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Duality and the Legendre Transform

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Abstract

We define a weak-strong coupling transformation based on the Legendre transformation of the effective action. In the case of $N = 2$ supersymmetric Yang-Mills theory, this coincides with the duality transform on the low energy effective action considered by Seiberg and Witten. This Legendre transform interpretation of duality generalizes directly to the full effective action, and in principle to other theories.

In 1977 Montonen and Olive (MO) [1] proposed that the strong coupling regime of certain Yang-Mills-Higgs theories with gauge group G, spontaneously broken to some compact subgroup H is equally well described by a dual weakly coupled theory, where topological and Noether charges exchange roles. MO duality is believed to operate in $N=4$ Yang-Mills (YM) theory [2, 3]. Recently Seiberg and Witten have constructed a version of MO duality on the light fields of $N=2$ YM theories [4, 5].

In this letter we observe that for $N=2$ theories the low energy effective action of the dual theory is the Legendre transform of the ordinary low energy effective action. That is, the dual effective action is the Schwinger functional for some (topological) current. This definition of the S duality transformation directly extends to the full effective action, and in principle to other theories. Here we address some issues associated with the full effective potential in $N = 2$ YM. We also discuss why this relation does not hold in the Sine-Gordon Thirring duality [6].

To see how the dual low energy effective action in pure $N=2$ YM with gauge group $SU(2)$ [5] is related to the generating functional for the topological current, couple a source term

$$\frac{1}{4\pi}\mathrm{Im}\int d^4x d^2\theta_1 d^2\theta_2\ \mathcal{A}^a\mathcal{J}^a \tag{1}$$

to the classical action. Here $\mathcal{A}^a$, $a=1,2,3$ is a $N=2$ chiral superfield, which satisfies a Bianchi constraint (see for example [7]), and $\mathcal{J}^a$ is a $N = 2$ chiral superfield. In component field language the $\mathcal{A}^a\mathcal{J}^a$ term contains a coupling $V_\mu j^\mu_{top} = V_\mu\epsilon^{\mu\nu\rho\lambda}\partial_\nu F_{\rho\lambda}$ to the abelian topological current. The Schwinger functional $\mathbf{W}[\mathcal{J}^a]$ obtained from (1) then leads to the effective action $\mathbf{\Gamma}[\mathcal{A}^a]$ after Legendre transformation[1]. In ref. [4, 5] attention was focussed on the *low energy* effective action $\Gamma[\mathcal{A}]$, where $\mathcal{A}$ refers to the light fields ($\mathcal{A} = \mathcal{A}^3$, say) and derivative terms (see e.g. [8]) are ignored. The low energy effective action for pure $N = 2$ $SU(2)$ YM can be written in $N = 2$ superspace [9] as follows

$$\Gamma[\mathcal{A}] = \frac{1}{4\pi}\mathrm{Im}\int d^4x d^2\theta_1 d^2\theta_2 \mathcal{F}(\mathcal{A}), \tag{2}$$

where $\mathcal{F}$ [2] is a holomorphic function. In ref. [4] an exact (although implicit) expression was given for $\mathcal{F}(\mathcal{A})$, and it was argued that the theory has an equivalent description in terms of the "dual" variables

$$\mathcal{A}_D = 8\pi i \frac{\delta\Gamma}{\delta\mathcal{A}} = \frac{\partial\mathcal{F}(\mathcal{A})}{\partial\mathcal{A}}, \quad \bar{\mathcal{A}}_D = -8\pi i \frac{\delta\Gamma}{\delta\bar{\mathcal{A}}} = \frac{\partial\bar{\mathcal{F}}(\bar{\mathcal{A}})}{\partial\bar{\mathcal{A}}}, \tag{3}$$

in which the the magnetically charged solitons are treated as local fields. The dual (low energy) effective action is now $4\pi\Gamma_D = \mathrm{Im}\int \mathcal{F}_D(\mathcal{A}_D)$ where the dual potential $\mathcal{F}_D$ satisfies (see [8] for a $N=2$ formulation)

$$\frac{\partial^2\mathcal{F}(\mathcal{A})}{\partial\mathcal{A}^2} = -\left(\frac{\partial^2\mathcal{F}_D(\mathcal{A}_D)}{\partial\mathcal{A}_D^2}\right)^{-1}, \tag{4}$$

and $\mathrm{Im}\mathcal{F}(z)$ is a convex function. Using (3) and (4) it is easy to see that an equivalent definition of $\Gamma_D(\mathcal{A}_D)$ is given by

$$\Gamma_D(\mathcal{A}_D, \bar{\mathcal{A}}_D) = \min_{\mathcal{A},\bar{\mathcal{A}}}\left[\Gamma[\mathcal{A}] - \frac{1}{4\pi}\mathrm{Im}\int \mathcal{A}_D\mathcal{A}\right]. \tag{5}$$

Having established that $\Gamma_D(\mathcal{A}_D)$ is the Legendre transform of $\Gamma(\mathcal{A})$, it follows at once that it must be (the convex hull of) the low energy generating functional obtained from (1). Of course the Schwinger functional is a well defined object for any theory, and in particular the so defined duality transformation extends immediately to the full effective action for $N=2$ YM. We now examine the full superfield effective potential. A gauge invariant extension of the low energy effective potential proposed in [5, 9] is given by

$$\mathcal{H}(\mathcal{A}^a\mathcal{A}^a) = \mathcal{F}(\sqrt{\mathcal{A}^a\mathcal{A}^a}), \tag{6}$$

and so one can write down a gauge invariant extension of the low energy effective action as[3] $4\pi\hat{\Gamma} = \mathrm{Im}\int\mathcal{H}$. As pointed out in [10]

[1] In order to define the effective action, $\mathbf{\Gamma}[\mathcal{A}^a]$, one must impose a constraint on $\mathcal{J}^a$ (which matches the Bianchi constraint on $\mathcal{A}^a$) chosen so that $\delta\mathbf{W}[\mathcal{J}^b]/\delta\mathcal{J}^a = \mathcal{A}^a$ is invertible.

[2] We can regard $\mathcal{F}$ as a N=2 superspace effective potential, although it will contain space-time derivative terms when expanded out in component fields.

[3] This is still not the full effective action, since we ignore higher derivative terms.

$$\mathrm{Im}\frac{\partial^2\mathcal{H}(\mathcal{A}\cdot\mathcal{A})}{\partial\mathcal{A}^i\partial\mathcal{A}^j} = \mathrm{Im}\left[\mathcal{F}''(\sqrt{\mathcal{A}\cdot\mathcal{A}})\frac{\mathcal{A}^i\mathcal{A}^j}{\mathcal{A}\cdot\mathcal{A}} + \frac{\mathcal{A}_D\cdot\mathcal{A}}{\mathcal{A}\cdot\mathcal{A}}\left(\delta^{ij} - \frac{\mathcal{A}^i\mathcal{A}^j}{\mathcal{A}\cdot\mathcal{A}}\right)\right], \tag{7}$$

is not positive definite in a certain region of the moduli space. On the other hand, we know that the effective action should be convex. So it seems the high energy effective potential given by (6) does not properly describe this region. One possibility is to replace $\hat{\Gamma}$ by its convex hull. We now define the dual high energy effective potential as in (5) by

$$\hat{\Gamma}_D(\mathcal{A}^a_D, \bar{\mathcal{A}}^a_D) = \min_{\mathcal{A}^a,\bar{\mathcal{A}}^a}\left[\hat{\Gamma}[\mathcal{A}^a, \bar{\mathcal{A}}^a] - \frac{1}{4\pi}\mathrm{Im}\int \mathcal{A}^a_D\mathcal{A}^a\right], \tag{8}$$

where it is understood that $\mathcal{A}^a$ satisfies the Bianchi constraint. Note that Γ_D is automatically convex, since we have defined it as a Legendre transform. In the region(s) of the moduli space where the Hessian (7) is positive definite, a solution to (8) is given by

$$\hat{\Gamma}_D(\mathcal{A}^a_D, \bar{\mathcal{A}}^a_D) = \frac{1}{4\pi}\mathrm{Im}\int d^2\theta_1 d^2\theta_2 \mathcal{H}_D(\mathcal{A}^a_D\mathcal{A}^a_D), \tag{9}$$

where

$$\mathcal{H}_D(\mathcal{A}^a_D\mathcal{A}^a_D) = \mathcal{F}_D(\sqrt{\mathcal{A}^a_D\mathcal{A}^a_D}) \tag{10}$$

and

$$\mathcal{A}^a_D = \frac{\partial\mathcal{H}(\mathcal{A}\cdot\mathcal{A})}{\partial\mathcal{A}^a} = \frac{\mathcal{F}'(\sqrt{\mathcal{A}\cdot\mathcal{A}})\mathcal{A}^a}{\sqrt{\mathcal{A}\cdot\mathcal{A}}}. \tag{11}$$

In the region where the matrix (7) is not positive definite we do not have an explicit expression for the dual effective action, but its stability is guaranteed by general properties of the Legendre transform.

Seiberg and Witten have also extended their work to $N=2$ gauge theory with $N=2$ matter multiplets [5]. In this case the matter fields seem to play a passive role in the duality. Therefore in order to obtain the dual effective action one would only Legendre transform with respect to the $N=2$ gauge fields.

The situation in the Thirring Sine-Gordon duality [6] is somewhat different. This duality leads to an identity between the Schwinger functionals for the respective conserved currents. Indeed, proceeding as in [6] one can show that

$$\begin{aligned}\exp\left(i\mathbf{W}_T[J_\mu]\right) &= \int d\bar{\psi}d\psi \exp\left(i\int d^2x\left[\mathcal{L}_T + J_\mu\bar{\psi}\gamma^\mu\psi\right]\right)\\ &= \int d\phi \exp\left(i\int d^2x\left[\mathcal{L}_{SG} + J_\mu j^\mu_{top}\right]\right) = \exp\left(i\mathbf{W}_{SG}[J_\mu]\right),\end{aligned} \tag{12}$$

where[4] $2\pi j^\mu_{top} = -\beta\epsilon^{\mu\nu}\partial_\nu\phi$ and identification of the couplings is made as in [6]. Hence in contrast to $N=2$ YM, the respective Schwinger functionals are identical. If there was

[4] Our notation follows that of [6]

to be a Legendre transform interpretation of Coleman duality we would have the "self duality" relation $\mathbf{W}_{SG}[J_\mu = 2\pi\epsilon_{\mu\nu}\partial^\nu\Phi/\beta] = \mathbf{\Gamma}_{SG}[\Phi]$, where $\mathbf{\Gamma}_{SG}[\Phi]$ is the usual effective action for the Sine-Gordon model. This is certainly not expected. Up to second order in perturbation theory we observed

$$\mathbf{W}_{SG}[J_\mu = 2\pi\epsilon_{\mu\nu}\partial^\nu\Phi/\beta] = \mathbf{\Gamma}_{SG}[\Phi] - \int d^2x\partial_\mu\Phi\partial^\mu\Phi.$$

The Thirring Sine-Gordon duality relates the Schwinger functional for the Noether current of the Thirring model with the same for the topological current in the Sine-Gordon model, whereas the $N = 2$ YM duality relates the Noether and topological sectors of the same theory.

To summarize, we have shown that in $N = 2$ YM theory the low energy effective action for the dual theory is the Schwinger functional for the topological current of the original theory. Here we have verified this weak-strong coupling relations for the low-energy effective action of $N = 2$ YM, although it might well extend directly to the full effective action as well as other theories. In particular, it would be interesting to see how these ideas extend to N=1 theories. If such a relation exists, it would reduce the computation of the S-matrix in the strongly coupled theory to the weak coupling expansion of the dual effective action.

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References

[1] C.Montonen and D.Olive, Phys. Lett. B72 (1977) 117.

[2] H.Osborn, Phys. Lett. B83 (1979) 321.

[3] E.Witten and C.Vafa, Nucl. Phys. B431 (1994) 3.

[4] N.Seiberg and E.Witten, Nucl. Phys. B426 (1994) 19, (E) B430 (1994) 485.

[5] N.Seiberg and E.Witten, Nucl. Phys. B431 (1994) 484.

[6] S.Coleman, Phys. Rev. D11 (1975) 2088.

[7] P.West, *Introduction to Supersymmetry and Supergravity*, extended 2nd edition, World Scientific (1990).

[8] M.Henningson, hep-th 9507135, YCTP-P12-95.

[9] N.Seiberg, Phys. Lett. B206 (1988) 75.

[10] U.Lindstrom and M.Rocek, Phys. Lett. B355 (1995) 492.