The trigonometric counterpart of the Haldane Shastry Model

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August 17, 2002

Abstract

The hierarchy of Integrable Spin Chain Hamiltonians, which are trigonometric analogs of the Haldane Shastry Model and the associated higher conserved charges, is derived by a reduction from the trigonometric Dynamical Models of Bernard-Gaudin-Haldane-Pasquier. The Spin Chain Hamiltonians have the property of $U'_q(\hat{\mathfrak{gl}}_2)$-invariance. The spectrum of the Hamiltonians and the $U'_q(\hat{\mathfrak{gl}}_2)$-representation content of their eigenspaces are found by a descent from the Dynamical Models.

0 Introduction

The Haldane Shastry Model [H,S] has been a subject of much attention in the past years. This Model is a version of the XXX Heisenberg Spin Chain where spins interact with a potential inversely proportional to the squared distance between the spins. Like the XXX Chain with nearest-neighbour interaction, the Haldane Shastry Model is Integrable in the sense that its Hamiltonian is a member of a family of mutually commuting, independent Integrals of Motion. The Haldane Shastry Model, however, has been solved in much greater detail than the nearest-neighbour XXX Model. The reason for this is a remarkably large algebra of symmetries found in the former Model. For the Haldane Shastry Spin Chain with the spins taking values in the fundamental representation of $\mathfrak{gl}_n$ the symmetry algebra is the $\mathfrak{gl}_n$-Yangian. This infinite-dimensional algebra of symmetries facilitated computation of explicit expressions for the energy levels and the eigenvectors! [HHTBP,BPS,H2]. Some of the corre

lation functions have been found as well [HZ].

An important reason for the attention attracted by the Haldane Shastry Model is the fractional statistics exhibited by elementary excitations over the antiferromagnetic ground state present in this Model [H2]. In the case of $\mathfrak{gl}_2$-Haldane Shastry Chain these excitations are spin-1/2 particles that are "semions" – particles with statistics exactly half-way between bosons and fermions in the sense of Haldane. Being exactly solvable the Haldane Shastry Model provides a valuable laboratory for a study of physical implications of the fractional statistics.

A remarkable connection exists between the Haldane Shastry Spin Chain in the limit of infinite number of sites and WZNW Conformal Field Theory at level 1 [BPS2,BLS]. This connection has led to a novel description of the space of states in WZNW level-1 CFT, where the states are organized into irreducible multiplets of Yangian symmetry algebra inhereted from the Haldane Shastry Model. This, in turn, provided an explanation for the Fermionic Virasoro character formulas for the level-1 integrable representations of $\hat{\mathfrak{sl}}_2$ that were earlier derived by [DKKMM].

In the paper [BGHP] it was realized that the Haldane Shastry Spin Chain is related to a more general class of Integrable Models – the so-called Dynamical Models that describe quantum particles with spin moving along a circle. These Dynamical Models can be thought of as generalzations of the Calogero-Sutherland Model with inverse squared sine potential. The precise way in which the Haldane Shastry Hamiltonian and associated higher conserved charges are obtained from the hierarchy of Dynamical Models was recently explained by Polychronakos in [P], and by Talstra and Haldane in [TH]. In the last paper the authors have shown how the hierarchy of Integrable Spin Models including the Haldane Shastry Model appears in the static limit of the Dynamical Models in which the coordinates of the particles are "frozen" along the circle in an equidistantly spaced lattice.

In the present paper we define trigonometric counterparts of the spin-1/2 Haldane Shastry Hamiltonian and the associated higher conserved charges. The hierarchy of Integrable Spin Chain Hamiltonians that we obtain has an infinite-dimensional symmetry algebra $U_q'(\hat{\mathfrak{gl}}_2)$ that takes over the role played by the Yangian in the Haldane Shastry Model. We compute the eigenvalue spectrum of the hierarchy and find that it has an additive, particle-like form. The space of states is decomposed into eigenspaces of the operators that form the hierarchy. Each of the eigenspaces is a highest-weight irreducible representation of $U_q'(\hat{\mathfrak{gl}}_2)$ parametrized by a sequence of integers –"magnon quasimomenta" or a "motif" in terminology of [HHTBP,BPS]. The eigenvalue that corresponds to such eigenspace is a q-deformation of the eigenvalue of the Haldane Shastry Hamiltonian parametrized by the same magnon quasimomenta.

The procedure that is used to derive the trigonometric hierarchy has been inspired by the Talstra-Haldane approach – we extract the Integrable Spin Models from a static limit of the trigonometric $U_q'(\hat{\mathfrak{gl}}_2)$-invariant Dynamical Models that were defined in [BGHP]. The spectrum of the Spin Models is obtained by a descent from the spectrum of these Dynamical Models.

0.1 A survey of the method and results

In this subsection we highlight the main steps of the procedure that is used to define the trigonometric hierarchy of Integrable Spin Models and formulate the results of this article. The details and proofs of the statements are contained in the main body of the paper starting with sec.1.

We derive the hierachy of trigonometric Spin Models by a two-step reduction from the trigonometric Dynamical Models that were introduced in [BGHP]. First of all we recall the definition of these trigonometric Dynamical Models.

0.1.1 Trigonometric Dynamical Models of [BGHP]

The trigonometric Dynamical Models are defined starting with two representations of the finite-dimensional Hecke Algebra $H_N(q)$ (Cf. **1.1**). The first of these representations is defined in the ring of polynomials in N variables $\mathbb{C}[z_1,\ldots,z_N]$. The generators $g_{i,i+1} \quad (i=1,\ldots,N-1)$ of $H_N(q)$ in this representation have the following form

$$g_{i,i+1} := \frac{q^{-1}z_i - qz_{i+1}}{z_i - z_{i+1}}(K_{i,i+1}-1)+q \qquad (i=1,\ldots,N-1). \tag{0.0.1}$$

Where $K_{i,j}$ is the exchange operator for variables z_i, z_j.
The second representation of $H_N(q)$ is defined in the N-fold tensor product of two-dimensional vector spaces $H := V^{\otimes N},\ V := \mathbb{C}^2 = \mathbb{C}\{v^+, v^-\}$. The Hecke generator $t_{i,i+1} \quad (i=1,\ldots,N-1)$ is a matrix acting in H according to the formula

$$t_{i,i+1} = I\otimes\cdots\otimes I\otimes \underset{i,i+1}{t}\otimes I\otimes\cdots\otimes I \quad (i=1,\ldots,N-1), \tag{0.0.2}$$

where t is the matrix which acts in $V\otimes V$:

$$tv^+\otimes v^- = (q-q^{-1})v^+\otimes v^- + v^-\otimes v^+ \tag{0.0.3}$$
$$tv^-\otimes v^+ = v^+\otimes v^- \tag{0.0.4}$$
$$tv^\pm\otimes v^\pm = qv^\pm\otimes v^\pm. \tag{0.0.5}$$

The two of these representations naturally extend to $\mathbb{C}[z_1,\ldots,z_N]\otimes H$.

The Hecke representation generated by $g_{i,i+1} \quad (i=1,\ldots,N-1)$ is enlarged to a representation of the Affine Hecke Algebra $\widehat{H_N(q)}$ by adjoining affine generators $Y_i \quad (i=1,\ldots,N)$:

$$Y_i \ := \ g_{i,i+1}^{-1}K_{i,i+1}\ldots g_{i,N}^{-1}K_{i,N}p^{D_i}K_{1,i}g_{1,i}\ldots K_{i-1,i}g_{i-1,i}. \tag{0.0.6}$$

Where p is a c-number , $D_i := z_i\frac{\partial}{\partial z_i}$ and

$$g_{i,j} := \frac{q^{-1}z_i - qz_j}{z_i - z_j}(K_{i,j}-1)+q \qquad (i,j=1,\ldots,N-1). \tag{0.0.7}$$

After introducing the objects just described the authors of [BGHP] define the hierarchy of Dynamical Models. The operators $\Delta^{(n)} \quad (n = 1, \dots, N)$ that constitute the hierarchy are coefficients of the polynomial $\Delta(u) = \sum_{n=0}^{N} u^n \Delta^{(n)}$ which generates elementary symmetric functions of $Y_1, Y_2, \dots, Y_N$:

$$\Delta(u) := \prod_{i=1}^{N} (1 + uY_i). \tag{0.0.8}$$

Due to the mutual commutativity of the operators $Y_1, Y_2, \dots, Y_N$ the hierachy of Dynamical Models is integrable:

$$[\Delta(u), \Delta(v)] = 0. \tag{0.0.9}$$

In the space $\mathbb{C}[z_1, \dots, z_N] \otimes H$ one defines a representation of the algebra $U'_q(\hat{\mathfrak{gl}_2})$ (Cf. **1.2**). In this representation the generators of $U'_q(\hat{\mathfrak{gl}_2})$ are obtained by expanding in the parameter u the monodromy matrix $T_a(u)$ which is defined in a standard way as a product of elementary L-operators [BGHP]:

$$T_a(u) := L_{a1}(uY_1) L_{a2}(uY_2) \dots L_{aN}(uY_N). \tag{0.0.10}$$

Where the elementary L-operator $L_{ai}(uY_i) \quad (i = 1, \dots, N)$ acts in the tensor product of an auxiliary copy of the two-dimensional vector space denoted by V_a and the space $\mathbb{C}[z_1, \dots, z_N] \otimes H$:

$$L_{ai}(uY_i) \quad := \quad \frac{uY_i t_{a,i} - t_{a,i}^{-1}}{uY_i - 1} P_{a,i} \quad (i = 1, \dots, N). \tag{0.0.11}$$

Here P is the permutation operator in $V \otimes V$ and $t_{a,i}$, $P_{a,i}$ are the usual extensions of t , P as operators in $V_a \otimes H$. The fact that $T_a(u)$ defines a representation of $U'_q(\hat{\mathfrak{gl}_2})$ follows from the $RTT = TTR$ relation which involves the trigonometric R-matrix:

$$\bar{R}_{ab}(u/v) T_a(u) T_b(v) = T_b(v) T_a(u)) \bar{R}_{ab}(u/v), \tag{0.0.12}$$

$$\bar{R}(z) := \frac{zt - t^{-1}}{qz - q^{-1}} P. \tag{0.0.13}$$

The hierarchy of Dynamical Models defined by $\Delta(u)$ is $U'_q(\hat{\mathfrak{gl}_2})$-invariant, that is

$$[\Delta(u), T_a(v)] = 0. \tag{0.0.14}$$

This again follows from the mutual commutativity of the operators $Y_1, Y_2, \dots, Y_N$.

Both $\Delta(u)$ and $T_a(u)$ act in the "bosonic" subspace of $\mathbb{C}[z_1, \dots, z_N] \otimes H$ as explained in [BGHP]. This subspace which we denote by $\mathcal{B}$ is defined by the requirement of the Hecke-invariance:

$$\mathcal{B} := \{ b \in \mathbb{C}[z_1, \dots, z_N] \otimes H | (g_{i,i+1} - t_{i,i+1}) b = 0 \quad (i = 1, \dots, N-1) \}. \tag{0.0.15}$$

In any operator O acting in $\mathcal{B}$ one can eliminate the coordinate exchange operators $K_{i,j}$ by carrying them one-by-one to the right of any expression in O and replacing a $K_{i,j}$ standing on the right of an expression in accordance with the rule: $g_{i,i+1} \rightarrow t_{i,i+1} \quad (i = 1, \dots, N-1)$. This leads to a uniquely defined operator $\widehat{O}$ which does not contain coordinate permutations, such that

$$O\mathcal{B} = \widehat{O}\mathcal{B}. \tag{0.0.16}$$

Where the notation means that the equality holds for any vector in $\mathcal{B}$.
With this definition the eq. (0.0.9,.12,.14) lead to

$$[\widehat{\Delta(u)}, \widehat{\Delta(v)}]\mathcal{B} = 0, \tag{0.0.17}$$

$$[\widehat{\Delta(u)}, \widehat{T_a(v)}]\mathcal{B} = 0, \tag{0.0.18}$$

$$(\bar{R}_{ab}(u/v) \widehat{T_a(u)} \widehat{T_b(v)} - \widehat{T_b(v)} \widehat{T_a(u)} \bar{R}_{ab}(u/v))\mathcal{B} = 0. \tag{0.0.19}$$

The set of relations (0.0.17-.19) constitutes the result of [BGHP] concerned witn the trigonometric Dynamical Models.

0.1.2 The hierarchy of trigonometric Dynamical Models at $p = 1$

The first step of reduction from the Dynamical to the Spin Models consists in taking the static limit $p \to 1$ in the construction described in the previous subsection. Following the idea of [TH] we expand the generating function for the commuting charges of the hierarchy around the point $p = 1$ up to the linear term in $p - 1$:

$$\Delta(u) = \Delta_0(u) + (p-1)\Delta_1(u) + O((p-1)^2). \tag{0.0.20}$$

In the symmetry generator $T_a(u)$ we retain the leading term only:

$$T_a(u) = T_a^0(u) + O(p-1). \tag{0.0.21}$$

The operators $\Delta_0(u)$ and $\Delta_1(u)$ turn out to have a very special form. First of all $\Delta_0(u)$ is a *constant* :

$$\Delta_0(u) = \prod_{i=1}^{N}(1 + uq^{2i-N-1}). \tag{0.0.22}$$

(**Remark** In the paper [TH] which deals with the rational case, the leading term of the generating function

$$\Delta_{rational}(u) := \prod_{i=1}^{N}(u - d_i),$$
$$(d_i := hD_i + \sum_{j>i} \frac{z_i}{z_i - z_j} K_{i,j} - \sum_{j<i} \frac{z_j}{z_j - z_j} K_{i,j} \quad (i = 1, \ldots, N))$$

in the static limit $h \to 0$ is not a constant but a function of $z_1, \ldots, z_N$. This is due to a choice of the Dunkl operators d_i (above) which do not act in the space of polynomials. The rational limit of the affine Hecke generators Y_i which we use gives the gauge transformed Dunkl operators acting in the space of polynomials (Cf. [BGHP]).)

Since $\Delta_0(u)$ is a constant, the first-order differential operator $\Delta_1(u)$ takes over the role of the generating function for Integrals of Motion:

$$[\Delta_1(u), \Delta_1(v)] = 0, \tag{0.0.23}$$
$$[\Delta_1(u), T_a^0(v)] = 0. \tag{0.0.24}$$

Secondly, we find that $\Delta_1(u)$ has the following structure:

$$\Delta_1(u) = \sum_{i=1}^{N} \theta(u; z)_i D_i + \Xi(u; z). \tag{0.0.25}$$

Where $\Xi(u; z)$ is a function of the operators $z_1, \ldots, z_N$ and $K_{i,j} \quad (i, j = 1, \ldots, N)$ only. On the other hand the coefficients $\theta(u; z)_i$ do not depend on the operators of coordinate permutation and are functions of the coordinates $z_1, \ldots, z_N \quad (i = 1, \ldots, N)$ only. This kind of separation of the differentials D_i and the operators of coordinate permutation was first observed by [TH] in the rational case.

For the differential part $\mathcal{D}(u) := \sum_{i=1}^{N} \theta(u; z)_i D_i$ of $\Delta_1(u)$ we obtain an explicit expression in terms of the generating function $D(u; p, t)$ for the Macdonald operators [M],[JKKMP] (Cf. **2.1**):

$$\mathcal{D}(u) = D_1(q^{N-1}u; q^{-2}). \tag{0.0.26}$$

Where $D_1(u; t)$ is the linear term in the expansion of the generating function $D(u; p, t)$ around the point $p = 1$:

$$D(u; p, t) = \Delta_0(u; t) + (p-1)D_1(u; t) + O((p-1)^2). \tag{0.0.27}$$

The zero-order part $\Xi(u; z)$ of the differential operator $\Delta_1(u)$ is the object which we use to define the hierarchy of Integrable Spin Models.

First of all we observe that $\Delta_1(u)$ lies in the centre of the Affine Hecke Algebra generated by $g_{i,i+1} \quad (i = 1, \ldots, N-1)$ and $y_i := Y_i|_{p=1} \quad (i = 1, \ldots, N)$. Therefore $\Delta_1(u)$ acts in the bosonic subspace $\mathcal{B}$ defined in the previous subsection, and in this subspace we have analogs of the relations (0.0.17,-.19):

$$[\widehat{\Delta_1(u)}, \widehat{\Delta_1(v)}]\mathcal{B} = 0, \tag{0.0.28}$$
$$[\widehat{\Delta_1(u)}, \widehat{T_a^0(v)}]\mathcal{B} = 0, \tag{0.0.29}$$
$$(\bar{R}_{ab}(u/v)\widehat{T_a^0(u)}\widehat{T_b^0(v)} - \widehat{T_b^0(v)}\widehat{T_a^0(u)}\bar{R}_{ab}(u/v))\mathcal{B} = 0. \tag{0.0.30}$$

The operator $\widehat{\Delta_1(u)}$ is a sum of two parts: the $\mathcal{D}(u)$ which is a first order differential operator and $\widehat{\Xi(u;z)}$ which is a matrix acting on H whose entries are rational functions of the coordinates $z_1, \ldots, z_N$. At the second step of the reduction from the Dynamical Models to the Spin Models we shall eliminate the differential part of $\widehat{\Delta_1(u)}$ by restricting the coordinates to special values in a way which leaves the Integrability and the $U'_q(\hat{\mathfrak{gl}}_2)$-invariance intact.

0.1.3 Definition of the hierarchy of trigonometric Spin Models

The way to "freeze" the coordinates while keeping the spins as dynamical variables goes trough the use of the *evaluation map* $Ev(v) : \mathbb{C}[z_1, \ldots, z_N] \otimes H \mapsto H$. This map is parametrized by complex numbers $v_1, \ldots, v_N$ and works by taking values of functions of $z_1, \ldots, z_N$ at the point $z_1 = v_1, \ldots, z_N = v_N$. We use this map at the special point $z = \omega : z_1 = \omega^1, \ldots, z_N = \omega^N$ where $\omega = e^{2\pi i/N}$.

In order to explain the relevance of this point we first of all observe, that at this point the coefficients of the differential operator $\mathcal{D}(u)$ are all equal one to another:

$$\theta(u;\omega)_i = \theta(u) \qquad (i = 1, \ldots, N). \tag{0.0.31}$$

Where the constant $\theta(u)$ is given by eq. (2.2.26) in the main text. Next, we observe, that in the expansion of $\widehat{\Delta_1(u)}$ in u: $\widehat{\Delta_1(u)} := \sum_{n=1}^{N} u^n \hat{\Delta}_1^{(n)}$ the term $\hat{\Delta_1^{(N)}}$ is the scale operator: $\hat{\Delta}_1^{(N)} = D_1 + D_2 + \cdots + D_N$. We can modify the generating function $\widehat{\Delta_1(u)}$ by subtracting from it the product of the constant $\theta(u)$ and the operator $\hat{\Delta}_1^{(N)}$. The equations (0.0.28 -.30) clearly still hold for this modified generating function $\widehat{\Delta_1(u)} - \theta(u)\hat{\Delta}_1^{(N)}$. Moreover due to (0.0.31) for any vector $b \in \mathcal{B}$ we have:

$$Ev(\omega)(\widehat{\Delta_1(u)} - \theta(u)\hat{\Delta}_1^{(N)})b = Ev(\omega)\widehat{\Xi(u;z)}b. \tag{0.0.32}$$

The map $Ev(\omega)$ naturally pulls through the operators $\widehat{\Xi(u;z)}$ and $\widehat{T_a^0(u)}$ since these operators are matrices acting in H, and the entries of these matrices are rational functions of $z \equiv (z_1, \ldots, z_N)$ non-singular at the point $z = \omega$. By pulling $Ev(\omega)$ through $\widehat{\Xi(u;z)}$ and $\widehat{T_a^0(u)}$ we define operators $\Xi(u;\omega)$ and $T_a^0(u;\omega)$ acting in the image of $Ev(\omega)$ in H by :

$$Ev(\omega)\widehat{\Xi(u;z)}\mathcal{B} = \Xi(u;\omega)Ev(\omega)\mathcal{B}, \tag{0.0.33}$$

$$Ev(\omega)\widehat{T_a^0(u)}\mathcal{B} = T_a^0(u;\omega)Ev(\omega)\mathcal{B}. \tag{0.0.34}$$

Taking (0.0.31) and (0.0.33,.34) into account we apply the evaluation map to the relations (0.0.28 - .30) and get:

$$[\Xi(u;\omega), \Xi(v;\omega)]H_{\mathcal{B}}(\omega) = 0, \tag{0.0.35}$$

$$[\Xi(u;\omega), T_a^0(v;\omega)]H_{\mathcal{B}}(\omega) = 0, \tag{0.0.36}$$

$$(\bar{R}_{ab}(u/v)T_a^0(u;\omega)T_b^0(v;\omega) - T_b^0(v;\omega)T_a^0(u;\omega)\bar{R}_{ab}(u/v))H_{\mathcal{B}}(\omega) = 0. \tag{0.0.37}$$

Where $H_{\mathcal{B}}(\omega) := Ev(\omega)\mathcal{B} \subset H$.

Next, we prove, that $H_{\mathcal{B}}(\omega) = H$ .Therefore the relations (0.0.35 -.37) express Integrability and $U'_q(\hat{\mathfrak{gl}}_2)$-invariance of the hierarchy of Spin Chain Models. The Hamiltonians of these Models act in H and are obtained by expanding the generating function $\Xi(u;\omega)$ in the parameter u:

$$\Xi(u;\omega) = \sum_{n=1}^{N-1} u^n \Xi^{(n)}(\omega). \tag{0.0.38}$$

While the generating function $\Xi(u;\omega)$ is completely defined by the relation (0.0.25), the computation of an explicit expression is still quite a difficult task. We have computed the explicit expression only for the first member of the hierarchy – the operator $\Xi^{(1)}(\omega)$. To give this expression we introduce several notations. For $i \neq j \in \{1, \ldots, N-1\}$ define the rational functions:

$$a_{i,j} := \frac{q^{-1}z_i - qz_j}{z_i - z_j}\ , \quad b_{i,j} := \frac{(q - q^{-1})z_i}{z_i - z_j}. \tag{0.0.39}$$

Define also the matrices:

$$Y_{i,i+1}(w) := \frac{wt_{i,i+1} - t^{-1}_{i,i+1}}{qw - q^{-1}},$$

$$M^{(j,i)}(x,y) := \left(Y_{i+1,i+2}(y/z_{i+1}) \dots Y_{j-1,j}(y/z_{j-1})\right)^{-1} Y_{i,i+1}(x/y) \times \times \left(Y_{i+1,i+2}(x/z_{i+1}) \dots Y_{j-1,j}(x/z_{j-1})\right) \quad (j > i).$$

With these notations we have:

$$\Xi^{(1)}(\omega) = \sum_{M=2}^{N} \frac{(-1)^M}{(q-q^{-1})} \sum_{N \geq i_M > \dots > i_1 \geq 1} \mathcal{H}(\omega)_{i_M, i_{M-1}, \dots, i_1} R(\omega)_{i_2,i_1} R(\omega)_{i_3,i_2 \dots} R(\omega)_{i_M, i_{M-1}} + cI;$$

$$\mathcal{H}(z)_{i_M, i_{M-1}, \dots, i_1} := \left(\prod_{i_1 < f < i_2} a_{i_1,f}\right)\left(\prod_{i_2 < f < i_3} a_{i_2,f}\right) \cdots \left(\prod_{i_M < f < N + i_1} a_{i_M, f(\mod N)}\right) \times$$

$$\times\, b_{i_M, i_{M-1}} b_{i_{M-1}, i_{M-2}} \dots b_{i_2,i_1} b_{i_1,i_M},$$

$$R(z)_{j,i} := M^{(j,i)}(z_{i_M}, z_i) \qquad (N \geq j > i \geq 1). \quad (0.0.40)$$

Where c is unimportant constant. In the limit $q \to 1$ we recover the Haldane Shastry Hamiltonian:

$$\lim_{q \to 1} \frac{\Xi^{(1)}(\omega)}{(q - q^{-1})} = H_{HS} := -\sum_{N \geq j > i \geq 1} \frac{\omega^i \omega^j}{(\omega^i - \omega^j)^2}(P_{i,j} - 1). \quad (0.0.41)$$

While the Hamiltonian $\Xi^{(1)}(\omega)$ looks rather intimidating, its spectrum, as well as the spectrum of the whole hierarchy generated by $\Xi(u;\omega)$ has a remarkably simple additive form. We describe this spectrum in the next subsection.

0.1.4 Eigenvalue spectrum of the hierarchy of Spin Hamiltonians

As in the case of the Haldane Shastry Model the common eigenspaces of the operators $\Xi^{(n)}(\omega) \quad (n = 1, \dots, N-1)$ are in one-to-one correspondence with $\mathfrak{sl}_2$ motifs (Cf. [HHTBP] or [BPS]). A sequence of integers $(m_1, m_2, \dots, m_M)$ is called an $\mathfrak{sl}_2$ motif iff :

$$1 \leq m_1 < m_2 < \dots < m_M \leq N - 1\,; \quad (0.0.42a.)$$

$$m_{i+1} - m_i \geq 2 \quad (i = 1, \dots, M-1). \quad (0.0.42b.)$$

For a fixed N we denote the set of all $\mathfrak{sl}_2$ motifs including the empty one by $\mathfrak{M}_N$. For the eigenspace of $\Xi(u;\omega)$ which correspons to a motif $(m_1, m_2, \dots, m_M)$ we use the notation $H^{(m_1,m_2,\dots,m_M)}_{\mathcal{B}}(\omega)$. We have:

$$\Xi(u;\omega) H^{(m_1,m_2,\dots,m_M)}_{\mathcal{B}}(\omega) = \left(\sum_{i=1}^{M} \xi^{(m_i)}(u)\right) H^{(m_1,m_2,\dots,m_M)}_{\mathcal{B}}(\omega). \quad (0.0.42)$$

Where the sum is understood to be zero for $M = 0$.
The elementary eigenvalue $\xi^{(m)}(u) \quad (m \in \{1, \dots, N-1\})$ is

$$\xi^{(m)}(u) = u \prod_{k=1}^{N} (1 + uq^{2k-N-1}) \left\{ \sum_{i=1}^{m} \frac{q^{2i-N-1}}{1 + uq^{2i-N-1}} - \frac{m}{N} \sum_{i=1}^{N} \frac{q^{2i-N-1}}{1 + uq^{2i-N-1}} \right\}. \quad (0.0.43)$$

In particular the elementary eigenvalue of the Hamiltonian $\Xi^{(1)}(\omega)$ is given by

$$\xi^{(m),(1)} = \frac{q^{-N}}{N}(Nq^m[m]_q - mq^N[N]_q). \quad (0.0.44)$$

Where we used the usual notation $[x]_q \equiv \frac{q^x - q^{-x}}{(q - q^{-1})}$. In the limit $q \to 1$ we recover the elementary eigenvalue of the Haldane Shastry Model [HHTBP,BPS]:

$$\lim_{q \to 1} \frac{\xi^{(m),(1)}}{(q - q^{-1})} = m(m - N). \quad (0.0.45)$$

The space of states of the Spin Models is represented as a direct sum of the eigenspaces $H^{(m_1,m_2,\dots,m_M)}(\omega)$:

$$H = \bigoplus_{(m_1,m_2,\dots,m_M)\in\mathfrak{M}_N} H_{\mathcal{B}}^{(m_1,m_2,\dots,m_M)}(\omega). \tag{0.0.46}$$

0.1.5 Structure of the common eigenspaces of the hierarchy of Spin Hamiltonians.

Each of the eigenspaces $H_{\mathcal{B}}^{(m_1,m_2,\dots,m_M)}(\omega)$ is an irreducible highest-weight representation of $U_q'(\hat{\mathfrak{gl}}_2)$. The Drinfeld polynomial [CP] $Q^{(m_1,m_2,\dots,m_M)}(u)$ of $H_{\mathcal{B}}^{(m_1,m_2,\dots,m_M)}(\omega)$ is

$$Q^{(m_1,m_2,\dots,m_M)}(u) = \prod_{\substack{1\le k\le N\\ k\notin\{m_i,m_i+1\}}} (1-q^{-2k+N+1}u). \tag{0.0.47}$$

(Cf. **1.2** for our conventions about Drinfeld polynomials).

To describe an eigenspace $H_{\mathcal{B}}^{(m_1,m_2,\dots,m_M)}(\omega) \quad (m_1,m_2,\dots,m_M)\in\mathfrak{M}_N$ in a more explicit way we give some facts (Cf. **4.1**) about eigenvectors of the operator $\Delta_1(u)$ which defines the hierarchy of Dynamical Models at $p=1$.

The linear space of polynomials $\mathbb{C}[z_1,\dots,z_N]$ is represented as a direct sum of eigenspaces of the operator $\Delta_1(u)$. There is a one-to-one correspondence between these eigenspaces and partitions with N parts. In an eigenspace which corresponds to a partition $\lambda := (\lambda_1\ge\lambda_2\ge\dots\lambda_N\ge 0)$ there exists a basis $\{\varphi_\sigma^\lambda(z)\}_{\sigma\in S_N^\lambda}$. The polynomials forming this basis are indexed by elements from symmetric group S_N. There is a one-to-one correspondence between the elements of the basis $\{\varphi_\sigma^\lambda(z)\}_{\sigma\in S_N^\lambda}$ and elements of the set $S_N^\lambda \quad (S_N^\lambda\subset S_N)$ defined in **1.4**. A polynomial $\varphi_\sigma^\lambda(z) \quad (\sigma\in S_N^\lambda)$ is completely specified by the three conditions:

$$\Delta_1(u)\varphi_\sigma^\lambda(z) = \left(\prod_{k=1}^{N}(1+uq^{2k-N-1})\left\{\sum_{i=1}^{N}\frac{uq^{2i-N-1}}{1+uq^{2i-N-1}}\lambda_i\right\}\right)\varphi_\sigma^\lambda(z), \tag{0.0.48a.}$$

$$y_i\varphi_\sigma^\lambda(z) = q^{2\sigma_i-N-1}\varphi_\sigma^\lambda(z) \quad (i=1,\dots,N), \tag{0.0.48b.}$$

$$\varphi_\sigma^\lambda(z) = z_1^{\lambda_{\sigma_1}} z_2^{\lambda_{\sigma_2}}\dots z_N^{\lambda_{\sigma_N}} + \text{smaller monomials}. \tag{0.0.48c.}$$

We remind, that $y_i := Y_i|_{p=1} \quad (i=1,\dots,N)$.
The "smaller monomials " means a linear combination of monomials that are smaller than the monomial $z_1^{\lambda_{\sigma_1}} z_2^{\lambda_{\sigma_2}}\dots z_N^{\lambda_{\sigma_N}}$ in the ordering described in **1.4(2)** (Cf. [BGHP]).

With any motif $(m_1,m_2,\dots,m_M)\in\mathfrak{M}_N$ associate the partition

$$(\underset{1}{M},\dots,\underset{m_1}{M},\underset{m_1+1}{M-1},\dots,\underset{m_2}{M-1},\underset{m_2+1}{M-2},\dots \\ \dots,\underset{m_M}{1},\underset{m_M+1}{0},\dots,\underset{N}{0}) \tag{0.0.48}$$

We use the same notation $(m_1,m_2,\dots,m_M)$ for a motif and the associated partition.

For an element σ of the symmetric group S_N define

$$\{\sigma_1,\sigma_2,\dots,\sigma_N\} := \sigma.\{1,2,\dots,N\}\,; \tag{0.0.49a.}$$

$$i := \sigma_{p_i^\sigma} \qquad (i=1,\dots,N). \tag{0.0.49b.}$$

For any $(m_1,m_2,\dots,m_M)\in\mathfrak{M}_N$ define the subset $S_{N,(m_1,m_2,\dots,m_M)}^{(m_1,m_2,\dots,m_M)}$ of S_N (Cf. **4.3(2)**):

$$S_{N,(m_1,m_2,\dots,m_M)}^{(m_1,m_2,\dots,m_M)} := \\ \left\{\sigma\in S_N \;\middle|\; \begin{matrix} p_i^\sigma < p_{i+1}^\sigma & (m_k<i<m_{k+1}) & \text{for all } k\in\{0,1,\dots,M\} \\ p_{m_k}^\sigma > p_{m_k+1}^\sigma & & \text{for all } k\in\{1,\dots,M-1\}\end{matrix}\right\}. \tag{0.0.49}$$

Where we adopt the convention: $m_0 := 0\,, m_{M+1} := N+1$.

Next, for any motif $(m_1,m_2,\dots,m_M)\in\mathfrak{M}_N$ we define the subspace $W^{(m_1,m_2,\dots,m_M)}$ of the space of states H (Cf. **6.1(2)**) as follows:

$$W^{(m_1,m_2,\dots,m_M)} := \\ S_q(\underset{1}{V}\otimes\dots\otimes\underset{m_1-1}{V})\otimes A_q(\underset{m_1}{V}\otimes\underset{m_1+1}{V})\otimes S_q(\underset{m_1+2}{V}\otimes\dots\otimes\underset{m_2-1}{V})\otimes A_q(\underset{m_2}{V}\otimes\underset{m_2+1}{V})\otimes\dots \\ \dots\otimes S_q(\underset{m_{M-1}+2}{V}\otimes\dots\otimes\underset{m_M-1}{V})\otimes A_q(\underset{m_M}{V}\otimes\underset{m_M+1}{V})\otimes S_q(\underset{m_M+2}{V}\otimes\dots\otimes\underset{N}{V}) \\ W^{(m_1,m_2,\dots,m_M)}\subset H := \underset{1}{V}\otimes\underset{2}{V}\otimes\dots\underset{N}{V}. \tag{0.0.50}$$

Where S_q and A_q mean q-symmetrization and q-antisymmetrization as defined in (1.1.14,.15,5.5.17). The space $W^{(m_1,m_2,\ldots,m_M)}$ $\quad (m_1, m_2, \ldots, m_M) \in \mathfrak{M}_N$ is an irreducible highest weight $U'_q(\hat{\mathfrak{gl}}_2)$-module. The Drinfeld polynomial of this module is given by (0.0.47). The $U'_q(\hat{\mathfrak{gl}}_2)$-action on $W^{(m_1,m_2,\ldots,m_M)}$ is given by

$$\mathcal{L}(u; \{q^{2\sigma[0]_i-N-1}\}) := L_{a1}(uq^{2\sigma[0]_1-N-1})L_{a2}(uq^{2\sigma[0]_2-N-1}) \ldots L_{aN}(uq^{2\sigma[0]_N-N-1}). \tag{0.0.51}$$

Where for a fixed $(m_1, m_2, \ldots, m_M)$ we have introduced the notation

$$\sigma[0] := (m_1, m_1 + 1)(m_2, m_2 + 1) \ldots (m_M, m_M + 1) \in S^{(m_1,m_2,\ldots,m_M)}_{N,(m_1,m_2,\ldots,m_M)} \subset S_N. \tag{0.0.52}$$

The eigenspace $H^{(m_1,m_2,\ldots,m_M)}_{\mathcal{B}}(\omega)$ $\quad ((m_1, m_2, \ldots, m_M) \in \mathfrak{M}_N)$ is isomorphic to $W^{(m_1,m_2,\ldots,m_M)}$ as $U'_q(\hat{\mathfrak{gl}}_2)$-module. This isomorphism is given by an invertible intertwiner $\check{\mathbb{U}}^{(m_1,m_2,\ldots,m_M)}(\omega)$

$$H^{(m_1,m_2,\ldots,m_M)}_{\mathcal{B}}(\omega) = \check{\mathbb{U}}^{(m_1,m_2,\ldots,m_M)}(\omega)W^{(m_1,m_2,\ldots,m_M)}. \tag{0.0.53}$$

The intertwiner $\check{\mathbb{U}}^{(m_1,m_2,\ldots,m_M)}(\omega)$ is defined by the expression

$$\check{\mathbb{U}}^{(m_1,m_2,\ldots,m_M)}(\omega) := (-(q^2 + 1))^M \sum_{\sigma \in S^{(m_1,m_2,\ldots,m_M)}_{N,(m_1,m_2,\ldots,m_M)}} \varphi^{(m_1,m_2,\ldots,m_M)}_\sigma(\omega)\mathbb{Y}(\sigma). \tag{0.0.54}$$

In this definition

$$\varphi^{(m_1,m_2,\ldots,m_M)}_\sigma(\omega) = \varphi^\lambda_\sigma(z)|_{z_1=\omega^1,\ldots,z_N=\omega^N}. \tag{0.0.55}$$

Where the partition λ is the one specified by $(m_1, m_2, \ldots, m_M)$ in accordance with (0.0.48). The matrix $\mathbb{Y}(\sigma)$ $\quad (\sigma \in S^{(m_1,m_2,\ldots,m_M)}_{N,(m_1,m_2,\ldots,m_M)})$ is an intertwiner which is defined by the recursion relations (Cf. **5.2(4)**):

$$\mathbb{Y}(\sigma[0]) := \mathrm{Id}\ ,$$

$$\mathbb{Y}((i, i + 1)\sigma) := \begin{cases} Y^+_{i,i+1}(q^{2\sigma_i-2\sigma_{i+1}})\mathbb{Y}(\sigma) & \text{if } \sigma_i - \sigma_{i+1} \geq 2, \\ Y^-_{i,i+1}(q^{2\sigma_i-2\sigma_{i+1}})\mathbb{Y}(\sigma) & \text{if } \sigma_i - \sigma_{i+1} \leq -2\ . \end{cases}$$

Where the matrices $Y^\pm_{i,i+1}(w)$ are

$$Y^\pm_{i,i+1}(w) := \varrho^\pm(w)\frac{wt_{i,i+1} - t^{-1}_{i,i+1}}{q^{-1}w - q}\ , \tag{0.0.56}$$

$$\varrho^+(w) := \frac{w - 1}{q^2w - 1}\ , \qquad \varrho^-(w) := \frac{w - q^2}{w - 1}.$$

Notice that $\mathbb{Y}(\sigma)$ $\quad (\sigma \in S^{(m_1,m_2,\ldots,m_M)}_{N,(m_1,m_2,\ldots,m_M)})$ is an invertible intertwiner.

In general we cannot claim to know the explicit expression for the intertwiner $\check{\mathbb{U}}^{(m_1,m_2,\ldots,m_M)}(\omega)$ since we have not found the eigenvectors $\varphi^{(m_1,m_2,\ldots,m_M)}_\sigma(z)$ explicitely. One exception is the case $q = 0$ when $\check{\mathbb{U}}^{(m_1,m_2,\ldots,m_M)}(\omega)$ becomes very simple. In this case we have

$$\check{\mathbb{U}}^{(m_1,m_2,\ldots,m_M),q=0}(\omega)|_{W^{(m_1,m_2,\ldots,m_M),q=0}} = \omega^{\frac{1}{2}\sum_{i=1}^M m_i(m_i+1)}\mathrm{Id}. \tag{0.0.57}$$

Therefore at $q = 0$ the eigenspace $H^{(m_1,m_2,\ldots,m_M)}_{\mathcal{B}}(\omega)$ is the linear span of the following vectors in H:

$$\left| \left\{ \begin{array}{ccccc} \overset{1}{+} & + & + & \ldots & \overset{m_1-1}{+} \\ - & + & + & \ldots & + \\ - & - & + & \ldots & + \\ \vdots & \vdots & \vdots & & \vdots \\ - & - & - & \ldots & - \end{array} \right\} \overset{m_1}{+} \overset{m_1+1}{-} \left\{ \begin{array}{ccccc} \overset{m_1+2}{+} & + & + & \ldots & \overset{m_2-1}{+} \\ - & + & + & \ldots & + \\ - & - & + & \ldots & + \\ \vdots & \vdots & \vdots & & \vdots \\ - & - & - & \ldots & - \end{array} \right\} \overset{m_2}{+} \overset{m_2+1}{-} \ldots \right.$$

$$\left. \ldots \overset{m_M}{+} \overset{m_M+1}{-} \left\{ \begin{array}{ccccc} \overset{m_M+2}{+} & + & + & \ldots & \overset{N}{+} \\ - & + & + & \ldots & + \\ - & - & + & \ldots & + \\ \vdots & \vdots & \vdots & & \vdots \\ - & - & - & \ldots & - \end{array} \right\} \right\rangle$$

Where we use the notation:

$$|\epsilon_1\epsilon_2\dots\epsilon_N\rangle := v^{\epsilon_1}\otimes v^{\epsilon_2}\otimes\cdots\otimes v^{\epsilon_N} \quad (\epsilon_i=\pm). \tag{0.0.58}$$

For example when N is even, $H_{\mathcal{B}}^{(1,3,\dots,N-1)}(\omega)$ is one-dimensional and at $q=0$ it is spanned by the vector (antiferromagnetic ground state):

$$|+-+-\cdots+-\rangle$$

In the rest of the paper we give a detailed exposition of the matters which were briefly recounted in this introduction. In the sec.1 we gather predominantly known facts about the trigonometric Dynamical Models of [BGHP] and explain the conventions about the algebra $U_q'(\hat{\mathfrak{gl}}_2)$ that we use. In the sec. 2 we discuss the hierarchy of Dynamical Models in the static limit $p=1$. In the sec. 3 we define the hierarchy of the Spin Models. The sec. 4. is concerned with properties of the eigenvectors of the hierarchy of Dynamical Models in the limit $p=1$. In the sec. 5 we construct the Hecke-invariant "bosonic" eigenspaces for the Dynamical Models with spin at $p=1$. The eigenvalue spectrum of the Spin Models and the $U_q'(\hat{\mathfrak{gl}}_2)$-representation content of their eigenspaces are derived in sec. 6.

Acknowledgments

I am most grateful to Drs. R. Kedem and R. Weston and to Professors M.Jimbo, M.Kashiwara and T.Miwa for numerous discussions and support.

1 The $U_q'(\hat{\mathfrak{gl}}_2)$-invariant Dynamical Models

In this section we summarize largely known facts about the trigonometric Dynamical Models defined by [BGHP]. We also recount several facts about the algebra $U_q'(\hat{\mathfrak{gl}}_2)$ and explain our notations.

1.1 The representations of the Affine Hecke Algebra

1.1.1 The representation of $\widehat{H_N(q)}$ in the space of polynomials

Following [BGHP] define the operators $g_{i,j}\in End(\mathbb{C}[z_1,\dots,z_N]) \quad (i,j=1,\dots,N)$:

$$g_{i,j} := a_{i,j}K_{i,j}+b_{i,j}$$

where

$$a_{i,j}=\frac{q^{-1}z_i-qz_j}{z_i-z_j}\,, \qquad b_{i,j}=\frac{(q-q^{-1})z_i}{z_i-z_j}=q-a_{i,j}.$$

$K_{i,j}$ is the interchange operator for variables z_i,z_j.

For $p\in\mathbb{C}$ and $D_i := z_i\frac{\partial}{\partial z_i}$ define operators $Y_i\in End(\mathbb{C}[z_1,\dots,z_N])(i=1,\dots,N)$:

$$Y_i \;:=\; g_{i,i+1}^{-1}K_{i,i+1}\dots g_{i,N}^{-1}K_{i,N}p^{D_i}K_{1,i}g_{1,i}\dots K_{i-1,i}g_{i-1,i} \tag{1.1.1}$$

Taken together with $g_{i,i+1}$ $(i=1,\dots,N-1)$ these operators satisfy the relations of the Affine Hecke Algebra $\widehat{H_N(q)}$ [BGHP] :

$$g_{i,i+1}^2 = (q-q^{-1})g_{i,i+1}+1 \tag{1.1.2}$$
$$g_{i,i+1}g_{k,k+1} = g_{k,k+1}g_{i,i+1}\,, \qquad |i-k|\ge 2 \tag{1.1.3}$$
$$g_{i,i+1}g_{i+1,i+2}g_{i,i+1} = g_{i+1,i+2}g_{i,i+1}g_{i+1,i+2} \tag{1.1.4}$$
$$Y_kg_{i,i+1} = g_{i,i+1}Y_k\,, \qquad k\ne i,i+1 \tag{1.1.5}$$
$$g_{i,i+1}Y_i = Y_{i+1}g_{i,i+1}^{-1} \tag{1.1.6}$$
$$Y_iY_j = Y_jY_i\,. \tag{1.1.7}$$

Any symmetric polynomial in Y_i $(i=1,\dots,N)$ belongs to the center of $\widehat{H_N(q)}$. All symmetric polynomials in Y_i $(i=1,\dots,N)$ are generated by the elementary symmetric polynomials which are obtained by expanding in the parameter u the generating function $\Delta(u)$:

$$\Delta(u) = \prod_{(i=1,\dots,N)}(1+uY_i) \tag{1.1.8}$$

1.1.2 The representation of the Hecke Algebra $H_N(q)$ in $(\mathbb{C}^2)^{\otimes N}$

Let $V := \mathbb{C}^2 = \mathrm{span}\{v^+, v^-\}$. Define $t \in End(V \otimes V)$ by

$$tv^+ \otimes v^- = (q - q^{-1})v^+ \otimes v^- + v^- \otimes v^+ \quad (1.1.9)$$
$$tv^- \otimes v^+ = v^+ \otimes v^- \quad (1.1.10)$$
$$tv^\pm \otimes v^\pm = qv^\pm \otimes v^\pm \quad (1.1.11)$$

The operators $\Pi^\pm(q)$:

$$\Pi^+(q) := \frac{q^{-1} + t}{q + q^{-1}} \quad (1.1.12)$$
$$\Pi^-(q) := \frac{q - t}{q + q^{-1}} \quad (1.1.13)$$

are orthogonal projectors on the subspaces:

$$S_q(V \otimes V) := \mathbb{C}\{v^+ \otimes v^+, v^- \otimes v^-, qv^+ \otimes v^- + v^- \otimes v^+\} \quad (1.1.14)$$
$$A_q(V \otimes V) := \mathbb{C}\{v^+ \otimes v^- - qv^- \otimes v^+\} \quad (1.1.15)$$

respectively.

Let $H := V^{\otimes N}$. For an $O \in End(V \otimes V)$ denote by $O_{i,j} \in End(H)$ the standard injection $End(V \otimes V) \to End(H)$ which acts trivially on all the factors except the i-th and j-th ones.

The matrices $t_{i,i+1}$ $(i = 1, \ldots, N-1)$ satisfy the defining relations (1.1.2-.4) of the finite-dimensional Hecke Algebra $H_N(q)$.

1.2 The Algebra $U'_q(\hat{\mathfrak{gl}}_2)$ at level 0 and some of its representations

In this subsection we summarize several facts about the algebra $U'_q(\hat{\mathfrak{gl}}_2)$ and its representations. Our conventions and notations mainly follow [JKKMP].

1.2.1 The Algebra $U'_q(\hat{\mathfrak{gl}}_2)$

In the L-operator formalism $U \equiv U'_q(\hat{\mathfrak{gl}}_2)$ at zero level is defined to be the associative algebra with unit generated by elements $l^\pm_{ij}[\pm n]$ $(i, j = 1, 2\,; n = 0, 1, \ldots)$.The L-operators $L^\pm(u) \in End(V \otimes U)$ are the generating series in the spectral parameter u:

$$L^\pm(u) := \sum_{\pm n \geq 0} u^n \begin{pmatrix} l_{11}[n] & l_{12}[n] \\ l_{21}[n] & l_{22}[n] \end{pmatrix} \quad (1.1.16)$$

The defining relations of U are written in the the form:

$$\bar{R}_{ab}(u/v)L^\pm_a(u)L^\pm_b(v) = L^\pm_b(v)L^\pm_a(u)\bar{R}_{ab}(u/v) \quad (1.1.17)$$
$$\bar{R}_{ab}(u/v)L^+_a(u)L^-_b(v) = L^-_b(v)L^+_a(u)\bar{R}_{ab}(u/v) \quad (1.1.18)$$
$$l^+_{ii}[0]l^-_{ii}[0] = 1 \quad (i = 1, 2)\,, \qquad l^+_{21}[0] = l^-_{12}[0] = 0 \quad (1.1.19)$$

where the R-matrix $\bar{R}_{ab}(z) \in End(V \otimes V := V_a \otimes V_b)$ is defined as follows:

$$\bar{R}(z) = \frac{zt - t^{-1}}{qz - q^{-1}}P \quad (1.1.20)$$

by P we denote the permutation operator in $V \otimes V$.

1.2.2 Some representations of U

A finite-dimensional highest weight module W of U contains non-zero vector Ω which satisfies the condition:

$$L^\pm(u)\Omega = \begin{pmatrix} A^\pm(u) & * \\ 0 & D^\pm(u) \end{pmatrix} \Omega \quad (1.1.21)$$

where $A^{\pm}(u)$ and $D^{\pm}(u)$ are $\mathbb{C}$-valued series in u.

If W is irreducible, it is specified up to equivalence by its Drinfeld polynomial $Q(u)$ whch is determined by the conditions: $Q(0) = 1$ and

$$q^{degQ}\frac{Q(q^{-2}u)}{Q(u)} = \frac{A^+(u^{-1})}{D^+(u^{-1})} \qquad (u \to 0) \tag{1.1.22}$$

$$= \frac{A^-(u^{-1})}{D^-(u^{-1})} \qquad (u \to \infty) \tag{1.1.23}$$

The example of such W is the 2-dimensional evaluation module $W(a)$ where $a \in \mathbb{C}\backslash\{0\}$ is the parameter. As a vector space $W(a)$ is isomorphic to V. The generators $l^{\pm}_{ij}[\pm n]$ $(i,j = 1,2; n = 0,1,\dots)$ are defined by expanding the L-operator:

$$L(ua) := \frac{uat - t^{-1}}{ua - 1}P \in End(V_a \otimes V) \tag{1.1.24}$$

into power series in $u^{\pm}$ around zero and infinity respectively.

The Drinfeld polynomial of $W(a)$ is: $Q(u;a) = 1 - a^{-1}u$.

A pair of tensor products $W(a) \otimes W(b)$, $W(b) \otimes W(a)$ is intertwined by the matrix

$$\bar{Y}(z) = zt - t^{-1} \quad (z \in \mathbb{C}) \in End(V \otimes V) \tag{1.1.25}$$

i.e.:

$$\bar{Y}_{12}(a/b)L_{a1}(ua)L_{a2}(ub) = L_{a1}(ub)L_{a2}(ua)\bar{Y}_{12}(a/b) \tag{1.1.26}$$

this relation holds in the tensor product of an auxiliary copy of V indicated by the subscript a and $V \otimes V$ indicated by subscripts 1 and 2.

The intertwiner $\bar{Y}(a/b)$ is invertible unless either $a = q^2 b$, in which case

$$\bar{Y}(q^2) = (q^2 - 1)(q + q^{-1})\Pi^+(q) \tag{1.1.27}$$

or $a = q^{-2}b$, in which case

$$\bar{Y}(q^{-2}) = (1 - q^{-2})(q + q^{-1})\Pi^-(q) \tag{1.1.28}$$

Together with (26) this leads to the invariance relations:

$$L_{a1}(q^2u)L_{a2}(u) : A_q(V \otimes V) \subset A_q(V \otimes V) \tag{1.1.29}$$

$$L_{a1}(q^2u)L_{a2}(u) : S_q(V \otimes V) \subset A_q(V \otimes V) \oplus S_q(V \otimes V) \tag{1.1.30}$$

$$L_{a1}(u)L_{a2}(q^2u) : S_q(V \otimes V) \subset S_q(V \otimes V) \tag{1.1.31}$$

$$L_{a1}(u)L_{a2}(q^2u) : A_q(V \otimes V) \subset A_q(V \otimes V) \oplus S_q(V \otimes V) \tag{1.1.32}$$

1.3 The hierarchy of U-invariant Dynamical Models

The central elements of $\widehat{H_N(q)}$ generated by $\Delta(u)$ were proposed in [BGHP] to define the hierarchy of integrable Dynamical Models which are trigonometric - that is U-invariant - generalizations of the Yangian-invariant Dynamical Models found by the same authors.

Define $T_a(u) \in End(\mathbb{C}[z_1,\dots,z_N] \otimes H)$ by taking the tensor product of the L-operators (24):

$$T_a(u) = L_{a1}(uY_1)L_{a2}(uY_2)\dots L_{aN}(uY_N) \tag{1.1.33}$$

After expansion in $u^{\pm}$, $T_a(u)$ gives rise to a representation of U in $\mathcal{P} := \mathbb{C}[z_1,\dots,z_N] \otimes H$. The action of $\Delta(u)$ naturally extends to $\mathcal{P}$. Retain the same notation $\Delta(u)$ for this extension. The U-invariance and integrability relations for $\Delta(u)$ are immediate:

$$[\Delta(u), T_a(v)] = 0 \tag{1.1.34}$$

$$[\Delta(u), \Delta(v)] = 0 \tag{1.1.35}$$

Both $\Delta(u)$ and $T_a(v)$ act in the ("bosonic") subspace $\mathcal{B}$:

$$\mathcal{B} := \{b \in \mathcal{P} | g_{i,i+1}b = t_{i,i+1}b \ (i = 1,\dots,N-1)\} \tag{1.1.36}$$

This allows to restrict $\Delta(u)$ and $T_a(v)$ on $\mathcal{B}$ where both of these operators can be rewritten in such a way that they do not depend explicitely on the operators $K_{i,j}$.

1.4 Eigenvalue spectrum of the operators Y_i $(i = 1, \ldots, N)$

1. We shall work in the monomial basis of $\mathbb{C}[z_1, \ldots, z_N]$.

Introduce a convenient parametrization of the elements of this basis. With a monomial $z^\nu := z_1^{\nu_1} z_2^{\nu_2} \ldots z_N^{\nu_N}$ $\quad (\nu_i \in \mathbb{Z}^+ (i = 1, \ldots, N))$ associate a partition $\lambda := (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0)$ such that $\{\nu_1, \nu_2, \ldots, \nu_N\} = \{\lambda_{\sigma_1}, \lambda_{\sigma_2}, \ldots, \lambda_{\sigma_N}\} := \sigma.\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ for some $\sigma \in S_N$ = symmetric group with $N-1$ generators. A partition λ is uniquely specified by ν, while in general σ is not.

For $\sigma \in S_N : \{\sigma_1, \sigma_2, \ldots, \sigma_N\} = \sigma.\{1, 2, \ldots, N\}$ define $p_i^\sigma \ : \ i = \sigma_{p_i^\sigma}$ $\quad (i = 1, \ldots, N)$. Let Λ_N be the set of all N-member partitions. For any $\lambda \in \Lambda_N$ write: $\lambda = (\lambda_1 = \lambda_2 = \cdots = \lambda_{m_1} > \lambda_{m_1+1} = \lambda_{m_1+2} = \cdots = \lambda_{m_2} > \ldots$
$\ldots > \lambda_{m_M+1} = \lambda_{m_M+2} = \cdots = \lambda_N)$.

Definition 1 *For any* $\lambda \in \Lambda_N$*:*

$$S_N^\lambda := \{\sigma \in S_N \mid p_1^\sigma < p_2^\sigma < \ldots p_{m_1}^\sigma ,\ p_{m_1+1}^\sigma < p_{m_1+2}^\sigma < \ldots p_{m_2}^\sigma , \ldots \\ \ldots , p_{m_M+1}^\sigma < p_{m_M+2}^\sigma < \ldots p_N^\sigma\}.$$

For a given λ the elements of the set S_N^λ are in one-to-one correspondence with distinct rearrangements of the sequence $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$. If $\sigma \in S_N^\lambda$ then $(i\, i+1)\sigma \in S_N^\lambda$ iff $\lambda_{\sigma_i} \neq \lambda_{\sigma_{i+1}}$ $\quad (i = 1, \ldots, N-1)$. The particular choice of S_N^λ as a subset of S_N parametrizing distinct rearrangements of a partition will be explained by the Proposition 1.

Denote $z^{\lambda_\sigma} := z_1^{\lambda_{\sigma_1}} z_2^{\lambda_{\sigma_2}} \ldots z_N^{\lambda_{\sigma_N}}$ $\quad (\lambda \in \Lambda_N\ ,\ \sigma \in S_N^\lambda)$. With this notation:

$$\mathbb{C}[z_1, \ldots, z_N] = \bigoplus_{\lambda \in \Lambda_N} \oplus_{\sigma \in S_N^\lambda} \mathbb{C} z^{\lambda_\sigma}. \tag{1.1.37}$$

2. Introduce an ordering on the set of monomials $\{z^{\lambda_\sigma}\}$ $\quad (\lambda \in \Lambda_N,\ \sigma \in S_N^\lambda)$.

We say that $\lambda > \tilde\lambda$ $\quad (\lambda, \tilde\lambda \in \Lambda_N)$ iff the first (counting from left) non-vanishing element of the sequence $\{\lambda_1 - \tilde\lambda_1, \lambda_2 - \tilde\lambda_2, \ldots, \lambda_N - \tilde\lambda_N\}$ is positive. Fix $\lambda \in \Lambda_N$. We say that $\sigma > \tilde\sigma$ $\quad (\sigma, \tilde\sigma \in S_N^\lambda)$ iff the last non-vanishing element of the sequence $\{\lambda_{\sigma_1} - \lambda_{\tilde\sigma_1}, \lambda_{\sigma_2} - \lambda_{\tilde\sigma_2}, \ldots \lambda_{\sigma_N} - \lambda_{\tilde\sigma_N}\}$ is negative.

For $\lambda, \tilde\lambda \in \Lambda_N$; $\sigma \in S_N^\lambda$, $\tilde\sigma \in S_N^{\tilde\lambda}$ define $\lambda_\sigma > \tilde\lambda_{\tilde\sigma}$ iff either $\lambda > \tilde\lambda$, or $\lambda = \tilde\lambda$, $\sigma > \tilde\sigma$. The ordering on monomials z^{λ_σ} is induced by the ordering on the exponents λ_σ $\quad (\lambda \in \Lambda_N,\ \sigma \in S_N^\lambda)$.

3. The action of $H_N(q)$ in the monomial basis is found by a straightforward computation to be as follows:

$$g_{i,j} z^{\lambda_\sigma} = (i < j) = \begin{cases} (q - q^{-1}) z^{\lambda_\sigma} + q z^{\lambda_{(ij)\sigma}} + \text{“s.p.”} & \text{if } \lambda_{\sigma_i} > \lambda_{\sigma_j} \ , \\ q z^{\lambda_\sigma} & \text{if } \lambda_{\sigma_i} = \lambda_{\sigma_j} \ , \\ q^{-1} z^{\lambda_{(ij)\sigma}} + \text{“s.p.”} & \text{if } \lambda_{\sigma_i} < \lambda_{\sigma_j} \ . \end{cases} \tag{1.1.38}$$

And

$$g_{i,j}^{-1} z^{\lambda_\sigma} = (i < j) = \begin{cases} q z^{\lambda_{(ij)\sigma}} + \text{“s.p.”} & \text{if } \lambda_{\sigma_i} > \lambda_{\sigma_j} \ , \\ q^{-1} z^{\lambda_\sigma} & \text{if } \lambda_{\sigma_i} = \lambda_{\sigma_j} \ , \\ (q - q^{-1}) z^{\lambda_\sigma} + q^{-1} z^{\lambda_{(ij)\sigma}} + \text{“s.p.”} & \text{if } \lambda_{\sigma_i} < \lambda_{\sigma_j} \ . \end{cases} \tag{1.1.39}$$

Where “s.p.” means a linear combination of monomials with smaller partitions.

It follows that:

$$K_{i,j} g_{i,j} z^{\lambda_\sigma} = (i < j) = \begin{cases} q z^{\lambda_\sigma} + \text{“s.m.”} & \text{if } \lambda_{\sigma_i} \geq \lambda_{\sigma_j} \ , \\ q^{-1} z^{\lambda_\sigma} + \text{“s.m.”} & \text{if } \lambda_{\sigma_i} < \lambda_{\sigma_j} \ . \end{cases} \tag{1.1.40}$$

And

$$g_{i,j}^{-1} K_{i,j} z^{\lambda_\sigma} = (i < j) = \begin{cases} q^{-1} z^{\lambda_\sigma} + \text{“s.m.”} & \text{if } \lambda_{\sigma_i} \geq \lambda_{\sigma_j} \ , \\ q z^{\lambda_\sigma} + \text{“s.m.”} & \text{if } \lambda_{\sigma_i} < \lambda_{\sigma_j} \ . \end{cases} \tag{1.1.41}$$

Where “s.m.” signifies a linear combination of smaller monomials.

4. The formulas of the preceding paragraph lead to the following proposition:

Proposition 1 *The operators Y_i $(i=1,\dots,N)$ are triangular in the monomial basis of $\mathbb{C}[z_1,\dots,z_N]$. The action of these operators on monomials is given by:*

$$Y_i z^{\lambda_\sigma} = p^{\lambda_{\sigma_i}} q^{l_{\sigma_i}} z^{\lambda_\sigma} + \text{“s.m.”} \quad (\lambda \in \Lambda_N,\ \sigma \in S_N^\lambda,\ (i=1,\dots,N))$$

where $l_i := 2i - N - 1 \quad (i=1,\dots,N)$.

This proposition shows that $\zeta_i^\lambda(\sigma) := p^{\lambda_{\sigma_i}} q^{l_{\sigma_i}} \quad (\lambda \in \Lambda_N,\ \sigma \in S_N^\lambda)$ constitute a complete set of characteristic numbers of the operator $Y_i \quad (i \in \{1,2,\dots,N\})$. In order to prove that the operators Y_i $(i=1,\dots,N)$ are simultaneously diagonalizable and that $\{\zeta_i^\lambda(\sigma)\} \quad (\lambda \in \Lambda_N,\ \sigma \in S_N^\lambda)$ form the complete set of eigenvalues of $Y_i \quad (i \in \{1,2,\dots,N\})$ we shall make use of Lemma 1 discussed in the next paragraph.

5. The aim of this paragraph is to recall the following (presumably well-known) result:

Lemma 1 *Let $\mathcal{V} = \mathbb{C}\{f_a\}_{a=1,2,\dots,d=\dim\mathcal{V}}$ be a finite-dimensional vector space. Let $\mathcal{Z}_i \in End(\mathcal{V}) \quad (i=1,2,\dots,\mathcal{N})$ and :*

$$[\mathcal{Z}_i, \mathcal{Z}_j] = 0 \quad (i,j = 1,2,\dots,\mathcal{N}), \tag{a}$$

$\mathcal{Z}_i \quad (i=1,2,\dots,\mathcal{N})$ *are simultaneously triangular in the basis* $\{f_a\}_{a=1,2,\dots,d}$:

$$\mathcal{Z}_i f_a = \xi_i^a f_a + \sum_{b<a} m_i^{b\,a} f_b \quad (i=1,2,\dots,\mathcal{N}), \tag{b}$$

where $m_i^{b\,a}$ are coefficients.

(c) *The joint set of characteristic numbers $\{\xi_i^a\} \quad (i=1,2,\dots,\mathcal{N}\ ;\ a=1,2,\dots,d)$ is multiplicity-free:*

$$\forall\ a \neq b\ (a,b = 1,2,\dots,d)\ \ \exists\, I(a,b) \subset \{1,2,\dots,\mathcal{N}\} :$$
$$\forall\ i \in I(a,b)\ \xi_i^a - \xi_i^b \neq 0.$$

Then $\exists$ a basis $\{\phi_a\}_{a=1,2,\dots,d}$:

$$\mathcal{Z}_i \phi_a = \xi_i^a \phi_a \quad (a=1,2,\dots,d;\ i=1,2,\dots,\mathcal{N})$$
$$\phi_a = f_a + \sum_{b<a} \phi_{b\,a} f_b \quad (a=1,2,\dots,d)$$

Where the coefficients $\phi_{b\,a}$ are recursively defined as follows:

$$\phi_{b\,a} = \frac{1}{\xi^a(w) - \xi^b(w)} (m_{b\,a}(w) + \sum_{b<c<a} m_{b\,c}(w) \phi_{c\,a}),$$
$$\text{here} \quad \xi^a(w) := \sum_{i=1}^{\mathcal{N}} w^{i-1} \xi_i^a\,, \qquad m_{a\,b}(w) \ := \ \sum_{i=1}^{\mathcal{N}} w^{i-1} m_i^{a\,b}\,.$$

$w \in \mathbb{C}$ and $\phi_{b\,a}$ does not depend on w.

6. The joint characteristic number spectrum $\{\zeta_i^\lambda(\sigma) = p^{\lambda_{\sigma_i}} q^{l_{\sigma_i}}\} \quad ((i=1,\dots,N); \quad \lambda \in \Lambda_N;\ \sigma \in S_N^\lambda)$ of the operators $Y_i \quad (i=1,\dots,N)$ is explicitely multiplicity-free.The operators $g_{i,j} \quad (i,j = 1,2,\dots,N-1)$ preserve the finite-dimensional subspaces of $\mathbb{C}[z_1,\dots,z_N]$ formed by homogeneous polynomials of any total degree. Therefore $g_{i,j} \quad (i,j = 1,2,\dots,N-1)$ and consequently $Y_i \quad (i=1,\dots,N)$ are direct sums of finite dimensional operators. So we can apply the result of Lemma 1 and arrive at the following proposition:

Proposition 2 *There exist polynomials $\Phi_\sigma^\lambda \quad (\lambda \in \Lambda_N;\ \sigma \in S_N^\lambda)$ s.t.:*

$$\mathbb{C}[z_1,\dots,z_N] = \bigoplus_{\lambda\in\Lambda_N} E^\lambda\ ,\ E^\lambda \ := \ \oplus_{\sigma\in S_N^\lambda} \mathbb{C}\Phi_\sigma^\lambda\,, \tag{i}$$

$$Y_i \Phi_\sigma^\lambda = \zeta_i^\lambda(\sigma) \Phi_\sigma^\lambda \quad (i=1,\dots,N), \tag{ii}$$

$$\Delta(u) \Phi_\sigma^\lambda = \prod_{i=1}^{N} (1 + u p^{\lambda_i} q^{l_i}) \Phi_\sigma^\lambda\,, \tag{ii'}$$

$$\Phi_\sigma^\lambda = z^{\lambda_\sigma} + \text{“s.m.”}. \tag{iii}$$

1.5 Action of the Hecke Algebra in the eigenspaces $E^\lambda \quad (\lambda \in \Lambda_N)$ of the operator $\Delta(u)$.

1. Fix any $\lambda \in \Lambda_N$. Since $[\widehat{H_N(q)}\ ,\ \Delta(u)] \ = \ 0$ with the action of $\widehat{H_N(q)}$ defined in **1.1.1**, we have:

$$\widehat{H_N(q)} :\ E^\lambda \to E^\lambda$$

The affine generators of $\widehat{H_N(q)}$ act in E^λ as given by (ii) in Proposition 2. In this section we find the action of the finite-dimensional Hecke Algebra generated by $g_{i,i+1} \quad (i = 1, ..., N-1)$ on the polynomials $\Phi^\lambda_\sigma \quad (\sigma \in S^\lambda_N)$ forming a basis in $E^\lambda \quad (\lambda \in \Lambda_N)$.

2. Introduce operators $T_{i,i+1} \ \in \ \widehat{H_N(q)} \quad (i = 1, \ldots, N-1)$:

$$T_{i,i+1} := g_{i,i+1}(Y_{i+1} - Y_i) - (q - q^{-1})Y_{i+1} \quad (i = 1, \ldots, N-1). \tag{1.1.42}$$

These operators satisfy the following relations:

$$T_{i,i+1}Y_i \ = \ Y_{i+1}T_{i,i+1}\ , \quad T_{i,i+1}Y_{i+1} \ = \ Y_iT_{i,i+1} \quad (i = 1, \ldots, N-1) \tag{1.1.43}$$

$$[T_{i,i+1}, Y_j] = 0 \qquad (j \neq i, i+1) \tag{1.1.44}$$

Since $T_{i,i+1} \ \in \ \widehat{H_N(q)}$, we have: $T_{i,i+1} : E^\lambda \to E^\lambda \quad (i = 1, \ldots, N-1)$.

3. Consider the vector $T_{i,i+1}\Phi^\lambda_\sigma \in E^\lambda \quad (\sigma \in S^\lambda_N\ ,\ i \in 1, 2, \ldots, N-1)$. Due to (1.1.43,44):

$$Y_kT_{i,i+1}\Phi^\lambda_\sigma \quad = \quad \xi^\lambda_k((i, i+1)\sigma)T_{i,i+1}\Phi^\lambda_\sigma \quad (k = 1, \ldots, N).$$

Since the spectrum of $Y_k \quad (k = 1, \ldots, N)$ on E^λ does not contain $\xi^\lambda_k(\sigma)$ s.t. $\sigma \notin S^\lambda_N$, we have:

$$T_{i,i+1}\Phi^\lambda_\sigma = 0 \quad \text{when } \sigma \in S^\lambda_N\, , (i, i+1)\sigma \notin S^\lambda_N \quad (i = 1, \ldots, N-1). \tag{1.1.45}$$

Since the joint spectrum of Y_i $(i = 1, \ldots, N)$ on E^λ is multiplicity-free, we have:

$$T_{i,i+1}\Phi^\lambda_\sigma = \tau^\lambda_{i,i+1}(\sigma)\Phi^\lambda_{(i,i+1)\sigma} \quad \text{when } \sigma, (i, i+1)\sigma \in S^\lambda_N \quad (i = 1, \ldots, N-1). \tag{1.1.46}$$

Where $\tau^\lambda_{i,i+1}(\sigma)$ is a coefficient.

Let $\sigma \in S^\lambda_N$ and i be s.t. $(i, i+1)\sigma \notin S^\lambda_N$. Then $\lambda_{\sigma_i} = \lambda_{\sigma_{i+1}}$, $\sigma_{i+1} = \sigma_i + 1$, and (1.1.45) gives:

$$(g_{i,i+1} - q)\Phi^\lambda_\sigma = 0. \tag{1.1.47}$$

Let $\sigma \in S^\lambda_N$ and i be s.t. $(i, i+1)\sigma \in S^\lambda_N$. Then we can recast (1.1.46) as follows:

$$g_{i,i+1}\Phi^\lambda_\sigma = \frac{(q - q^{-1})\zeta^\lambda_{i+1}(\sigma)}{\zeta^\lambda_{i+1}(\sigma) - \zeta^\lambda_i(\sigma)}\Phi^\lambda_\sigma + \frac{\tau^\lambda_{i,i+1}(\sigma)}{\zeta^\lambda_{i+1}(\sigma) - \zeta^\lambda_i(\sigma)}\Phi^\lambda_{(i,i+1)\sigma}\,. \tag{1.1.48}$$

In order to find $\tau^\lambda_{i,i+1}(\sigma)$ we shall equate coefficients standing in front of monomials z^{λ_σ} , $z^{\lambda_{(i\, i+1)\sigma}}$ in the both sides of eq. (1.1.48). Recall that σ, $(i, i+1)\sigma \in S^\lambda_N$ entails in particular $\lambda_{\sigma_i} \neq \lambda_{\sigma_{i+1}}$.
Let $\lambda_{\sigma_i} > \lambda_{\sigma_{i+1}} \ \Rightarrow \ z^{\lambda_\sigma} > z^{\lambda_{(i\, i+1)\sigma}}$.
According to (1.1.38) the monomial z^{λ_σ} which is the maximal monomial in Φ^λ_σ appears in $g_{i,i+1}\Phi^\lambda_\sigma$ from two sources: from $g_{i,i+1}z^{\lambda_\sigma}$ and from $g_{i,i+1}z^{\lambda_{(i\, i+1)\sigma}}$. Denote by x the coefficient at the monomial $z^{\lambda_{(i\, i+1)\sigma}}$ in Φ^λ_σ.
In the RHS of (1.1.48) z^{λ_σ} appears as the maximal monomial in Φ^λ_σ and does not appear in $\Phi^\lambda_{(i,i+1)\sigma}$.

Using (1.1.38) to compute the contributions from $g_{i,i+1}\Phi$ we equate the coefficients in front of z^{λ_σ} in the both sides of (1.1.48):

$$(q - q^{-1}) + q^{-1}x = \frac{(q - q^{-1})\zeta^\lambda_{i+1}(\sigma)}{\zeta^\lambda_{i+1}(\sigma) - \zeta^\lambda_i(\sigma)}. \tag{1.1.49}$$

Computing the contribution from $z^{\lambda_{(i\, i+1)\sigma}}$ we find that in the LHS of (1.1.48) this monomial appears only in $g_{i,i+1}z^{\lambda_\sigma}$, while in the RHS it appears with coefficient x in Φ^λ_σ and as the maximal monomial in $\Phi^\lambda_{(i,i+1)\sigma}$. Equating the coefficients we get:

$$q = \frac{(q - q^{-1})\zeta^\lambda_{i+1}(\sigma)}{\zeta^\lambda_{i+1}(\sigma) - \zeta^\lambda_i(\sigma)}x + \frac{\tau^\lambda_{i,i+1}(\sigma)}{\zeta^\lambda_{i+1}(\sigma) - \zeta^\lambda_i(\sigma)}\,. \tag{1.1.50}$$

Combining (1.1.49) and (1.1.50) we obtain:

$$\tau^{\lambda}_{i,i+1}(\sigma) = q\frac{(q^{-1}\zeta^{\lambda}_{i+1}(\sigma) - q\zeta^{\lambda}_{i}(\sigma))(q\zeta^{\lambda}_{i+1}(\sigma) - q^{-1}\zeta^{\lambda}_{i}(\sigma))}{\zeta^{\lambda}_{i+1}(\sigma) - \zeta^{\lambda}_{i}(\sigma)} \quad (\lambda_{\sigma_i} > \lambda_{\sigma_{i+1}}) \tag{1.1.51}$$

Let $\lambda_{\sigma_i} < \lambda_{\sigma_{i+1}} \Rightarrow z^{\lambda_\sigma} < z^{\lambda_{(i\,i+1)\sigma}}$. Equate the coefficients in front of the monomial $z^{\lambda_{(i\,i+1)\sigma}}$ in the both sides of (1.1.48).
In the LHS the contribution comes only from $g_{i,i+1}z^{\lambda_\sigma}$. In the RHS only $\Phi^{\lambda}_{(i,i+1)\sigma}$ contributes $z^{\lambda_{(i\,i+1)\sigma}}$ as its maximal monomial. Application of (1.1.38) leads to:

$$q^{-1} = \frac{\tau^{\lambda}_{i,i+1}(\sigma)}{\zeta^{\lambda}_{i+1}(\sigma) - \zeta^{\lambda}_{i}(\sigma)} \quad (\lambda_{\sigma_i} < \lambda_{\sigma_{i+1}}). \tag{1.1.52}$$

4. To summarize, we have obtained the following proposition:

Proposition 3 *Let $\sigma \in S^{\lambda}_N \quad (\lambda \in \Lambda_N)$, then $H_N(q)$ acts in $E^{\lambda} = \mathbb{C}\{\Phi^{\lambda}_{\sigma}\}_{\sigma\in S^{\lambda}_N}$ as follows:*

$$g_{i,i+1}\Phi^{\lambda}_{\sigma} = \frac{(q-q^{-1})\zeta^{\lambda}_{i+1}(\sigma)}{\zeta^{\lambda}_{i+1}(\sigma) - \zeta^{\lambda}_{i}(\sigma)}\Phi^{\lambda}_{\sigma} + \begin{cases} q\frac{(q^{-1}\zeta^{\lambda}_{i+1}(\sigma)-q\zeta^{\lambda}_{i}(\sigma))(q\zeta^{\lambda}_{i+1}(\sigma)-q^{-1}\zeta^{\lambda}_{i}(\sigma))}{(\zeta^{\lambda}_{i+1}(\sigma)-\zeta^{\lambda}_{i}(\sigma))(\zeta^{\lambda}_{i+1}(\sigma)-\zeta^{\lambda}_{i}(\sigma))}\Phi^{\lambda}_{(i,i+1)\sigma} \\ \text{when } \lambda_{\sigma_i} > \lambda_{\sigma_{i+1}} \Rightarrow (i,i+1)\sigma \in S^{\lambda}_N, \\ 0 \quad \text{when } \lambda_{\sigma_i} = \lambda_{\sigma_{i+1}} \Leftrightarrow (i,i+1)\sigma \notin S^{\lambda}_N, \\ q^{-1}\Phi^{\lambda}_{(i,i+1)\sigma} \\ \text{when } \lambda_{\sigma_i} < \lambda_{\sigma_{i+1}} \Rightarrow (i,i+1)\sigma \in S^{\lambda}_N. \end{cases} \tag{1.1.53}$$

2 The limit $p \to 1$ of the hierarchy of Dynamical Models.

2.1 Few facts about Macdonald operators.

1.The Macdonald operators $D^n_N(p,t) \quad (n = 0,\ldots,N)$ [M] act in the subspace of $\mathbb{C}[z_1,\ldots,z_N]$ formed by symmetric polynomials. In notation of [JKKMP] these operators are defined as follows:

$$D^n_N(p,t) := t^{n(n-1)/2}\sum_{I_n}\prod_{\substack{i\in I_n\\ j\notin I_n}}\frac{tz_i - z_j}{z_i - z_j}\prod_{k\in I_n}p^{D_k} \quad (n = 0,\ldots,N), \tag{2.2.1}$$

where the summation is over all subsets I_n of $\{1,2,\ldots,N\}$ which contain n elements. Using the formula:

$$det\left\|\frac{(t-1)w_i}{tw_i - w_j}\right\|_{1\le i,j\le m} = t^{m(m-1)/2}\prod_{1\le i\ne j\le m}\frac{w_i - w_j}{tw_i - w_j},$$

where $w_i \quad (i = 1,\ldots,m)$ are numbers; we can rewrite the definition of the operators $D^n_N(p,t) \quad (n = 0,\ldots,N)$ in another form:

$$D^n_N(p,t) = \sum_{I_n} detA_{I_n}(t)\prod_{k\in I_n}p^{D_k} \quad (n = 0,\ldots,N), \tag{2.2.2}$$

where $A_{I_n}(t)$ is a submatrix of the matrix:

$$A(t) = \|A_{ij}(t)\|_{1\le i,j\le N}\ ,\ A_{ij}(t) := \frac{(t-1)z_i}{tz_i - z_j}\prod_{\substack{1\le k\le N\\ k\ne i}}\frac{tz_i - z_k}{z_i - z_k}; \tag{2.2.3}$$

which is defined as follows:$A_{I_n}(t) := \|A_{ij}(t)\|_{i,j\in I_n}$.

The Macdonald polynomials $P_\lambda(p,t) \quad (\lambda \in \Lambda_N)$ are eigenfunctions of the operators $D^n_N(p,t) \quad (n = 0,\ldots,N)$:

$$D(v;p,t)P_\lambda(p,t) = \prod_{i=1}^{N}(1 + t^{N-i}p^{\lambda_i}v)\,P_\lambda(p,t) \quad (\lambda \in \Lambda_N), \tag{2.2.4}$$

here $D(v;p,t)$ is the generating function of Macdonald operators:

$$D(v;p,t) := \sum_{n=0}^{N} v^n D_N^n(p,t)\ .$$

2. In the limit $p \to 1$ one finds [M]:

$$P_\lambda(p,t) = e_{\lambda'} + O(p-1) \quad (\lambda \in \Lambda_N), \tag{2.2.5}$$

where λ' is the conjugate partition of λ and for a partition $\pi\ : (\pi_1 \geq \pi_2 \geq \dots)\ ,\ (\pi_1 \leq N) \quad e_\pi := e_{\pi_1} e_{\pi_2} \dots$; where e_r is the elementary symmetric polynomial:

$$e_r := \sum_{1 \leq i_1 \leq \dots \leq i_r \leq N} z_{i_1} z_{i_2} \dots z_{i_r}.$$

Consider the limit $p \to 1$ of the generating function $D(v;p,t)$:

$$D(v;p,t) \overset{p \to 1}{=} D_0(v;t) + (p-1) D_1(v;t) + O((p-1)^2). \tag{2.2.6}$$

From (2.2.1,.4,.5) it follows that $D_0(v;t)$ is a multiplication by a constant:

$$D_0(v;t) = \prod_{i=1}^{N} (1 + t^{N-i} v)\ . \tag{2.2.7}$$

The first-order term $D_1(v;t)$ is a differential operator:

$$D_1(v;t) = \sum_{i=1}^{N} \left(\sum_{n=1}^{N} v^n \sum_{I_n : i \in I_n} det A_{I_n}(t) \right) D_i \tag{2.2.8}$$

Expanding the eq. (2.2.4) up to the first order in $p-1$ and using (2.2.5,.7) we get:

$$D_1(v;t) e_{\lambda'} = \left(v \sum_{j=1}^{N} \prod_{\substack{1 \leq k \leq N \\ k \neq j}} (1 + v t^{N-k}) t^{N-j} \lambda_j \right) e_{\lambda'} \quad (\lambda \in \Lambda_N). \tag{2.2.9}$$

2.2 Taking the limit $p \to 1$ in the hierarchy of Dynamical Models

1. The following fact was established in the paper [JKKMP]. Let S be any symmetric polynomial ($S \in \mathbb{C}[z_1, \dots, z_N]$). The action of the operator $\Delta(u)$ (cf. **1.1**) on such S coincides with the action of the generating function of Macdonald operators:

$$\Delta(u) S = D(q^{N-1} u; p, q^{-2}) S\ . \tag{2.2.10}$$

There is another connection between $\Delta(u)$ and Macdonald operators. For an operator O which is a function of operators D_i, z_i $(i = 1, \dots, N), K_{i,i+1}$ $(i = 1, \dots, N-1)$ introduce a normal ordering : :. The normal ordering is described as follows: in O bring all the operators D_i to the right *without* taking commutators between D_i and z_j, but taking into account the commutation relations between D_i and $K_{j,j+1}$. For instance:

$$:\ p^{D_1} \frac{q^{-1} z_1 - q z_2}{z_1 - z_2} (K_{12} - 1)\ := \frac{q^{-1} z_1 - q z_2}{z_1 - z_2} (K_{12}\, p^{D_2} - p^{D_1}).$$

We formulate the following Lemma:

Lemma 2

$$:\ \Delta(u)\ := D(q^{N-1} u; p, q^{-2})\ . \tag{2.2.11}$$

Proof. To facilitate the proof we introduce an extension of the algebra generated by $z_1, z_2, \dots, z_N$ and $K_{1,2}, K_{2,3}, \dots, K_{N-1,N}$ by symbols $\xi_1, \xi_2, \dots, \xi_N$. These symbols are defined by the commutation relations

$$\begin{aligned}&[\xi_i, \xi_j] = 0\ ,\ [\xi_i, z_j] = 0 \qquad (i, j \in \{1, 2, \dots, N\}),\\ &K_{i,j}\xi_j = \xi_i K_{i,j}\ ,\ [K_{i,j}, \xi_k] = 0 \quad (k \neq i, j) \qquad (i \neq j \in \{1, 2, \dots, N\}).\end{aligned}$$

One can take $\xi_i := f(z_i)$, where f is any function of one variable, as a realization for ξ_i.
Together with $g_{i,i+1}$ the operators

$$Y_i(\xi) := g_{i,i+1}^{-1} K_{i,i+1} \dots g_{i,N}^{-1} K_{i,N} \xi_i K_{1,i} g_{1,i} \dots K_{i-1,i} g_{i-1,i}$$

still satisfy the defining relations of $\widehat{H_N(q)}$.
This implies in particular that for the operator

$$\Delta(u; \xi) := \prod_{i=1}^{N} (1 + uY_i(\xi))$$

we have

$$[\Delta(u; \xi), g_{i,i+1}] = 0 \qquad (i = 1, \dots, N-1). \tag{i}$$

Using the commutation relations with $z_1, z_2, \dots, z_N$ and $K_{i,j}$ we can bring the symbols ξ_i to the right of all expressions in $\Delta(u; \xi)$. Denote $\Delta(u; \xi)$ with all ξ_i brought to the right by $\Delta(u; \xi)'$.
We have

$$: \Delta(u) := \Delta(u; \xi)'|_{\xi_i \to p^{D_i}}.$$

Therefore in order to prove the statement of the lemma we compute the coefficients standing in front of monomials $\xi_{i_1} \xi_{i_2} \dots \xi_{i_n} \quad (1 \le i_1 \le i_2 \le \dots \le i_n \le N)$ in the symmetric functions

$$\Delta(\xi)^{(n)} := \sum_{N \ge k_1 > k_2 > \dots > k_n \ge 1} Y_{k_1}(\xi) Y_{k_2}(\xi) \dots Y_{k_n}(\xi) \quad (1 \le n \le N).$$

With notation of **1.1.1** we have

$$\begin{aligned} r_{i,j} &:= K_{i,j} g_{i,j} = a_{j,i} + b_{j,i} K_{i,j},\\ r_{i,j}^{-1} &= a_{i,j} - b_{j,i} K_{i,j},\\ Y_i(\xi) &= r_{i,i+1}^{-1} \dots r_{i,N}^{-1} \xi_i r_{1,i} \dots r_{i-1,i}.\end{aligned}$$

Let us compute the terms in $\Delta(\xi)^{(n)'} \quad (1 \le n \le N)$ which contain symbols $\xi_1, \xi_2, \dots, \xi_n$ only. By inspection we find that such terms can appear only in $Y_n(\xi) Y_{n-1}(\xi) \dots Y_1(\xi)$. Furthermore the relevant contributions from the individual factors in the last expression are

$$\begin{aligned} Y_1(\xi) &\to r_{1,2}^{-1} \dots r_{1,n}^{-1} a_{1,n+1} \dots a_{1,N} \xi_1,\\ Y_2(\xi) &\to r_{2,3}^{-1} \dots r_{2,n}^{-1} a_{2,n+1} \dots a_{2,N} \xi_2 r_{1,2},\\ &\vdots\\ Y_k(\xi) &\to r_{k,k+1}^{-1} \dots r_{k,n}^{-1} a_{k,n+1} \dots a_{k,N} \xi_k r_{1,k} \dots r_{k-1,k},\\ &\vdots\\ Y_{n-1}(\xi) &\to r_{n-1,n}^{-1} a_{n-1,n+1} \dots a_{n-1,N} \xi_{n-1} r_{1,n-1} \dots r_{n-2,n-1},\\ Y_n(\xi) &\to a_{n,n+1} \dots a_{n,N} \xi_n r_{1,n} \dots r_{n-1,n}.\end{aligned}$$

Multiplying these contributions we find that there is only one term in $\Delta(\xi)^{(n)'}$ which contains $\xi_1, \xi_2, \dots, \xi_n$ only; and this term is

$$(a_{1,n+1} \dots a_{1,N})(a_{2,n+1} \dots a_{2,N}) \dots (a_{n,n+1} \dots a_{n,N}) \xi_1 \xi_2 \dots \xi_n.$$

Next we use the Hecke-invariance relation *(i)* and find

$$\Delta(\xi)^{(n)} = \sum_{I_n} \left(\prod_{\substack{i\in I_n \\ j\notin I_n}} a_{i,j} \right) \prod_{i\in I_n} \xi_i .$$

Where the summation is over all n-element subsets of $\{1,2,\dots,N\}$. Comparing this expression with *(2.2.1)* we obtain the statement of the lemma. ■

2. Let us take the limit $p \to 1$ in the operators $\Delta(u), T_a(u), Y_i \quad (i=1,\dots,N)$. Expanding around $p=1$ and keeping the first two terms of the expansion we write:

$$Y_i \stackrel{p\to 1}{=} y_i + (p-1)x_i + O((p-1)^2) \quad ((i=1,\dots,N)), \tag{2.2.12}$$

$$\Delta(u) \stackrel{p\to 1}{=} \Delta_0(u) + (p-1)\Delta_1(u) + O((p-1)^2)\,, \tag{2.2.13}$$

$$T_a(u) \stackrel{p\to 1}{=} T_a^0(u) + (p-1)T_a^1(u) + O((p-1)^2)\,. \tag{2.2.14}$$

Here we introduced the operators:

$$y_i := g_{i,i+1}^{-1}K_{i,i+1}\dots g_{i,N}^{-1}K_{i,N}K_{1,i}g_{1,i}\dots K_{i-1,i}g_{i-1,i} \quad ((i=1,\dots,N)), \tag{2.2.15}$$

$$x_i := g_{i,i+1}^{-1}K_{i,i+1}\dots g_{i,N}^{-1}K_{i,N}D_iK_{1,i}g_{1,i}\dots K_{i-1,i}g_{i-1,i} \quad ((i=1,\dots,N)), \tag{2.2.16}$$

$$\Delta_0(u) := \prod_{i=1}^{N}(1+uy_i)\,, \tag{2.2.17}$$

$$\Delta_1(u) := u\sum_{j=1}^{N} \prod_{1\le i<j} (1+uy_i)\, x_j \prod_{j<k\le N} (1+uy_k)\,, \tag{2.2.18}$$

$$T_a^0(u) := L_{a1}(uy_1)L_{a2}(uy_2)\dots L_{aN}(uy_N) \quad \in End\mathbb{C}[z_1,\dots,z_N]\otimes H\,. \tag{2.2.19}$$

The operators $y_i, g_{i,i+1}$ satisfy the relations (cf. **1.1**) of Affine Hecke Algebra and $T_a^0(u)$ defines a representation of U.

3. From Lemma 2 it follows that the operators $\Delta_0(u)\,,\ \Delta_1(u)$ have a rather special form. We have the proposition:

Proposition 4 *The following statements hold:*
Let $D_0(v;t)\,,\ D_1(v;t)$ be those defined in (2.2.6,.8)*; then:*

$$\Delta_0(u) = D_0(q^{N-1}u;q^{-2}) \ = \prod_{i=1}^{N}(1+uq^{2i-N-1})I\,. \tag{i}$$

I.e. $\Delta_0(u)$ is a multiplication by a constant.

$$\Delta_1(u) = D_1(q^{N-1}u;q^{-2}) + \Xi(u)\,. \tag{ii}$$

Where operator $\Xi(u)$ is a function of operators $z_i\,,\ K_{i,j} \quad (i,j=1,\dots,N)$ only (and not *of D_i).*

Let $\Delta(u)_1 := \sum_{i=1}^{N} u^i\Delta_1^{(i)}$ and $\Xi(u) := \sum_{i=1}^{N} u^i\Xi^{(i)}$. We have computed explicit expressions for the operators $\Delta_1^{(N)}$ and $\Delta_1^{(1)}$. In notation of section **1.1** one has:

$$\Delta_1^{(N)} = D_1 + D_2 + \dots + D_N\,, \tag{2.2.20}$$

$$\Delta_1^{(1)} = \sum_{i=1}^{N} \left(\prod_{\substack{1\le k\le N \\ k\ne i}} a_{i,k} \right) D_i \ + \ \Xi^{(1)} \tag{2.2.21}$$

where:

$$\Xi^{(1)} =$$

$$\sum_{M=2}^{N} \frac{(-1)^M}{(q-q^{-1})} \sum_{N\geq i_M>\cdots>i_1\geq 1} \mathcal{A}_{i_M,i_{M-1},\ldots,i_1} \mathcal{B}_{i_M,i_{M-1},\ldots,i_1} K_{i_M,i_{M-1}} \ldots K_{i_2,i_1} \; + \; \varphi^{(1)} \; ,$$

$$\mathcal{A}_{i_M,i_{M-1},\ldots,i_1} = \left(\prod_{i_1<f<i_2} a_{i_1,f} \right) \left(\prod_{i_2<f<i_3} a_{i_2,f} \right) \cdots \left(\prod_{i_M<f<N+i_1} a_{i_M,f} \pmod{N} \right) ,$$

$$\mathcal{B}_{i_M,i_{M-1},\ldots,i_1} = b_{i_M,i_{M-1}} b_{i_{M-1},i_{M-2}} \ldots b_{i_2,i_1} b_{i_1,i_M} \; ,$$

$$\varphi^{(1)} = - \sum_{1\leq k<i\leq N} \frac{a_{i,i+1} \ldots a_{i,N} a_{i,1} \ldots a_{i,k-1} a_{i,k+1} \ldots a_{i,i-1} b_{k,i} b_{i,k}}{(q-q^{-1})} \; .$$

4. Let us fix the notation:

$$\mathcal{D}(u) := D_1(q^{N-1}u; q^{-2}) = \sum_{i=1}^{N} \theta_i(u) \, D_i \; . \tag{2.2.23}$$

Where according to (2.2.8) :

$$\theta_i(u) := \sum_{n=1}^{N} u^n q^{n(N-1)} \sum_{I_n : i\in I_n} det A_{I_n}(q^{-2}) \quad ((i=1,\ldots,N)). \tag{2.2.24}$$

(Cf. sec. **2.1** for the definition of $A_{I_n}(t)$). The functions $\theta_i(u) \quad ((i=1,\ldots,N))$ can be written in the following form:

$$\theta_i(u) = \frac{\partial}{\partial\gamma_i} det(I + u\Gamma q^{N-1} A(q^{-2}))|_{\Gamma=I} \; , \tag{2.2.25}$$

where we have introduced an auxiliary matrix: $\Gamma := \mathrm{diag}\{\gamma_1, \ldots, \gamma_N\}$. Using this representation we compute $\theta_i(u) \quad ((i=1,\ldots,N))$ at the point $z_1 = \omega^1, \ldots, z_N = \omega^N$, where $\omega := \exp(2\pi i/N)$. The computation yields:

$$\theta_i(u)|_{z_1=\omega^1,\ldots,z_N=\omega^N} = \frac{1}{N} \prod_{k=1}^{N} (1+uq^{l_k}) \sum_{n=1}^{N} \frac{uq^{l_n}}{1+uq^{l_n}} \equiv \theta(u) \quad (i=1,\ldots,N). \tag{2.2.26}$$

$$l_i := 2i - N - 1$$

Thus the point $z_1 = \omega^1, \ldots, z_N = \omega^N$ (or any point obtained from it by a permutation of coordinates) is special in that at this point all the $\{z_i\}$-dependent coefficients $\theta_i(u)$ of the first-order differential operator $\mathcal{D}(u)$ become equal one to another.

3 Definition of the Hierarchy of Integrable, U-invariant Spin Models

3.1 Preliminaries

1. Let us expand the relations (1.1.34,.35) around the point $p=1$ using the definitions (2.2.12-.14) and the fact that $\Delta_0(u)$ is a constant. For $\Delta_1(u)$, $T_a^0(v) \in End(\mathbb{C}[z_1,\ldots,z_N] \otimes H)$ we obtain:

$$[\Delta(u)_1, T_a^0(v)] = 0 \; , \tag{3.3.1}$$

$$[\Delta_1(u), \Delta_1(v)] = 0. \tag{3.3.2}$$

Expanding the relations:

$$[\Delta(u), Y_i] = 0 \quad (i=1,\ldots,N), \quad [\Delta(u), g_{i,i+1}] = 0 \quad (i=1,\ldots,N-1), \tag{3.3.3}$$

we get:

$$[\Delta_1(u), y_i] = 0 \quad (i=1,\ldots,N), \quad [\Delta_1(u), g_{i,i+1}] = 0 \quad (i=1,\ldots,N-1). \tag{3.3.4}$$

Due to (3.3.4) and the fact that $y_i \quad (i=1,\dots,N)$, $g_{i,i+1} \quad (i=1,\dots,N-1)$ satisfy the Affine Hecke Algebra relations, the operators $\Delta_1(u), T^0_a(v)$ act in the "bosonic" subspace $\mathcal{B}$ (cf. 1.1.36 and 3.3.7).

2. Introduce several definitions. Let

$$\mathcal{R} := \mathbb{C}[\{\frac{1}{z_i - z_j}\}_{1\le i\neq j\le N}\,, \{\frac{1}{z_i - q^2 z_j}\}_{1\le i\neq j\le N}\,,\ z_1,\dots,z_N]\otimes H. \tag{3.3.5}$$

Let $\mathcal{P}$ and $\mathcal{B}$ be the subspaces of $\mathcal{R}$:

$$\mathcal{P} := \mathbb{C}[z_1,\dots,z_N]\otimes H\,, \tag{3.3.6}$$

$$\mathcal{B} := \{b\in\mathcal{P} | (g_{i,i+1} - t_{i,i+1})b = 0 \quad (i=1,\dots,N-1)\}. \tag{3.3.7}$$

For $v_1,\dots,v_N \in \mathbb{C}$ such that $v_i \neq v_j\,,\, q^2 v_j \quad (i\neq j\,; i,j = 1,\dots,N)$ define the evaluation map: $Ev(v) : \mathcal{R}\mapsto H$ by taking values of rational functions at the point $z_1 = v_1,\dots,z_N = v_N$. ($\Leftrightarrow z = v$).
For any $O\in End(\mathcal{R})$ define an operator $\hat{O}\in End(\mathcal{R})$ by the rule [BGHP]:

Using the commutation relations bring all the permutation operators $K_{i,j} \quad (i,j\in\{1,2,\dots,N\})$ to the right of an expression in O; replace the rightmost of $K_{i,j}$ using the substitution:

$$g_{i,i+1}\to t_{i,i+1} \Rightarrow K_{i,i+1} \to \frac{z_i t^{-1}_{i,i+1} - z_{i+1}t_{i,i+1}}{q^{-1}z_i - q z_{i+1}} \quad (i=1,\dots,N-1). \tag{3.3.8}$$

Repeat the procedure until there are no operators $K_{i,j}$ left. The result is $\hat{O}$.

In what follows we adopt the following notational convention: if $\mathcal{L}$ is a linear space and A, B are linear operators defined on $\mathcal{L}$, we write:

$$A\mathcal{L} = B\mathcal{L} \quad \text{meaning} \quad Al = Bl \quad \forall l\in\mathcal{L}.$$

In particular for $O,\hat{O} \quad \in End(\mathcal{R})$ defined above we have:

$$O\mathcal{B} = \hat{O}\mathcal{B}. \tag{3.3.9}$$

Let $O, O'\in End(\mathcal{P})$ be s.t.:

$$[O,O']\mathcal{P} = 0 \quad \text{and} \quad O,O' : \mathcal{B}\mapsto\mathcal{B}, \tag{3.3.10}$$

then

$$[\hat{O},\hat{O}']\mathcal{B} = 0. \tag{3.3.11}$$

3.With notation of (2.2.20,.23,.26) let us consider the following differential operator $\widetilde{\mathcal{D}(u)}\in End(\mathcal{R})$:

$$\widetilde{\mathcal{D}(u)} := \mathcal{D}(u) - \theta(u)\Delta_1^{(N)}. \tag{3.3.12}$$

Let $Ev(\omega)$ be the evaluation map $Ev(v)$ taken at the special point $v_1 = \omega^1,\dots,v_N = \omega^N \quad (\Leftrightarrow v = \omega)$. Then in virtue of (2.2.26) we obtain the following property of $\widetilde{\mathcal{D}(u)}$:

$$Ev(\omega)\widetilde{\mathcal{D}(u)}\mathcal{R} = 0. \tag{3.3.13}$$

Let us introduce the modified generating function $\widetilde{\Delta_1(u)}$ by subtracting the product of the constant $\theta(u)$ and the operator $\Delta_1^{(N)}$:

$$\widetilde{\Delta_1(u)} := \Delta_1(u) - \theta(u)\Delta_1^{(N)} = \widetilde{\mathcal{D}(u)} + \Xi(u). \tag{3.3.14}$$

Since $\Delta_1^{(N)}$ is a member of the hierarchy of commuting operators defined by $\Delta_1(u)$, the equations (3.3.1,.2),(3.3.4) still hold if we replace in these equations $\Delta_1(u)$ by $\widetilde{\Delta_1(u)}$:

$$[\widetilde{\Delta_1(u)},\widetilde{\Delta_1(v)}]\mathcal{P} = 0, \tag{3.3.15}$$

$$[\widetilde{\Delta_1(u)}, T^0_a(v)]\mathcal{P} = 0, \tag{3.3.16}$$

$$\widetilde{\Delta_1(u)} : \mathcal{B}\mapsto\mathcal{B}. \tag{3.3.17}$$

3.2 Definition of the hierarchy of Spin Models

1. Let $H_{\mathcal{B}}(\omega)$ be the image of $\mathcal{B}$ under the action of the evaluation map $Ev(\omega)$:

$$Ev(\omega)\mathcal{B} = H_{\mathcal{B}}(\omega) \subset H. \tag{3.3.18}$$

Since $\widehat{T^0_a(u)}$, $\widehat{\Xi(u)}$ do not depend on the diffrential operators D_i $(i = 1, \ldots, N)$ and the operators of coordinate permutation, we can define the operators $T^0_a(u;\omega), \Xi(u;\omega)$ as follows:

$$Ev(\omega)\widehat{T^0_a(u)} = T^0_a(u;\omega)Ev(\omega), \quad Ev(\omega)\widehat{\Xi(u)} = \Xi(u;\omega)Ev(\omega). \tag{3.3.19}$$

Applying $Ev(\omega)$ to the relations :

$$\widehat{T^0_a(u)} : \mathcal{B} \mapsto \mathcal{B}, \tag{3.3.20}$$

$$\widehat{\widehat{\Delta_1(u)}} : \mathcal{B} \mapsto \mathcal{B}, \tag{3.3.21}$$

and using (3.3.13) we get:

$$T^0_a(u;\omega) : H_{\mathcal{B}}(\omega) \mapsto H_{\mathcal{B}}(\omega), \tag{3.3.22}$$

$$\Xi(u;\omega) : H_{\mathcal{B}}(\omega) \mapsto H_{\mathcal{B}}(\omega). \tag{3.3.23}$$

2. Apply the evaluation map $Ev(\omega)$ to the relations (3.3.15,.16) taking (3.3.13) and (3.3.23) into account. As the result we find that the operator $\Xi(u;\omega)$ is a generating function of the commuting, U-invariant integrals of motion which are operators in $H_{\mathcal{B}}(\omega)$:

$$[\Xi(u;\omega), \Xi(v;\omega)]H_{\mathcal{B}}(\omega) = 0\,, \tag{3.3.24}$$

$$[\Xi(u;\omega), T^0_a(v;\omega)]H_{\mathcal{B}}(\omega) = 0, \tag{3.3.25}$$

$$\left(\bar{R}_{ab}(u/v)T^0_a(u;\omega)T^0_b(v;\omega) - T^0_b(v;\omega)T^0_a(u;\omega)\bar{R}_{ab}(u/v)\right) H_{\mathcal{B}}(\omega) = 0. \tag{3.3.26}$$

In sec. 6 we shall show that $H_{\mathcal{B}}(\omega) = H$. This completes the definition of the hierarchy $\Xi(\omega)^{(1)}, \ldots, \Xi(\omega)^{(N-1)}$ $(\Xi(u;\omega) = \sum_{n=1}^{N-1} u^n \Xi^{(n)}(\omega))$ of Spin Models.

4 Eigenvalue spectrum of the operators $\Delta_1(u)$, y_i $(i = 1, \ldots, N)$

4.1 Characteristic numbers and eigenvalues of $\Delta_1(u)$, y_i

1. To find the action of the operators $\Delta_1(u)$, y_i $(i = 1, \ldots, N)$ in the monomial basis of $\mathbb{C}[z_1, \ldots, z_N]$ we can take the limit $p \to 1$ in the formulas of Proposition 1. This gives the following proposition:

Proposition 5 *The operators $\Delta_1(u)$, y_i $(i = 1, \ldots, N)$ are triangular in the monomial basis of $\mathbb{C}[z_1, \ldots, z_N]$. The action of these operators on monomials is given by:*

$$y_i z^{\lambda_\sigma} = q^{l_{\sigma_i}} z^{\lambda_\sigma} + \text{“s.m”} \quad (\lambda \in \Lambda_n, \sigma \in S^\lambda_N,\ (i = 1, \ldots, N)), \tag{i}$$

$$\Delta_1(u) z^{\lambda_\sigma} = \delta^\lambda(u) z^{\lambda_\sigma} + \text{“s.m”} \quad (\lambda \in \Lambda_N, \sigma \in S^\lambda_N). \tag{ii}$$

where $l_i := 2i - N - 1 \quad (i = 1, \ldots, N)$ and

$$\delta^\lambda(u) := u \sum_{j=1}^{N} \left(\prod_{\substack{1 \le k \le N \\ k \ne j}} (1 + u q^{l_k}) \right) q^{l_j} \lambda_j.$$

Since $\Delta_1(u)$ and y_i $(i = 1, \ldots, N)$ commute among themselves (cf. 3.3.4) and the joint spectrum of characteristic numbers of $\Delta_1(u)$ and y_i $(i = 1, \ldots, N)$ given by Proposition 5 is explicitely multiplicity-free, we apply Lemma 1 and claim that $\Delta_1(u)$, y_i $(i = 1, \ldots, N)$ are simultaneously diagonalizable:

Proposition 6 *There exist polynomials* $\varphi_\sigma^\lambda \quad (\lambda \in \Lambda_N, \sigma \in S_N^\lambda)$ *s.t.:*

$$\mathbb{C}[z_1, \ldots, z_N] = \bigoplus_{\lambda \in \Lambda_N} \mathcal{E}^\lambda \,, \; \mathcal{E}^\lambda := \oplus_{\sigma \in S_N^\lambda} \mathbb{C}\varphi_\sigma^\lambda \,, \tag{i}$$

$$y_i \varphi_\sigma^\lambda = q^{l_{\sigma_i}} \varphi_\sigma^\lambda \quad (i = 1, \ldots, N), \tag{ii}$$

$$\Delta_1(u)\varphi_\sigma^\lambda = \delta^\lambda(u)\varphi_\sigma^\lambda \,, \tag{ii'}$$

$$\varphi_\sigma^\lambda = z^{\lambda_\sigma} + \text{“s.m”} \,. \tag{4.4.1}$$

2. Let us show that $\varphi_\sigma^\lambda = \lim_{p\to 1} \Phi_\sigma^\lambda \quad (\lambda \in \Lambda_N, \sigma \in S_N^\lambda)$ where $\Phi_\sigma^\lambda \quad (\lambda \in \Lambda_N, \sigma \in S_N^\lambda)$ are eigenfunctions of the operators $Y_i \quad (i = 1, \ldots, N)$ (Cf. Proposition 2).

Expanding $\Phi_\sigma^\lambda \quad (\lambda \in \Lambda_N, \sigma \in S_N^\lambda)$ around $p = 1$ let us write:

$$\Phi_\sigma^\lambda \overset{p\to 1}{=} (p-1)^s \psi_\sigma^\lambda + O((p-1)^{s+1}), \tag{4.4.2}$$

where ψ_σ^λ is a non-zero polynomial. Since $\Phi_\sigma^\lambda = z^{\lambda_\sigma} +$ “ *s.m.*”, we have: $s \leq 0$.

Let us show that $s = 0$. Suppose $s < 0$. The satement (c) of Lemma 1 when applied to the eigenvectors Φ_σ^λ enables us to detect which coefficients in the decomposition of Φ_σ^λ into monomials are potentially singular in the limit $p \to 1$. The singularities may arise because of the presence of denominators of the form

$$\frac{1}{p^{\lambda_{\sigma_i}} q^{l_{\sigma_i}} - p^{\mu_{\sigma_i}} q^{l_{\sigma_i}}} \tag{4.4.3}$$

where μ is a partition *smaller* than λ and $\sigma \in S_N^\lambda, S_N^\mu$.Therefore if s in (4.4.2) is negative, the maximal monomial in ψ_σ^λ is *smaller* than any of the monomials $z^{\lambda_\sigma} \quad (\sigma \in S_N^\lambda)$.

Expanding the equations (ii),(ii') of the Proposition 2 around the point $p = 1$ and taking into account that $\Delta_0(u) = \Delta(u)|_{p=1}$ is a constant, we arrive at the following equations:

$$\Delta_1(u)\psi_\sigma^\lambda = \delta^\lambda(u)\psi_\sigma^\lambda \,,$$
$$y_i \psi_\sigma^\lambda = q^{l_{\sigma_i}} \psi_\sigma^\lambda.$$

Since the joint spectrum of the operators $\Delta_1(u)$ and y_i $(i = 1, \ldots, N)$ is multiplicity-free, we must have:

$$\psi_\sigma^\lambda \propto \varphi_\sigma^\lambda \,.$$

This contradicts the observation that the maximal monomial of ψ_σ^λ is smaller than any of the monomials $z^{\lambda_\sigma} \quad (\sigma \in S_N^\lambda)$.

Thus $s = 0$ therefore $\psi_\sigma^\lambda = \varphi_\sigma^\lambda$ and consequently $\varphi_\sigma^\lambda = \lim_{p\to 1} \Phi_\sigma^\lambda \quad (\lambda \in \Lambda_N, \sigma \in S_N^\lambda)$.

4.2 Action of the Hecke Algebra in the eigenspaces $\mathcal{E}^\lambda$ $(\lambda \in \Lambda_N)$ of the operator $\Delta_1(u)$.

1. According to (3.3.4) the Hecke Algebra $H_N(q)$ generated by $g_{i,i+1}$ $(i = 1, \ldots, N-1)$ acts in each eigenspace $\mathcal{E}^\lambda$ $(\lambda \in \Lambda_N)$ of the operator $\Delta_1(u)$. To compute this action explicitely in the basis $\{\varphi_\sigma^\lambda\}_{\sigma \in S_N^\lambda}$ we can either repeat almost word-by-word the derivation described in **1.5** or take the limit $p \to 1$ in the result of Proposition 3. Either way we arrive at the following proposition:

Proposition 7 *Let* $\sigma \in S_N^\lambda \quad (\lambda \in \Lambda_N)$, *then* $H_N(q)$ *generated by* $g_{i,i+1}$ $(i = 1, \ldots, N-1)$ *acts in* $\mathcal{E}^\lambda = \mathbb{C}\{\varphi_\sigma^\lambda\}_{\sigma \in S_N^\lambda}$ *as follows:*

$$g_{i,i+1}\varphi_\sigma^\lambda = \frac{(q-q^{-1})q^{l_{\sigma_{i+1}}}}{q^{l_{\sigma_{i+1}}} - q^{l_{\sigma_i}}}\varphi_\sigma^\lambda + \begin{cases} q\frac{(q^{-1}q^{l_{\sigma_{i+1}}} - qq^{l_{\sigma_i}})(qq^{l_{\sigma_{i+1}}} - q^{-1}q^{l_{\sigma_i}})}{(q^{l_{\sigma_{i+1}}} - q^{l_{\sigma_i}})(q^{l_{\sigma_{i+1}}} - q^{l_{\sigma_i}})}\varphi_{(i,i+1)\sigma}^\lambda \\ \text{when } \lambda_{\sigma_i} > \lambda_{\sigma_{i+1}} \Rightarrow (i,i+1)\sigma \in S_N^\lambda, \\ 0 \quad \text{when } \lambda_{\sigma_i} = \lambda_{\sigma_{i+1}} \Leftrightarrow (i,i+1)\sigma \notin S_N^\lambda, \\ q^{-1}\varphi_{(i,i+1)\sigma}^\lambda \\ \text{when } \lambda_{\sigma_i} < \lambda_{\sigma_{i+1}} \Rightarrow (i,i+1)\sigma \in S_N^\lambda. \end{cases} \tag{4.4.4}$$

4.3 "Motifs" and associated partitions

1. Following [HHTBP,BPS] introduce the definition:

Definition 2 *Call a sequence of M integers $(m_1, m_2, \ldots, m_M)$ a* motif *iff:*

$$1 \le m_1 < m_2 < \ldots < m_M \le N-1\,, \tag{i}$$
$$m_{i+1} \ge m_i + 2 \quad (i = 1, \ldots, M-1). \tag{ii}$$

With any motif $(m_1, m_2, \ldots, m_M)$ associate a partition μ:

$$\mu = (M, \ldots, \underset{m_1}{M}, M-1, \ldots, \underset{m_2}{M-1}, \ \ldots\ , \underset{m_{M-1}+1}{1}, \ldots, \underset{m_M}{1}, 0, \ldots, \underset{N}{0}).$$

The subscripts in the last equation indicate positions of numbers in the partition.

In what follows we shall identify motifs with the partitions they define. We shall indiscriminately use the notation $(m_1, m_2, \ldots, m_M)$ for both a motif and the corresponding partition. Let $\mathfrak{M}_N$ be the set of all motifs for a fixed N. We use the same notation for the corresponding subset of all partitions.

2. Let $(m_1, m_2, \ldots, m_M)$ be a partition from the set $\mathfrak{M}_N$. We subdivide the set $S_N^{(m_1,m_2,\ldots,m_M)}$ (cf. **1.4**) into disjoint subsets:

Definition 3 *For any subset $\{i_1, i_2, \ldots, i_L\} \subset \{1, 2, \ldots, M\}$ $(0 \le L \le M)$ define $S^{(m_1,m_2,\ldots,m_M)}_{N,(m_{i_1},m_{i_2},\ldots,m_{i_L})} \subset S_N^{(m_1,m_2,\ldots,m_M)} \subset S_N$ as follows:*

$$S^{(m_1,m_2,\ldots,m_M)}_{N,(\emptyset)} := \{\mathrm{id}\},$$
$$\text{for } \ 1 \le L \le M \qquad S^{(m_1,m_2,\ldots,m_M)}_{N,(m_{i_1},m_{i_2},\ldots,m_{i_L})} :=$$
$$\left\{ \sigma \in S_N^{(m_1,m_2,\ldots,m_M)} \ \middle| \ \begin{array}{ll} p^\sigma_{m_{i_k}} > p^\sigma_{m_{i_k}+1} & \forall\, 1 \le k \le L \\ p^\sigma_{m_j} < p^\sigma_{m_j+1} & \forall\, j \in \{1,2,\ldots,M\} \setminus \{i_1, i_2, \ldots, i_L\} \end{array} \right\}$$

Recall (**1.4**) that for $\sigma \in S_N$ we define:

$$\{\sigma_1, \ldots, \sigma_N\} := \sigma.\{1, 2, \ldots, N\}\,, \quad i = \sigma_{p_i^\sigma} \quad (i = 1, \ldots, N).$$

Example Let $N = 4$, $M = 2$ and the motif is: $(m_1, m_2) = (1, 3)$. The corresponding partition is: $(2, 1, 1, 0)$. In this case the set $S_N^{(m_1,m_2)} = S_4^{(1,3)}$ contains altogether twelve elements. This set is subdivided into four subsets: $S^{(1,3)}_{4,(\emptyset)}$, $S^{(1,3)}_{4,(1)}$, $S^{(1,3)}_{4,(3)}$, $S^{(1,3)}_{4,(1,3)}$:

$$S^{(1,3)}_{4,(\emptyset)} = \{\{1234\}\}\,;$$
$$S^{(1,3)}_{4,(1)} = \{\{2134\}, \{2314\}, \{2341\}\}\,;$$
$$S^{(1,3)}_{4,(3)} = \{\{1243\}, \{1423\}, \{4123\}\}\,;$$
$$S^{(1,3)}_{4,(1,3)} = \{\{2143\}, \{2413\}, \{4213\}, \{2431\}, \{4231\}\}.$$

3. Let us describe several properties of the sets $S^{(m_1,m_2,\ldots,m_M)}_{N,(m_{i_1},m_{i_2},\ldots,m_{i_L})}$ $\quad((m_1, m_2, \ldots, m_M) \in \mathfrak{M}_N$, $\{i_1, i_2, \ldots, i_L\} \subset \{1, 2, \ldots, M\}$ $(0 \le L \le M))$. Throughout this paragraph we fix such a set $S^{(m_1,m_2,\ldots,m_M)}_{N,(m_{i_1},m_{i_2},\ldots,m_{i_L})}$. Let μ be the partition that corresponds to $(m_1, m_2, \ldots, m_M)$.

Lemma 3 *Let $\sigma \in S^{(m_1,m_2,\ldots,m_M)}_{N,(m_{i_1},m_{i_2},\ldots,m_{i_L})}$.*

then either:

$$\exists\, i \in \{1, 2, \ldots, N-1\}\,, \sigma\prime \in S^{(m_1,m_2,\ldots,m_M)}_{N,(m_{i_1},m_{i_2},\ldots,m_{i_L})} \quad \text{s.t.} \tag{i}$$
$$\sigma = (i, i+1)\sigma\prime\,,\ \mu_{\sigma_i} < \mu_{\sigma_{i+1}}.$$
$$\text{and therefore } \sigma\prime > \sigma \ (\Leftrightarrow \mu_{\sigma\prime} > \mu_\sigma);$$

or

$$\sigma = \sigma[0] := (m_{i_1}, m_{i_1}+1)(m_{i_2}, m_{i_2}+1)\ldots(m_{i_L}, m_{i_L}+1). \tag{ii}$$

Proof.
Let $\sigma = \{\sigma_1, \sigma_2, \dots, \sigma_N\}$. Examine the pairs σ_i, σ_{i+1} $(i = 1, \dots, N-1)$ step by step starting with $i = 1$ and increasing i by 1 at every next step.
At each step i one has the two possibilities:

$$\text{I.} \quad \mu_{\sigma_i} \geq \mu_{\sigma_{i+1}} \qquad\qquad \text{II.} \quad \mu_{\sigma_i} < \mu_{\sigma_{i+1}}$$

If *I.*, go to the next step. If at each step $(i = 1, \dots, N-1)$ holds *I.*, then $\sigma = \text{id}$ and therefore $L = 0$, $\sigma[0] = \text{id}$. The poof is finished.
If *II.*, then one has the further two possibilities:

$$1.\ (i, i+1)\sigma \notin S^{(m_1, m_2, \dots, m_M)}_{N,(m_{i_1}, m_{i_2}, \dots, m_{i_L})} \qquad\qquad 2.\ (i, i+1)\sigma \in S^{(m_1, m_2, \dots, m_M)}_{N,(m_{i_1}, m_{i_2}, \dots, m_{i_L})}.$$

If *1.* go to the next step. If for all cases where *II.* holds we have *1.*, then $\sigma = \sigma[0]$. The poof is finished.
If *2.* denote $\sigma\prime := (i, i+1)\sigma$. Since $\mu_{\sigma_i} < \mu_{\sigma_{i+1}}$, we have $\mu_{\sigma\prime} > \mu_\sigma \Leftrightarrow \sigma\prime > \sigma$. The poof is finished. ■
The element $\sigma[0] := (m_{i_1}, m_{i_1} + 1)(m_{i_2}, m_{i_2} + 1) \dots (m_{i_L}, m_{i_L} + 1)$
$\in S^{(m_1, m_2, \dots, m_M)}_{N,(m_{i_1}, m_{i_2}, \dots, m_{i_L})}$ is the maximal element in the set $S^{(m_1, m_2, \dots, m_M)}_{N,(m_{i_1}, m_{i_2}, \dots, m_{i_L})}$ (Cf. **1.4.2** for the definition of the ordering of elements of $S^{(m_1, m_2, \dots, m_M)}_N$).
We summarize the properties of the subset $S^{(m_1, m_2, \dots, m_M)}_{N,(m_{i_1}, m_{i_2}, \dots, m_{i_L})}$ in the following proposition:

Proposition 8 *Let* $\{i_1, i_2, \dots, i_L\} \subset \{1, 2, \dots, M\}$, $(0 \leq L \leq M)$.
Then:

$$S^{(m_1, m_2, \dots, m_M)}_{N,(m_{i_1}, m_{i_2}, \dots, m_{i_L})} = S^{(m_{i_1}, \dots, m_{i_L})}_{N,(m_{i_1}, \dots, m_{i_L})} , \tag{i}$$

$$\forall\, \sigma \in S^{(m_1, m_2, \dots, m_M)}_{N,(m_{i_1}, m_{i_2}, \dots, m_{i_L})} \quad \exists\, \{j_1, j_2, \dots, j_r\} \subset \{1, \dots, N-1\} \quad (r \geq 0) \text{ s.t.} \tag{ii}$$

the elements $\sigma[r], \sigma[r-1], \dots, \sigma[1]$ *defined by*

$$\sigma[0] := (m_{i_1}, m_{i_1} + 1)(m_{i_2}, m_{i_2} + 1) \dots (m_{i_L}, m_{i_L} + 1),$$
$$\sigma[k] := (j_k, j_k + 1)\sigma[k-1] \quad (k = 1, 2, \dots, r)$$

belong to the set $S^{(m_1, m_2, \dots, m_M)}_{N,(m_{i_1}, m_{i_2}, \dots, m_{i_L})}$,
satisfy

$$\sigma[k] < \sigma[k-1] \qquad (k = 1, 2, \dots, r),$$

and $\sigma[r] = \sigma$.

(iii)
If there exists $i \in \{1, 2, \dots, N\}$ *such that*

$$\sigma\ ,\ (i, i+1)\sigma \ \in\ S^{(m_1, m_2, \dots, m_M)}_{N,(m_{i_1}, m_{i_2}, \dots, m_{i_L})},$$
$$\textit{then} \quad |\sigma_{i+1} - \sigma_i| \geq 2.$$

If there exists $i \in \{1, 2, \dots, N\}$ *such that*

$$\sigma \ \in\ S^{(m_1, m_2, \dots, m_M)}_{N,(m_{i_1}, m_{i_2}, \dots, m_{i_L})},\ (i, i+1)\sigma \ \in\ S^{(m_1, m_2, \dots, m_M)}_{N,(m_{i_2}, \dots, m_{i_L})},$$
$$\textit{then} \quad \sigma_i \sigma_{i+1} + 1.$$

Proof.
(i) is a direct consequence of the *Definitions 1 (***1.4***) and 3.*
(ii) follows from Lemma 3 and the observation that $\sigma[0] = (m_{i_1}, m_{i_1} + 1)(m_{i_2}, m_{i_2} + 1) \dots (m_{i_L}, m_{i_L} + 1)$ is the maximal element in $S^{(m_1, m_2, \dots, m_M)}_{N,(m_{i_1}, m_{i_2}, \dots, m_{i_L})}$.

(iii) For $\sigma \in S_{N,(m_{i_1},m_{i_2},\ldots,m_{i_L})}^{(m_1,m_2,\ldots,m_M)}$ assume that $\sigma_{i+1} = \sigma_i + 1$. One has the two possibilities:

1. $i = p_j^\sigma$, $i+1 = p_{j+1}^\sigma$ where $m_{s-1}+1 \le j, j+1 \le m_s$ for some $s \in \{1,2,\ldots,M\}$ ($m_0 := 0$).
2. $i = p_{m_s}^\sigma$, $i+1 = p_{m_s+1}^\sigma$ for some $s \in \{1,2,\ldots,M\}$.

In the case *1.* $(i,i+1)\sigma \notin S_N^{(m_1,m_2,\ldots,m_M)}$.
In the case *2.* $(i,i+1)\sigma \in S_{N,(m_s,m_{i_1},\ldots,m_{i_L})}^{(m_1,m_2,\ldots,m_M)}$.
Assume that $\sigma_i = \sigma_{i+1} + 1$.
In this case $\exists\ s \in \{i_1,i_2,\ldots,i_L\}$ *s.t.*

$$\sigma_i = m_s + 1 \ , \ \sigma_{i+1} = m_s \quad \Leftrightarrow \quad i = p_{m_s+1}^\sigma \ , \ i+1 = p_{m_s}^\sigma .$$

Since by definition of $S_{N,(m_{i_1},m_{i_2},\ldots,m_{i_L})}^{(m_1,m_2,\ldots,m_M)}$:

$$i < p_{m_s+2} < \ldots < p_{m_{s+1}} \ ,$$
$$p_{m_{s-1}} < \ldots < p_{m_s-2} < p_{m_s-1} < i+1;$$

we have:

$$(i,i+1)\sigma \in S_{N,(m_{i_1},\ldots,m_{i_L})\backslash m_s}^{(m_1,m_2,\ldots,m_M)} .$$

The first statement in *(iii)* is proven.
The second statement in *(iii)* is proven in a similar way. ■

4.4 A property of the eigenvectors φ_σ^λ for $\lambda \in \mathfrak{M}_N$

1. In this section we shall derive a certain proprerty of the eigenspaces $\mathcal{E}^\mu$ ($\mu \in \mathfrak{M}_N$) of the operators $\Delta_1(u), y_i (i = 1,\ldots,N)$. First of all, we notice that the eigenvalue $\delta_1^\mu(u)$ of $\Delta_1(u)$ associated with $\mathcal{E}^\mu$ can be represented in the additive "particle" form:

$$\delta_1^\mu(u) := \delta_1^{(m_1,m_2,\ldots,m_M)}(u) = \sum_{k=1}^{M} \delta_1^{(m_k)}(u), \tag{4.4.5}$$

where $\delta_1^{(m)}(u)$ $m \in \{1,\ldots,N-1\}$ is the one-particle eigenvalue:

$$\delta_1^{(m)}(u) := u \sum_{i=1}^{m} \prod_{\substack{1\le j\le N \\ j\ne i}} (1+uq^{l_j}) q^{l_i}. \tag{4.4.6}$$

(Cf. 2.2.26 for the definition of l_i).
2. Let $\mu = (m_1,m_2,\ldots,m_M) \in \mathfrak{M}_N$ then the conjugate partition $\mu\prime$ is: $\mu\prime = (m_M,m_{M-1},\ldots,m_1)$. Due to (2.2.9,.10) we have:

$$\Delta_1(u) e_{m_1} e_{m_2} \ldots e_{m_M} = \left(\sum_{k=1}^{M} \delta_1^{(m_k)}(u) \right) e_{m_1} e_{m_2} \ldots e_{m_M} . \tag{4.4.7}$$

While for any symmetric polynomial S one has:

$$y_i S = q^{l_i} S \quad (i = 1,\ldots,N). \tag{4.4.8}$$

Now we have the following proposition:

Proposition 9 *Let* $(m_1,m_2,\dots,m_M)\in\mathfrak{M}_N\,,\ \{i_1,i_2,\dots,i_L\}\subset\{1,2,\dots,M\}$ $(0\le L\le M)$, *and* $\{j_1,\dots,j_{M-L}\}=\{1,2,\dots,M\}\setminus\{i_1,i_2,\dots,i_L\}$.
Then:

$$\varphi_\sigma^{(m_1,m_2,\dots,m_M)}\ (\sigma\in S^{(m_1,m_2,\dots,m_M)}_{N,(m_{i_1},m_{i_2},\dots,m_{i_L})})\ =\varphi_\sigma^{(m_{i_1},\dots,m_{i_L})}e_{m_{j_1}}e_{m_{j_2}}\dots e_{m_{j_{M-L}}}. \tag{4.4.9}$$

Proof. Compute the action of the operators $\Delta_1(u), y_i\ (i=1,\dots,N)$. on the polynomial $\varphi_\sigma^{(m_{i_1},\dots,m_{i_L})}e_{m_{j_1}}e_{m_{j_2}}\dots e_{m_{j_{M-L}}}$.

$$y_i\varphi_\sigma^{(m_{i_1},\dots,m_{i_L})}S=q^{l_{\sigma_i}}\varphi_\sigma^{(m_{i_1},\dots,m_{i_L})}S\quad(i=1,\dots,N). \tag{a}$$

For any symmetric polynomial S.

$$\begin{aligned}\Delta_1(u)\varphi_\sigma^{(m_{i_1},\dots,m_{i_L})}e_{m_{j_1}}e_{m_{j_2}}\dots e_{m_{j_{M-L}}}&=\\ &=[\Delta_1(u)\,,\,\varphi^{(m_{i_1},\dots,m_{i_L})}]e_{m_{j_1}}e_{m_{j_2}}\dots e_{m_{j_{M-L}}}+\\ &\quad+\varphi^{(m_{i_1},\dots,m_{i_L})}\Delta_1(u)e_{m_{j_1}}e_{m_{j_2}}\dots e_{m_{j_{M-L}}}.\end{aligned} \tag{b}$$

Due to the special form of $\Delta_1(u)$ (Cf. *Proposition 4*) the commutator in the last formula is a zero-order differential operator. Therefore if S is a symmetric polynomial we have:

$$[\Delta_1(u)\,,\,\varphi^{(m_{i_1},\dots,m_{i_L})}]S=f(u)S, \tag{4.4.10}$$

where $f(u)$ is a function of $z_i\ (i=1,\dots,N)$ independent of S. Furthermore:

$$\Delta_1(u)1=0,$$

and therefore:

$$\begin{aligned}[\Delta_1(u)\,,\,\varphi^{(m_{i_1},\dots,m_{i_L})}]1=\Delta_1(u)\varphi^{(m_{i_1},\dots,m_{i_L})}&=\\ =\sum_{1\le k\le L}^{M}\delta_1^{(m_{i_k})}(u)\varphi^{(m_{i_1},\dots,m_{i_L})}&.\end{aligned}$$

Taking *(b),(4.4.7,.11)* into account we get:

$$\begin{aligned}\Delta_1(u)\varphi_\sigma^{(m_{i_1},\dots,m_{i_L})}e_{m_{j_1}}e_{m_{j_2}}\dots e_{m_{j_{M-L}}}&=\\ =\left(\sum_{1\le k\le L}\delta^{(m_{i_k})}+\sum_{1\le s\le M-L}\delta^{(m_{j_s})}\right)\varphi_\sigma^{(m_{i_1},\dots,m_{i_L})}e_{m_{j_1}}e_{m_{j_2}}\dots e_{m_{j_{M-L}}}&=\\ =\sum_{1\le i\le M}\delta^{(m_i)}\varphi_\sigma^{(m_{i_1},\dots,m_{i_L})}e_{m_{j_1}}e_{m_{j_2}}\dots e_{m_{j_{M-L}}}&.\end{aligned}$$

Since the joint spectrum of $\Delta_1(u)\,,\,y_i\ (i=1,\dots,N)$ is multiplicity-free, we conclude from *(a)* and the last equation that:

$$\varphi_\sigma^{(m_{i_1},\dots,m_{i_L})}e_{m_{j_1}}e_{m_{j_2}}\dots e_{m_{j_{M-L}}}=\mathrm{const}\,\varphi_\sigma^{(m_1,m_2,\dots,m_M)}\quad(\sigma\in S^{(m_1,m_2,\dots,m_M)}_{N,(m_{i_1},m_{i_2},\dots,m_{i_L})}). \tag{4.4.11}$$

By comparison of the maximal monomials in the both sides of the equation $const=1$. ■

Since $e_r(z_1=\omega^1,\dots,z_N=\omega^N)=0\quad(1\le r\le N-1)$ we have the following corollary to Proposition 9:

Corollary 1 *Let* $(m_1,m_2,\dots,m_M)\in\mathfrak{M}_N\,,\ M>0\,,\ \sigma\in S_N^{(m_1,m_2,\dots,m_M)}$.
Then:

$$\varphi_\sigma^{(m_1,m_2,\dots,m_M)}(z_1=\omega^1,\dots,z_N=\omega^N)=0\qquad\text{unless}\quad\sigma\in S^{(m_1,m_2,\dots,m_M)}_{N,(m_1,m_2,\dots,m_M)}.$$

4.5 Limit $q \to 0$ of the eigenfunctions of $\Delta_1(u), y_i \quad (i = 1, \dots, N)$

1. The aim of this subsection is to compute some of the eigenfunctions $\varphi_\sigma^\lambda \quad (\lambda \in \mathfrak{M}_N, \sigma \in S_N^\lambda)$ in the limit $q \to 0$ *under assumption* that this limit is well-defined. We do not have a proof of the last statement. In particular examples where N is small this statement holds.

Introduce operators:

$$\gamma_{i,j} := q g_{i,j}|_{q=0} = \frac{z_i}{z_i - z_j}(K_{i,j} - 1) \quad (i \neq j \in \{1, \dots, N\}). \tag{4.4.12}$$

Lemma 4 *Let* $(m_1, m_2, \dots, m_M) \in \mathfrak{M}_N$*, and* $\psi \in \mathbb{C}[z_1, \dots, z_N]$ *satisfies the equations:*

$$(\gamma_{m_i, m_i+1} + 1)\psi = 0 \quad (i = 1, \dots, M), \tag{4.4.13}$$

$$\gamma_{n,n+1}\psi = 0 \quad (n \in \{1, \dots, N-1\} \setminus \{m_1, \dots, m_M\}). \tag{4.4.14}$$

Then

$$\psi = (z_1 \dots z_{m_1})^M (z_{m_1+1} \dots z_{m_2})^{M-1} \dots (z_{m_{M-1}+1} \dots z_{m_M}) S(z_1, \dots, z_N), \tag{4.4.15}$$

where S is a symmetric polynomial.

Proof. Let $N = 2$, $\psi = \psi(z_1, z_2)$.

The equation $(\gamma_{1,2} + 1)\psi = 0$ implies that

$$\frac{z_1\psi(z_2, z_1) - z_2\psi(z_1, z_2)}{z_1 - z_2} = 0. \tag{4.4.16}$$

This leads to $\psi(z_1, z_2) = z_1 S(z_1, z_2)$, where S is a symmetric polynomial.

The equation

$$\gamma_{1,2}\psi = \frac{z_1(\psi(z_2, z_1) - \psi_(z_1, z_2))}{z_1 - z_2} = 0, \tag{4.4.17}$$

yields $\psi(z_1, z_2) = S(z_1, z_2)$ where S is a symmetric polynomial.

The case of arbitrary N reduces to the case $N = 2$ by consideration of conseqtive pairs of coordinates. ■

2. We *conjecture* that the limit $q \to 0$ of φ_σ^λ is well-defined:

$$\varphi_\sigma^\lambda|_{q=0} := \varphi_\sigma^{\lambda,0} = z^{\lambda_\sigma} + \text{“s.m.”} \qquad (\lambda \in \mathfrak{M}_N,\ \sigma \in S_N^\lambda). \tag{4.4.18}$$

In what follows we assume that the statement of the conjecture is valid.

Fix $(m_1, m_2, \dots, m_M) \in \mathfrak{M}_N$. Our purpose is to find $\varphi_{\sigma[0]}^{(m_1,m_2,\dots,m_M),0}$ for $\sigma[0] := (m_1, m_1+1) \dots (m_M, m_M+1)$. Take the limit $q \to 0$ in the eq. (4.4.4) of Proposition 9. This yields

$$(\gamma_{m_i,m_i+1} + 1)\varphi_{\sigma[0]}^{(m_1,m_2,\dots,m_M),0} = \varphi_{(m_i,m_i+1)\sigma[0]}^{(m_1,m_2,\dots,m_M),0} \quad (i = 1, \dots, M), \tag{4.4.19}$$

$$\gamma_{n,n+1}\varphi_{\sigma[0]}^{(m_1,m_2,\dots,m_M),0} = 0 \quad (n \in \{1, \dots, N-1\} \setminus \{m_1, \dots, m_M\}). \tag{4.4.20}$$

According to Proposition 12, we have

$$\varphi_{(m_i,m_i+1)\sigma[0]}^{(m_1,m_2,\dots,m_M),0} = \varphi_{(m_1,m_1+1)\dots\widehat{(m_i,m_i+1)}\dots(m_M,m_M+1)}^{(m_1,m_2,\dots,m_M),0} = \varphi_{(m_1,m_1+1)\dots\widehat{(m_i,m_i+1)}\dots(m_M,m_M+1)}^{(m_1,\dots,\hat{m}_i,\dots,m_M),0} e_{m_i} \quad (i = 1, \dots, M). \tag{4.4.21}$$

Where we put a hat over terms that are omitted.

The pair of equations (4.4.20,.21) provides a set of recurrent relations for the eigenfunctions $\varphi_{\sigma[0]}^{(m_1,m_2,\dots,m_M),0}$. Notice that when $M = 0$ we have $\varphi_{\sigma[0]}^{(\emptyset),0} = 1$. Taking into account Lemma 4 we write the general solution of these recurrent relations:

$$\varphi_{\sigma[0]}^{(\emptyset),0} = 1, \tag{4.4.22}$$

$$\begin{aligned}\varphi_{\sigma[0]}^{(m_1,m_2,\dots,m_M),0} &= (e_{m_1} - z_1 \dots z_{m_1})(e_{m_2} - z_1 \dots z_{m_2}) \dots (e_{m_M} - z_1 \dots z_{m_M}) + \\ &+ (z_1 \dots z_{m_1})^M (z_{m_1+1} \dots z_{m_2})^{M-1} \dots (z_{m_{M-1}+1} \dots z_{m_M}) \times \\ &\times S^{(m_1,m_2,\dots,m_M)}(z_1, \dots, z_N) \quad (M \geq 1).\end{aligned}$$

Where $S^{(m_1,m_2,\ldots,m_M)}(z_1,\ldots,z_N)$ is an arbitrary symmetric polynomial.

Observe now that the total degree of the homogeneous polynomial $\varphi_{\sigma[0]}^{(m_1,m_2,\ldots,m_M),0}$ is equal to $m_1+\cdots+m_M \quad ((m_1,m_2,\ldots,m_M)\in\mathfrak{M}_N)$. Therefore we must have $S^{(m_1,m_2,\ldots,m_M)}(z_1,\ldots,z_N)=\text{const}$. Observe further that

$$(z_1\ldots z_{m_1})^M(z_{m_1+1}\ldots z_{m_2})^{M-1}\ldots(z_{m_{M-1}+1}\ldots z_{m_M}) >$$
$$> \max(\varphi_{\sigma[0]}^{(m_1,m_2,\ldots,m_M),0}) \quad (\sigma[0]:=(m_1,m_1+1)\ldots(m_M,m_M+1)). \quad (4.4.23)$$

Where $\max(\varphi_{\sigma[0]}^{(m_1,m_2,\ldots,m_M),0})$ is the maximal monomial of $\varphi_{\sigma[0]}^{(m_1,m_2,\ldots,m_M),0}$. Therefore we must have: $S^{(m_1,m_2,\ldots,m_M)}(z_1,\ldots,z_N)=0$.

Thus for $(m_1,m_2,\ldots,m_M)\in\mathfrak{M}_N$:

$$\varphi_{\sigma[0]}^{(\emptyset),0}=1,$$
$$\varphi_{\sigma[0]}^{(m_1,m_2,\ldots,m_M),0}=(e_{m_1}-z_1\ldots z_{m_1})(e_{m_2}-z_1\ldots z_{m_2})\ldots(e_{m_M}-z_1\ldots z_{m_M})$$
$$(M\geq 1)$$

Notice that

$$\varphi_{\sigma[0]}^{(m_1,m_2,\ldots,m_M),0}(z_1=\omega^1,\ldots,z_N=\omega^N)=(-1)^M\omega^{\frac{1}{2}\sum_{i=1}^M m_i(m_i+1)},$$
$$((m_1,m_2,\ldots,m_M)\in\mathfrak{M}_N,\ \sigma[0]=(m_1,m_1+1)\ldots(m_M,m_M+1)). \quad (4.4.25)$$

5 Hecke-invariant ("bosonic") subspaces of $\mathcal{E}^\lambda\otimes H$ for $\lambda\in\mathfrak{M}_N$

5.1 Preliminaries

1. Let $\mathcal{E}^\lambda\subset\mathbb{C}[z_1,\ldots,z_N]$ be the eigenspace of the operators $\Delta_1(u)$, $y_i \quad (i=1,\ldots,N)$ parametrized by a partition λ and $H:=(\mathbb{C}^2)^{\otimes N}$. The bosonic subspace $\mathcal{B}^\lambda$ of $\mathcal{E}^\lambda\otimes H$ is defined as follows:

$$B^\lambda:=\{b\in\mathcal{E}^\lambda\otimes H|(g_{i,i+1}-t_{i,i+1})b=0 \quad (i=1,\ldots,N-1)\}. \quad (5.5.1)$$

Since $\mathcal{P}:=\mathbb{C}[z_1,\ldots,z_N]\otimes H=\oplus_\lambda(\mathcal{E}^\lambda\otimes H)$ and $g_{i,i+1}:\mathcal{E}^\lambda\mapsto\mathcal{E}^\lambda \quad (i=1,\ldots,N-1)$; we have: $\mathcal{B}=\oplus_\lambda\mathcal{B}^\lambda$. (Cf. (1.1.36), (3.3.7) for the definition of $\mathcal{B}$).

2. Any vector ψ from $\mathcal{E}^\lambda\otimes H$ is represented as follows:

$$\psi=\sum_{\sigma\in S_N^\lambda}\varphi_\sigma^\lambda\chi_\sigma, \quad (5.5.2)$$

where $\chi_\sigma \quad (\sigma\in S_N^\lambda)\in H$.

The condition $(g_{i,i+1}-t_{i,i+1})\psi=0 \quad (i=1,\ldots,N-1)$ gives a set of linear equations which must be satisfied by the vectors $\chi_\sigma \quad (\sigma\in S_N^\lambda)\in H$. In order to derive these equations we can apply the result of Proposition 9 to find out the action of $g_{i,i+1} \quad (i=1,\ldots,N-1)$ on ψ, and then use the linear-independence of the polynomials $\varphi_\sigma^\lambda \quad (\sigma\in S_N^\lambda)$. In this way we arrive at the following proposition:

Proposition 10 *A vector $\psi\in\mathcal{E}^\lambda\otimes H$; $\psi=\sum_{\sigma\in S_N^\lambda}\varphi_\sigma^\lambda\chi_\sigma$ belongs to $\mathcal{B}^\lambda$ iff $\chi_\sigma \quad (\sigma\in S_N^\lambda)\in H$ satisfy the following set of equations $(i=1,\ldots,N-1)$:*

$$q\frac{(q^{-1}q^{l_{\sigma_{i+1}}}-qq^{l_{\sigma_i}})(q^{-1}q^{l_{\sigma_i}}-qq^{l_{\sigma_{i+1}}})}{q^{l_{\sigma_i}}-q^{l_{\sigma_{i+1}}}}\chi_{(i,i+1)\sigma}= \quad (a)$$
$$=\left((q^{l_{\sigma_{i+1}}}-q^{l_{\sigma_i}})t_{i,i+1}-(q-q^{-1})q^{l_{\sigma_{i+1}}}\right)\chi_\sigma$$
$$\text{when} \quad \lambda_{\sigma_{i+1}}>\lambda_{\sigma_i}\ ,$$
$$(t_{i,i+1}-q)\chi_\sigma=0 \quad (b)$$
$$\text{when} \quad \lambda_{\sigma_{i+1}}=\lambda_{\sigma_i}\Leftrightarrow(i,i+1)\sigma\notin S_N^\lambda\Rightarrow\sigma_{i+1}=\sigma_i+1\ ,$$
$$\chi_{(i,i+1)\sigma}=\frac{(q^{l_{\sigma_{i+1}}}-q^{l_{\sigma_i}})t_{i,i+1}-(q-q^{-1})q^{l_{\sigma_{i+1}}}}{q^{-1}(q^{l_{\sigma_{i+1}}}-q^{l_{\sigma_i}})}\chi_\sigma \quad (c)$$
$$\text{when} \quad \lambda_{\sigma_{i+1}}<\lambda_{\sigma_i}\ .$$

5.2 Spaces $\mathcal{B}^\mu$ for $\mu \in \mathfrak{M}_N$

1. Let us fix a partition $\mu \in \mathfrak{M}_N$ parametrized by a motif $(m_1, m_2, \ldots, m_M)$. We use the same notation $(m_1, m_2, \ldots, m_M)$ for both the motf and the partition.
For $\mu \in \mathfrak{M}_N$ let us further analyse the equations (a)-(c) obtained in the Proposition 10. In section **4.3** (Definition 3) we introduced the decomposition of the set $S_N^{(m_1,m_2,\ldots,m_M)}$ into disjoint subsets:

$$S_N^{(m_1,m_2,\ldots,m_M)} = \bigsqcup_{\{i_1,i_2,\ldots,i_L\}\in\{1,2,\ldots,M\}} S_{N,(m_{i_1},m_{i_2},\ldots,m_{i_L})}^{(m_1,m_2,\ldots,m_M)} \qquad (0 \le L \le M) \tag{5.5.3}$$

This decomposition is reflected in the equations (a)-(c) of the Proposition 10.
2. Let in these equations σ, i be such that σ, $(i, i+1)\sigma \in S_{N,(m_{i_1},m_{i_2},\ldots,m_{i_L})}^{(m_1,m_2,\ldots,m_M)}$ for some $\{i_1, i_2, \ldots, i_L\} \in \{1, 2, \ldots, M\}$. According to Proposition 8,(iii) $|\sigma_i - \sigma_{i+1}| \ge 2$.

Let $\sigma_i - \sigma_{i+1} \ge 2$ and consequently $\mu_{\sigma_i} < \mu_{\sigma_{i+1}}$. Since $\sigma_i - \sigma_{i+1} \ge 2$ the coefficient in front of $\chi_{(i,i+1)\sigma}$ in Pr.10(a) is not equal to zero. Therefore we have:

$$\chi_{(i,i+1)\sigma} = \frac{(q^{l_{\sigma_i}} - q^{l_{\sigma_{i+1}}})((q^{l_{\sigma_{i+1}}} - q^{l_{\sigma_i}})t_{i,i+1} - (q-q^{-1})q^{l_{\sigma_{i+1}}})}{q(q^{-1}q^{l_{\sigma_{i+1}}} - qq^{l_{\sigma_i}})(q^{-1}q^{l_{\sigma_i}} - qq^{l_{\sigma_{i+1}}})}\chi_\sigma$$
$$(\sigma, (i, i+1)\sigma \in S_{N,(m_{i_1},m_{i_2},\ldots,m_{i_L})}^{(m_1,m_2,\ldots,m_M)}\,,\ \sigma_i - \sigma_{i+1} \ge 2). \tag{5.5.4}$$

Let $\sigma_i - \sigma_{i+1} \le -2$ and consequently $\mu_{\sigma_i} > \mu_{\sigma_{i+1}}$. Pr.10(c) gives:

$$\chi_{(i,i+1)\sigma} = \frac{((q^{l_{\sigma_{i+1}}} - q^{l_{\sigma_i}})t_{i,i+1} - (q-q^{-1})q^{l_{\sigma_{i+1}}})}{q^{-1}(q^{l_{\sigma_{i+1}}} - q^{l_{\sigma_i}})}\chi_\sigma$$
$$(\sigma, (i, i+1)\sigma \in S_{N,(m_{i_1},m_{i_2},\ldots,m_{i_L})}^{(m_1,m_2,\ldots,m_M)}\,,\ \sigma_i - \sigma_{i+1} \le -2). \tag{5.5.5}$$

Introduce a pair of U-intertwiners:

$$Y^\pm(z) := \varrho^\pm(z)\frac{zt - t^{-1}}{q^{-1}z - q} \ \in End(V \otimes V), \tag{5.5.6}$$

where

$$\varrho^+(z) := \frac{z-1}{q^2z-1}\,, \qquad \varrho^-(z) := \frac{z-q^2}{z-1}. \tag{5.5.7}$$

The eq. (5.5.4,.5) can be reformulated as follows: Let $\sigma, (i, i+1)\sigma \in S_{N,(m_{i_1},m_{i_2},\ldots,m_{i_L})}^{(m_1,m_2,\ldots,m_M)}$ $(\{i_1, i_2, \ldots, i_L\} \in \{1, 2, \ldots, M\} \quad (0 \le L \le M))$ then:

$$\chi_{(i,i+1)\sigma} = \begin{cases} Y^+_{i,i+1}(q^{l_{\sigma_i}-l_{\sigma_{i+1}}})\chi_\sigma & \text{when } \sigma_i - \sigma_{i+1} \ge 2 \Rightarrow \mu_{\sigma_i} < \mu_{\sigma_{i+1}}\ , \\ Y^-_{i,i+1}(q^{l_{\sigma_i}-l_{\sigma_{i+1}}})\chi_\sigma & \text{when } \sigma_i - \sigma_{i+1} \le -2 \Rightarrow \mu_{\sigma_i} > \mu_{\sigma_{i+1}}. \end{cases} \tag{5.5.8}$$

Notice that when $\sigma, (i, i+1)\sigma \in S_{N,(m_{i_1},m_{i_2},\ldots,m_{i_L})}^{(m_1,m_2,\ldots,m_M)}$ the intertwiners $Y^\pm_{i,i+1}(q^{l_{\sigma_i}-l_{\sigma_{i+1}}})$ are invertible.

3. Now consider the situation when σ and $(i, i+1)\sigma$ in Proposition 10 belong to different subsets $S_{N,(m_{i_1},m_{i_2},\ldots,m_{i_L})}^{(m_1,m_2,\ldots,m_M)}$.
According to Proposition 8 this situation takes place when $|\sigma_i - \sigma_{i+1}| = 1$.
Let $\sigma_{i+1} = \sigma_i + 1$ and consequently $\mu_{\sigma_i} > \mu_{\sigma_{i+1}}$. In this case $\exists\ s \in \{1, 2, \ldots, M\}$ s.t. $\sigma_i = m_s$, $\sigma_{i+1} = m_s + 1$ or , equivalently , $i = p^\sigma_{m_s}$, $i + 1 = p^\sigma_{m_s+1}$; and if $\sigma \in S_{N,(m_{i_1},m_{i_2},\ldots,m_{i_L})}^{(m_1,m_2,\ldots,m_M)}$ then $s \notin \{i_1, i_2, \ldots, i_L\}$. Since $((i, i+1)\sigma)_i = m_s + 1$, $((i, i+1)\sigma)_{i+1} = m_s$; we have $(i, i+1)\sigma \in S_{N,(m_s,m_{i_1},\ldots,m_{i_L})}^{(m_1,m_2,\ldots,m_M)}$.
Substituting $\sigma_{i+1} = \sigma_i + 1$ into Pr.10(c) we find:

$$\chi_{(i,i+1)\sigma} = q(t_{i,i+1} - q)\chi_\sigma = -(q^2 + 1)\Pi^-_{i,i+1}(q)\chi_\sigma$$
$$(\sigma \in S_{N,(m_{i_1},m_{i_2},\ldots,m_{i_L})}^{(m_1,m_2,\ldots,m_M)}\,,\ (i, i+1)\sigma \in S_{N,(m_s,m_{i_1},\ldots,m_{i_L})}^{(m_1,m_2,\ldots,m_M)}\,,$$
$$\sigma_i = m_s, \sigma_{i+1} = m_s + 1(s \in \{1, 2, \ldots, M\} \setminus \{i_1, i_2, \ldots, i_L\})). \tag{5.5.9}$$

(Cf. (1.1.12,.13) for the definition of projectors $\Pi^{\pm}_{i,i+1}(q)$).

Let $\sigma_i = \sigma_{i+1}+1$ and consequently $\mu_{\sigma_i} < \mu_{\sigma_{i+1}}$. In this case $\exists\, s \in \{1,2,\ldots,M\}$ s.t. $\sigma_i = m_s+1$, $\sigma_{i+1} = m_s$; and since $\sigma \in S^{(m_1,m_2,\ldots,m_M)}_{N,(m_{i_1},m_{i_2},\ldots,m_{i_L})}$; $s \in \{i_1,i_2,\ldots,i_L\}$. On the other hand $(i,i+1)\sigma \in S^{(m_1,m_2,\ldots,m_M)}_{N,(m_{i_1},\ldots,m_{i_L})\setminus m_s}$.
Here $(m_{i_1},\ldots,m_{i_L})\setminus m_s$ signifies the motif obtained from $(m_{i_1},\ldots,m_{i_L})$ by removing m_s.
Substituting $\sigma_i = \sigma_{i+1}+1$ into Pr.10(a) we find:

$$(t_{i,i+1}+q^{-1})\chi_\sigma = 0 \quad \Rightarrow \quad \Pi^{+}_{i,i+1}(q)\chi_\sigma = 0$$
$$(\sigma \in S^{(m_1,m_2,\ldots,m_M)}_{N,(m_{i_1},m_{i_2},\ldots,m_{i_L})}\ ,\ (i,i+1)\sigma \in S^{(m_1,m_2,\ldots,m_M)}_{N,(m_{i_1},\ldots,m_{i_L})\setminus m_s}\ ,$$
$$\sigma_i = m_s+1, \sigma_{i+1} = m_s (s \in \{i_1,i_2,\ldots,i_L\} \subset \{1,2,\ldots,M\})). \quad (5.5.10)$$

4. Now we are in position to reformulate Propositon 10 in a more suitable form in the case when $\lambda \equiv \mu \equiv (m_1,m_2,\ldots,m_M) \in \mathfrak{M}_N$.

Proposition 11 *Let $\psi \in \mathcal{E}^\mu \otimes H$, i.e.*

$$\psi = \sum_{\sigma \in S^{(m_1,m_2,\ldots,m_M)}_N} \varphi^{(m_1,m_2,\ldots,m_M)}_\sigma \chi_\sigma \qquad (\chi_\sigma \in H). \quad (5.5.11)$$

Then $\psi \in \mathcal{B}^{(m_1,m_2,\ldots,m_M)}$ iff χ_σ $(\sigma \in S^{(m_1,m_2,\ldots,m_M)}_N)$ satisfy the following linear relations:

$$\forall\, \{i_1,i_2,\ldots,i_L\} \subset \{1,2,\ldots,M\} \quad (0 \le L \le M)$$
$$\chi_{(m_{i_1},m_{i_1}+1)\ldots(m_{i_L},m_{i_L}+1)} =$$
$$= -(q^2+1)\Pi^{-}_{m_{i_k},m_{i_k}+1}(q)\chi_{(m_{i_1},m_{i_1}+1)\ldots\widehat{(m_{i_k},m_{i_k}}+1)\ldots(m_{i_L},m_{i_L}+1)}$$
$$(k = 1,2,\ldots,L). \quad (5.5.12)$$

Where $\widehat{\ }$ means that the corresponding factor is omitted from the product.

$$\forall\, \{i_1,i_2,\ldots,i_L\} \subset \{1,2,\ldots,M\} \quad (0 \le L \le M)$$
$$\text{and } j \in \{1,\ldots,N-1\} \text{ s.t.}$$
$$\{j,j+1\} \cap \{m_1,m_1+1,m_2,m_2+1,\ldots,m_M,m_M+1\} = \emptyset\ ;$$
$$\Pi^{-}_{j,j+1}(q)\chi_{(m_{i_1},m_{i_1}+1)\ldots(m_{i_L},m_{i_L}+1)} = 0. \quad (5.5.13)$$

$$\forall\, \sigma \in S^{(m_1,m_2,\ldots,m_M)}_{N,(m_{i_1},m_{i_2},\ldots,m_{i_L})} \quad (\{i_1,i_2,\ldots,i_L\} \subset \{1,2,\ldots,M\} \quad (0 \le L \le M)) \quad (5.5.14)$$
$$\chi_\sigma = \mathbb{Y}(\sigma)\chi_{(m_{i_1},m_{i_1}+1)\ldots(m_{i_L},m_{i_L}+1)}.$$

Where invertible $\mathbb{Y}(\sigma) \in End(H)$ is recursively defined as follows:

$$\mathbb{Y}((m_{i_1},m_{i_1}+1)\ldots(m_{i_L},m_{i_L}+1)) := \mathrm{Id}\,,$$
$$\text{for } (i,i+1)\sigma \in S^{(m_1,m_2,\ldots,m_M)}_{N,(m_{i_1},m_{i_2},\ldots,m_{i_L})}$$
$$\mathbb{Y}((i,i+1)\sigma) = \begin{cases} Y^{+}_{i,i+1}(q^{l_{\sigma_i}-l_{\sigma_{i+1}}})\mathbb{Y}(\sigma) & \text{if } \sigma_i - \sigma_{i+1} \ge 2\,, \\ Y^{-}_{i,i+1}(q^{l_{\sigma_i}-l_{\sigma_{i+1}}})\mathbb{Y}(\sigma) & \text{if } \sigma_i - \sigma_{i+1} \le -2\,. \end{cases}$$

It is possible to give more explicit expression for the matrix $\mathbb{Y}(\sigma)$ that appears in (5.5.14). In notation of Proposition 8 (ii) we have:

$$\text{For } \sigma \in S^{(m_1,m_2,\ldots,m_M)}_{N,(m_{i_1},m_{i_2},\ldots,m_{i_L})} \qquad (\{i_1,i_2,\ldots,i_L\} \subset \{1,2,\ldots,M\}\ (0 \le L \le M))$$
$$\mathbb{Y}(\sigma) = Y^{-}_{j_r,j_r+1}(q^{l_{\sigma[r-1]_{j_r}}-l_{\sigma[r-1]_{j_r+1}}})\ldots Y^{-}_{j_1,j_1+1}(q^{l_{\sigma[0]_{j_1}}-l_{\sigma[0]_{j_1+1}}}). \quad (5.5.15)$$

Where:

$$\begin{aligned}
&\sigma[k] := (j_k, j_k+1)\dots(j_1, j_1+1)\sigma[0] \in S^{(m_1,m_2,\dots,m_M)}_{N,(m_{i_1},m_{i_2},\dots,m_{i_L})} \ (1\le k\le r),\\
&\sigma[0] := (m_{i_1}, m_{i_1}+1)\dots(m_{i_L}, m_{i_L}+1),\\
&\sigma[r] = \sigma\\
&\sigma[k-1]_{j_k} - \sigma[k-1]_{j_k+1} \le -2 \quad (1\le k\le r).
\end{aligned} \tag{5.5.16}$$

5. Introduce two definitions.
For $(m_1, m_2, \dots, m_M) \in \mathfrak{M}_N$:

$$\mathcal{Z}^{(m_1,m_2,\dots,m_M)} := \left\{ v\in H \,\middle|\, \begin{matrix} \Pi^-_{j,j+1}(q)v = 0 \quad \text{for all} \quad j\in\{1,\dots,N-1\} \text{ s.t.} \\ \{j,j+1\}\cap\{m_1, m_1+1,\dots,m_M,m_M+1\} = \emptyset \end{matrix} \right\} \tag{5.5.17}$$

More explicitely:

$$\mathcal{Z}^{(m_1,m_2,\dots,m_M)} = S_q(\underset{1}{V}\otimes\dots\otimes\underset{m_1-1}{V})\otimes\underset{m_1}{V}\otimes\underset{m_1+1}{V}\otimes S_q(\underset{m_1+2}{V}\otimes\dots\otimes\underset{m_2-1}{V})\otimes\underset{m_2}{V}\otimes\underset{m_2+1}{V}\otimes$$
$$\dots\otimes S_q(\underset{m_M+2}{V}\otimes\dots\otimes\underset{N}{V}) \;\subset\; H := \underset{1}{V}\otimes\underset{2}{V}\otimes\dots\otimes\underset{N}{V}.$$

Where S_q means q-symmetrization and the subscripts indicate positions of the factors in the tensor product $V^{\otimes N}$.

For any $\{i_1, i_2, \dots, i_L\} \in \{1, 2, \dots, M\}$ $\quad (0 \le L \le M)$ define the following projector:

$$\Pi^{-,(m_1,m_2,\dots,m_M)}_{(m_{i_1},\dots,m_{i_L})} := \begin{cases} \Pi^-_{m_{i_1},m_{i_1}+1}(q)\dots\Pi^-_{m_{i_L},m_{i_L}+1}(q) & (1\le L\le M),\\ \mathrm{I} & (L=0). \end{cases} \tag{5.5.18}$$

6. Proposition 11 yields the following expression for $\mathcal{B}^\mu \equiv \mathcal{B}^{(m_1,m_2,\dots,m_M)}$:

$$\mathcal{B}^{(m_1,m_2,\dots,m_M)} = \mathbb{U}^{(m_1,m_2,\dots,m_M)}(z)\mathcal{Z}^{(m_1,m_2,\dots,m_M)}. \tag{5.5.19}$$

Where

$$\mathbb{U}^{(m_1,m_2,\dots,m_M)}(z) :=$$
$$\sum_{\substack{I\subset\{1,2,\dots,M\}\\ I:=\{i_1,i_2,\dots,i_L\}}} (-(q^2+1))^L \left\{ \sum_{\sigma\in S^{(m_1,m_2,\dots,m_M)}_{N,(m_{i_1},m_{i_2}\dots,m_{i_L})}} \varphi^{(m_1,m_2,\dots,m_M)}_\sigma(z)\mathbb{Y}(\sigma) \right\} \Pi^{-,(m_1,m_2,\dots,m_M)}_{(m_{i_1},\dots,m_{i_L})},$$
$$\mathbb{U}^{(m_1,m_2,\dots,m_M)}(z) : H \mapsto \mathbb{C}[z_1,\dots,z_N]\otimes H. \tag{5.5.20}$$

In the last formula we explicitely indicated z-dependence of the polynomials $\varphi^{(m_1,m_2,\dots,m_M)}_\sigma$.

7. The space $\mathcal{B}^{(m_1,m_2,\dots,m_M)}$ is a U-module where the action of U is given by (2.2.19):

$$T^0_a(u) = L_a(u;\{y_i\}) := L_{a1}(uy_1)L_{a2}(uy_2)\dots L_{aN}(uy_N) \;\in\; End(\mathbb{C}[z_1,\dots,z_N]\otimes H). \tag{5.5.21}$$

The space $\mathcal{Z}^{(m_1,m_2,\dots,m_M)}$ is a (reducible,indecomposable) U-module as well, with the U-action:

$$L_a(u;\{q^{l_i}\}) := L_{a1}(uq^{l_1})L_{a2}(uq^{l_2})\dots L_{aN}(uq^{l_N}) \;\in\; End(H). \tag{5.5.22}$$

The operator $\mathbb{U}^{(m_1,m_2,\dots,m_M)}(z)$ in (5.5.19,.20) is an U-intertwiner of these two modules. To see this let us consider the product $T^0_a(u)\mathbb{U}^{(m_1,m_2,\dots,m_M)}(z)$.Since $\varphi^{(m_1,m_2,\dots,m_M)}_\sigma(z)$ are eigenvectors of y_i $(i=1,\dots,N)$ (Cf.

Proposition 6) we get:

$$T_a^0(u)\mathbb{U}^{(m_1,m_2,\ldots,m_M)}(z) =$$

$$\sum_{\substack{I\subset\{1,2,\ldots,M\}\\ I:=\{i_1,i_2,\ldots,i_L\}}} (-(q^2+1))^L \times$$

$$\times \left\{ \sum_{\sigma\in S^{(m_1,m_2,\ldots,m_M)}_{N,(m_{i_1},m_{i_2},\ldots,m_{i_L})}} \varphi_\sigma^{(m_1,m_2,\ldots,m_M)}(z) L_a(u;\{q^{l_{\sigma_i}}\})\mathbb{Y}(\sigma) \right\} \Pi^{-,(m_1,m_2,\ldots,m_M)}_{(m_{i_1},\ldots,m_{i_L})}. \quad (5.5.23)$$

It follows from the recursive definition of $\mathbb{Y}(\sigma)$ $(\sigma \in S^{(m_1,m_2,\ldots,m_M)}_{N,(m_{i_1},m_{i_2},\ldots,m_{i_L})})$ given in Proposition 11 that $\mathbb{Y}(\sigma)$ is an U-intertwiner:

$$L_a(u;\{q^{l_{\sigma_i}}\})\mathbb{Y}(\sigma) = \mathbb{Y}(\sigma)L_a(u;\{q^{l_{\sigma[0]_i}}\}). \quad (5.5.24)$$

Where $\sigma[0] = (m_{i_1}, m_{i_1}+1)\ldots(m_{i_L}, m_{i_L}+1)$.
Furthermore the projector $\Pi^{-,(m_1,m_2,\ldots,m_M)}_{(m_{i_1},\ldots,m_{i_L})}$ is an interwiner as well (Cf. 1.1.28):

$$L_a(u;\{q^{l_{\sigma[0]_i}}\}\Pi^{-,(m_1,m_2,\ldots,m_M)}_{(m_{i_1},\ldots,m_{i_L})} = \Pi^{-,(m_1,m_2,\ldots,m_M)}_{(m_{i_1},\ldots,m_{i_L})} L_a(u;\{q^{l_i}\}). \quad (5.5.25)$$

Hence we obtain:

$$T_a^0(u)\mathbb{U}^{(m_1,m_2,\ldots,m_M)}(z) = \mathbb{U}^{(m_1,m_2,\ldots,m_M)}(z)L_a(u;\{q^{l_i}\}). \quad (5.5.26)$$

Thus the intertwinig property of $\mathbb{U}^{(m_1,m_2,\ldots,m_M)}(z)$ is established.

6 Spectrum and eigenspaces of the operators $\Xi^{(n)}(\omega)$ $(n = 1,\ldots,N-1)$ forming the hierarchy of U-invariant Spin Models.

6.1 Eigenspaces $H^\mu_{\mathbf{B}}(\omega)$ of the generating function $\Xi(u;\omega)$

1. Let $\mu \equiv (m_1,m_2,\ldots,m_M) \in \mathfrak{M}_N$. Define a subspace $H^\mu_{\mathcal{B}}(\omega) \subset H_{\mathcal{B}}(\omega) \subset H$ by applying the evaluation map $Ev(\omega)$ (Cf. **3.1**) to the "bosonic" subspace $\mathcal{B}^{(m_1,m_2,\ldots,m_M)}$ introduced in the previous section:

$$H^{(m_1,m_2,\ldots,m_M)}_{\mathcal{B}}(\omega) := Ev(\omega)\mathcal{B}^{(m_1,m_2,\ldots,m_M)} \quad ((m_1,m_2,\ldots,m_M) \in \mathfrak{M}_N). \quad (6.6.1)$$

From (3.3.13,.14,.19) we obtain

$$\Xi(u;\omega)H^{(m_1,m_2,\ldots,m_M)}_{\mathcal{B}}(\omega) = \left(\sum_{i=1}^M (\delta_1^{(m_i)}(u) - \theta(u)m_i)\right) H^{(m_1,m_2,\ldots,m_M)}_{\mathcal{B}}(\omega)$$

$$((m_1,m_2,\ldots,m_M) \in \mathfrak{M}_N). \quad (6.6.2)$$

Where $\delta_1^{(m_i)}(u)$ was defined in (4.4.6) and $\theta(u)$ was defined in (2.2.6). The last equation says that $H^{(m_1,m_2,\ldots,m_M)}_{\mathcal{B}}(\omega)$ is an eigenspace of $\Xi(u;\omega)$ unless $H^{(m_1,m_2,\ldots,m_M)}_{\mathcal{B}}(\omega) \equiv 0$.

2. Application of the evaluation map $Ev(\omega)$ to $\mathcal{B}^{(m_1,m_2,\ldots,m_M)}$ (5.5.19) yields the explicit expression for the space $H^{(m_1,m_2,\ldots,m_M)}_{\mathcal{B}}(\omega)$:

$$H^{(m_1,m_2,\ldots,m_M)}_{\mathcal{B}}(\omega) = \mathbb{U}^{(m_1,m_2,\ldots,m_M)}(\omega)\mathcal{Z}^{(m_1,m_2,\ldots,m_M)} \quad ((m_1,m_2,\ldots,m_M) \in \mathfrak{M}_N). \quad (6.6.3)$$

Where

$$\mathbb{U}^{(m_1,m_2,\ldots,m_M)}(\omega) := \mathbb{U}^{(m_1,m_2,\ldots,m_M)}(z_1 = \omega^1,\ldots,z_N = \omega^N) =$$
$$= (-(q^2+1))^M \sum_{\sigma\in S^{(m_1,m_2,\ldots,m_M)}_{N,(m_1,m_2,\ldots,m_M)}} \varphi_\sigma^{(m_1,m_2,\ldots,m_M)}(\omega)\mathbb{Y}(\sigma)\Pi^{-,(m_1,m_2,\ldots,m_M)}_{\{1,2,\ldots,M\}},$$

$$\mathbb{U}^{(m_1,m_2,\ldots,m_M)}(\omega) : H \mapsto H. \quad (6.6.4)$$

To get the expression for $\mathbb{U}^{(m_1,m_2,\ldots,m_M)}(\omega)$ we used the Corollary 1 to Proposition 9.

Let us introduce the notation

$$W^{(m_1,m_2,\ldots,m_M)} := \Pi^{-,(m_1,m_2,\ldots,m_M)}_{\{1,2,\ldots,M\}} \mathcal{Z}^{(m_1,m_2,\ldots,m_M)} \quad ((m_1,m_2,\ldots,m_M) \in \mathfrak{M}_N). \tag{6.6.5}$$

The definition of $\mathcal{Z}^{(m_1,m_2,\ldots,m_M)}$ given in (5.5.17) leads to a more explicit form of $W^{(m_1,m_2,\ldots,m_M)}$:

$$W^{(m_1,m_2,\ldots,m_M)} = \\ S_q(\underset{1}{V} \otimes \cdots \otimes \underset{m_1-1}{V}) \otimes A_q(\underset{m_1}{V} \otimes \underset{m_1+1}{V}) \otimes S_q(\underset{m_1+2}{V} \otimes \cdots \otimes \underset{m_2-1}{V}) \otimes A_q(\underset{m_2}{V} \otimes \underset{m_2+1}{V}) \otimes \\ \ldots \otimes S_q(\underset{m_M+2}{V} \otimes \cdots \otimes \underset{N}{V}) \subset H := \underset{1}{V} \otimes \underset{2}{V} \otimes \ldots \otimes \underset{N}{V}. \tag{6.6.6}$$

Where A_q signifies q-antisymmetrization (1.1.15).

Observe that $W^{(m_1,m_2,\ldots,m_M)}$ $((m_1,m_2,\ldots,m_M) \in \mathfrak{M}_N)$ is an irreducible highest-wight U-module with the U-action is given by

$$L_a(u;\{q^{l_{\sigma[0]_i}}\}) := L_{a1}(uq^{l_{\sigma[0]_1}})L_{a2}(uq^{l_{\sigma[0]_2}})\ldots L_{aN}(uq^{l_{\sigma[0]_N}}) \in End(H). \tag{6.6.7}$$

Where we used the notation

$$\sigma[0] := (m_1, m_1+1)(m_2, m_2+1)\ldots(m_M, m_M+1) \quad ((m_1,m_2,\ldots,m_M) \in \mathfrak{M}_N). \tag{6.6.8}$$

The Drinfeld polynomial of this module is

$$Q^{(m_1,m_2,\ldots,m_M)}(u) = \prod_{\substack{1\le n\le N \\ n\neq m_i, m_i+1}} (1 - q^{-l_n} u). \tag{6.6.9}$$

According to (6.6.3) we have

$$H_{\mathcal{B}}^{(m_1,m_2,\ldots,m_M)}(\omega) = \check{\mathbb{U}}^{(m_1,m_2,\ldots,m_M)}(\omega) W^{(m_1,m_2,\ldots,m_M)} \quad ((m_1,m_2,\ldots,m_M) \in \mathfrak{M}_N), \\ \check{\mathbb{U}}^{(m_1,m_2,\ldots,m_M)}(\omega) := (-(q^2+1))^M \sum_{\sigma \in S^{(m_1,m_2,\ldots,m_M)}_{N,(m_1,m_2,\ldots,m_M)}} \varphi_\sigma^{(m_1,m_2,\ldots,m_M)}(\omega)\mathbb{Y}(\sigma). \tag{6.6.10}$$

3. The space $H_{\mathcal{B}}^{(m_1,m_2,\ldots,m_M)}(\omega)$ $((m_1,m_2,\ldots,m_M) \in \mathfrak{M}_N)$ is a U-module with the U-action given by $T_a^0(u;\omega)$ defined in (3.3.19). Explicitely (Cf. 5.5.23):

$$T_a^0(u;\omega) H_{\mathcal{B}}^{(m_1,m_2,\ldots,m_M)}(\omega) = T_a^0(u;\omega)\check{\mathbb{U}}^{(m_1,m_2,\ldots,m_M)}(\omega) W^{(m_1,m_2,\ldots,m_M)} = \\ = (-(q^2+1))^M \sum_{\sigma \in S^{(m_1,m_2,\ldots,m_M)}_{N,(m_1,m_2,\ldots,m_M)}} \varphi_\sigma^{(m_1,m_2,\ldots,m_M)}(\omega) L_a(u;\{q^{l_{\sigma_i}}\})\mathbb{Y}(\sigma), \\ L_a(u;\{q^{l_{\sigma_i}}\}) := L_{a1}(uq^{l_{\sigma_1}})L_{a2}(uq^{l_{\sigma_2}})\ldots L_{aN}(uq^{l_{\sigma_N}}). \tag{6.6.11}$$

Applying (5.5.24) we find that $\check{\mathbb{U}}^{(m_1,m_2,\ldots,m_M)}$ is an intertwiner of the modules $H_{\mathcal{B}}^{(m_1,m_2,\ldots,m_M)}(\omega)$ and $W^{(m_1,m_2,\ldots,m_M)}$:

$$T_a^0(u;\omega) H_{\mathcal{B}}^{(m_1,m_2,\ldots,m_M)}(\omega) = T_a^0(u;\omega)\check{\mathbb{U}}^{(m_1,m_2,\ldots,m_M)}(\omega) W^{(m_1,m_2,\ldots,m_M)} = \\ \check{\mathbb{U}}^{(m_1,m_2,\ldots,m_M)}(\omega) L_a(u;\{q^{l_{\sigma[0]_i}}\}) W^{(m_1,m_2,\ldots,m_M)} \quad ((m_1,m_2,\ldots,m_M) \in \mathfrak{M}_N). \tag{6.6.12}$$

Since $W^{(m_1,m_2,\ldots,m_M)}$ is irreducible so is $H_{\mathcal{B}}^{(m_1,m_2,\ldots,m_M)}(\omega)$. The highest-weight vector of $W^{(m_1,m_2,\ldots,m_M)}$ is (Cf. 6.6.6)

$$\tilde{\Omega}^{(m_1,m_2,\ldots,m_M)} := \\ v^+ \otimes \cdots \otimes v^+ \otimes (\underset{m_1}{v^+} \otimes \underset{m_1+1}{v^-} - qv^- \otimes v^+) \otimes \quad \ldots \quad \otimes v^+ \otimes \cdots \otimes v^+. \tag{6.6.13}$$

If the vector $\Omega^{(m_1,m_2,\ldots,m_M)} := \check{\mathbb{U}}^{(m_1,m_2,\ldots,m_M)} \tilde{\Omega}^{(m_1,m_2,\ldots,m_M)} \in H_{\mathcal{B}}^{(m_1,m_2,\ldots,m_M)}(\omega)$ is not zero, it is the highest-weight vector, and the modules $H_{\mathcal{B}}^{(m_1,m_2,\ldots,m_M)}(\omega)$ and $W^{(m_1,m_2,\ldots,m_M)}$ are isomorphic , with $\check{\mathbb{U}}^{(m_1,m_2,\ldots,m_M)}$ defining the isomorphism explicitely.

4. In order to show that $\Omega^{(m_1,m_2,\dots,m_M)}$ $((m_1,m_2,\dots,m_M)\in\mathfrak{M}_N)$ is not zero we compute this vector at $q=0$.
Consider the matrix $\mathbb{Y}(\sigma)$ $(\sigma\in S^{(m_1,m_2,\dots,m_M)}_{N,(m_1,m_2,\dots,m_M)})$ that enters the definition of $\check{\mathbb{U}}^{(m_1,m_2,\dots,m_M)}$. According to (5.5.15,.16):

$$\mathbb{Y}(\sigma)=\mathrm{Id}\quad(\sigma=\sigma[0]:=(m_1,m_1+1)\dots(m_M,m_M+1)), \tag{6.6.14}$$
$$\mathbb{Y}(\sigma)=\mathbb{Y}(\sigma)'Y^-_{i,i+1}(q^{l_{\sigma[0]_i}-l_{\sigma[0]_{i+1}}})\quad(\sigma\neq\sigma[0]). \tag{6.6.15}$$

Where depending on σ , i takes one of the values in the set $\{m_k-1,m_k+1\}_{k\in\{1,2,\dots,M\}}$. For any such i we have $\sigma[0]_i-\sigma[0]_{i+1}=-2$. $\mathbb{Y}(\sigma)'$ is either identity or a product of intertwiners of the form $Y^-_{j,j+1}(q^{-2r})$ $j\in(\{1,\dots,N-1\})$ where $r\geq 2$.
For $r\geq 2$ we find

$$Y^-_{j,j+1}(q^{-2r})|_{q=0}=-\Pi^-_{j,j+1}(0)=-(|+-><+-|)_{j,j+1}\quad(j=1,\dots,N-1). \tag{6.6.16}$$

Thus

$$\lim_{q\to 0}\mathbb{Y}(\sigma)=-\mathbb{Y}(\sigma)'|_{q=0}\Pi^-_{i,i+1}(0)\quad(i\in\{m_k-1,m_k+1\}_{k\in\{1,2,\dots,M\}}),$$
$$(\sigma\in S^{(m_1,m_2,\dots,m_M)}_{N,(m_1,m_2,\dots,m_M)},\ \sigma\neq\sigma[0]). \tag{6.6.17}$$

The highest-weight vector $\tilde\Omega^{(m_1,m_2,\dots,m_M)}$ in the limit $q\to 0$ is

$$\tilde\Omega^{(m_1,m_2,\dots,m_M),q=0}:=$$
$$v^+\otimes\dots\otimes\underset{m_1}{v^+}\otimes\underset{m_1+1}{v^-}\otimes v^+\dots\otimes\underset{m_2}{v^+}\otimes\underset{m_2+1}{v^-}\otimes\quad\dots\quad\otimes v^+\otimes\dots\otimes v^+. \tag{6.6.18}$$

Therefore taking into account (6.6.17) and (4.4.26) we arrive at the following expression for $\Omega^{(m_1,m_2,\dots,m_M)}\in H^{(m_1,m_2,\dots,m_M)}_{\mathcal{B}}(\omega)$ at $q=0$:

$$\Omega^{(m_1,m_2,\dots,m_M),q=0}=(-1)^M\varphi^{(m_1,m_2,\dots,m_M)}_{\sigma[0]}(\omega)\tilde\Omega^{(m_1,m_2,\dots,m_M),q=0}=$$
$$=\omega^{\frac{1}{2}\sum_{i=1}^M m_i(m_i+1)}\tilde\Omega^{(m_1,m_2,\dots,m_M),q=0}. \tag{6.6.19}$$

Since $\Omega^{(m_1,m_2,\dots,m_M),q=0}$ is not zero we can argue that same holds for any generic value of q (not a root of unity) and therefore $H^{(m_1,m_2,\dots,m_M)}_{\mathcal{B}}(\omega)$ and $W^{(m_1,m_2,\dots,m_M)}$ are isomorphic U-modules for any generic q and $(m_1,m_2,\dots,m_M)$
$\in\mathfrak{M}_N$.

5. From consideration of the case $q=0$ we deduce that

$$H=\bigoplus_{(m_1,m_2,\dots,m_M)\in\mathfrak{M}_N}W^{(m_1,m_2,\dots,m_M)}. \tag{6.6.20}$$

Since the U-modules $H^{(m_1,m_2,\dots,m_M)}_{\mathcal{B}}(\omega)$ have different Drinfeld polynomials (6.6.9) for different motifs $(m_1,m_2,\dots,m_M)$ any two of these modules do not intersect except at zero vector. Therefore we can take the direct sum of all these modules. Since $\dim H^{(m_1,m_2,\dots,m_M)}_{\mathcal{B}}(\omega)=\dim W^{(m_1,m_2,\dots,m_M)}$, we conclude from (6.6.20) that

$$H=\bigoplus_{(m_1,m_2,\dots,m_M)\in\mathfrak{M}_N}H^{(m_1,m_2,\dots,m_M)}_{\mathcal{B}}(\omega). \tag{6.6.21}$$

Thus we have found the complete decomposition of the space of states into eigenspaces of the operator $\Xi(u;\omega)$, as well as the U-representation content of this decomposition.

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