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Bern-Kosower Rule for Scalar QED

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Abstract

We derive a full Bern-Kosower-type rule for scalar QED starting from quantum field theory: we derive a set of rules for calculating S-matrix elements for any processes at any order of the coupling constant. Gauge-invariant set of diagrams in general is first written in the worldline path-integral expression. Then we integrate over $x(\tau)$, and the resulting expression is given in terms of correlation function on the worldline $\langle x(\tau)x(\tau')\rangle$. Simple rules to decompose the correlation function into basic elements are obtained. Gauge transformation known as integration by parts technique can be used to reduce the number of independent terms before integration over proper-time variables. The surface terms can be omitted provided the external scalars are on-shell. Also, we clarify correspondence to the conventional Feynman rule, which enabled us to avoid any ambiguity coming from the infinite dimensionality of the path-integral approach.

1 Introduction

Recently, Bern and Kosower derived from superstring theory a powerful method for calculating one-loop S-matirx elements for QCD processes.[1] Although the new rule had reduced the amount of work required in the calculation greatly, it had little resemblance to the conventional Feynman rule, and to date, the complete Bern-Kosower rule has not been derived from quantum field theory (QCD). The equivalence of the Bern-Kosower rule and the conventional Feynman rule has been shown only in some concrete examples.[2] Moreover, practical problems are that since the Bern-Kosower rule has been derived from the string theory, it is difficult to include massive particles and also multi-loop generalizations do not readily lead to simple calculational tools.[3]

As for the approach from the quantum field theory, there has been some progress. Bern-Kosower-type rules for calculating one-loop effective actions for both abelian and non-abelian gauge theories have been derived from quantum field theories and studied extensively by Strassler.[4, 5] Also, multiloop diagrams with one-fermion-loop and multiple propagator insertions has been cast into Bern-Kosower-type rule by Schmidt and Schubert, and they applied the rule to the calculation of two-loop QED β function.[6] On the other hand, a quite different approach was developed by Lam, where he showed that expressions similar to Bern-Kosower rule can be obtained starting from the conventional Feynman parameter formula in abelian gauge theories even beyond one-loop order.[7]

In this paper we refine the ideas in the above approaches from field theory, and derive a full Bern-Kosower-type rule for scalar QED: we derive a set of rules for calculating S-matrix elements for any processes at any order of the coupling constant. Also we clarify correspondence to the conventional Feynman rule. (The method we show in this paper can straightforwardly be extended to the case of spinor QED.)

The main idea is:

1. Express a set of diagrams connected by gauge transformation (see Fig.3 below) by a single worldline path-integral.

2. Use gauge transformation (known as integration by parts technique[1, 5]) to simplify calculation.

For those unfamiliar with worldline path-integral formalism, relation to the conventional Feynman rule may be seen as follows. Let us express the Feynman propagator in coordinate space using Feynman parameter*:

$$
\begin{aligned}
i\Delta_F(x-y) &= \int \frac{d^D p}{(2\pi)^D} \frac{i\, e^{ip\cdot(x-y)}}{p^2-m^2+i\epsilon} && (1.1)\\
&= \int_0^\infty d\alpha \int \frac{d^D p}{(2\pi)^D}\, e^{ip\cdot(x-y)+i\alpha(p^2-m^2+i\epsilon)} && (1.2)\\
&= \int_0^\infty d\alpha\; i\left(\frac{1}{4\pi i\alpha}\right)^{D/2} \exp\left[-\frac{i}{4\alpha}(x-y)^2 - i\alpha(m^2-i\epsilon)\right]. && (1.3)
\end{aligned}
$$

Note that (part of) the integrand in eqs.(1.2) and (1.3) has a similar form to the propagator of a non-relativistic free particle if $\alpha(>0)$ is identified with the time interval of propagation:

$$
\begin{aligned}
K(x-y;\alpha) &\equiv \int \frac{d^D p}{(2\pi)^D}\, e^{ip\cdot(x-y)+i\alpha p^2} && (1.4)\\
&= i\left(\frac{1}{4\pi i\alpha}\right)^{D/2} \exp\left[-\frac{i}{4\alpha}(x-y)^2\right]. && (1.5)
\end{aligned}
$$

Namely, it satisfies

$$
\left(i\frac{\partial}{\partial\alpha} - \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\right) K(x-y;\alpha) = 0, \qquad (1.6)
$$

$$
K(x-y;+0) = \delta(x-y). \qquad (1.7)
$$

Hence, the associativity relation

$$
\int d^D z\; K(x-z;\alpha_1)\, K(z-y;\alpha_2) = K(x-y;\alpha_1+\alpha_2) \qquad (1.8)
$$

holds as an important property of K (see Fig.1), which can be shown easily from eq.(1.4). This property allows one to insert arbitrary number of vertices along the propagator lines of a given diagram, and if infinitely many are inserted, the integral expression reduces to the path-integral.

In section 2, we derive the path-integral expression for a general set of diagrams starting from quantum field theory, and derive the general expression after integration over $x(\tau)$. Section 3 clarifies correspondence of the proper time integral formula obtained in the previous section and the Feynman parameter integral formula obtained from the conventional Feynman rule. This enables one to express the two-point function (correlation function) $\langle x(\tau)x(\tau')\rangle$ on the general diagram in terms of basic elements. Section 4 explains a general prescription for integration by parts and discuss relation to

*Throughout the paper we work in D dimensional space-time with the metric tensor $g_{\mu\nu} = \mathrm{diag}(+1, \underbrace{-1,\ldots,-1}_{D-1})$.

Figure 1: A diagrammatical representation of the associativity relation satisfied by $K(x-y;\alpha)$.

the gauge transformation on worldline. The gauge-fixing parameter dependence of a set of diagrams is discussed in section 5. The Bern-Kosower-type rule for a general set of diagrams is summarized in section 6. The rule for calculating a set of diagrams including other than gauge interactions is demonstrated in section 7. Concluding remarks are given in section 8.

In Appendix A, details of calculation required in section 3 are shown. Some properties of (counterpart of) the two-point function are listed in Appendix B with proofs. A sample calculation using the Bern-Kosower-type rule is shown in Appendix C.

2 General Expression

We consider scalar QED theory, whose Lagrangian is given by

$$\mathcal{L}(\phi, A_\mu) = (D_\mu\phi)^*(D^\mu\phi) - m^2|\phi|^2 - \frac{\lambda}{4}|\phi|^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.1}$$

with

$$D_\mu(A) = \partial_\mu - ieA_\mu(x). \tag{2.2}$$

We set $\lambda = 0$ in most of the paper since the simplification of calculation occurs regarding the gauge interactions. The method for including $|\phi|^4$ interaction will be demonstrated in section 7. As for the gauge-fixing term, we take Feynman gauge

$$\mathcal{L}_{gf}(A_\mu) = -\frac{1}{2}(\partial^\mu A_\mu)^2 \tag{2.3}$$

in the following, and discuss other gauge fixing conditions in section 5.

We start by defining a generating functional of connected Green functions, which is *amputated* with respect to external photons and *unamputated* with respect to external scalars:

$$e^{W(J,J^*,A_\mu)} \equiv \int \mathcal{D}\phi\, \mathcal{D}Q_\mu \, \exp\, i \int dx \left[\mathcal{L}(\phi, Q_\mu) + \mathcal{L}_{gf}(Q_\mu) + J^*\phi + J\phi^* + j^\mu Q_\mu\right]\Big|_{j_\mu \to -\Box A_\mu}, \quad (2.4)$$

where Q_μ denotes quantum gauge field. Integrating out the scalar field, and then rewriting the integral over Q_μ by functional derivatives, we obtain

$$\begin{aligned} e^{W(J,J^*,A_\mu)} &= \int \mathcal{D}Q_\mu \; e^{i\int dx\, [\frac{1}{2}(A_\mu - Q_\mu)\Box(A^\mu - Q^\mu) - \frac{1}{2}A_\mu \Box A^\mu]} \\ &\quad \times \exp[-\mathrm{Tr\,Log}(D(Q)^2 + m^2) + i \textstyle\iint dxdy\, J^*(x) \left(\frac{1}{D(Q)^2+m^2}\right)_{xy} J(y)] \quad (2.5) \\ &= e^{-\frac{i}{2}\int dx A_\mu \Box A^\mu} \exp[\tfrac{i}{2} \textstyle\iint dxdy \frac{\delta}{\delta A_\mu(x)} (\Box^{-1})_{xy} \frac{\delta}{\delta A^\mu(y)}] \\ &\quad \times \exp[-\mathrm{Tr\,Log}(D(A)^2 + m^2) + i \textstyle\iint dxdy\, J^*(x) \left(\frac{1}{D(A)^2+m^2}\right)_{xy} J(y)], \quad (2.6) \end{aligned}$$

where we used functional analogue of an identity[†]

$$\int_{-\infty}^{\infty} \frac{d\eta}{\sqrt{2\pi i a}}\; e^{i\frac{(\xi-\eta)^2}{2a}} f(\eta) = e^{\frac{1}{2}ia\frac{d^2}{d\xi^2}} f(\xi). \quad (2.7)$$

Interaction terms in eq.(2.6), which functional derivatives operate, can be represented by path-integrals of a particle interacting with the background gauge field A_μ, respectively, as

$$-\mathrm{Tr\,Log}(D(A)^2 + m^2) = \int_0^\infty \frac{dT}{T}\, e^{-im^2T} \int\limits_{x(0)=x(T)} \mathcal{D}x(\tau)\, \exp\Big[-i\int_0^T d\tau \Big(\frac{\dot{x}^2}{4} - eA(x)\cdot\dot{x}\Big)\Big], \quad (2.8)$$

$$-i\left(\frac{1}{D(A)^2+m^2}\right)_{wz} = \int_0^\infty dT\, e^{-im^2T} \int\limits_{x(0)=z}^{x(T)=w} \mathcal{D}x(\tau)\, \exp\Big[-i\int_0^T d\tau \Big(\frac{\dot{x}^2}{4} - eA(x)\cdot\dot{x}\Big)\Big]. \quad (2.9)$$

Derivation of the first equation is given in Ref.[4], and the second expression can be shown similarly. The above interaction terms, respectively, correspond to a closed scalar chain (making a loop) and an open scalar chain (whose both ends are connected to external scalars) in the background gauge field. Each term corresponds to the sum of Feynman diagrams with different location of photons along the scalar chain, including arbitrary number of three-point vertices and seagull vertices; see Fig.2. Eq.(2.6) has a simple form of connecting the two kinds of scalar chains by photon propagators $ig_{\mu\nu}(\Box^{-1})_{xy}$, which serves for deriving path-integral expression for (a set of) diagrams.

[†] To derive the integral form (left-hand-side) from the differential form (right-hand-side), substitute

$$f(\xi) = \int d\eta\, \delta(\xi - \eta) f(\eta) = \int \frac{dp\, d\eta}{2\pi}\, e^{ip(\xi-\eta)} f(\eta)$$

Figure 2: The path-integral representation of a scalar particle interacting with the background gauge field a) where the scalar line is making a loop, corresponding to eq.(2.8), and b) where the scalar line is connected to external lines,corresponding to eq.(2.9).

Consider first a specific example. We will find a convenient expression for the contribution of the set of diagrams shown in Fig.3 (hereafter referred to as set I diagrams) to the momentum space Green function defined by

$$\begin{aligned} G(k_1, k_4; k_3, \epsilon_3, k_6, \epsilon_6) \;\; &\equiv \;\; \int dx dx' dw dz \; e^{i(k_1 \cdot z + k_4 \cdot w + k_3 \cdot x + k_6 \cdot x')} \\ &\times \tfrac{\delta}{\delta J(z)} \tfrac{\delta}{\delta J^*(w)} \, \epsilon_{3\mu} \tfrac{\delta}{\delta A_\mu(x)} \, \epsilon_{6\nu} \tfrac{\delta}{\delta A_\nu(x')} \, W(J, J^*, A) \Big|_{J=J^*=A=0} . \end{aligned} \tag{2.10}$$

All external momenta are taken to be outgoing.

Let us choose the first diagram in the set I as the representative, and extract step by step the relevant terms in eq.(2.10); following procedure is sufficient for including all contributions from the set I diagrams. After substituting (2.8) and (2.9) into (2.6), we keep the term including one open scalar chain, one closed scalar chain, and one internal photon propagator:

$$\begin{aligned} W \;\; \sim \;\; & \frac{i}{2} \int\!\!\int dx dy \, \tfrac{\delta}{\delta A_\mu(x)} (\Box^{-1})_{xy} \tfrac{\delta}{\delta A^\mu(y)} \\ & \times \int_0^\infty dT \, e^{-im^2 T} \int dw dz \, J^*(w) J(z) \int_z^w \mathcal{D}x \, \exp\Big[-i \int_0^T d\tau \, \Big(\frac{1}{4} \dot{x}^2 - eA(x) \cdot \dot{x} \Big) \Big] \\ & \times \int_0^\infty \frac{dT'}{T'} \, e^{-im^2 T'} \oint \mathcal{D}x' \, \exp\Big[-i \int_0^{T'} d\tau' \, \Big(\frac{1}{4} \dot{x}'^2 - eA(x') \cdot \dot{x}' \Big) \Big]. \end{aligned} \tag{2.11}$$

and integrate over p after replacing $d/d\xi$ by ip.

Figure 3: The set I diagrams, which includes diagrams interrelated to one another by the gauge transformation of internal and external photons.

We expand the integrand in powers of the coupling e, and extract the term corresponding to two photon insertions in each scalar chain:

$$\frac{(ie)^2}{2}\int_0^T dt_2\, A(x_2)\cdot\dot{x}_2 \int_0^T dt_3\, A(x_3)\cdot\dot{x}_3 \times \frac{(ie)^2}{2}\int_0^{T'} dt_5\, A(x'_5)\cdot\dot{x}'_5 \int_0^{T'} dt_2\, A(x'_6)\cdot\dot{x}'_6, \tag{2.12}$$

where $x_i \equiv x(t_i)$ and $x'_j \equiv x'(t_j)$. Then connect the internal photon propagator by taking derivative as

$$\frac{\delta}{\delta A_\mu(x)}\frac{\delta}{\delta A^\mu(y)}\,[A(x_2)\cdot\dot{x}_2]\,[A(x'_5)\cdot\dot{x}'_5] = \dot{x}_2\cdot\dot{x}'_5\,\left[\delta(x_2-x)\delta(x'_5-y)+(x\leftrightarrow y)\right]. \tag{2.13}$$

There are also terms in which $A(x_3)$ and $A(x'_6)$ are differentiated instead of $A(x_2)$ and $A(x'_5)$, respectively, so the factor 1/4 in (2.12) gets cancelled. According to the definition (2.10), the Green function is obtained by substituting[‡]

$$J^*(w) = e^{ik_4\cdot w}, \quad J(z) = e^{ik_1\cdot z}, \quad A^\mu(x_3) = \epsilon_3^\mu e^{ik_3\cdot x_3}, \quad A^\mu(x'_6) = \epsilon_6^\mu e^{ik_6\cdot x'_6} \tag{2.14}$$

[‡]Note that in the case where n external photon vertices are on some chain, one should multiply by $n!$ after substituting $A^\mu(x(t_i)) = \epsilon_i^\mu e^{ik_i\cdot x(t_i)}$.

to eq.(2.11). Thus,

$$
\begin{aligned}
G_{\mathrm{I}}(k,\epsilon) &= ie^4\int dxdy\,(\Box^{-1})_{xy}\int_0^\infty dT\,e^{-im^2T}\int_0^\infty\frac{dT'}{T'}\,e^{-im^2T'}\int_0^T dt_2dt_3\int_0^{T'}dt_5dt_6\\
&\quad\times\int dwdz\int_z^w\mathcal{D}x\,e^{-i\int_0^T d\tau\,\frac{1}{4}\dot{x}^2}\oint\mathcal{D}x'\,e^{-i\int_0^{T'}d\tau'\,\frac{1}{4}\dot{x}'^2}\delta(x_2-x)\delta(x_5'-y)\\
&\qquad\times e^{i(k_1\cdot z+k_4\cdot w)}\,(\dot{x}_2\cdot\dot{x}_5')\left(\epsilon_3\cdot\dot{x}_3e^{ik_3\cdot x_3}\right)\left(\epsilon_6\cdot\dot{x}_6'e^{ik_6\cdot x_6'}\right) \qquad (2.15)\\
&= e^4\int_0^\infty d\alpha\int_0^\infty dT\,e^{-im^2T}\int_0^\infty\frac{dT'}{T'}\,e^{-im^2T'}\int_0^T dt_2dt_3\int_0^{T'}dt_5dt_6\\
&\quad\times\int_{\mathrm{I}}\mathcal{D}x(\tau)\,e^{-i\int d\tau\,\frac{1}{4}\dot{x}(\tau)^2}\,e^{i(k_1\cdot z+k_4\cdot w)}\,(-\dot{x}_2\cdot\dot{x}_5')\left(\epsilon_3\cdot\dot{x}_3e^{ik_3\cdot x_3}\right)\left(\epsilon_6\cdot\dot{x}_6'e^{ik_6\cdot x_6'}\right), \qquad (2.16)
\end{aligned}
$$

where we have expressed the photon propagator using Feynman parameter, and defined a "path-integral over the set I diagrams"[§] as

$$
\begin{aligned}
\int_{\mathrm{I}}\mathcal{D}x(\tau)\,e^{-i\int d\tau\frac{1}{4}\dot{x}(\tau)^2} &\equiv \int dxdy\,i\left(\frac{1}{4\pi i\alpha}\right)^{D/2}e^{-\frac{i}{4\alpha}(x-y)^2}\\
&\quad\times\int dwdz\int_z^w\mathcal{D}x\,e^{-i\int_0^T d\tau\,\frac{1}{4}\dot{x}^2}\oint\mathcal{D}x'\,e^{-i\int_0^{T'}d\tau'\,\frac{1}{4}\dot{x}'^2}\\
&\qquad\times\delta(x_2-x)\delta(x_5'-y). \qquad (2.17)
\end{aligned}
$$

Since the path-integral over $x(\tau)$ is Gaussian, it is straightforward (at least formally) to perform the integration. For convenience, we assign an outgoing momentum k_i and a polarization vector ϵ_i to every vertex ($x_1\equiv z$, $x_4\equiv w$), and replace the vertex factors by an exponential factor:

$$
e^{i(k_1\cdot z+k_4\cdot w)}\,(-\dot{x}_2\cdot\dot{x}_5')\left(\epsilon_3\cdot\dot{x}_3e^{ik_3\cdot x_3}\right)\left(\epsilon_6\cdot\dot{x}_6'e^{ik_6\cdot x_6'}\right)\longrightarrow\exp\left[\sum_{i=1}^{6}(ik_i\cdot x_i+\epsilon_i\cdot\dot{x}_i)\right]. \qquad (2.18)
$$

At the end of the calculation, to recover the correct result:

1) We set $k_2=k_5=0$ and $\epsilon_1=\epsilon_4=0$.

2) Only the terms in which each polarizatoin vector $\epsilon_2,\epsilon_3,\epsilon_5,\epsilon_6$ appears precisely once (multi-linear in each polarization vector) are retained.

3) We replace the internal photon wave function as

$$
\epsilon_2^\mu\epsilon_5^\nu\to-g^{\mu\nu}. \qquad (2.19)
$$

[§]To be precise, we have expressed scalar chains in path-integrals and photon propagators in Feynman parameter integrals.

The replacement (2.18) simplifies the integration over $x(\tau)$. Hence, we obtain

$$\begin{aligned} G_{\mathrm{I}}(k,\epsilon) &= e^4 \int_0^\infty d\alpha \int_0^\infty dT\, e^{-im^2T} \int_0^\infty \frac{dT'}{T'}\, e^{-im^2T'} \int_0^T dt_2 dt_3 \int_0^{T'} dt_5 dt_6 \\ &\quad \times \mathcal{N} \exp\Big[\frac{1}{2}\sum_{i,j=1}^{6}\Big\{-ik_i\cdot k_j G_B^{ij} - 2k_i\cdot\epsilon_j \partial_j G_B^{ij} + i\epsilon_i\cdot\epsilon_j \partial_i\partial_j G_B^{ij}\Big\}\Big], \end{aligned} \tag{2.20}$$

where the normalization factor is defined by

$$\mathcal{N} \equiv \int_{\mathrm{I}} \mathcal{D}x(\tau)\, e^{-i\int d\tau \frac{1}{4}\dot{x}(\tau)^2}, \tag{2.21}$$

and the two-point functions are given by

$$\begin{aligned} g^{\mu\nu}\, G_B^{ij} &= -i\,\langle x^\mu(t_i) x^\nu(t_j)\rangle\,, \\ g^{\mu\nu}\, \partial_j G_B^{ij} &= -i\,\langle x^\mu(t_i) \dot{x}^\nu(t_j)\rangle\,, \\ g^{\mu\nu}\, \partial_i\partial_j G_B^{ij} &= -i\,\langle \dot{x}^\mu(t_i) \dot{x}^\nu(t_j)\rangle\,, \end{aligned} \tag{2.22}$$

with the expectation value taken with respect to the path-integral average over the set I diagrams:

$$\langle \mathcal{O}(x)\rangle \equiv \mathcal{N}^{-1} \int_{\mathrm{I}} \mathcal{D}x(\tau)\, \mathcal{O}(x)\, e^{-i\int d\tau \frac{1}{4}\dot{x}(\tau)^2}. \tag{2.23}$$

We remind the reader that $\partial_j G_B^{ij}$ *differs* from the differentiation of G_B^{ij} with respect to t_j. Precise definition will be made clear in the next section.

So far we considered a specific example. The steps that led to eq.(2.20) can be generalized to an arbitrary set of diagrams: A set of diagrams consists of those which can be transformed to one another by sliding photon legs along the scalar chains, where any two three-point vertices on a same chain may combine to become a seagull vertex. Any single set contains all diagrams that are interrelated to one another by the gauge transformation of external and internal photons. In other words, each set constitutes a gauge-invariant subamplitude if the external scalar propagators are amputated and taken to be on-shell, $k_s^2 \to m^2$.¶ Thus, the Green function

$$G(k,\epsilon) = \int \prod_i dx_i\, e^{i\sum k_i\cdot x_i} \Big[\prod \tfrac{\delta}{i\delta J(w_i)} \prod \tfrac{\delta}{i\delta J^*(z_i)} \prod \epsilon_i^\mu \tfrac{\delta}{i\delta A^\mu(y_i)}\, W(J,J^*,A)\Big]_{J=J^*=A=0} \tag{2.24}$$

at each order of the coupling e can be decomposed to the sub-Green functions corresponding to the sets S of diagrams as

$$G(k,\epsilon) = \sum_S G_S(k,\epsilon), \tag{2.25}$$

¶This is true only for the renormalized Green function.

where the decomposition is accomplished naturally by expanding eq.(2.6) in powers of e, taking functional derivatives, and then substituting the external wave functions; see eqs.(2.11)-(2.16).

Following similar steps as in the former example, it is easy to see that the sub-Green function for a set S with $2n_s$ external scalars at $\mathcal{O}(e^n)$ is given generally by

$$\begin{aligned} G_S(k,\epsilon) &= (ie)^n\, C \int_0^\infty \prod_r d\alpha_r \prod_{chain\, l} \left(\int_0^\infty [dT_l]\, e^{-im^2 T_l} \int_0^{T_l} \prod_{i_l} dt_{i_l} \right) \\ &\quad \times \mathcal{N} \exp\left[\frac{1}{2} \sum_{i,j=1}^{n+2n_s} \left\{ -ik_i\cdot k_j G_B^{ij} - 2k_i\cdot\epsilon_j \partial_j G_B^{ij} + i\epsilon_i\cdot\epsilon_j \partial_i\partial_j G_B^{ij} \right\} \right], \end{aligned} \tag{2.26}$$

where C is the combinatorial factor[‖], α_r denotes the Feynman parameter of the r-th photon propagator. The chain l represents open or closed scalar chain, and the integral measure for its length T_l is

$$[dT_l] = \begin{cases} dT_l & \text{for } l = \text{open} \\ dT_l/T_l & \text{for } l = \text{closed} \end{cases} . \tag{2.27}$$

i_l represents photon vertex on the chain l. For convenience, we assigned an outgoing external momentum k_i and a polarization vector ϵ_i to every vertex i. Normalization factor $\mathcal{N}$ and two-point functions G_B^{ij}, $\partial_j G_B^{ij}$, and $\partial_i\partial_j G_B^{ij}$ are defined similarly as eqs.(2.21)-(2.23), but for the path-integral over the set S diagrams. The exponential factor is common to all S once the numbers of external scalars and photons as well as the order of e are fixed. (Explicit forms of G_B^{ij}'s depend on S, though.)

Furthermore, one should manipulate following processes (dependent on the set S) to the above $G_S(k,\epsilon)$:

1) If the vertex i is internal, we set corresponding $k_i = 0$.

2) If the vertex i is an endpoint of an open scalar chain, we set corresponding $\epsilon_i = 0$.

3) Only the terms multi-linear in each remaining polarization vector are kept.

4) We replace the polarization vectors at both ends (i_r and j_r) of every photon propagator r as

$$\epsilon_{i_r}^{\mu}\epsilon_{j_r}^{\nu} \to -g^{\mu\nu}. \tag{2.28}$$

[‖]The combinatorial factor C in general differs from (symmetry factor) $\times$ (statistical factor) of the corresponding Feynman diagrams, since certain diagrams do not distinguish the interchange of photon legs. e.g. $C = 1/2$ for the scalar self-energy at one-loop.

At this stage, one could directly evaluate the integrals in eq.(2.26) once the explicit forms of $\mathcal{N}$ and G_B^{ij}, $\partial_j G_B^{ij}$, and $\partial_i\partial_j G_B^{ij}$ are known. It already has advantages that a set of diagrams is cast into one single expression, and that the expressions for different sets of diagrams can be obtained in similar simple manners. Also, the spinor helicity technique [8, 9] can be used, so the number of independent dot products in the exponent can be reduced. Moreover, the Bern-Kosower-type rule allows use of partial integration technique, which simplify the calculation further. After that, one will integrate over α_r, t_i, and T_l.

In order to understand the remaining part of the rule, one needs a close study of the two-point function

$$g^{\mu\nu}\, G_B(\tau,\tau') \equiv -i\,\langle x^\mu(\tau) x^\nu(\tau')\rangle\,. \tag{2.29}$$

In principle, $G_B(\tau,\tau')$ is obtained by solving

$$\frac{\partial^2}{\partial\tau^2}\, G_B(\tau,\tau') = 2\,\delta(\tau-\tau') \tag{2.30}$$

after removing the zero mode, where appropriate boundary condition should be imposed at each internal vertex of the diagram[6]. We take, however, an alternative approach. It is possible to find simple rules to express $G_B(\tau,\tau')$ for a general diagram in terms of basic elements.

3 Relation to Feynman Parameter Formula and Decomposition of G_B

In this section, we derive the Feynman parameter formula for a scalar QED diagram (rather than for a set of diagrams considered in the previous section). In this formula a matrix Z_{ij} appears, which is identified to be the counterpart of G_B^{ij}. Z_{ij} is defined through integral over finite number of variables instead of the path-integral formulation, which enables us to investigate its properties in an unambiguous way. We deal with a general ϕ^3 diagram in subsection 3.a, followed by an extension to scalar QED diagrams in subsection 3.b. Then subsection 3.c will clarify the relation between the Feynman parameter integral formula and the general expression for $G_S(k,\epsilon)$ obtained in the last section. Finally, we show how to decompose $\mathcal{N}$ and G_B^{ij} to simpler elements in subsection 3.d.

3.a Scalar ϕ^3 Diagram

For the calculation of a general ϕ^3 diagram, it has long been known how to write down the Feynman parameter formula[10]. We rederive the formula in a manner convenient for application to the case of scalar QED diagram.

A general connected ϕ^3 diagram with n vertices and N internal lines can be written using Feynman rule in coordinate space as

$$iT = (ie)^n \int \prod_{i=1}^{n} d^D x_i \, e^{i \sum_i k_i \cdot x_i} \left[\prod_{r=1}^{N} i\Delta_F(x_{i_r} - x_{j_r}) \right], \tag{3.1}$$

where e is the ϕ^3 coupling constant. i_r and j_r represent the vertices at both ends of the r-th internal line. For convenience an outgoing external momentum k_i is assigned to every vertex. If the vertex is internal, we set the corresponding $k_i = 0$ at the end of the calculation. Combinatorial factor, if any, is suppressed for simplicity.

Substituting the propagator given in eq.(1.3), we have

$$iT = (ie)^n \int_0^\infty \prod_{r=1}^{N} d\alpha_r \, e^{-i(m^2 - i\epsilon)\sum_r \alpha_r} \, I(\alpha), \tag{3.2}$$

where

$$I(\alpha) \equiv \int [dx_i] \, \exp\left[-\frac{i}{4} \sum_{i,j=1}^{n} x_i \cdot x_j \, A_{ij}(\alpha) + i \sum_{i=1}^{n} k_i \cdot x_i \right], \tag{3.3}$$

and

$$\sum_{i,j=1}^{n} x_i \cdot x_j \, A_{ij}(\alpha) \equiv \sum_{r=1}^{N} \frac{(x_{i_r} - x_{j_r})^2}{\alpha_r}. \tag{3.4}$$

The matrix $A_{ij}(\alpha)$ represents the topoplogy of the diagram (how the vertices are connected). We have absorbed the factor before exponential in eq.(1.5) into the integral measure:

$$[dx_i] \equiv \left[\prod_{r=1}^{N} i \left(\frac{1}{4\pi i \alpha_r} \right)^{D/2} \right] \cdot \prod_{i=1}^{n} d^D x_i. \tag{3.5}$$

Note that it depends on Feynman parameters.

Then, after Gaussian integration over x_i's in $I(\alpha)$, we will be left with the desired Feynman parameter integral formula. Reflecting the invariance of the quadratic form (3.4) under translation

$$x_i^\mu \to x_i^\mu + c^\mu, \tag{3.6}$$

the matrix $A_{ij}(\alpha)$ has a zero eigenvalue. Namely, $I(\alpha)$ will be proportional to the δ-function representing momentum conservation. Indeed, after integration over x_i's, we obtain

$$I(\alpha) = (2\pi)^D \delta\left(\sum_{i=1}^n k_i\right) \cdot i^l \left(\frac{1}{4\pi i}\right)^{Dl/2} \Delta(\alpha)^{-D/2} \exp\left[i \sum_{i,j=1}^n k_i \cdot k_j Z_{ij}(\alpha)\right] \tag{3.7}$$

with

$$\Delta(\alpha) = \frac{1}{n}\left(\prod_{r=1}^N \alpha_r\right) \det{}' A(\alpha). \tag{3.8}$$

Here, $l = N - n + 1$ is the number of loop of the diagram. $\det'$ denotes the product of eigenvalues but zero. $Z_{ij}(\alpha)$ is the inverse of $A_{ij}(\alpha)$ after the zero mode is removed, or, fixing the center of gravity of vertices. Derivation of eqs.(3.7) and (3.8) is given in Appendix A.

In eq.(3.7), $Z_{ij}(\alpha)$ is not uniquely determined. This is because one can readily confirm the invariance of $I(\alpha)$ under the transformation of Z,

$$Z_{ij}(\alpha) \to Z_{ij}(\alpha) + f_i(\alpha) + f_j(\alpha) \qquad \text{for } \forall f_i(\alpha), \tag{3.9}$$

due to the momentum conservation. Among the class of $Z(\alpha)$'s connected by the transformation, there is a specific choice of $Z(\alpha)$ most convenient to the following argument. We choose

$$g^{\mu\nu} Z_{ij}(\alpha) \equiv -\frac{i}{4} \langle\!\langle\, (x_i - x_j)^\mu (x_i - x_j)^\nu \,\rangle\!\rangle \tag{3.10}$$

with $\langle\!\langle \ldots \rangle\!\rangle$ defined by

$$\langle\!\langle \mathcal{O} \rangle\!\rangle \equiv \frac{\int [dx_i]\, \mathcal{O}\, \exp[-\frac{i}{4}\sum x_i \cdot x_j A_{ij}]}{\int [dx_i]\, \exp[-\frac{i}{4}\sum x_i \cdot x_j A_{ij}]}. \tag{3.11}$$

The numerator and the denominator of eq.(3.11), respectively, are ill-defined due to the zero eigenvalue of $A(\alpha)$, so one has to first remove the zero mode in the integrals. Because $x_i - x_j$ in eq.(3.10) is invariant under the translation (3.6), $Z(\alpha)$ thus defined is independent of how one removes the zero mode.** Lam has pointed out[7] that this choice of $Z(\alpha)$ is characterized by the condition

$$Z_{ii}(\alpha) = 0 \qquad \text{for } 1 \le i \le n, \tag{3.12}$$

and is called zero-diagonal level scheme.

We list some important properties of Z_{ij} together with their proofs in Appendix B.

**Naively, $Z(\alpha)$ being the inverse of $A(\alpha)$, one may consider a natural definition would be $g^{\mu\nu} Z'_{ij}(\alpha) \equiv \frac{i}{2} \langle\!\langle x_i^\mu x_j^\nu \rangle\!\rangle$. Z' and Z given by eq.(3.10) are equivalent under the transformation (3.9) with $f_i = -Z'_{ii}/2$. The disadvantage of Z' is that it depends on how one removes the zero mode in calculating $\langle\!\langle x_i^\mu x_j^\nu \rangle\!\rangle$ since $x_i^\mu x_j^\nu$ is not translationally invariant.

Figure 4: A scalar QED diagram including only three-point gauge vertices, which contributes to the Green function amputated with respect to external photons and unamputated with respect to external scalars.

3.b Scalar QED Diagram

Now we derive the Feynman parameter intergral formula for a scalar QED diagram. We consider diagrams contributing to the Green function (2.24) which is amputated with respect to external photons and unamputated with respect to external scalars.

First, let us consider a diagram without seagull vertex; see Fig.4:

$$G_D(k,\epsilon) = (ie)^n \int \prod_i d^D x_i \, e^{i\sum k_i\cdot x_i} \left[\prod_{chain\, l} \left\{ \prod_{i_l=1}^{n_l} i\Delta_F(x_{i_l+1} - x_{i_l}) \overset{\leftrightarrow}{V}_{i_l} \right\} \right] \times \prod_{photon\, r} i\Delta_F(x_{i_r} - x_{j_r}) \Bigg|_{\epsilon^\mu_{i_r}\epsilon^\nu_{j_r} \to -g^{\mu\nu}}, \tag{3.13}$$

with the vertex operator

$$\overset{\leftrightarrow}{V}_j \equiv \epsilon^\mu_j \left(i\,\frac{\overrightarrow{\partial}}{\partial x^\mu_j} - i\,\frac{\overleftarrow{\partial}}{\partial x^\mu_j} \right). \tag{3.14}$$

Here, i_l's ($1 \le i_l \le n_l$) denote vertices on the scalar propagator chain l, labelled in increasing order along the charge flow on that chain. For an open chain we suppressed one additional scalar propagator $i\Delta_F(x_1 - x_0)$ on the right of the vertex operator $\overset{\leftrightarrow}{V}_1$ in eq.(3.13). i_r and j_r represent the vertices at both ends of the photon propagator r. Again, we assign an outgoing external momentum k_i and a polarization vector ϵ_i to every vertex i. At the end of the calculation, we set $k_i = 0$ for internal

Figure 5: The Feynman gauge photon propagator is obtained by replacing internal photon polarization vectors at both ends of every photon propagator by $-g_{\mu\nu}$.

vertices, $\epsilon_i = 0$ at the endpoints of open scalar chains, and also replace the polarization vectors at both ends of every internal photon line as $\epsilon^{\mu}_{i_r}\epsilon^{\nu}_{j_r} \to -g^{\mu\nu}$ (corresponding to taking Feynman gauge for photon propagator); see Fig.5.

Introducing Feynman parameter for every propagator, we have

$$G_D(k,\epsilon) = (ie)^n \prod_l \left(\int_0^\infty \prod_{i_l} d\alpha_{i_l} \right) \int_0^\infty \prod_r d\alpha_r \, e^{-i\sum_l T_l (m^2 - i\epsilon)} I(\alpha), \tag{3.15}$$

where

$$I(\alpha) \equiv \int \prod_i d^D x_i \, e^{i\sum k_i \cdot x_i} \left[\prod_l \left\{ \prod_{i_l} K(x_{i_l+1} - x_{i_l}; \alpha_{i_l}) \overset{\leftrightarrow}{V}_{i_l} \right\} \right] \prod_r K(x_{i_r} - x_{j_r}; \alpha_r). \tag{3.16}$$

K is the propagator defined in eq.(1.5); α_{i_l} is the Feynman parameter between the vertices i_l and $i_l - 1$, and $T_l = \sum_{i_l} \alpha_{i_l}$.

Before integrating over x_i's in $I(\alpha)$, we would like to replace the vertex operator $\overset{\leftrightarrow}{V}_i$ by some simple factor *associated with the vertex i*. To this end, we insert, on both sides of every vertex i, dummy vertices i' and i'' on the scalar line in the order $i'' < i < i'$ using the associativity relation (1.8); see Fig.6. Then we can replace the vertex operators acting on scalar propagators as

$$\overset{\leftrightarrow}{V}_i \quad \longrightarrow \quad \frac{1}{2}\epsilon_i \cdot \left(\tfrac{x'_i - x_i}{u'_i} + \tfrac{x_i - x''_i}{u''_i} \right). \tag{3.17}$$

Hence, we have

$$I(\alpha) \;\; = \;\; \int [dx_a] \prod_i \frac{1}{2}\epsilon_i \cdot \left(\tfrac{x'_i - x_i}{u'_i} + \tfrac{x_i - x''_i}{u''_i} \right) \exp\Bigl[-\frac{i}{4} \sum_{a,b} x_a \cdot x_b A_{ab}(\alpha, u', u'') + i \sum_i k_i \cdot x_i \Bigr]. \tag{3.18}$$

Figure 6: The dummy vertices i' and i'' inserted on both sides of every vertex i in the order $i'' < i < i'$ along the charge flow on the scalar line. The Feynman parameter between vertices i' and i (i and i'') is denoted as u'_i (u''_i).

Here, a, b denote vertices including dummy vertices (i, i', and i''). The matrix A_{ab} and the measure $[dx_a]$, respectively, are defined similarly as in eqs.(3.4) and (3.5), but depend also on u' and u''. Note that $I(\alpha)$ is independent of u'_i and u''_i, since it is completely arbitrary where to insert dummy vertices as long as the order $i'' < i < i'$ is preserved.

To perform Gaussian integration over x_a's, we exponentiate the polarization vectors as in eq.(2.18). Defining a source

$$J_a^\mu = \sum_i \left[k_i^\mu \delta_{ia} - \frac{i}{2} \epsilon_i^\mu \left(\tfrac{\delta_{i'a} - \delta_{ia}}{u'_i} + \tfrac{\delta_{ia} - \delta_{i''a}}{u''_i} \right) \right], \tag{3.19}$$

we have

$$\begin{aligned} I(\alpha) &= \int [dx_a] \exp\Big[-\frac{i}{4} \sum_{a,b} x_a \cdot x_b A_{ab}(\alpha, u', u'') + i \sum_a J_a \cdot x_a \Big]_{\text{linear in each } \epsilon} \qquad (3.20) \\ &= (2\pi)^D \delta\left(\sum_{i=1}^n k_i \right) \cdot i^l \left(\frac{1}{4\pi i} \right)^{Dl/2} \Delta(\alpha)^{-D/2} \\ &\quad \times \exp\Big[\sum_{i,j=1}^n \{ i\, k_i \cdot k_j Z_{ij} + 2 k_i \cdot \epsilon_j (\triangle_j Z_{ij}) - i \epsilon_i \cdot \epsilon_j (\triangle_i \triangle_j Z_{ij}) \} \Big]_{\text{linear in each } \epsilon} \qquad (3.21) \end{aligned}$$

for an l-loop diagram with

$$\begin{aligned} \triangle_j Z_{ij} &= \tfrac{Z_{ij'} - Z_{ij}}{2u'_j} + \tfrac{Z_{ij} - Z_{ij''}}{2u''_j}, \qquad (3.22) \\ \triangle_i \triangle_j Z_{ij} &= \frac{1}{4} \sum_{a,b} \left(\tfrac{\delta_{i'a} - \delta_{ia}}{u'_i} + \tfrac{\delta_{ia} - \delta_{i''a}}{u''_i} \right) \left(\tfrac{\delta_{j'b} - \delta_{jb}}{u'_j} + \tfrac{\delta_{jb} - \delta_{j''b}}{u''_j} \right) Z_{ab} \qquad (3.23) \end{aligned}$$

$$= \frac{1}{4u_i'u_j'}(Z_{i'j'} - Z_{ij'} - Z_{i'j} + Z_{ij}) + \dots \quad . \tag{3.24}$$

In the above expressions, $\Delta(\alpha)$ and Z_{ij} are the same as those appeared in eq.(3.7) for the ϕ^3 diagram of the same topology, since we recover exactly eq.(3.3) if we set all $\epsilon_i = 0$ and integrate out the dummy vertices in eq.(3.20). $Z_{ij'}$, etc. are defined similarly as in (3.10):

$$g^{\mu\nu} Z_{ab}(\alpha) \equiv -\frac{i}{4} \langle\!\langle\, (x_a - x_b)^\mu (x_a - x_b)^\nu \,\rangle\!\rangle \,, \tag{3.25}$$

but now $\langle\!\langle \dots \rangle\!\rangle$ includes integral over dummy vertices.

Remembering that $I(\alpha)$ is independent of u_i' and u_i'', we can take the limit $u_i', u_i'' \to +0$. Due to the fact

$$\lim_{u_i' \to 0} Z_{i'a} = \lim_{u_i'' \to 0} Z_{i''a} = Z_{ia}, \tag{3.26}$$

we can replace $\triangle_j Z_{ij}$ and $\triangle_i \triangle_j Z_{ij}$ as

$$\triangle_j Z_{ij} = \frac{1}{2} \lim_{u_j', u_j'' \to 0} \left(\frac{\partial}{\partial u_j'} Z_{ij'} - \frac{\partial}{\partial u_j''} Z_{ij''} \right) \tag{3.27}$$

$$\triangle_i \triangle_j Z_{ij} = \frac{1}{4} \lim_{\substack{u_i', u_i'' \to 0 \\ u_j', u_j'' \to 0}} \left(\frac{\partial}{\partial u_i'} \frac{\partial}{\partial u_j'} Z_{i'j'} - \frac{\partial}{\partial u_i'} \frac{\partial}{\partial u_j''} Z_{i'j''} - \frac{\partial}{\partial u_i''} \frac{\partial}{\partial u_j'} Z_{i''j'} + \frac{\partial}{\partial u_i''} \frac{\partial}{\partial u_j''} Z_{i''j''} \right) \tag{3.28}$$

At the same time, we can drop all diagonal terms $(i = j)$ in (3.21) using

$$\lim_{u_i' \to +0} \frac{\partial}{\partial u_i'} Z_{i'i} = \lim_{u_i'' \to +0} \frac{\partial}{\partial u_i''} Z_{i''i} = -\frac{1}{2} \tag{3.29}$$

and noting that only the terms multi-linear in each ϵ_i should be kept. See Appendix B for proofs of eqs.(3.26)-(3.29).

So far we considered a diagram without seagull vertex. The contribution of a seagull vertex can be incorporated through the process known as "pinching" from the corresponding diagram without seagull vertex. Any diagram containing a seagull vertex has the following factor (see Fig.7):

$$G_D(k, \epsilon) \propto i\Delta_F(y - x)\, \epsilon^\mu e^{ik\cdot x}\, \epsilon'_\mu e^{ik'\cdot x}\, i\Delta_F(x - z) \tag{3.30}$$

$$= \int dx'\, i\Delta_F(y - x)\, \epsilon^\mu e^{ik\cdot x}\, \delta(x - x')\, \epsilon'_\mu e^{ik'\cdot x'}\, i\Delta_F(x' - z). \tag{3.31}$$

The last line corresponds diagramatically to pinching the propagator between the two adjacent three-point vertices x and x'; see Fig.7. Noting that $\delta(x - x')$ is obtained by taking the $\alpha \to +0$ limit of

Figure 7: The seagull vertex can be incorporated by pinching the propagator between two adjacent three-point vertices with vertex factors $\epsilon^\mu e^{ik\cdot x}$ and $\epsilon'_\mu e^{ik'\cdot x}$.

the propagator in question (see eq.(1.7)), one can incorporate the contribution of a seagull vertex by replacing

$$\epsilon_i \cdot \epsilon_j \triangle_i \triangle_j Z_{ij} \to 2\epsilon_i \cdot \epsilon_j \, \delta(\alpha_{ij} - 0) \tag{3.32}$$

in eq.(3.21) of the diagram without seagull vertex, where α_{ij} is the Feynman parameter between the two adjacent three-point vertices i and j. If there are two or more seagull vertices in a diagram, one should pinch as many propagators of the corresponding diagram without seagull vertex.

3.c Relation between General Expression and Feynman Parameter Formula

Path-integral expression for $G_S(k, \epsilon)$ such as eq.(2.16) can be obtained from the finite dimensional integral (3.18) by inserting infinitely many dummy vertices along scalar chains using the associativity relation (1.8). The advantage of the path-integral expression lies in that it combines in a single expression sum of different diagrams that are related to one another by sliding photon legs along the scalar chains. Different orderings of photon legs correspond to different orderings of the proper time t_i's of the vertices.

Once the ordering of t_{i_l}'s is fixed along the scalar chain l, relations between t_{i_l}'s and Feynman parameters α_{i_l} are given by:

- For l =open, and $0 < t_1 < t_2 < \ldots < t_{n_l} < T_l$,

$$\begin{array}{lcl} t_1 & = & \alpha_1 \\ t_2 - t_1 & = & \alpha_2 \\ & \vdots & \\ t_{n_l} - t_{n_l - 1} & = & \alpha_{n_l} \\ T_l - t_{n_l} & = & \alpha_{n_l + 1} \end{array} \tag{3.33}$$

- For l =closed, and $0 < t_1 < t_2 < \ldots < t_{n_l} < T_l$,

$$\begin{array}{lcl} t_1 - t_{n_l} + T_l & = & \alpha_1 \\ t_2 - t_1 & = & \alpha_2 \\ & \vdots & \\ t_{n_l} - t_{n_l - 1} & = & \alpha_{n_l} \end{array} \tag{3.34}$$

With these relations, constituents of the general expression (2.26) and of the Feynman parameter formula (3.21) are identified as follows:

$$\mathcal{N} \;=\; (2\pi)^D \delta\left(\sum_{i=1}^{n} k_i\right) \cdot i^l \left(\frac{1}{4\pi i}\right)^{Dl/2} \Delta^{-D/2} \tag{3.35}$$

and

$$\begin{array}{rcr} G_B^{ij} & = & -2\, Z_{ij}, \\ \partial_j G_B^{ij} & = & -2\, \triangle_j Z_{ij}, \\ \partial_i \partial_j G_B^{ij} & = & -2\, \triangle_i \triangle_j Z_{ij}. \end{array} \tag{3.36}$$

We take the convention $G_B^{ii} = 0$ in accord with the zero-diagonal level scheme of Z_{ab}. As $\mathcal{N}$ and G_B^{ij}'s are defined for a set of diagrams, for a different ordering of t_{i_l}'s, Δ and Z_{ij} of a different diagram should be taken on the right-hand-side.

It is more subtle how the contributions of seagull vertices are contained in the general expression (2.26). They are contained in the $\partial_i \partial_j G_B^{ij}$ term when the two vertices t_i and t_j come to the same point. To see this, we consider the two-point function $G_B(\tau, \tau')$ defined in eq.(2.29) when τ and τ' are arbitrary points along a same scalar chain. One may, if necessary, identify it with Z_{ab}, where x_a and x_b are the dummy vertices inserted at the position of τ and τ', respectively. Due to eqs.(3.27), (3.28) and (3.36), one may express G_B^{ij}'s as

$$G_B^{ij} \;=\; G_B(t_i, t_j) \tag{3.37}$$

$$\partial_j G_B^{ij} \;=\; \frac{1}{2}\left[\lim_{\tau' \to t_j + 0} + \lim_{\tau' \to t_j - 0}\right] \frac{\partial}{\partial \tau'}\, G_B(t_i, \tau') \tag{3.38}$$

$$\partial_i \partial_j G_B^{ij} \;=\; \frac{1}{4}\left[\lim_{\tau \to t_i + 0} + \lim_{\tau \to t_i - 0}\right]\left[\lim_{\tau' \to t_j + 0} + \lim_{\tau' \to t_j - 0}\right] \frac{\partial}{\partial \tau} \frac{\partial}{\partial \tau'}\, G_B(\tau, \tau') \tag{3.39}$$

for $i \neq j$, and we may omit all terms where $i = j$; see discussion after eq.(3.28). Then using the identity[††]

$$\lim_{\tau \to \tau' \pm 0} \frac{\partial}{\partial \tau'}\, G_B(\tau, \tau') = \mp 1 \tag{3.40}$$

[††]The corresponding identity of Z_{ab} is shown in Appendix B, eq.(B.12).

which holds for any diagram, it can be shown that

$$\int_{t_j-u''}^{t_j+u'} dt_i\ \partial_i \partial_j G_B^{ij} = -2 + \Bigg(\int_{t_j+0}^{t_j+u'} dt_i + \int_{t_j-u''}^{t_j-0} dt_i \Bigg) \partial_i \partial_j G_B^{ij} \qquad (u', u'' > 0). \tag{3.41}$$

Thus, we see δ-function contribution as

$$\partial_i \partial_j G_B^{ij} \sim -2\delta(t_i - t_j) \qquad \text{for} \quad t_j - 0 < t_i < t_j + 0, \tag{3.42}$$

so that the contributions of seagull vertices are included as in eq.(3.32). (The factor 2 is accounted for by the interchange of i and j.) It is interesting how gauge symmetry takes advantage of the property of $G_B(\tau, \tau')$ which is an intrinsic quantity to any diagram.

Finally we comment on the integral variables of the two formulas (2.26) and (3.15). Note that along a closed scalar chain we have one more time variables to integrate over $(t_1, \ldots, t_{n_l}, T_l)$ than the corresponding Feynman parameters. In fact, one proper time variable can be integrated trivially; after the first $n_l - 1$ integrals over t_{i_l}'s, there remains no dependence on t_{n_l}[‡‡], so the last integral just gives a factor of T_l, which compensates T_l^{-1} in the integral measure (2.27).

3.d Decomposition of G_B and $\mathcal{N}$

Up to now we dealt with $G_B(\tau, \tau')$ and $\mathcal{N}$ for a general set of diagrams. We show that these quantities can be decomposed and written in terms of those for the basic sets of diagrams, namely, $G_B(\tau, \tau')$ and $\mathcal{N}$ for an open scalar chain and for a closed scalar chain; see Fig.8.

Let us first find the explicit forms of these basic $G_B(\tau, \tau')$ and $\mathcal{N}$. They are obtained from Z_{ij} and $\Delta(\alpha)$ for the corresponding diagrams (Fig.9). According to the calculation method described in Appendix B, one obtains for these diagrams

$$Z_{12}^{(open)} = -\frac{1}{2}\,\alpha_2, \qquad \Delta^{(open)} = 1, \tag{3.44}$$

$$Z_{12}^{(closed)} = -\frac{1}{2}\,\frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2}, \qquad \Delta^{(closed)} = \alpha_1 + \alpha_2. \tag{3.45}$$

‡‡ Any function of the form

$$f(t_{n_l}) = \int_0^{T_l} dt_{n_l - 1} \cdots \int_0^{T_l} dt_1\, F(G_B^{ij}, \mathcal{N}) \qquad (l : \text{closed chain}) \tag{3.43}$$

is invariant under translation $t_{n_l} \to t_{n_l} + c$ since G_B^{ij} and $\mathcal{N}$ are periodic functions of t_{i_l}'s and depend only on $t_{i_l} - t_{j_l}$; see eqs.(3.35) and (3.36). This means $f'(t) = 0$ so that $f(t)$ is independent of t.

Figure 8: The basic diagrams: a) an open scalar chain, and b) a closed scalar chain. Two-point function for an arbitrary set of diagrams can be decomposed and written in terms of $G_B^{(open)}$ and $G_B^{(closed)}$.

Figure 9: The basic diagrams corresponding to Fig.8 but parametrized by Feynman parameters.

It follows

$$G_B^{(open)}(\tau,\tau') = |\tau-\tau'|, \qquad \Delta^{(open)} = 1, \tag{3.46}$$

$$G_B^{(closed)}(\tau,\tau') = |\tau-\tau'| - \frac{(\tau-\tau')^2}{T}, \qquad \Delta^{(closed)} = T, \tag{3.47}$$

where the normalization factor $\mathcal{N}$ is given by eq.(3.35).

We deal with finite dimensional integral, and start from the defining equation of Z_{ab} and Δ for a diagram D:

$$\begin{aligned} I &= \int [dx_a]\, \exp\Big[-\frac{i}{4}\sum_{a,b} x_a\cdot x_b A_{ab} + i\sum_a J_a\cdot x_a\Big] && (3.48)\\ &= (2\pi)^D\,\delta\Big(\sum_a J_a\Big)\cdot i^l\left(\frac{1}{4\pi i}\right)^{Dl/2}\Delta^{-D/2}\,\exp\Big[i\sum_{a,b}J_a\cdot J_b Z_{ab}\Big]. && (3.49)\end{aligned}$$

We would like to know how the above expression changes when the vertices i and j in D are connected by a propagator whose Feynman parameter is α. (The diagram thus obtained is denoted as D'.) This is achieved if we multiply the integrand in (3.48) by

$$K(x_i-x_j;\alpha) \;=\; i\left(\frac{1}{4\pi i\alpha}\right)^{D/2}\exp\left[-\frac{i}{4\alpha}(x_i-x_j)^2\right] \tag{3.50}$$

before integration over $[dx_a]$. But it is an equivalent manipulation if we shift

$$J_a \to J_a + p\,(\delta_{ai}-\delta_{aj}), \tag{3.51}$$

multiply by $\exp(i\alpha p^2)$, and then integrate over p; see eq.(1.4). Applying this manipulation to (3.49), one obtains

$$\begin{aligned} I \;\to\; I' &= (2\pi)^D\,\delta\Big(\sum_a J_a\Big)\cdot i^{l+1}\left(\frac{1}{4\pi i}\right)^{D(l+1)/2}\left[\Delta\cdot(\alpha-2\,Z_{ij})\right]^{-D/2} && (3.52)\\ &\quad\times\exp\Big[i\sum_{a,b}J_a\cdot J_b\Big\{Z_{ab}+\frac{(Z_{ia}-Z_{ja}-Z_{ib}+Z_{jb})^2}{2\,(\alpha-2\,Z_{ij})}\Big\}\Big]. && (3.53)\end{aligned}$$

This expression defines Δ and Z_{ab} for D', and correspondingly we find the following rule* for obtaining $\mathcal{N}$ and G_B for the diagram D':

$$\begin{aligned} \Delta' &= \Delta\cdot(\alpha+G_B(t_i,t_j))\,, && (3.54)\\ G'_B(\tau,\tau') &= G_B(\tau,\tau') - \frac{(G_B(\tau,t_i)-G_B(\tau,t_j)-G_B(\tau',t_i)+G_B(\tau',t_j))^2}{4\,(\alpha+G_B(t_i,t_j))}. && (3.55)\end{aligned}$$

*Eq.(3.55) differs slightly from the expression obtained by Schmidt and Schubert[6] since they do not take the zero-diagonal level scheme. The difference is accounted for by the transformation (3.9).

Next we consider the case where two diagrams $D_1(\ni i)$ and $D_2(\ni j)$ are sewn together by a propagator (ij). In this case, we shift

$$J_a^{(1)} \to J_a^{(1)} + p\,\delta_{ia}, \qquad J_a^{(2)} \to J_a^{(2)} - p\,\delta_{ja} \tag{3.56}$$

in $I^{(1)}$ and $I^{(2)}$, respectively, multiply by $\exp(i\alpha p^2)$, and then integrate over p. It is straightforward to find the following rule:

$$\Delta' = \Delta^{(1)} \cdot \Delta^{(2)}, \tag{3.57}$$

$$G'_B(\tau,\tau') = \begin{cases} \alpha + G_B^{(1)}(\tau,t_i) + G_B^{(2)}(\tau',t_j) & \tau \in D_1,\ \tau' \in D_2 \\ G_B^{(1)}(\tau,\tau') & \tau,\tau' \in D_1 \\ G_B^{(2)}(\tau,\tau') & \tau,\tau' \in D_2 \end{cases}. \tag{3.58}$$

Any set S of diagrams can be constructed by connecting scalar chains with photon propagators. Then one may express G_B ($\mathcal{N}$) for S in terms of $G_B^{(open)}$ ($\mathcal{N}^{(open)}$) and $G_B^{(closed)}$ ($\mathcal{N}^{(closed)}$) either by using the above rules recursively, or, by applying similar manipulation for multiple photon propagator insertions at once.

Now we find an important property of two-point functions $\partial_j G_B^{ij}$ and $\partial_i\partial_j G_B^{ij}$. Writing $G_B(\tau,\tau')$ for an arbitrary set of diagrams in terms of the basic elements, we notice that ∂_i (∂_j) can be replaced by $\partial/\partial t_i$ ($\partial/\partial t_j$) if the vertex i (j) is external[7] or if the diagram is one-particle-reducible with respect to the photon propagator connected to the vertex i (j). (cf. eqs.(3.38) and (3.39).)

4 Integration By Parts

Now we are in place to explain the integration by parts technique, first introduced to field theoretical calculation by Bern and Kosower, which enables non-trivial reshuffling of various terms in eq.(2.26) *before* integrating over α_r, t_{i_l}, and T_l. This technique reduces the number of independent terms, and consequently reduces the labor in the evaluation of integrals.

4.a Example

Consider a simplest example [4]. According to eq.(2.26) and the manipulation 1)-4), the photon vacuum polarization at one-loop (Fig.10) is given by

$$\begin{aligned} G_S &= (2\pi)^D \delta(k_1+k_2)\cdot(ie)^2\cdot i\left(\frac{1}{4\pi i}\right)^{D/2}\int_0^\infty \frac{dT}{T}\int_0^T dt_1\,dt_2 \\ &\quad \times T^{-D/2}\, e^{-ik_1\cdot k_2 G_B^{12}}\,(k_1\cdot\epsilon_2\, k_2\cdot\epsilon_1\, \partial_1 G_B^{12}\, \partial_2 G_B^{12} + i\epsilon_1\cdot\epsilon_2\, \partial_1\partial_2 G_B^{12}), \end{aligned} \tag{4.1}$$

Figure 10: The one-loop diagrams contributing to the photon vacuum polarization.

where we used $\Delta = T$. Note that ∂_1 (∂_2) can be identified with $\partial/\partial t_1$ ($\partial/\partial t_2$) since vertices 1 and 2 are external vertices. We integrate by parts the second term with respect to t_1. The surface term vanishes due to the periodicity of G_B^{ij}. Thus,

$$\begin{aligned} G_S &= -(2\pi)^D \delta(k_1 + k_2) \cdot ie^2 \cdot \left(\frac{1}{4\pi i}\right)^{D/2} (k_1 \cdot \epsilon_2 \, k_2 \cdot \epsilon_1 \, - \epsilon_1 \cdot \epsilon_2 \, k_1 \cdot k_2) \\ & \quad \times \int_0^\infty dT \, T^{-1-D/2} \int_0^T dt_1 \, dt_2 \, e^{-ik_1 \cdot k_2 G_B^{12}} \, \partial_1 G_B^{12} \, \partial_2 G_B^{12}, \end{aligned} \tag{4.2}$$

and we find G_S is gauge-invariant *before* integration over t_1, t_2 and T. Note that the number of independent terms reduced from two to one.

To see the relation between gauge transformation and the integration by parts technique, we remember

$$G_S \propto \left\langle \int_0^T dt_1 \, \epsilon_1 \cdot \dot{x}(t_1) \, e^{ik_1 \cdot x(t_1)} \times \int_0^T dt_2 \, \epsilon_2 \cdot \dot{x}(t_2) \, e^{ik_2 \cdot x(t_2)} \right\rangle, \tag{4.3}$$

where $\langle \ldots \rangle$ denotes the path-integral average. Gauge transformation of photon 1 is achieved by replacing ϵ_1 by k_1. Then the vertex operator changes as

$$\epsilon_1 \cdot \dot{x}(t_1) \, e^{ik_1 \cdot x(t_1)} \to k_1 \cdot \dot{x}(t_1) \, e^{ik_1 \cdot x(t_1)} = -i \frac{d}{dt_1} e^{ik_1 \cdot x(t_1)}, \tag{4.4}$$

and

$$\begin{aligned} \delta G_S &\propto \left\langle \int_0^T dt_1 \, \frac{d}{dt_1} e^{ik_1 \cdot x(t_1)} \times \int_0^T dt_2 \, \epsilon_2 \cdot \dot{x}(t_2) \, e^{ik_2 \cdot x(t_2)} \right\rangle \\ &= \int_0^T dt_1 \, dt_2 \, \frac{\partial}{\partial t_1} \left\langle e^{ik_1 \cdot x(t_1)} \, \epsilon_2 \cdot \dot{x}(t_2) \, e^{ik_2 \cdot x(t_2)} \right\rangle \\ &= \int_0^T dt_1 \, dt_2 \, \frac{\partial}{\partial t_1} \left(-k_1 \cdot \epsilon_2 \, \partial_2 G_B^{12} \, e^{-ik_1 \cdot k_2 G_B^{12}} \right). \end{aligned} \tag{4.5}$$

Gauge transform of the integrand is given by total derivative, so G_S is obviously gauge-invariant whereas the integrand itself is not. We may add, however, to the integrand of G_S in eq.(4.3) a term which transforms equally but in opposite sign under the replacement $\epsilon_1 \to k_1$:

$$\frac{\partial}{\partial t_1}\left(\epsilon_1\cdot\epsilon_2\,\partial_2 G_B^{12}\,e^{-ik_1\cdot k_2 G_B^{12}}\right). \tag{4.6}$$

Being total derivative, addition of this term does not alter G_S. Now the integrand itself is gauge-invariant, and the above term is exactly the surface term of the partial integration in eq.(4.2).

4.b External Photon

We now show a general prescription for integration by parts with respect to the external gauge vertices.

First, if the external photons are on-shell and for a fixed helicity states, one can use spinor helicity technique[8, 9] to reduce the number of dot products in the exponent of the general expression (2.26). On the other hand, if the external photons are off-shell, one can replace each polarization vector as

$$\epsilon_i^\mu \to \epsilon_i'^\mu = \epsilon_i^\mu - \frac{\epsilon_i\cdot k_a}{k_i\cdot k_a}\,k_i^\mu = (\epsilon_i^\mu k_i^\nu - k_i^\mu \epsilon_i^\nu)\,k_{a\nu}\,\frac{1}{k_i\cdot k_a}. \tag{4.7}$$

The amplitude is invariant under this replacement, and also the resulting expression is manifestly gauge-invariant before integration over proper time variables. One may choose any k_a for each polarization vector ϵ_i. Since $k_a\cdot\epsilon_i' = 0$, appropriate choices of k_a's for all i's will reduce the number of terms in the exponent.

After reducing the terms in the exponent, and after manipulation 1)-4) above eq.(2.28), one integrates by parts with respect to the proper time of external vertices to reduce the number of independent terms in the integrand. In this procedure, one may omit surface terms for a closed scalar chain since the surface terms cancel with each other due to the periodicity of G_B. Also for an open scalar chain, surface terms can be neglected if one is interested in the S-matrix element, since each surface term cancel the propagator pole of the external scalars in the unamputated Green function; see Fig.11.

4.c Internal Photon

One may also apply integration by parts technique to the internal gauge vertices.[6] Using the decomposition rule derived in the previous section, one can write Δ, $\partial_j G_B$ and $\partial_i\partial_j G_B$ using $G_B^{(open)}$, $G_B^{(closed)}$, and their derivatives. One can always integrate by parts to eliminate all second derivatives. This corresponds to simplifying the expression using gauge transformation of the internal vertices.

Figure 11: The surface terms originating from the gauge transformation of an external photon along an open chain. (Some of) The propagator poles of external scalars get cancelled, so these surface terms do not contribute to the S-matrix element.

Figure 12: The surface terms originating from the gauge transformation of an internal photon whose both ends are attached to a same open scalar chain. (Some of) The surface terms cannot be omitted since they still contain the propagator poles of external scalars.

There is one exception for this procedure. The integration by parts with respect to any of the internal vertices whose the other end of the photon propagator is on a same *open* scalar chain does not lead to simplification. The surface terms of such partial integration still comprise the poles of external scalars as seen in Fig.12. Thus, one cannot omit the surface terms in this case.

5 Covariant Gauge for Internal Photons

From a field theoretical point of view it is interesting to know how the general expression changes if one used covariant gauge for internal photon propagators instead of Feynman gauge. Let i and j be the vertices at the both ends of photon propagator whose Feynman parameter is α; see Fig.13. In momentum space it can be written as

$$\frac{-i}{p^2 + i\epsilon}\left[g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right]. \tag{5.1}$$

Figure 13: The covariant gauge photon propagator whose Feynman parameter is α.

The $g_{\mu\nu}$ part is the Feynman gauge propagator, and appears in the path-integral formalism as

$$\dot{x}_i^\mu\,\dot{x}_j^\nu\;g_{\mu\nu}\exp\Big[-\frac{i}{4\alpha}(x_i-x_j)^2\Big] \tag{5.2}$$

with $x_i \equiv x(t_i)$ and $x_j \equiv x(t_j)$. Meanwhile, $p_\mu p_\nu$ part can be written as

$$\dot{x}_i^\mu\,\dot{x}_j^\nu\;i\alpha\frac{\partial}{\partial x_i^\mu}\frac{\partial}{\partial x_j^\nu}\exp\Big[-\frac{i}{4\alpha}(x_i-x_j)^2\Big]=i\alpha\,\frac{\partial}{\partial t_i}\frac{\partial}{\partial t_j}\exp\Big[-\frac{i}{4\alpha}(x_i-x_j)^2\Big], \tag{5.3}$$

where we used

$$i\int_0^\infty d\alpha\,\alpha\int\frac{d^Dp}{(2\pi)^D}\,\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\,e^{ip\cdot(x-y)+i\alpha p^2}=-i\int\frac{d^Dp}{(2\pi)^D}\,\frac{p_\mu p_\nu}{p^4}\,e^{ip\cdot(x-y)}, \tag{5.4}$$

cf. eq.(1.2). Therefore, we obtain the $p_\mu p_\nu$ part of photon (ij) by operating

$$(1-\xi)\,i\alpha\frac{\partial}{\partial t_i}\frac{\partial}{\partial t_j} \tag{5.5}$$

to the integrand of eq.(2.26) after setting $\epsilon_i = \epsilon_j = 0$. Again this is given by total derivative, so changing gauge parameter ξ can be regarded as a kind of gauge transformation.

From this we see that if one calculates a set of diagrams in different values of ξ, the difference of results is proportional to the surface term on each scalar chain. In particular, a set of diagrams without external scalars is independent of ξ (if expressed in terms of bare coupling and bare gauge parameter) since $G_B(\tau,\tau')$ is periodic function on each closed scalar chain.

6 Rule

Let us summarize the Bern-Kosower-type rule for calculating a set of diagrams in Scalar QED (*amputated* with respected to external photons and *unamputated* with respect to external scalars). The gauge-invariant sub-Green function for a set S with $2n_s$ external scalars at $\mathcal{O}(e^n)$ and for l loop is given by

$$\begin{aligned} G_S(k,\epsilon) &= (2\pi)^D\delta(\sum k_i)\cdot i^l\left(\frac{1}{4\pi i}\right)^{Dl/2}(ie)^n\, C \\ &\times\int_0^\infty\prod_r d\alpha_r\prod_{chain\, l}\left(\int_0^\infty [dT_l]\, e^{-i(m^2-i0)T_l}\int_0^{T_l}\prod_{i_l} dt_{i_l}\right)\mathcal{K}_{red}, \end{aligned} \tag{6.1}$$

where C is the combinatorial factor, α_r denotes the Feynman parameter of the r-th photon propagator. The chain l represents open or closed scalar chain, and the integral measure for its length T_l is

$$[dT_l] = \begin{cases} dT_l & \text{for } l = \text{open} \\ dT_l/T_l & \text{for } l = \text{closed} \end{cases} . \tag{6.2}$$

i_l represents photon vertex on the chain l.

The so-called reduced generating kinematical factor $\mathcal{K}_{red}$ is obtained from the generating kinematical factor

$$\mathcal{K} = \Delta^{-D/2}\cdot\exp\left[\frac{1}{2}\sum_{i\neq j}^{n+2n_s}\left\{-ik_i\cdot k_j G_B^{ij} - 2k_i\cdot\epsilon_j\partial_j G_B^{ij} + i\epsilon_i\cdot\epsilon_j\partial_i\partial_j G_B^{ij}\right\}\right] \tag{6.3}$$

after the following manipulation.

1) If the vertex i is internal, we set corresponding $k_i = 0$.

2) If the vertex i is an endpoint of an open scalar chain, we set corresponding $\epsilon_i = 0$.

3) If the external photons are on-shell and for a fixed helicity states, use spinor helicity technique to reduce the number of dot products in the exponent; if the external photons are off-shell use replacement (4.7) to reduce the number of dot products (written in terms of ϵ_i''s).

4) Only the terms multi-linear in each remaining polarization vector are kept.

5) We replace the polarization vectors at both ends of every photon propagator r as

$$\epsilon_{i_r}^\mu\epsilon_{j_r}^\nu \to -g^{\mu\nu}. \tag{6.4}$$

Figure 14: A set of one-loop diagrams containing a ϕ^4-operator insertion.

Again some of the Lorentz contractions vanish.

Then integrate by parts with respect to the proper-times of external vertices. Also, integrate by parts with respect to the proper-times of internal vertices after writing Δ, $\partial_j G_B$ and $\partial_i \partial_j G_B$ in terms of $G_B^{(open)}$, $G_B^{(closed)}$, and their derivatives. (Use decomposition rules (3.54), (3.55), (3.57) and (3.58), and also eqs.(3.37)-(3.39) for this purpose.) Surface terms can be omitted except for the special case described in subsection 4.c. The partial integrations generally reduce the number of independent terms.

In order to integrate over α_r, t_i, and T_l, it is sometimes convenient to transform the variables to the conventional Feynman parameter at this stage using relations (3.33) and (3.34).

7 Operator Insertion

So far we have considered sets of diagrams containing only gauge interactions. In practical calculations, however, one will need to calculate diagrams containing both gauge interactions and other interactions, or more generally, operator insertions to the sets of diagrams considered above. We show in two examples how to calculate such diagrams. The idea is to replace any operator $\mathcal{O}(\phi)$ by the functional derivatives $\delta/\delta J(x)$ and $\delta/\delta J^*(x)$.

Let us see how to calculate the set of diagrams in Fig.14 contributing to the Green function with a $|\phi|^4$ operator insertion:

$$\int \mathcal{D}\phi\, \mathcal{D}Q_\mu \int dz\, \frac{i\lambda}{4} |\phi(z)|^4 \exp i \int dx\, [\mathcal{L} + \mathcal{L}_{gf} + J^*\phi + J\phi^* + j^\mu Q_\mu] \Big|_{j_\mu \to -\Box A_\mu} \tag{7.1}$$

$$= \frac{i\lambda}{4} \int dz \left(\frac{\delta}{\delta J(z)} \right)^2 \left(\frac{\delta}{\delta J^*(z)} \right)^2 e^{W(J,J^*,A_\mu)}. \tag{7.2}$$

Figure 15: Any set of diagrams with ϕ^4-operator insertion can be obtained by pinching a dummy photon propagator by setting the Feynman parameter $\alpha \to 0$.

Following similar steps as in eqs.(2.10)-(2.20), we find

$$\begin{aligned} G(k,\epsilon) &= (2\pi)^D \delta(\sum k_i) \cdot i \left(\frac{1}{4\pi i}\right)^{D/2} \cdot (i\lambda)(ie)^2 \int_0^\infty dT\, e^{-im^2 T} \int_0^T dt_1 dt_2 \\ &\quad \times \Delta \,\exp\Big[\frac{1}{2}\sum_{i\neq j}\Big\{-ik_i \cdot k_j G_B^{ij} - 2k_i \cdot \epsilon_j \partial_j G_B^{ij} + i\epsilon_i \cdot \epsilon_j \partial_i \partial_j G_B^{ij}\Big\}\Big], \end{aligned} \tag{7.3}$$

where $k_0 = p + p'$ and $\epsilon_0 = 0$. The two-point function $G_B(\tau, \tau')$ is obtained using the decomposition rule described in subsection 3.d with a little modification. Namely, we can compute G_B by connecting both ends of an open scalar chain with a dummy photon propagator, and then pinching the photon propagator by setting its Feynman parameter as $\alpha \to 0$; see Fig.15 and eq.(1.7). Therefore, we find using (3.55)

$$\begin{aligned} G_B(\tau,\tau') &= |\tau - \tau'| - \frac{[\tau - (T-\tau) - \tau' + (T-\tau')]^2}{4T} \\ &= |\tau - \tau'| - \frac{(\tau-\tau')^2}{T} \end{aligned} \tag{7.4}$$

and

$$\Delta = T. \tag{7.5}$$

The above two-point function coincides with $G_B^{(closed)}$, which is a reasonable result. Note, however, that the integral measure dT differs from that of a closed scalar chain since the zeroth vertex is not that of gauge interaction. Compare the discussion in the last paragraph in subsection 3.c.

Figure 16: A set of two-loop diagrams with a ϕ^4-operator insertion.

The second example is the set of diagrams in Fig.16. Also starting from eq.(7.2), we obtain

$$\begin{aligned} G(k,\epsilon) &= (2\pi)^D\delta(\sum k_i)\cdot i^2\left(\frac{1}{4\pi i}\right)^D\cdot(i\lambda)(ie)^4 \\ &\quad\times\int_0^\infty dT_1\int_0^\infty dT_2\, e^{-im^2(T_1+T_2)}\int_0^{T_1}dt_1dt_2\int_0^{T_2}dt_3dt_4 \\ &\quad\times\Delta\,\exp\Big[\frac{1}{2}\sum_{i\neq j}\Big\{-ik_i\cdot k_jG_B^{ij}-2k_i\cdot\epsilon_j\partial_jG_B^{ij}+i\epsilon_i\cdot\epsilon_j\partial_i\partial_jG_B^{ij}\Big\}\Big] \end{aligned} \tag{7.6}$$

with $k_0 = p + p'$ and $\epsilon_0 = 0$. This time the two-point function is obtained by sewing together two scalar loops and pinching the photon propagator as in Fig.17. Thus,

$$G_B(\tau,\tau') = \begin{cases} |\tau-\tau'| - \frac{(\tau-\tau')^2}{T_1} & \tau,\tau'\in loop\,1 \\ \tau-\frac{\tau^2}{T_1}+\tau'-\frac{\tau'^2}{T_2} & \tau\in loop\,1,\quad \tau'\in loop\,2 \\ |\tau-\tau'| - \frac{(\tau-\tau')^2}{T_2} & \tau,\tau'\in loop\,2 \end{cases} \tag{7.7}$$

and

$$\Delta = T_1\,T_2. \tag{7.8}$$

8 Conclusion

First of all, we conceive a set of diagrams connected by gauge transformation as an entity expressed by a single path-integral. The point is to assign proper time to the set of diagrams along the charge flow and also express each photon propagator by Feynman parameter integral in coordinate

Figure 17: The two-point function $G_B(\tau, \tau')$ of the diagrams in Fig.16 can be obtained by sewing together two one-loop diagrams by a dummy photon propagator and taking $\alpha \to 0$.

space. This enables one to find a general path-integral expression for any set of diagrams starting from the quantum field theory. At this stage, the resulting expression after integrating out $x(\tau)$ is equivalent to the Feynman parameter integral formula. Simple rules for constructing the two-point function (correlation function on the worldline) $G_B(\tau, \tau') \sim \langle x(\tau)\, x(\tau')\rangle$ for a general set of diagrams is obtained.

Secondly, the path-integral expression allows us to use integration by parts technique both for external and internal gauge vertices. Manifestly gauge invariant form with respect to external photons can be obtained before integrating over the proper time variables. Surface terms can be neglected if the external scalars are on-shell. The integration by parts technique reduces the number of independent integrals, which can be interpreted as a non-trivial reshuffling of the original Feynman diagrams.

We have extended former trials to derive Bern-Kosower-type rule from quantum field theory to the general diagrams for scalar QED, in particular to the diagrams including external scalar particles. We have shown clear correspondence to the conventional Feynman rule, which enabled us to avoid any ambiguity coming from the infinite dimensionality of the path-integral approach.

The method for deriving the general path-integral expression in section 2 can be straightforwardly extended to the case of spinor QED.

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Appendix A: Derivation of Eqs.(3.7) and (3.8)

We show how to integrate over x_i's in eq.(3.3):

$$I(\alpha) \equiv \int [dx_i] \, \exp\left[-\frac{i}{4} \sum_{i,j=1}^{n} x_i \cdot x_j \, A_{ij}(\alpha) + i \sum_{i=1}^{n} k_i \cdot x_i \right]. \tag{A.1}$$

First, insert the identity

$$1 = \int d^D c \; \delta\left(\sum_{i=1}^{n} x_i - c \right), \tag{A.2}$$

and shift all vertices as $x_i^\mu \to x_i^\mu + c^\mu / n$. We have

$$\begin{aligned} I(\alpha) &= \int [dx_i] \int d^D c \; \delta\left(\sum x_i\right) \, \exp\left[-\frac{i}{4} \sum x_i \cdot x_j \, A_{ij} + i \sum k_i \cdot x_i + \frac{i}{n} \sum k_i \cdot c \right] & \text{(A.3)} \\ &= (2\pi)^D \delta\left(\sum k_i\right) \cdot n^D \int [dx_i] \, \delta\left(\sum x_i\right) \exp\left[-\frac{i}{4} \sum x_i \cdot x_j \, A_{ij} + i \sum k_i \cdot x_i \right] & \text{(A.4)} \end{aligned}$$

We may further shift $x_i^\mu \to x_i^\mu - y^\mu / n$:

$$I(\alpha) = (2\pi)^D \delta\left(\sum k_i\right) \cdot n^D \int [dx_i] \, \delta\left(\sum x_i - y\right) \, \exp\left[-\frac{i}{4} \sum x_i \cdot x_j \, A_{ij} + i \sum k_i \cdot x_i \right]. \tag{A.5}$$

It is independent of y. Again insert

$$1 = i \left(\frac{\beta}{4\pi i} \right)^{D/2} \int d^D y \, e^{-i\beta y^2/4}, \tag{A.6}$$

and integrate over y. Thus,

$$\begin{aligned} I(\alpha) &= (2\pi)^D \delta\left(\sum k_i\right) \cdot i \left(\frac{\beta}{4\pi i} \right)^{D/2} n^D \\ &\quad \times \int [dx_i] \, \exp\left[-\frac{i}{4} \sum x_i \cdot x_j \, A'_{ij} + i \sum k_i \cdot x_i \right], & \text{(A.7)} \end{aligned}$$

where $A'_{ij} = A_{ij} + \beta$. Now the zero-mode is removed. We may integrate over x_i's, and noting the fact $\det A' = n\beta \cdot \det' A$, we obtain eqs.(3.7) and (3.8) with $Z = A'^{-1}$. (It is necessary to transform Z_{ij} appropriately for obtaining Z in zero-diagonal level scheme; see Appendix B.)

Appendix B: Properties of Z_{ab}

Definition

For a given scalar QED diagram without seagull vertex, Z_{ab} is defined by

$$g^{\mu\nu} Z_{ab} \equiv \left(-\frac{i}{4}\right) \frac{\int [d^D x_c]\, (x_a - x_b)^\mu (x_a - x_b)^\nu \, \exp\left[-\frac{i}{4} \sum_{c,d} x_c \cdot x_d A_{cd}\right]}{\int [d^D x_c] \, \exp\left[-\frac{i}{4} \sum_{c,d} x_c \cdot x_d A_{cd}\right]}. \tag{B.1}$$

On both sides of each vertex i dummy vertices i' and i'' are inserted as shown in Fig.6. Here, a, b, c, d denote vertices including dummy vertices (i, i', and i''). The matrix A represents the topology of the diagram, and is defined by

$$\sum_{c,d} x_c \cdot x_d \, A_{cd} \equiv \sum_{(cd)} \frac{(x_c - x_d)^2}{\alpha_{(cd)}}, \tag{B.2}$$

where $\alpha_{(cd)}$ denotes the Feynman parameter of the propagator connecting the vertices c and d.

Methods for Calculating Z_{ab}†

In order to obtain Z_{ab} from the matrix A, first one may as well reduce the size of A by eliminating all external vertices in the diagram (but a and/or b if it is external) using the associativity property (1.8) of propagator K. Then, there are several ways to calculate Z_{ab} from the reduced A. We exemplify two such methods here.

(Method 1) Let T be a matrix defined by

$$T_{ab} = 1 \qquad \text{for } \forall a, b, \tag{B.3}$$

and define $Z' \equiv (A + \beta T)^{-1}$. Z' is well-defined as long as $\beta \neq 0$. Then Z_{ab} can be obtained using (3.9) as

$$Z_{ab} = Z'_{ab} - \frac{1}{2}(Z'_{aa} + Z'_{bb}). \tag{B.4}$$

Obviously the diagonal elements vanish. Z is independent of β so one may simplify calculation by taking $\beta \to \infty$ after getting Z'.

† Z_{ab} can also be computed using graph-theoretical formula[7].

(Method 2) Let $\tilde{A}$ be a submatrix of A obtained by deletion of the c-th row and c-th column. One may choose any vertex c for this purpose. (This corresponds to fixing the coordinate of c to be $x_c = 0$ in eq.(B.1).) $\tilde{A}$ can be inverted, so define

$$Z'_{ab} = \begin{cases} (\tilde{A}^{-1})_{ab} & \text{for } a, b \neq c \\ \\ 0 & \text{otherwise} \end{cases} . \tag{B.5}$$

Then Z_{ab} can be obtained as

$$Z_{ab} = Z'_{ab} - \frac{1}{2}(Z'_{aa} + Z'_{bb}). \tag{B.6}$$

Properties

$$Z_{ab} = Z_{ba} \tag{B.7}$$

$$Z_{aa} = 0 \tag{B.8}$$

$$\lim_{u'_i \to +0} Z_{i'a} = \lim_{u''_i \to +0} Z_{i''a} = Z_{ia} \tag{B.9}$$

$$\lim_{u'_i \to +0} \frac{Z_{i'a} - Z_{ia}}{u'_i} = \lim_{u'_i \to +0} \frac{\partial}{\partial u'_i} Z_{i'a} \tag{B.10}$$

$$\lim_{u''_i \to +0} \frac{Z_{ia} - Z_{i''a}}{u''_i} = - \lim_{u''_i \to +0} \frac{\partial}{\partial u''_i} Z_{i''a} \tag{B.11}$$

$$\lim_{u'_i \to +0} \frac{\partial}{\partial u'_i} Z_{i'i} = \lim_{u''_i \to +0} \frac{\partial}{\partial u''_i} Z_{i''i} = -\frac{1}{2} \tag{B.12}$$

$$\lim_{\substack{u'_i \to +0 \\ u'_j \to +0}} \frac{Z_{i'j'} - Z_{ij'} - Z_{i'j} + Z_{ij}}{u'_i u'_j} = \begin{cases} \displaystyle\lim_{\substack{u'_i \to +0 \\ u'_j \to +0}} \frac{\partial^2}{\partial u'_i \partial u'_j} Z_{i'j'} & \text{for } i \neq j \\ \\ \infty & \text{for } i = j \end{cases} \tag{B.13}$$

(Proof)

Eq.(B.9): Use eq.(1.7).

Eqs.(B.10) and (B.11): Use eq.(B.9).

Eq.(B.12):

$$\lim_{u_i' \to +0} \frac{\partial}{\partial u_i'} Z_{i'i} = \lim_{u_i' \to +0} \frac{Z_{i'i} - Z_{ii}}{u_i'} = \lim_{u_i' \to +0} \frac{Z_{i'i}}{u_i'}. \tag{B.14}$$

Then substituting the definition (B.1), the integrand will be

$$\begin{aligned} &\frac{(x_i' - x_i)^\mu (x_i' - x_i)^\nu}{u_i'} \exp\Big[-\frac{i}{4}\frac{(x_i' - x_i)^2}{u_i'}\Big] = (x_i' - x_i)^\mu \, 2i \, \frac{\partial}{\partial x'_{i\,\nu}} \exp\Big[-\frac{i}{4}\frac{(x_i' - x_i)^2}{u_i'}\Big] \\ &= -2i \, g^{\mu\nu} \exp\Big[-\frac{i}{4}\frac{(x_i' - x_i)^2}{u_i'}\Big], \end{aligned} \tag{B.15}$$

where in the last line we integrated by parts with respect to $x_i'^\nu$. Thus, the numerator will be proportional to the donominator in (B.1).

Appendix C: Sample Calculation

In this appendix we apply the Bern-Kosower-type rule to calculation of the set of diagrams shown in Fig.18. According to eq.(6.1), the Green function is given by

$$\begin{aligned} G_S(k_0, k_2, k_4, \epsilon_2) &= (2\pi)^D \delta\Big(\sum_{i=0}^{4} k_i\Big) \cdot i \left(\frac{1}{4\pi i}\right)^{D/2} \frac{1}{2} (ie)^3 \\ &\quad \times \int_0^\infty d\alpha \, e^{-i(\lambda^2 - i0)\alpha} \int_0^\infty dT \, e^{-i(m^2 - i0)T} \int_0^T dt_1 dt_2 dt_3 \, \mathcal{K}_{red}, \end{aligned} \tag{C.1}$$

where λ is the photon mass. $\mathcal{K}_{red}$ is obtained from $\mathcal{K}$ in eq.(6.3) after the manipulation 1)-5):

$$\begin{aligned} \mathcal{K}_{red} &= \Delta^{-D/2} \Big[-\sum_{i=0}^{4} k_i^\mu \partial_1 G_B^{i1} \sum_{j=0}^{4} k_{j\mu} \partial_3 G_B^{j3} \sum_{l=0}^{4} \epsilon_2' \cdot k_l \partial_2 G_B^{l2} + i \partial_1 \partial_2 G_B^{12} \sum_{j=0}^{4} \epsilon_2' \cdot k_j \partial_3 G_B^{j3} \\ &\quad + i \partial_2 \partial_3 G_B^{23} \sum_{j=0}^{4} \epsilon_2' \cdot k_i \partial_1 G_B^{i1} + i D \partial_1 \partial_3 G_B^{13} \sum_{l=0}^{4} \epsilon_2' \cdot k_l \partial_2 G_B^{l2} \Big] \exp\Big[-\frac{i}{2} \sum_{i \neq j} k_i \cdot k_j G_B^{ij}\Big]. \end{aligned} \tag{C.2}$$

Here, we choose

$$\epsilon_2'^\mu = \epsilon_2^\mu - \frac{\epsilon_2 \cdot k_2}{k_2^2} k_2^\mu, \tag{C.3}$$

so that $\epsilon_2' \cdot k_2 = 0$.

Figure 18: The set of diagrams calculated in Appendix C.

Now we integrate by parts with respect to t_2:

$$
\begin{aligned}
\mathcal{K}_{red} \to \quad \Delta^{-D/2}\Big[\quad & (k_0\partial_1 G_B^{01} + k_2\partial_1 G_B^{21} + k_4\partial_1 G_B^{41}) \cdot (k_0\partial_3 G_B^{03} + k_2\partial_3 G_B^{23} + k_4\partial_3 G_B^{43}) \\
& \times \epsilon_2' \cdot (k_0\partial_2 G_B^{02} + k_4\partial_2 G_B^{42}) \\
& -\partial_1 G_B^{12}\, \epsilon_2' \cdot (k_0\partial_3 G_B^{03} + k_4\partial_3 G_B^{43})\, k_2 \cdot (k_0\partial_2 G_B^{02} + k_4\partial_2 G_B^{42}) \\
& -\partial_3 G_B^{23}\, \epsilon_2' \cdot (k_0\partial_1 G_B^{01} + k_4\partial_1 G_B^{41})\, k_2 \cdot (k_0\partial_2 G_B^{02} + k_4\partial_2 G_B^{42}) \\
& +iD\, \partial_1\partial_3 G_B^{13} \epsilon_2' \cdot (k_0\partial_2 G_B^{02} + k_4\partial_2 G_B^{42})\Big] \\
& \quad \times \exp[-i(k_0 \cdot k_2 G_B^{02} + k_0 \cdot k_4 G_B^{04} + k_2 \cdot k_4 G_B^{24})]
\end{aligned} \tag{C.4}
$$

We do not integrate by parts with respect to t_1 or t_3; compare the discussion in subsection 4.c. The delta function part in $\partial_1\partial_3 G_B$ corresponds to the tadpole diagrams (Fig.18(f)(g)).

Then we substitute the explicit forms of Δ, G_B^{ij}, and their derivatives:

$$
\Delta = \alpha + |t_3 - t_1|, \tag{C.5}
$$

$$
G_B^{ij} = |t_i - t_j| - \frac{[|t_i - t_1| - |t_i - t_3| - |t_j - t_1| + |t_j - t_3|]^2}{4\Delta}, \tag{C.6}
$$

$$\partial_j G_B^{ij} = -\mathrm{sign}(t_i - t_j) + \frac{1}{2\Delta}[|t_i - t_1| - |t_i - t_3| - |t_j - t_1| + |t_j - t_3|]$$
$$\times [\mathrm{sign}(t_j - t_1) - \mathrm{sign}(t_j - t_3)], \tag{C.7}$$
$$\partial_1 \partial_3 G_B^{13} = -2\delta(t_1 - t_3) + \frac{1}{2\Delta}, \tag{C.8}$$

where $t_0 = 0$ and $t_4 = T$. It is understood that $\mathrm{sign}(0) = 0$ in eq.(C.7). Once the time ordering of t_1, t_2, and t_3 is fixed, we can transform the integral variables using eq.(3.33). The rest is same as the usual Feynman parameter integral. We obtain, for example,

$$G_S(t_1 < t_2 < t_3) = (2\pi)^D \delta(\sum k_i) \cdot i \left(\frac{1}{4\pi i}\right)^{D/2} (ie)^3 \left[\frac{i}{k_0^2 - m^2} \frac{i}{k_4^2 - m^2}\right]$$
$$\times i\epsilon_2' \cdot (k_4 - k_0) \left[(1-\omega)\, I_1 + \omega\, I_2 + (-i)^{-D/2}\, \Gamma(2 - \tfrac{D}{2})\, I_3\right] \tag{C.9}$$

where $\omega = -k_0 \cdot k_4 / m^2 > 1$, and

$$I_1 = \int_0^1 dx \int_0^{1-x} dy\, 2(1-2x)(y^2 - 2y) \times [x^2 + y^2 + 2\omega xy]^{-1}$$
$$= \frac{7}{6}\frac{1}{\omega - 1} - \frac{1}{\sqrt{\omega^2 - 1}}\Big(\frac{3}{2} + \frac{7}{6}\frac{1}{\omega - 1}\Big)\mathrm{arccosh}\,\omega \tag{C.10}$$
$$I_2 = \int_0^1 dx \int_0^{1-x} dy\, (1-x-y)(x+y-2)^2 \times \left[x^2 + y^2 + 2\omega xy + \frac{\lambda^2}{m^2}(1-x-y)\right]^{-1}$$
$$= -\frac{1}{\sqrt{\omega^2 - 1}}\left[\frac{35}{6} + 2\log\frac{\lambda^2}{m^2}\right]\mathrm{arccosh}\,\omega + \frac{8}{\sqrt{\omega^2 - 1}} \int_0^{\frac{1}{2}\mathrm{arccosh}\,\omega} d\varphi\, \varphi \tanh\varphi \tag{C.11}$$
$$I_3 = \int_0^1 dx \int_0^{1-x} dy\, \frac{1}{2}(1-x-y) \times [m^2(x^2 + y^2 + 2\omega xy)]^{D/2-2}$$
$$= \frac{1}{12} + \frac{D-4}{4}\Big(-\frac{11}{18} + \frac{1}{6}\log\frac{m^2}{\mu^2} - \frac{1}{6}\sqrt{\frac{\omega+1}{\omega-1}}\mathrm{arccosh}\,\omega\Big). \tag{C.12}$$

We set the external scalars on-shell $k_0^2 = k_4^2 = m^2$ except for the propagator factors in the above expressions. G_S for other time orderings can be calculated similarly. (See below.)

Finally, if we are interested in the vertex function, we should amputate the external scalars in the above example. For this purpose, one should add the counter term for the wave function correction first, which needs to be calculated separately. After adding the counter term and amputating the external propagators, we find the vertex function at one-loop (for on-shell external scalars) to be

$$\epsilon_2^\mu\, \Gamma_\mu^{\text{1-loop}}(k_0, k_4) = -\frac{e^2}{16\pi^2}\,\epsilon_2 \cdot (k_4 - k_0)\left[\frac{9}{2(4-D)} - \frac{9}{4}(\log\frac{m^2}{4\pi\mu^2} + \gamma_E) + \frac{19}{4}\right.$$
$$+\frac{1}{\sqrt{\omega^2 - 1}}\Big(\frac{19}{12} - \frac{17}{4}\omega - 2\omega\log\frac{\lambda^2}{m^2}\Big)\mathrm{arccosh}\,\omega$$
$$\left.+\frac{8\omega}{\sqrt{\omega^2 - 1}}\int_0^{\frac{1}{2}\mathrm{arccosh}\,\omega} d\varphi\, \varphi \tanh\varphi\right]. \tag{C.13}$$

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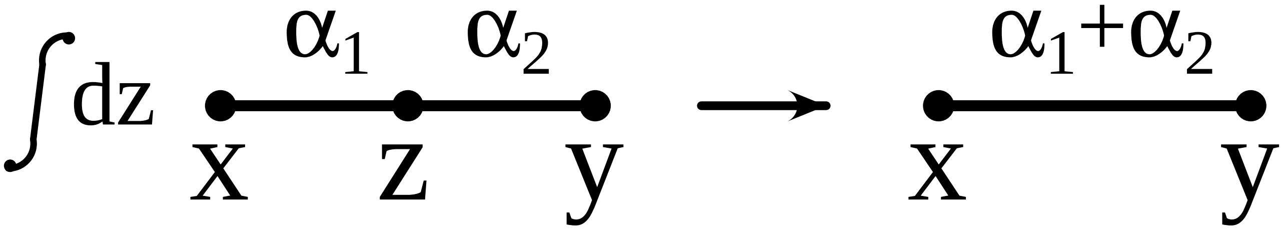
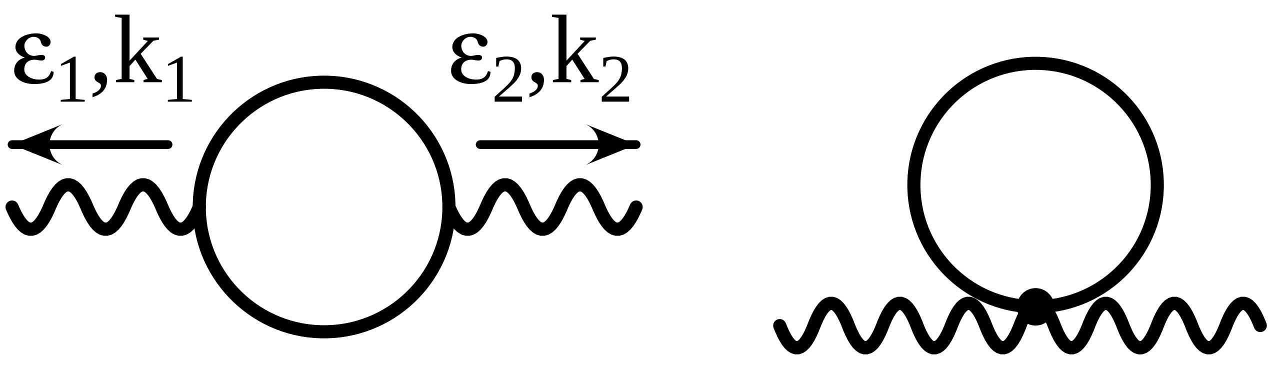
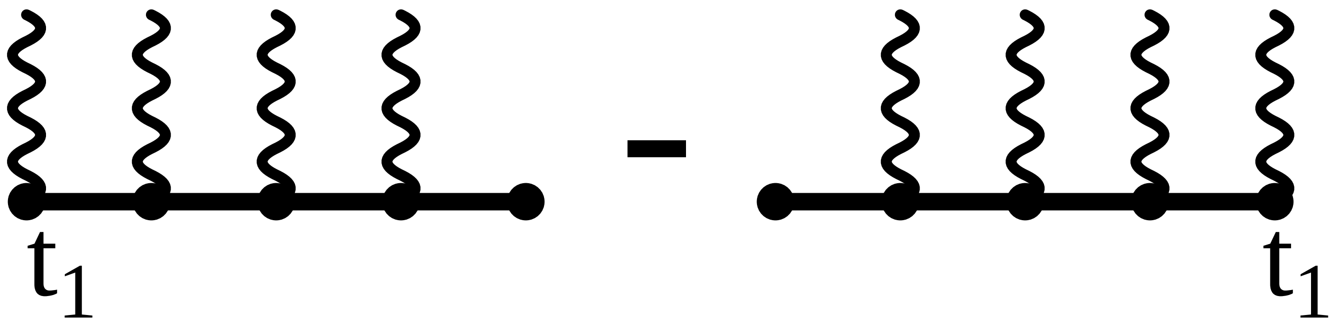
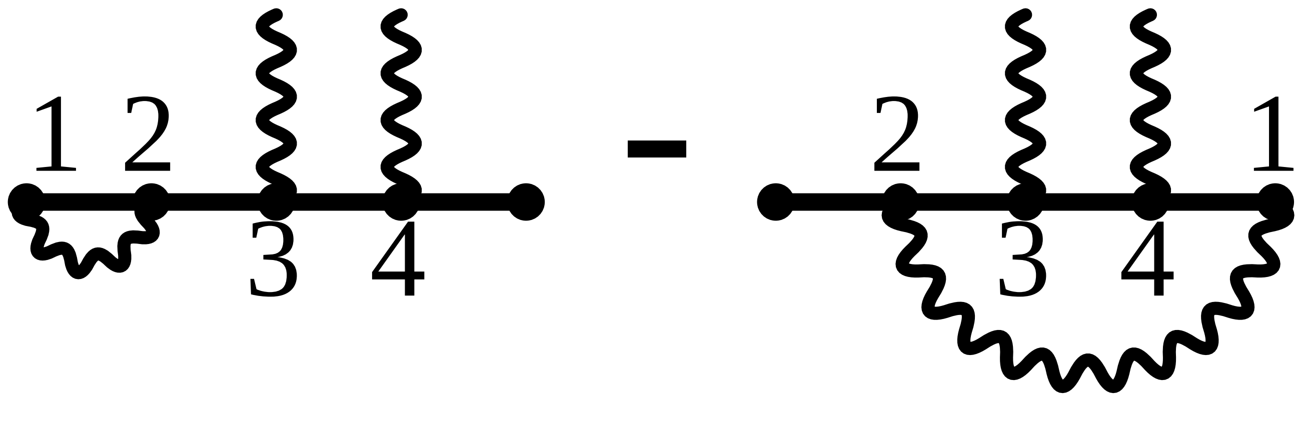
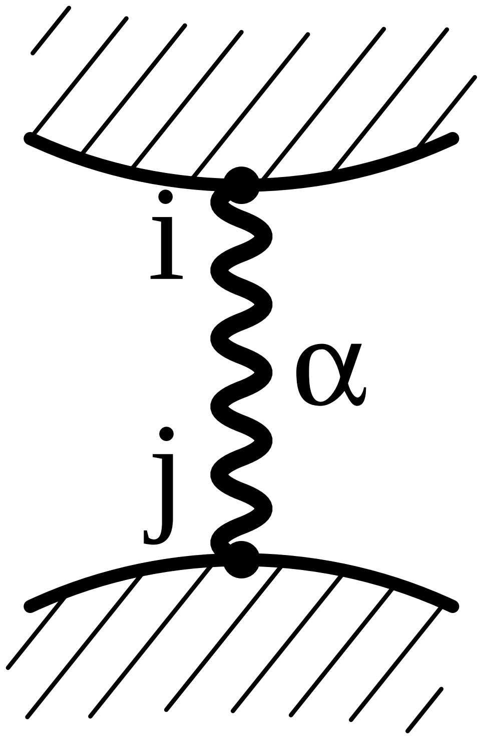
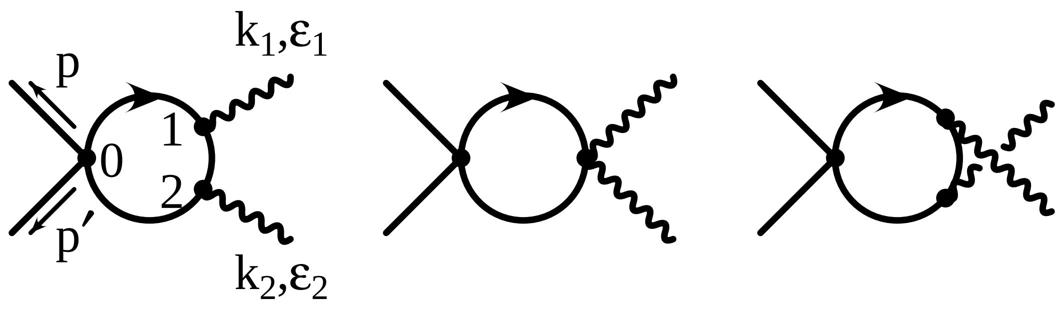

∫dz
α1
α2
x
z
y
α1+α2
x
y

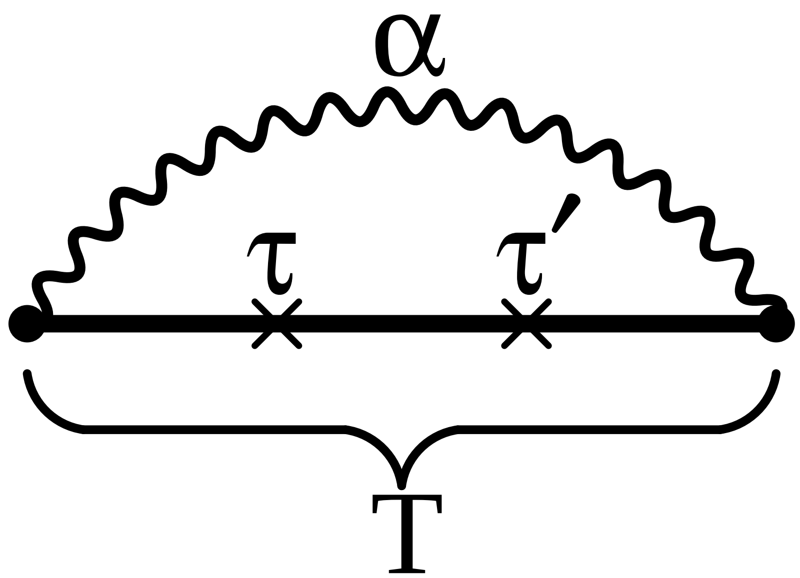

ε1,k1
ε2,k2

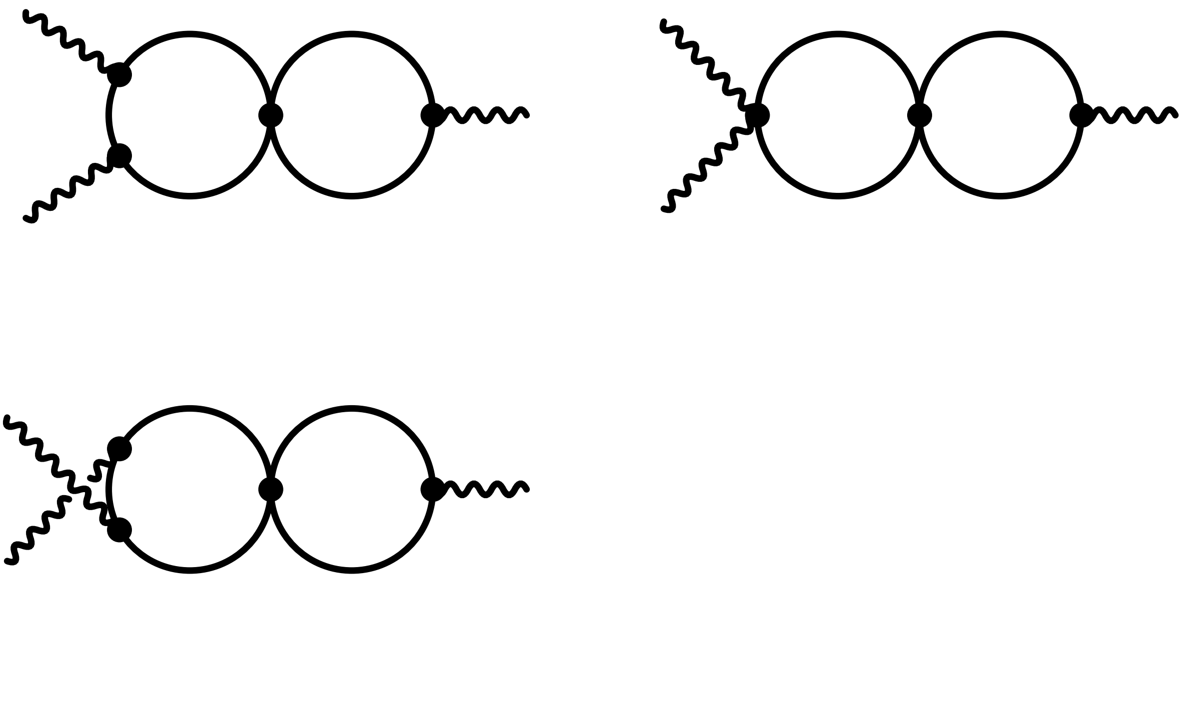

t1
-
t1

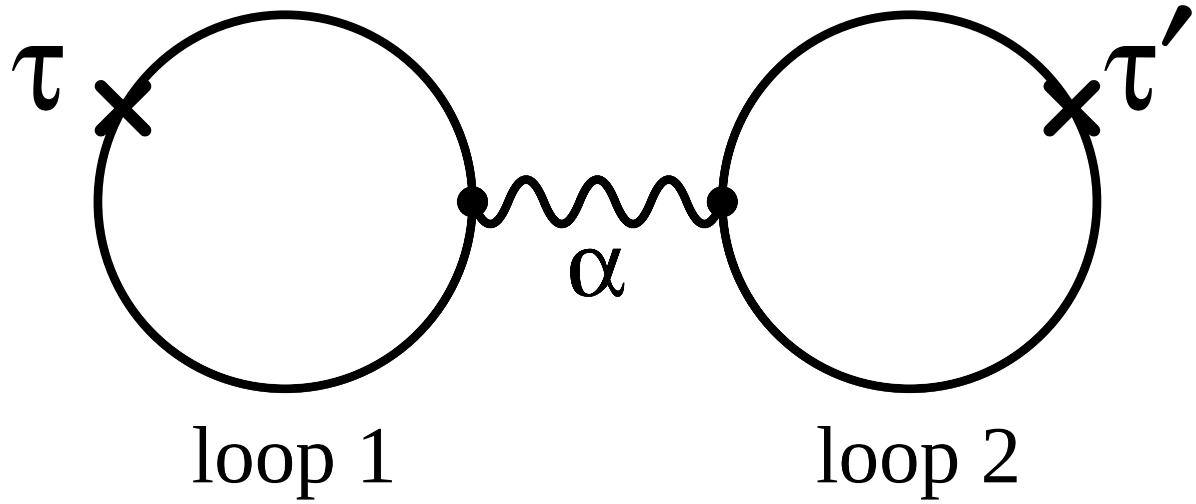

1
2
3
4
–
2
3
4
1

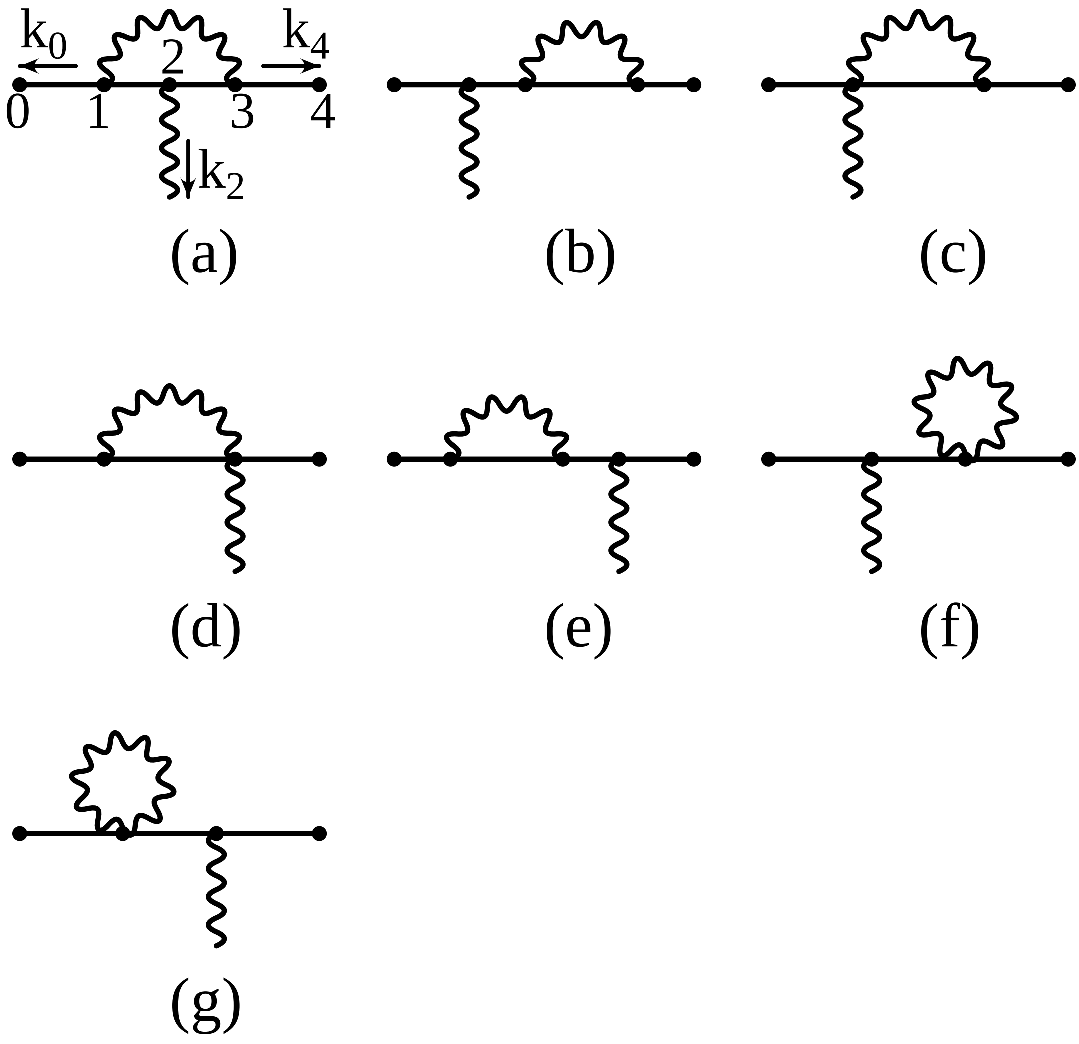

i
α
j


k_1, ε_1
p
1
0
2
p′
k_2, ε_2


α
τ
τ′
T

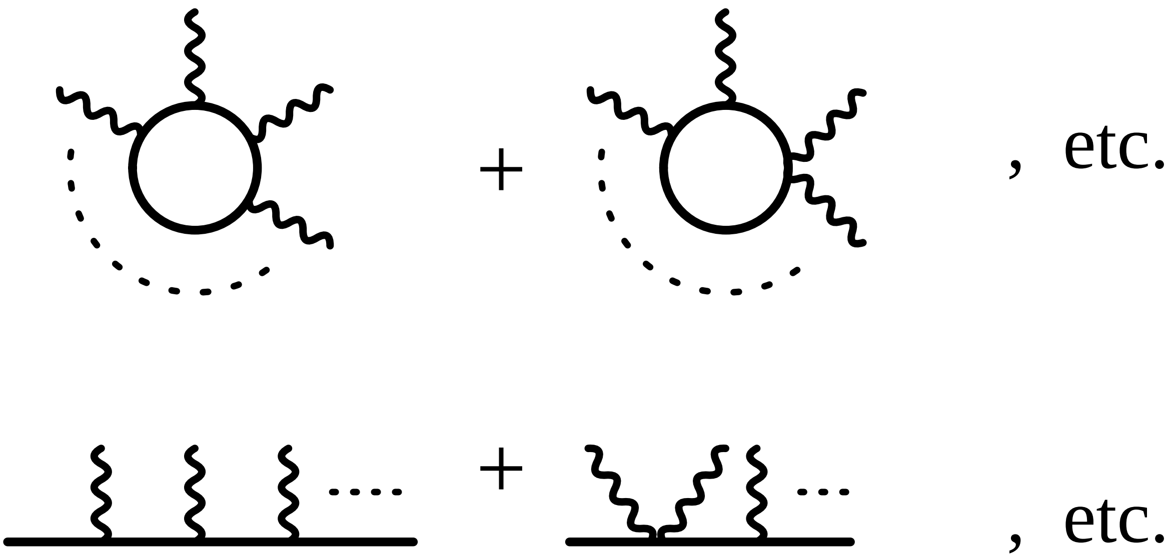


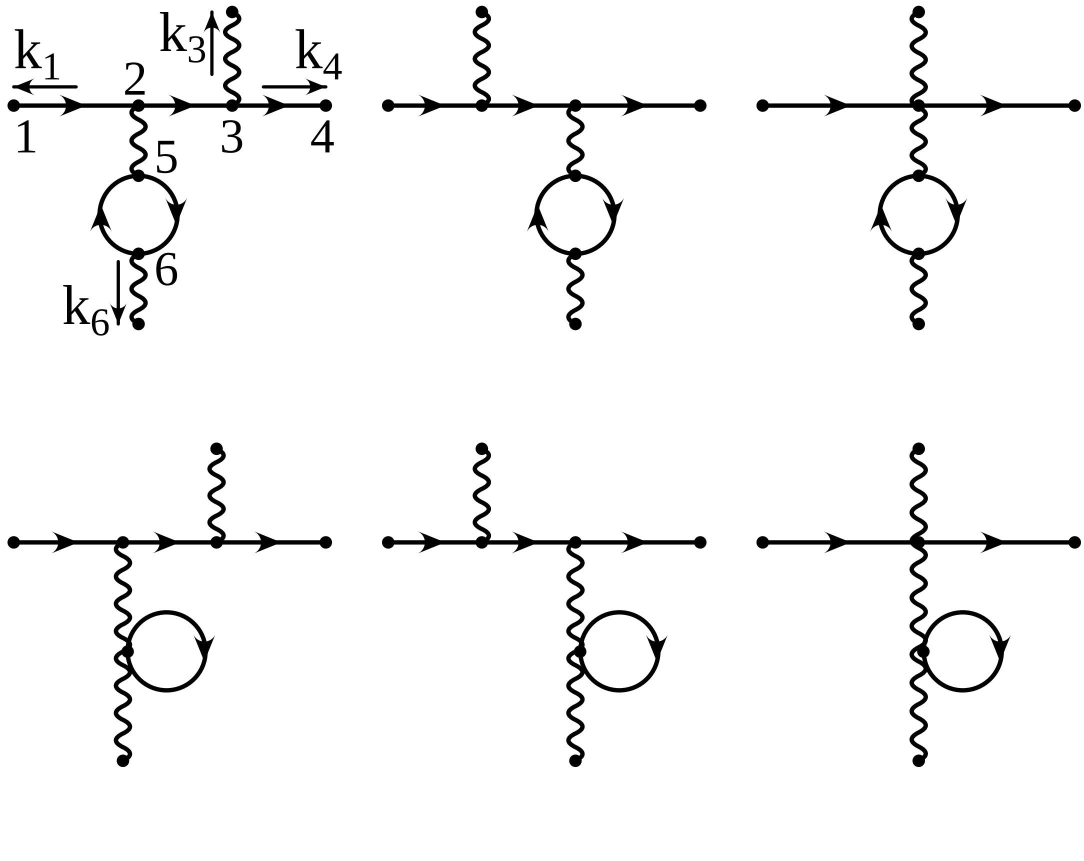

τ
τ′
α
loop 1
loop 2


k0
k4
k2
0
1
2
3
4
(a)
(b)
(c)
(d)
(e)
(f)
(g)

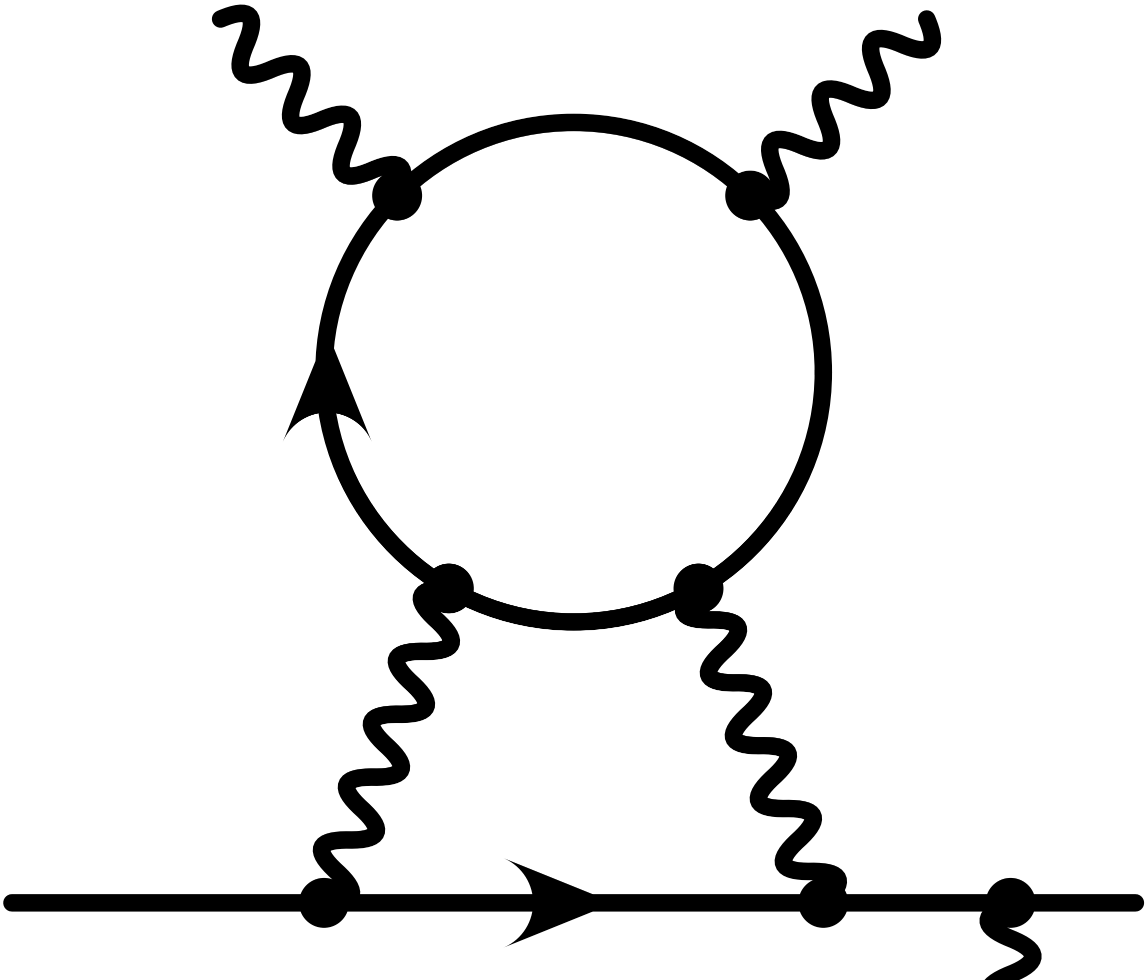
(a)

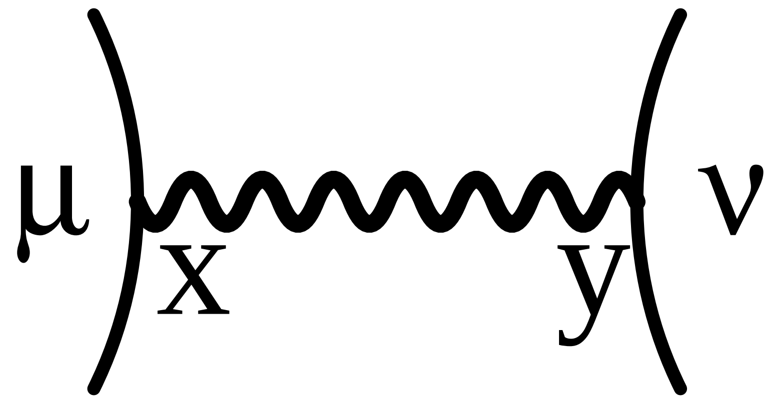
(b)



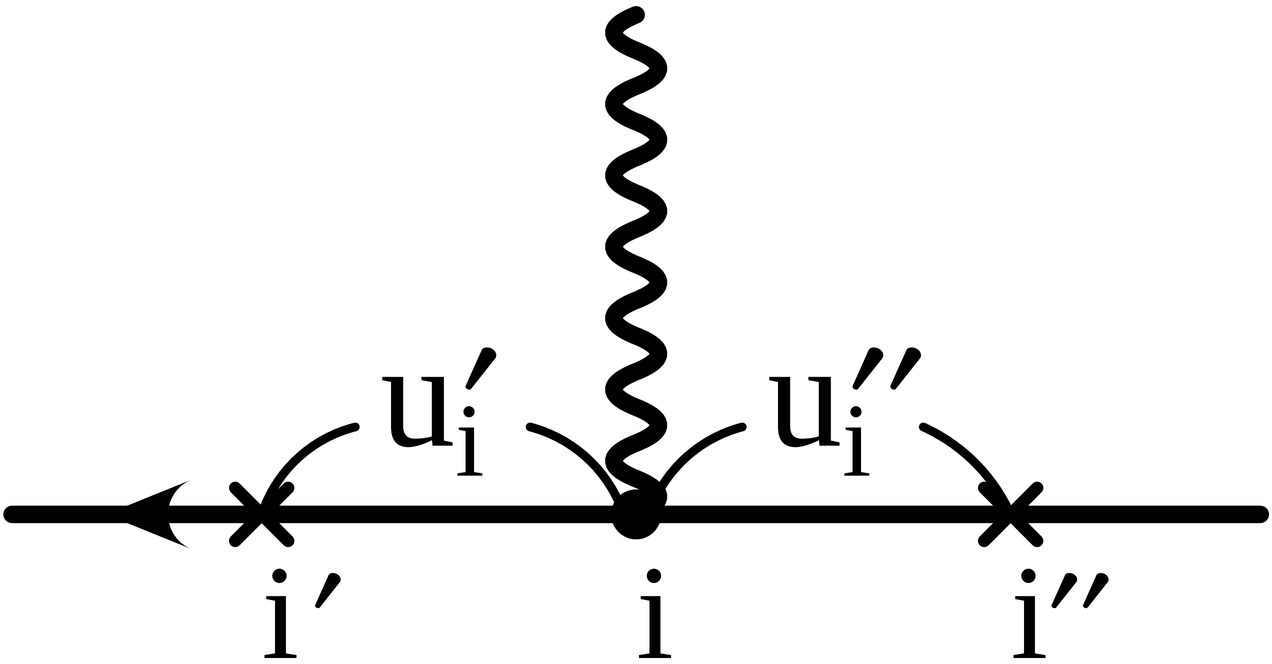


k_1
k_3
k_4
k_6
1
2
3
4
5
6

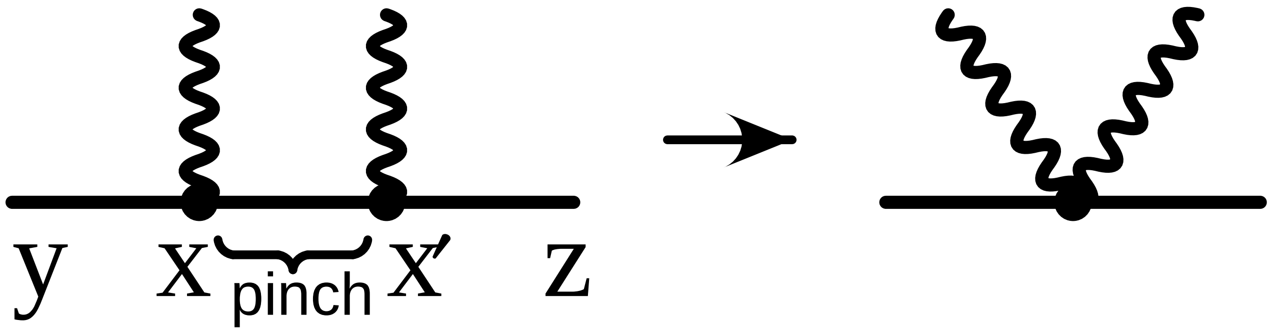




$$\varepsilon_\mu \varepsilon_\nu \, i\Delta_F(x-y) \longrightarrow -ig_{\mu\nu}\Delta_F(x-y)$$

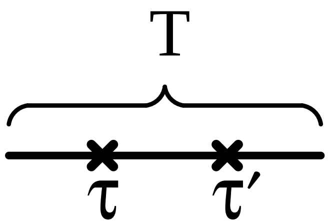


u'_i
u''_i
i'
i
i''



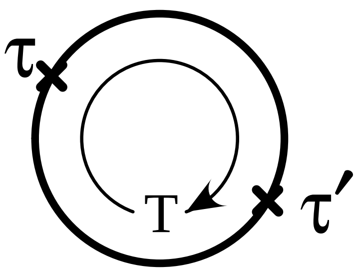
y
x
pinch
x′
z

a)

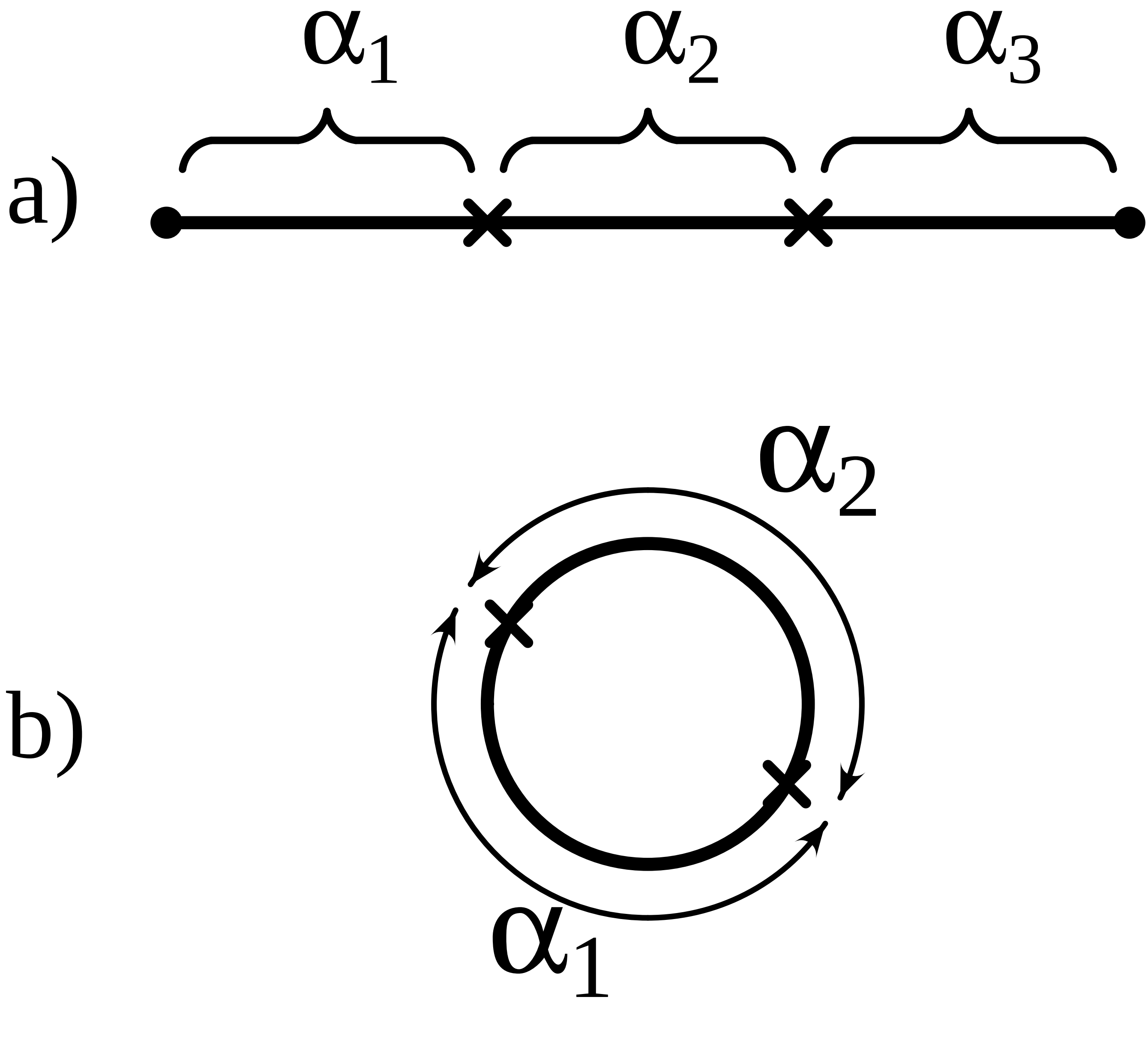


$G_B^{(\mathrm{open})}(\tau,\tau')$, $\mathcal{N}^{(\mathrm{open})}$

b)



$G_B^{(\mathrm{closed})}(\tau,\tau')$, $\mathcal{N}^{(\mathrm{closed})}$



α_1
α_2
α_3
a)
α_2
b)
α_1