conformal factor Φ is given by

$$\Phi(r, \theta; \mu_0) = 1 + 2(1 + M/2R)^{-1} \sum_{l=2,4,\ldots}^{\infty} \kappa_\ell \left(\frac{M}{R}\right)^{\ell+1} P_\ell(\cos\theta), \tag{26}$$

where

$$R \equiv (\sqrt{r} + \sqrt{r - 2M})^2 / 4$$

and

$$\kappa_\ell \equiv \left(\frac{1}{4\sum_{n=1}^{\infty}(\sinh n\mu_0)^{-1}}\right)^{\ell+1} \sum_{n=1}^{\infty} \frac{(\coth n\mu_0)^\ell}{\sinh n\mu_0} \,. \tag{27}$$

For this exercise, we pretend that the initial geometry is known only numerically, so no explicit formal linearization can be done. The odd parity perturbations vanish in the Misner solution. We compute the even parity gauge invariant wavefunction for $\ell = 2$ using numerical evaluations of (26) - (27). Specifically, we compute K and H_2 of (12) from

$$K = H_2 = \int d\Omega \, \Phi^4 Y_{20} \,. \tag{28}$$

All the other moments in (12) vanish for the conformally Schwarzschild metric of (25). The function Q_{20}^+ is evaluated at values of r corresponding to the range $r_* = -20M$ to $r_* = 50M$. The initial value of Q_{20}^+ (along with its time-derivative which is zero for the Misner time-symmetric initial data) provides initial values for integration of (20). At large radius, $r = 100M$, the value of $\partial Q_{20}^+ / \partial t$ is used in (24) to compute the radiated energy.

First, in Fig. 3 we show the result of directly computing the gauge invariant function Q_{20}^+ from the nonlinear initial data, integrating the Zerilli equation, and computing the radiated energy. For small values of μ_0 the agreement with the explicitly linearized data of Ref. [11] is excellent. At about $\mu_0 \simeq 1.2$ the agreement breaks down and the qualitative behavior becomes dramatically different. It is interesting to note that the apparent horizon encompassing both black holes does not exist for $\mu_0 > 1.36$, close to the dramatic reversal in the energy curve.

In Fig. 4 the violation of the linearized constraint by the nonlinear data is shown as a function of radius. We plot the ratio of the constrained gauge invariant function, q_1 to the radiative function q_2 scaled in such a way as to compensate for large violation at $r = 2M$. The value of q_2 clearly grows much faster than the radiative variable q_1 as the separation is increased.

As discussed in Sec. II, it is possible to obtain the results of formal perturbation theory directly from the numerical data without ever making reference to the analytic solution. In Fig. 5 we demonstrate the application of the forced linearization procedure to the nonlinear Misner data for various values of the fiducial radius $r_{\rm fid}$. For very small values of μ_0, such as $\mu_0 = 0.5$, the geometry outside the event horizon is everywhere well approximated by (4) and forced linearization works even for small values of $r_{\rm fid}/M$. When μ_0 is larger than around 1.5, on the other hand, the initial geometry near the horizon contains significant nonlinear effects, and large values of $r_{\rm fid}/M$ must be used to get results equivalent to those of formal linearized theory.

As $r_{\rm fid}$ gets large, the results become indistinguishable from those of formal perturbation theory reported in Ref. [11]. For $r_{\rm fid} = 30M$ the difference in radiated energy for $\mu_0 = 3.0$ is less than $10^{-3}\%$. This high-accuracy equivalence deserves some explanation. In particular, why is forced linearization equivalent to formal linearization with expansion parameter κ_2? Why is that expansion parameter singled out? The equivalence is a result of two features of the way in which the linearizations were done: First, both the formal linearization of Ref. [11], and the forced linearization results in Fig. 5, use precisely the same coordinates. (The forced linearization results, in fact, are not based on initial values that were generated by genuinely numerical means. Rather, the closed form solutions for the Misner metric functions were used. The "almost-Schwarzschild" coordinates of the forced linearization, were precisely the same as the "almost-Schwarzschild" coordinates in Ref. [11]). Secondly, in the "almost-Schwarzschild" coordinate system, the parameter κ_2 is, to all perturbation orders, the coefficient of the dominant nonsphericity at large radius. Forced linearization (in the limit of large $r_{\rm fid}$) results in a parameterization based on a gauge invariant measure of nonsphericity at large radius. It therefore must be proportional to κ_2 and produce results equivalent to those of the formal linearization of Ref. [11], in which κ_2 was the expansion parameter.

It should be understood that this does not imply that the parameter κ_2 is *physically* singled out. A first order change in the "almost-Schwarzschild" coordinates will change the coefficient of the dominant large-radius nonsphericity. We might, for example, transform from the "almost-Schwarzschild" radial coordinate r of (25) to a new coordinate $r' \equiv r[1 + \kappa_2 P_2(\cos\theta)]$. In this case the coefficient of the leading large r' term in the metric will be $\kappa_2 + \mathcal{O}(\kappa_2^2)$, and the results of forced linearization with the resulting "numerical" data will differ, when perturbations are large, from the results in Ref. [11]. The forced linearization will have induced an expansion parameter different from κ_2.

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FIGURES

FIG. 1. Spacetime regions for coalescence. The "legs of the trousers" represent the world tubes of two compact objects before coalescence; region I cannot be considered to be nearly spherical. The objects coalesce in region II which is also highly nonspherical, but lies inside a horizon. Region III, above the hypersurface and outside the horizon, can be treated as a nearly spherical spacetime.

FIG. 2. The effect of a change of expansion parameter. Results are given for the energy radiated, as a function of μ_0, during the head-on collision of two black holes (Misner initial data). The results of numerical relativity are compared with linearized theory for different choices of expansion parameters.

FIG. 3. Radiated energies from nonlinear Misner data. Energies computed by integration of the Zerilli equation are compared for initial perturbations calculated by explicit linearization of the Misner data (solid line) and initial perturbations extracted directly from the nonlinear Misner data (dashed line).

FIG. 4. Violation of the linearized hamiltonian constraint. The ratio of gauge invariant functions q_1/q_2 scaled by the factor $r - 2M$ is plotted as a function of tortoise coordinate r_*. Curves are shown for $\mu_0 = 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4$. The largest constraint violation occurs for $\mu_0 = 1.4$.

FIG. 5. Radiated energies from forced linearization procedure. Radiated energy is plotted as a function of Misner separation parameter μ_0 for various values of $r_{\rm fid}$. The curve for $r_{\rm fid} = 30M$ is indistinguishable from the formal perturbation theory result of Price and Pullin.