Propagators and Path Integrals

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Abstract

Path-integral expressions for one-particle propagators in scalar and fermionic field theories are derived, for arbitrary mass. This establishes a direct connection between field theory and specific classical point-particle models. The role of world-line reparametrization invariance of the classical action and the implementation of the corresponding BRST-symmetry in the quantum theory are discussed. The presence of classical world-line supersymmetry is shown to lead to an unwanted doubling of states for massive spin-1/2 particles. The origin of this phenomenon is traced to a 'hidden' topological fermionic excitation. A different formulation of the pseudo-classical mechanics using a bosonic representation of γ_5 is shown to remove these extra states at the expense of losing manifest supersymmetry.

1   Introduction

Because of their conceptual simplicity, path-integral methods [1, 2] often provide convenient procedures to obtain insight in field theoretical problems. In recent work by Strassler, McKeon, Schmidt, Schubert and others [3]-[6] world-line path integrals have been applied to a reformulation of standard Feynman perturbation theory from which useful information on the structure of perturbative Green's functions is extracted.

A basic question in this context is the representation of propagators in quantum field theory by path integrals for relativistic particles of various kind. In particular one would like to know the classical actions to be used in these path-integrals, as well as the precise meaning of the functional measure in some regularized form, e.g. by discretization. Answers to these questions establish firm connections between the so-called first- and second quantized formulations of relativistic quantum theory.

This paper addresses both of these problems. It pursues them from two complementary points of view. First of all, there is the pragmatic question of how to convert a given field-theory propagator to a path integral expression. Such a procedure is discussed in this paper for both spin-0 and spin-1/2 particles. Starting at the other end, one can also ask what kind of quantum field theory is associated with a given classical action. The kind of actions considered in this context are usually taken to possess some desirable properties like reparametrization invariance and supersymmetry, which pose however additional difficulties to quantization procedures since gauge fixing then becomes necessary. Using previously established BRST-procedures [7, 8] this is done in an extended state space with a number of ghost- and auxiliary degrees of freedom. The propagators in these extended state spaces are obtained, and it is then shown how to reduce them to standard expressions in terms of physical variables only.

One result which deserves to be emphasized is that the irreducible path-integral expression for the propagator of a massive Dirac fermion is not based on a manifestly supersymmetric action, although world-line supersymmetry is realized algebraically in the quantum theory by the Dirac operator. Indeed, it is shown that the manifestly supersymmetric version of the theory contains a 'hidden' topological fermionic degree of freedom which doubles the number of components of the physical states. This theory therefore describes a degenerate doublet of fermions, rather than a single massive Dirac particle. A path-integral expression for a single massive Dirac fermion is also obtained (sect.7). As in other cases, the difference between the two models can be traced to the representation of γ_5.

This paper is organized as follows. In sects.(2)-(6) the free field theory of a scalar spin-0 particle is considered. A simple and well-known path-integral expression is obtained, which is subsequently rederived from the reparametrization-invariant classical model. These calculations also help to explain the general

procedures used.

In sect.(7) a path-integral for a spin-1/2 particle is derived. For massless particles it agrees with results in the literature [9, 10]. For massive fermions a new term is present in the action which has not been considered before, and which is based on a bosonic representation of γ_5. The manifestly supersymmetric theory, which uses a fermionic representation for γ_5, is analyzed in sects.(8)-(10), and is shown to describe a doublet of spinors if the mass is non-zero. Our conclusions are presented in sect.(11).

2 Free particle propagators

The Feynman propagator for a free scalar particle of mass m in D-dimensional space-time is a specific solution of the inhomogeneous Klein-Gordon equation

$$\left(-\Box_x + m^2\right) \Delta_F(x - y) = \delta^D(x - y), \tag{1}$$

such that positive frequencies propagate forward in time, and negative frequencies backward. An explicit expression in terms of a Fourier integral is

$$\Delta_F(x - y) = \int \frac{d^D p}{(2\pi)^D} \frac{e^{ip \cdot (x-y)}}{p^2 + m^2 - i\varepsilon}. \tag{2}$$

In the limit $\varepsilon \to 0^+$ the simple real pole at positive $p^0 = E(\vec{p}) = \sqrt{\vec{p}^2 + m^2}$ gives the mass of the particle as $m = E(0)$. The arbitrarily small imaginary part $i\varepsilon$ implements the causality condition.

There is a straightforward connection between this propagator and the classical mechanics of a relativistic point-particle through the path-integral formalism [1]. To establish this, let us first consider the very general problem of finding the inverse of a non-singular hermitean operator $\hat{H}$. Following Schwinger [11], we construct the formal solution

$$\hat{H}^{-1} = \lim_{\varepsilon \to 0^+} i \int_0^\infty dT\, e^{-iT\left(\hat{H} - i\varepsilon\right)}. \tag{3}$$

Here the exponential operator

$$\hat{K}_\varepsilon(T) = e^{-iT\left(\hat{H} - i\varepsilon\right)} \tag{4}$$

is the solution of the Schrödinger equation

$$i\frac{\partial \hat{K}_\varepsilon(T)}{\partial T} = \left(\hat{H} - i\varepsilon\right) \hat{K}_\varepsilon(T), \tag{5}$$

with the special properties[1]

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[1] $\hat{I}$ denotes the unit operator.

$$\hat{K}_\varepsilon(0) = \hat{1}, \qquad \lim_{T\to\infty} \hat{K}_\varepsilon(T) = 0. \tag{6}$$

If the operator $\hat{H}$ acts on the single-particle state-space with a complete co-ordinate basis $\{|x\rangle\}$, the matrix elements of the operator in the co-ordinate basis are

$$K_\varepsilon(x - y|T) = \langle x|\hat{K}_\varepsilon(T)|y\rangle. \tag{7}$$

Completeness of the basis then implies Huygens' composition principle

$$\int d^D x'\, K_{\varepsilon'}(x - x'|T')\, K_{\varepsilon''}(x' - x''|T'') = K_\varepsilon(x - x''|T' + T''). \tag{8}$$

Note that $\varepsilon = (\varepsilon'T' + \varepsilon''T'')/(T' + T'')$ stays arbitrarily small if ε' and ε'' are small enough.

Repeated use of eq.(8) now allows one to write

$$K_\varepsilon(x - y|T) = \int \prod_{n=1}^{N} d^D x_n \prod_{m=0}^{N} K_\varepsilon\left(x_{m+1} - x_m|\Delta T\right), \tag{9}$$

with $\Delta T = T/(N+1)$, and $x_0 = y$, $x_{N+1} = x$. Keeping T fixed, the limit $N \to \infty$ becomes an integral over continuous (but generally non-differentiable) paths in co-ordinate space-time between points y and x. (Observe that $K_\varepsilon(x - y|\Delta T)$ depends only on the difference $(x - y)$ and converges to $\delta^D(x - y)$ for $\Delta T \to 0$.)

If the operator $\hat{H}$ is an ordered expression in terms of a canonical set of operators $(\hat{x}^\mu, \hat{p}_\mu)$:

$$\hat{H} = \sum_{k,l} \hat{p}_{\mu_1} ... \hat{p}_{\mu_k} H^{\mu_1 ... \mu_k}_{\nu_1 ... \nu_l} \hat{x}^{\nu_1} ... \hat{x}^{\nu_l}, \tag{10}$$

then we can expand the co-ordinate path-integral expression (9) further to a phase-space path-integral

$$K_\varepsilon(x - y|T) = \frac{1}{(2\pi)^D} \int d^D p_0 \int \prod_{n=1}^{N} \frac{d^D x_n d^D p_n}{(2\pi)^D} e^{i\sum_{k=0}^{N}(p_k\cdot(x_{k+1}-x_k) - \Delta T H(p_k, x_k))}$$

$$\to \int_y^x \mathcal{D}p(\tau)\mathcal{D}x(\tau)\, e^{i\int_0^T d\tau(p\cdot\dot{x} - H(p,x))}. \tag{11}$$

Here $H(p, x)$ is the c-number symbol of the ordered operator $\hat{H}$, and we have tacitly assumed that the ordered symbol of the exponential can be replaced by the exponential of the ordered symbol. This is certainly correct for the main applications we consider in this paper, as may be checked by explicit calculations.

It is now clear, that one may interpret the symbol $H(p, x)$ as the hamiltonian of some classical system, and the argument of the exponential as the classical

action. Integration over the momentum variables $p(\tau)$ then in general leads to the lagrangian form of this action

$$K_\varepsilon(x - y|T) = \int_y^x \mathcal{D}x(\tau) \, e^{i \int_0^T d\tau L(\dot{x}, x)},\tag{12}$$

where the precise meaning of the integration measure can be recovered either from the phase-space expression (11), or from requiring the path-integral to satisfy Huygens' composition principle (8).

Returning to eq.(1), it states that $\Delta_F(x - y)$ is the inverse of the Klein-Gordon operator (in the space of square-integrable functions). Rescaling it for later convenience by a factor $1/2m$, we consider the evolution operator

$$\hat{K}_\varepsilon(T) = \exp\left\{ -\frac{iT}{2m} \left(-\Box + m^2 - i\varepsilon \right) \right\}.\tag{13}$$

In the co-ordinate representation the explicit expression for the matrix element of this operator is

$$K_\varepsilon(x - y|T) = -i \left(\frac{m}{2\pi T} \right)^{D/2} e^{i\frac{m}{2T}(x-y)^2 - \frac{iT}{2}(m - i\varepsilon)}.\tag{14}$$

The Feynman propagator can then be written as

$$\Delta_F(x - y) = \frac{i}{2m} \int_0^\infty dT \, K_\varepsilon(x - y|T),\tag{15}$$

and using the previous results it can be cast in the form of a path integral [12]

$$\Delta_F(x - y) = \frac{i}{2m} \int_0^\infty dT \int_y^x \mathcal{D}x(\tau) \exp\left\{ \frac{im}{2} \int_0^T d\tau \left(\dot{x}_\mu^2 - 1 \right) \right\}.\tag{16}$$

The same expression is also obtained directly by iteration of (14) using the product formula (9). For massless particles yet another derivation, based on the concepts of moduli space, has been discussed in [9, 10]. The result shows, that the scalar propagator is connected to a classical particle model with lagrangian

$$L = \frac{m}{2} \left(\dot{x}_\mu^2 - 1 \right),\tag{17}$$

related by Legendre transform to the simple hamiltonian

$$H(p, x) = \frac{p_\mu^2 + m^2}{2m}.\tag{18}$$

Because $H(p, x)$ is quadratic in p_μ and independend of x^μ, the integration measure in the path integral (16) is just the free gaussian measure of ref.[1].

The path-integral representation of the propagator establishes a simple connection between scalar quantum field theory and the classical point particle model

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(17). As is well-known, this connection does not only hold at the level of the action or hamiltonian, it also extends to the dynamics, in the sense that paths in the neighborhood of the solutions of the classical equations of motion give the dominant contribution to the path integral, certainly for the simple model discussed where the path integral is a pure Gaussian.

3 Reparametrization invariance

The classical equations of motion which follow from the lagrangian (17) state that the momentum is constant along the particle worldline:

$$\dot{p}^\mu = m\ddot{x} = 0, \tag{19}$$

Therefore this classical theory seems to contain less information than the quantum theory from which we started: we have to recover the condition that the momentum should lie on the mass shell:

$$p_\mu^2 + m^2 = 0. \tag{20}$$

This condition is equivalent to the vanishing of the hamiltonian (18). Since the hamiltonian is the generator of translations in the worldline parameter τ, the mass-shell condition is recovered by requiring the dynamics of the particle to be independent of the world-line parametrization.

As is well-known [13, 14], this can be achieved by introducing a gauge variable $e(\tau)$ for reparametrizations of the world line (the einbein). Under local reparametrizations $\tau \to \tau' = f(\tau)$, let the co-ordinates and einbein transform as

$$x^\mu(\tau) \to x'^\mu(\tau') = x^\mu(\tau), \qquad e(\tau) \to e'(\tau') = e(\tau)\frac{d\tau}{d\tau'}. \tag{21}$$

Then a reparametrization-invariant action can be constructed of the form

$$S_{cl}[x^\mu(\tau), e(\tau)] = \frac{m}{2} \int_0^T d\tau \left(\frac{1}{e}\dot{x}_\mu^2 - e\right). \tag{22}$$

The equations of motion one derives from this action are

$$\frac{1}{e}\frac{d}{d\tau}\frac{1}{e}\frac{dx^\mu}{d\tau} = 0, \qquad \left(\frac{1}{e}\frac{dx^\mu}{d\tau}\right)^2 + 1 = 0. \tag{23}$$

Identifying $d\tilde{\tau} = ed\tau$ with the proper time, and as a consequence $p^\mu = mdx^\mu/d\tilde{\tau}$ with the proper momentum, this reproduces both the world-line equation of motion and the mass-shell condition.

For the new hamiltonian we can take

$$H = \frac{e}{2m}\left(p_\mu^2 + m^2\right). \tag{24}$$

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This generates proper-time translations on special phase-space functions $F(x(\tau), p(\tau))$ by the Poisson-brackets:

$$\frac{dF}{d\tilde{\tau}} = \frac{1}{e}\frac{dF}{d\tau} = \frac{1}{e}\{F, H\}. \tag{25}$$

Note, that one cannot impose the constraint that H vanishes before we compute the brackets.

The difficulty with this formulation is evidently the additional dynamical variable $e(\tau)$, for which the evolution is not fixed by the Euler-Lagrange equations derived from $S_{cl}[x^\mu, e]$, and which has no conjugate momentum. The origin of these difficulties is precisely the reparametrization invariance (21). As a result, the hamiltonian evolution equation (25) does not hold for arbitrary functions on the *complete* phase space, $F(x^\mu, p^\mu, e)$. Clearly, to recover the results of sect.2 it is necessary to fix $e(\tau)$ to a constant value and change $\tau \to \tilde{\tau}$, which amounts to a rescaling of the unit of time on the worldline such that the particle's internal clock and the laboratory clock tick at the equal rates when the particle is at rest.

Therefore one chooses a gauge

$$\dot{e} = 0, \tag{26}$$

implying that e is constant, and adds this condition to the equations of motion. But this can only be done after the variation of e in the action has produced the mass-shell constraint (20). It is therefore of interest to have a formalism in which one can impose the gauge condition (26) from the start, and still keep track of all the constraints imposed by the reparametrization invariance (21). An appropriate formalism to solve this problem is the BRST procedure (for an introduction, see for example refs.[15, 16, 17]), which we describe here for the case at hand [7].

4 BRST formulation

We impose the gauge condition (26) by adding it to the action with a Lagrange multiplier λ; at the same time we introduce a corresponding Faddeev-Popov ghost action, using (real) anti-commuting ghost variables (b, c), in such a way that the complete action is invariant under the Grassmann-odd, nilpotent BRST transformations

$$\delta_\Lambda x^\mu = c\dot{x}^\mu, \qquad \delta_\Lambda e = c\dot{e} + \dot{c}e, \qquad \delta_\Lambda c = c\dot{c},$$

$$\delta_\Lambda b = -i\lambda, \qquad \delta_\Lambda \lambda = 0. \tag{27}$$

After a convenient rescaling $ec \to c$, turning the ghost c into a world-line scalar density such that $\delta_\Lambda e = \dot{c}$, the BRST-invariant gauge fixed action reads

$$S_{gf} = S_{cl}[x(\tau), e(\tau)] + \int_0^T d\tau \left(\lambda \dot{e} + i \dot{b} \dot{c} \right). \tag{28}$$

Actually, because a partial integration has been performed in the ghost term, the action is invariant only modulo a total time derivative. This is sufficient. Note, that the Lagrange multiplier λ now plays the role of momentum p_e conjugate to the einbein e.

The action S_{gf} has no local invariances left and can be treated by the usual procedures of canonical hamiltonian analysis. The canonical momenta are

$$p^\mu = \frac{m}{e} \dot{x}^\mu, \qquad p_e = \lambda, \tag{29}$$

$$\pi_c = -i\dot{b}, \qquad \pi_b = i\dot{c}.$$

Note that the ghost-momenta (π_c, π_b) are imaginary. The gauge-fixed hamiltonian is

$$H_{gf} = \frac{e}{2m} \left(p_\mu^2 + m^2 \right) - i\pi_b \pi_c. \tag{30}$$

The equations of motion are given by the Poisson brackets

$$\frac{dF}{d\tau} = \{F, H_{gf}\}, \tag{31}$$

where the brackets are defined by

$$\{F, G\} = \frac{\partial F}{\partial x^\mu} \frac{\partial G}{\partial p_\mu} - \frac{\partial F}{\partial p_\mu} \frac{\partial G}{\partial x^\mu} + \frac{\partial F}{\partial e} \frac{\partial G}{\partial p_e} - \frac{\partial F}{\partial p_e} \frac{\partial G}{\partial e}$$

$$+ (-1)^{a_F} \left(\frac{\partial F}{\partial c} \frac{\partial G}{\partial \pi_c} + \frac{\partial F}{\partial \pi_c} \frac{\partial G}{\partial c} + \frac{\partial F}{\partial b} \frac{\partial G}{\partial \pi_b} + \frac{\partial F}{\partial \pi_b} \frac{\partial G}{\partial b} \right). \tag{32}$$

Here all derivatives are taken from the left, and a_F is the Grassmann parity of F.

To recover the constraints imposed by local reparametrization invariance, one constructs the (Grassmann-odd) conserved BRST charge

$$\Omega = \frac{c}{2m} \left(p_\mu^2 + m^2 \right) - i\pi_b p_e, \qquad \{\Omega, H_{gf}\} = 0. \tag{33}$$

This BRST charge is nilpotent in the sense that

$$\{\Omega, \Omega\} = 0. \tag{34}$$

The BRST principle makes use of this property, and states that the physical observables of the theory are the cohomology classes of the BRST operator on the phase-space functions $F(x, p; e, p_e; c, \pi_c; b, \pi_b)$ with ghost number[2] $N_{gh} = 0$

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[2] We assign ghost number $+1$ to (c, π_b), and -1 to (b, π_c).

[18, 19]. More precisely, physical quantities are BRST invariant:

$$\{F, \Omega\} = 0, \tag{35}$$

but this allows an ambiguity in F as a result of (34); this ambiguity is resolved by associating obervables F with equivalence classes of functions differing only by a BRST transformation:

$$F' \sim F \qquad \Leftrightarrow \qquad F' = F + \{\Lambda, \Omega\}, \tag{36}$$

with Λ a phase-space function of Grassmann parity opposite to F, and one unit in ghost-number lower. In particular, any function which can be written as a pure BRST transform

$$G = \{\Lambda, \Omega\}, \tag{37}$$

is in the equivalence class of zero and may be taken to vanish. In the case of the relativistic particle, this happens to be true for the hamiltonian itself:

$$H_{gf} = -\{e\pi_c, \Omega\}, \tag{38}$$

and for the classical hamiltonian as well:

$$p_\mu^2 + m^2 = -2m\{\pi_c, \Omega\}. \tag{39}$$

Hence the classical and ghost terms in the hamiltonian H_{gf} vanish separately and the mass-shell condition follows. With

$$p_e = -i\{b, \Omega\}, \tag{40}$$

we also recover the vanishing of the momentum conjugate to the einbein e.

Before turning to the quantum theory, we draw attention to a peculiarity of the action (28): it is invariant under SO(2) rotations in the ghost variables (b, c). Therefore this theory possesses a second BRST invariance with conserved nilpotent charge

$$\tilde{\Omega} = \frac{b}{2m}\left(p_\mu^2 + m^2\right) - i\pi_c p_e, \qquad \left\{\tilde{\Omega}, H_{gf}\right\} = 0. \tag{41}$$

The algebra of these two BRST charges has the property that

$$\left\{\Omega, \tilde{\Omega}\right\} = \frac{i}{m}\left(p_\mu^2 + m^2\right) p_e. \tag{42}$$

Note that both factors on the right-hand side are separately conserved, and vanish on the physical hyperplane in phase space.

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5   BRST quantization

Having formulated a complete, BRST-invariant (pseudo-)classical mechanics for the relativistic point particle, we now return to the quantum theory and study the relation between this model and quantum field theory. Various aspects of this problem have been discussed in [20, 21, 7]. To obtain additional insight, we construct a quantum theory corresponding to the classical hamiltonian (30) in an extended state-space, compute the propagator and establish the relation with the usual Feynman propagator (2), thereby showing the physical equivalence of these different formulations of scalar field theory.

The quantum theory of interest is obtained by replacing the phase-space variables $(x^\mu, p_\mu; e, p_e; c, \pi_c; b, \pi_b)$ by operators, and postulating (anti-)commutation relations between them in direct correspondence with the Poisson-brackets (32):

$$[\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu_\nu, \quad [\hat{e}, \hat{p}_e] = i,$$
$$\{\hat{c}, \hat{\pi}_c\} = -i, \quad \left\{\hat{b}, \hat{\pi}_b\right\} = -i. \tag{43}$$

In the co-ordinate representation these algebraic relations hold on making the identification

$$\hat{p}_\mu = -i\frac{\partial}{\partial x^\mu}, \quad \hat{p}_e = -i\frac{\partial}{\partial e},$$
$$\hat{\pi}_c = -i\frac{\partial}{\partial c}, \quad \hat{\pi}_b = -i\frac{\partial}{\partial b}. \tag{44}$$

The bosonic momenta are self adjoint, and the fermionic ones anti self-adjoint with respect to the inner product

$$(\Phi, \Psi) = i\int d^D x^\mu de \int db\, dc\, \Phi^* \Psi, \tag{45}$$

where wave functions like Ψ are polynomials in the ghost variables, with co-efficients depending on the remaining co-ordinates (x^μ, e):

$$\Psi(b, c) = \psi - ib\psi_b + c\psi_c - icb\psi_{cb}. \tag{46}$$

The inner product (45) then reads in components

$$(\Phi, \Psi) = \phi^* \psi_{cb} + \phi_b^* \psi_c + \phi_c^* \psi_b + \phi_{cb}^* \psi. \tag{47}$$

Clearly this form is not positive definite. However, with respect to this inner product the hamiltonian operator,

$$\hat{H}_{gf} = \frac{e}{2m}\left(-\Box + m^2\right) + i\frac{\partial^2}{\partial b \partial c}, \tag{48}$$

and the nilpotent BRST operator,

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$$\hat{\Omega} \; = \; \frac{c}{2m} \left(-\Box + m^2 \right) \; + \; i\frac{\partial^2}{\partial e \partial b}, \qquad \hat{\Omega}^2 \; = \; 0, \tag{49}$$

are self adjoint. The indefinite metric implicit in this inner product is a general and necessary feature of a model with a nilpotent self-adjoint operator like $\hat{\Omega}$ [8].

It is also possible to define a positive-definite inner product on the state space: introduce a duality operation [22]

$$\Psi \; \rightarrow \; \tilde{\Psi} \; = \; \psi_{cb} - ib\psi_c + c\psi_b - icb\psi, \tag{50}$$

and define

$$\langle \Phi, \Psi \rangle \;\; = \;\; \left(\Phi, \tilde{\Psi} \right)$$
$$\tag{51}$$
$$= \;\; \phi^*\psi + \phi_b^*\psi_b + \phi_c^*\psi_c + \phi_{cb}^*\psi_{cb}.$$

With respect to this positive definite inner product the fermionic momenta and the BRST charge are no longer (anti-)self adjoint. In particular $\hat{c}^* = i\hat{\pi}_c$, $\hat{b}^* = i\hat{\pi}_b$, and

$$\langle \hat{\Omega}\Phi, \Psi \rangle \; = \; \langle \Phi, \hat{\Omega}^*\Psi \rangle, \tag{52}$$

where the adjoint BRST charge, also known as the co-BRST operator, is

$$\hat{\Omega}^* \; = \; \frac{1}{2m} \left(-\Box + m^2 \right) \frac{\partial}{\partial c} \; + \; ib\frac{\partial}{\partial e}. \tag{53}$$

Like the BRST charge itself, $\hat{\Omega}^*$ is nilpotent; however, it is not conserved:

$$\left[\hat{\Omega}^*, \hat{H}_{gf} \right] \; = \; \tilde{\Omega}, \tag{54}$$

where $\tilde{\Omega}$ is the operator corresponding to the dual BRST charge we have encountered before in (41):

$$\tilde{\Omega} \; = \; \frac{ib}{2m} \left(-\Box + m^2 \right) \; + \; \frac{\partial^2}{\partial e \partial c}. \tag{55}$$

We also note in passing, that the BRST operator $\hat{\Omega}$ itself is obtained from the commutator of the hamiltonian with the adjoint of the dual BRST charge:

$$\left[\tilde{\Omega}^*, \hat{H}_{gf} \right] \; = \; -\hat{\Omega}, \tag{56}$$

with

$$\tilde{\Omega}^* \; = \; -\frac{i}{2m} \left(-\Box + m^2 \right) \frac{\partial}{\partial b} \; - \; c\frac{\partial}{\partial e}. \tag{57}$$

We can now show how the physical states of the scalar particle are reobtained in the BRST formalism through the cohomology of $\hat{\Omega}$. One identifies the physical

states with equivalence classes of states which are BRST invariant and differ only by a BRST-exact term:

$$\hat{\Omega}\,\Psi_{phys}\,=\,0, \qquad \Psi_{phys}\,\sim\,\Psi'_{phys}\,=\,\Psi_{phys}\,+\,\hat{\Omega}\,\Lambda. \tag{58}$$

To obtain exaxctly one representative of each BRST invariance class, consider the zero-modes of the BRST laplacian [8, 7]

$$\begin{aligned} \Delta_{BRST} &=& \hat{\Omega}\hat{\Omega}^*\,+\,\hat{\Omega}^*\hat{\Omega} \\[2mm] &=& \frac{1}{4m^2}\,\left(-\Box+m^2\right)^2\,-\,\frac{\partial^2}{\partial e^2}. \end{aligned} \tag{59}$$

This operator is positive definite, and its zero-modes are zero-modes of the two terms on the right-hand side separately. Therefore

$$\Delta_{BRST}\,\Psi_{phys}\,=\,0 \quad\Leftrightarrow\quad \left\{\left(-\Box+m^2\right)\Psi_{phys}\,=\,0\,\wedge\,\frac{\partial}{\partial e}\Psi_{phys}\,=\,0\right\}. \tag{60}$$

Hence the wave functions Ψ_{phys} indeed satisfy the Klein-Gordon equation and are independend of the gauge variable e.

6   Gauge-fixed propagator and path integral

The next step is to construct the propagator for the scalar particle as described by the wave functions (46) in the BRST-extended state space, using the formalism of sect.(2). Let us label the co-ordinates (x^μ, e, b, c) collectively by Z. Then the Schrödinger equation for the kernel $K_{gf}(Z; Z'|T)$ in the co-ordinate picture becomes:

$$i\frac{\partial}{\partial T}K_{gf}(Z; Z'|T)\,=\,\left(\frac{e}{2m}\left(-\Box+m^2-i\varepsilon\right)+i\frac{\partial^2}{\partial b\partial c}\right)K_{gf}(Z; Z'|T). \tag{61}$$

This kernel is required to satisfy the initial condition

$$\begin{aligned} K_{gf}(Z; Z'|0) &=& \delta(Z-Z') \\[2mm] &=& \delta^D(x-x')\delta(e-e')\delta(c-c')\delta(b-b'). \end{aligned} \tag{62}$$

The solution to this equation is

$$K_{gf}(Z; Z'|T)\,=\,-iT\delta(e-e')\left(\frac{m}{2\pi eT}\right)^{D/2}e^{i\left[\frac{m}{2eT}(x-x')^2-\frac{eT}{2}(m-i\varepsilon)+\frac{i}{T}(b-b')(c-c')\right]}. \tag{63}$$

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This is the direct counterpart of eq.(14). It is straightforward to verify that the initial condition (62) is satisfied, as is the composition principle:

$$\int dZ'\, K_{gf}(Z; Z'|T')\, K_{gf}(Z'; Z''|T'') \ = \ K_{gf}(Z; Z''|T' + T''), \qquad (64)$$

where the integration measure reads

$$\int dZ \ = \ i \int d^D x\, de \int db\, dc. \qquad (65)$$

The propagator in the extended state space then becomes

$$\Delta_{gf}(Z; Z') \ = \ \frac{ie}{2m} \int_0^\infty dT\, K_{gf}(Z; Z'|T). \qquad (66)$$

It is the solution of the inhomogeneous extended Klein-Gordon equation

$$\left(-\Box + m^2 - i\varepsilon + \frac{2im}{e} \frac{\partial^2}{\partial b \partial c} \right) \Delta_{gf}(Z; Z') \ = \ \delta(Z - Z'). \qquad (67)$$

Substitution of the expression (63) for the kernel $K_{gf}(Z, Z'|T)$ in eq.(66) for the generalized propagator gives a closed expression for the propagator in co-ordinate space which is conveniently represented in terms of a momentum integral. Switching to the Fourier representation of the co-ordinate part and performing the integral over T yields

$$\Delta_{gf}(x, e, b, c; x', e', b', c') \ = \ \frac{2m}{ie}\, \delta\left(e - e'\right) \int \frac{d^D p}{(2\pi)^D}\, e^{ip \cdot (x - x')} \left(\frac{e^{-\frac{ie}{2m}\left(p^2 + m^2 - i\varepsilon\right)(b - b')(c - c')}}{[p^2 + m^2 - i\varepsilon]^2} \right). \qquad (68)$$

From this expression it is straightforward to obtain the Feynman propagator by integration over the unphysical degrees of freedom:

$$\begin{aligned}
\Delta_F(x - x') \ &= \ \int_\infty^\infty de \int db \int dc\, \Delta_{gf}(x, e, b, c; x', e', b', c') \\
&= \ \int \frac{d^D p}{(2\pi)^D} \frac{e^{ip \cdot (x - x')}}{p^2 + m^2 - i\varepsilon}.
\end{aligned} \qquad (69)$$

Alternatively, repeated use of the composition principle can be used to construct a configuration space path integral representation of the propagator. Introducing the Fourier representation for the delta function $\delta(e - e')$ repeated use of eq.(64) gives in explicit notation:

$$\Delta_{gf}(x, e, b, c; x', e', b', c') \ = \ \frac{ie}{2m} \int_0^\infty dT \int_\Gamma [\mathcal{D}x(\tau)\mathcal{D}\lambda(\tau)\mathcal{D}e(\tau)\mathcal{D}c(\tau)\mathcal{D}b(\tau)]\ e^{iS_{gf}[x, \lambda, e, b, c]}, \qquad (70)$$

where S_{gf} is the classical gauge-fixed action (28) and the functional integral is over all paths Γ in the configuration space between (x', e', b', c') and (x, e, b, c). From the construction a consistent discrete regularization, specifying the measure to be used, is obtained:

$$
\int_\Gamma \left[\mathcal{D}x(\tau)\mathcal{D}\lambda(\tau)\mathcal{D}e(\tau)\mathcal{D}c(\tau)\mathcal{D}b(\tau) \right] \, e^{iS_{gf}[x,\lambda,e,b,c]} =
$$

$$
= \lim_{N\to\infty} \int \prod_{n=1}^{N} \left[d^D x_n \, d\lambda_n \, de_n \, dc_n \, db_n \, \frac{i\Delta T}{2\pi} \left(\frac{m}{2\pi i e_n \Delta T} \right)^{D/2} \right] \times \tag{71}
$$

$$
e^{i\sum_{k=0}^{N} \Delta T \left[\frac{m}{2e_k} \left(\frac{x_{k+1}^\mu - x_k^\mu}{\Delta T} \right)^2 - \frac{m}{2}e_k + \lambda_{k+1} \frac{(e_{k+1}-e_k)}{\Delta T} + i \left(\frac{b_{k+1}-b_k}{\Delta T} \right) \left(\frac{c_{k+1}-c_k}{\Delta T} \right) \right]},
$$

where again $\Delta T = T/(N+1)$, with T fixed. Note that the measure contains a factor $\sqrt{m/2\pi i e\Delta T}$ for each integral over a co-ordinate x^μ, and a factor $\sqrt{i\Delta T}$ for integration over a ghost b or c. Eqs.(70) and (71) give a precise meaning to relation between the path-integral representation of the propagator in the extended state space and the BRST-invariant gauge-fixed classical action (28).

7 Fermions

The Dirac-Feynman propagator for a free fermion is the solution of the inhomogeneous Dirac equation

$$
(\gamma \cdot \partial_x + m) \, S_F(x - y) = \delta^D(x - y), \tag{72}
$$

with the same causal boundary conditions as for the scalar particle. The Dirac matrices γ^μ form a D-dimensional Clifford algebra and are normalized to satisfy the anti-commutation relations

$$
\{\gamma^\mu, \gamma^\nu\} = 2\,\eta^{\mu\nu}. \tag{73}
$$

The standard Fourier integral representation of the Dirac-Feynman propagator is

$$
S_F(x - y) = \int \frac{d^D p}{(2\pi)^D} \frac{(-i\gamma \cdot p + m)}{p^2 + m^2 - i\varepsilon} e^{ip\cdot(x-y)}. \tag{74}
$$

It is possible to use the Schwinger procedure described in sect.2 to construct a path-integral representation of this propagator as a Clifford-algebra valued object. There is however another method that is often preferred in applications, especially when interactions are introduced into the model. This method uses

anti-commuting variables to represent the Clifford algebra [23, 24, 25]; the application to spinning particles has been studied for example in refs. [10, 13, 14], [26]-[29]. The use of this method here allows a straightforward path-integral representation of the Feynman propagator for fermions in terms of bosonic co-ordinates x^μ and a matching set of fermionic (Grassmann-odd) co-ordinates ψ^μ. However, as the details of the procedure depend on the number of dimensions, we will from now on choose $D = 4$, and limit ourselves to that physically relevant case. Modifications to treat the same problem in another number of dimensions are straightforward to make.

In order to achieve the transition to anti-commuting variables, we define a representation of the Clifford algebra in terms of two anti-commuting variables (ξ^1, ξ^2) by

$$\hat{\gamma}^1 = \xi^1 + \frac{\partial}{\partial \xi^1}, \quad \hat{\gamma}^2 = -i\left(\xi^1 - \frac{\partial}{\partial \xi^1}\right),$$

$$\hat{\gamma}^3 = \xi^2 + \frac{\partial}{\partial \xi^2}, \quad \hat{\gamma}^0 = -i\hat{\gamma}^4 = \xi^2 - \frac{\partial}{\partial \xi^2}. \tag{75}$$

In addition to the $\hat{\gamma}^\mu$ we also need the usual pseudo-scalar element of the Clifford algebra:

$$\hat{\gamma}_5 = -\frac{i}{4!}\,\varepsilon_{\mu\nu\kappa\lambda}\hat{\gamma}^\mu\hat{\gamma}^\nu\hat{\gamma}^\kappa\hat{\gamma}^\lambda$$

$$= \left(2\xi^1\frac{\partial}{\partial \xi^1} - 1\right)\left(2\xi^2\frac{\partial}{\partial \xi^2} - 1\right). \tag{76}$$

Note that in contrast to the other $\hat{\gamma}^\mu$, in this realization $\hat{\gamma}_5$ is represented by a Grassmann-even (bosonic) operator. Therefore we refer to this representation as the *bosonic* form of the $\hat{\gamma}_5$. In later sections we will also encounter a representation of $\hat{\gamma}_5$ in terms of a Grassmann-odd operator, which is appropriately refered to as the *fermionic* representation. Now the following algebraic relations are satisfied by these operators:

$$\{\hat{\gamma}^\mu, \hat{\gamma}^\nu\} = 2\eta^{\mu\nu}, \tag{77}$$

and

$$\hat{\gamma}_5^2 = 1, \qquad \{\hat{\gamma}_5, \hat{\gamma}^\mu\} = 0. \tag{78}$$

These operators therefore realize the Clifford algebra of the Dirac matrices including γ_5.

The operators $(\hat{\gamma}^\mu, \hat{\gamma}_5)$ act on spinors $\Phi(\xi^1, \xi^2)$, here defined as functions with a Grassmann polynomial structure

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$$\Phi(\xi^1, \xi^2) = \phi_2 + \xi^1 \phi_3 - \xi^2 \phi_4 - \xi^1 \xi^2 \phi_1, \tag{79}$$

where all co-efficients are functions either of co-ordinates or momentum. One can define the usual positive-definite inner product for spinors:

$$\langle \Phi, \Psi \rangle = \sum_\alpha \phi_\alpha^* \psi_\alpha. \tag{80}$$

This inner product can be written in the Grassmann representation as

$$\langle \Phi, \Psi \rangle = \int \prod_k \left(d\xi^k d\bar{\xi}^k \right) e^{\bar{\xi} \cdot \xi} \Phi^*(\bar{\xi}) \Psi(\xi), \tag{81}$$

where a second set of Grassmann variables $\bar{\xi}^{1,2}$ has been introduced as argument of the conjugate wave function, and the star denotes complex conjugation of the c-number co-efficients of $\Phi(\xi)$ plus reversal of the order of the ξ^k. It is straightforward to check that w.r.t. this inner product

$$\langle \Phi, \frac{\partial}{\partial \xi^k} \Psi \rangle = \langle \xi^k \Phi, \Psi \rangle. \tag{82}$$

This result implies that in contrast to the other $\hat{\gamma}^i$ ($i = 1, 2, 3$) and $\hat{\gamma}^4$, the time-like operator $\hat{\gamma}^0$ is not real w.r.t. $\langle \Phi, \Psi \rangle$. This is of course to be expected from the lorentzian signature of space-time. On the other hand, hermiticity can be restored by defining a (lorentz invariant) indefinite-metric scalar product $\langle \overline{\Phi}, \Psi \rangle$ using the Pauli-conjugate spinorial wave function

$$\overline{\Phi}(\xi^1, \xi^2) = \phi_4^* + \xi^1 \phi_1^* - \xi^2 \phi_2^* - \xi^1 \xi^2 \phi_3^*. \tag{83}$$

In general the co-efficients of Φ are independent complex numbers, in which case they represent the components of a Dirac spinor ϕ_α. Irreducible Weyl spinors can be obtained as eigenfunctions of $\hat{\gamma}_5$. As

$$\hat{\gamma}_5 \Phi(\xi^1, \xi^2) = \Phi(-\xi^1, -\xi^2), \tag{84}$$

it follows that the Grassmann-even components define a Weyl spinor of positive chirality, whilst the Grassmann-odd components define a Weyl spinor of negative chirality. In the representation chosen here the chirality can therefore be identified with the Grassmann parity of the spinor Φ.

It is also possible to represent Majorana spinors by requiring $C\Phi = \overline{\Phi}$, where $\overline{\Phi}$ is the Pauli-conjugate spinor and C is the (anti-hermitean) charge-conjugation operator

$$C = \left(\xi^1 - \frac{\partial}{\partial \xi^1} \right) \left(\xi^2 - \frac{\partial}{\partial \xi^2} \right). \tag{85}$$

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The Majorana constraint results in the component relations $\phi_4 = \phi_1^*$ and $\phi_3 = -\phi_2^*$. In the following we consider Dirac spinors unless explicitly stated otherwise.

With the above definitions, the Dirac equation can now be transcribed as follows: let

$$\Phi'(\xi^1, \xi^2) = (\hat{\gamma} \cdot p + m\hat{\gamma}_5)\, \Phi(\xi^1, \xi^2); \tag{86}$$

then the components of Φ' are then related to those of Φ by

$$\phi'_\alpha = [(-i\gamma \cdot p + m)\, \gamma_5]_{\alpha\beta}\, \phi_\beta, \tag{87}$$

in the representation of the Dirac matrices defined by

$$\gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}, \quad \gamma^4 = i\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{88}$$

Since the matrix γ_5 is unitary, the equation $\Phi' = 0$ is completely equivalent with the free Dirac equation in momentum space, and we infer that $S_F(x - y)\gamma_5$ can be identified with the inverse of the Dirac operator $(-i\hat{\gamma} \cdot \partial + m\hat{\gamma}_5)$ in co-ordinate space. In the process of translation we have obtained the following correspondence

$$\hat{\gamma}^\mu \mapsto -i\gamma^\mu \gamma_5, \qquad \hat{\gamma}_5 \mapsto \gamma_5. \tag{89}$$

The next step is to write the inverse of the Dirac operator in the Grassmann co-ordinate representation. This is achieved by introducing the ordered symbol for the operator [23, 24, 25]. A quick way to derive the necessary results is the following [30]. For a single Grassmann variable ξ the most general form of a differential operator is

$$A = a_0 + a_1 \xi + a_2 \frac{\partial}{\partial \xi} + (a_3 - a_0)\, \xi \frac{\partial}{\partial \xi}. \tag{90}$$

Consider the action of this operator on an arbitrary function $f(\xi) = f_0 + f_1 \xi$:

$$Af(\xi) = a_0 f_0 + a_2 f_1 + (a_1 f_0 + a_3 f_1)\, \xi. \tag{91}$$

The operation of A on f can be represented equivalently in terms of an integral. First observe that

$$f(\xi) = \int d\xi'\, \delta(\xi' - \xi) f(\xi') = \int d\xi' d\bar{\xi}\, e^{\bar{\xi}(\xi' - \xi)} f(\xi'). \tag{92}$$

Here we have introduced the Fourier representation of the anti-commuting δ-function in terms of a conjugate Grassmann variable $\bar{\xi}$. The ordered symbol $\bar{A}(\xi, \bar{\xi})$ of A is defined by the relation

$$Af(\xi) = \int d\xi' d\bar{\xi} \, \bar{A}(\xi, \bar{\xi}) e^{\bar{\xi}(\xi'-\xi)} f(\xi'). \tag{93}$$

Our construction shows, that it is obtained by replacing every $\partial/\partial\xi$ in A by a $\bar{\xi}$:

$$\bar{A}(\xi, \bar{\xi}) = a_0 + a_1\xi + a_2\bar{\xi} + (a_3 - a_0)\xi\bar{\xi}. \tag{94}$$

In the following it is useful to have an expression for the symbol of a product of operators in terms of the product of their symbols. The relation is given by the equation

$$\overline{[AB]}(\xi, \bar{\xi}) = \int d\xi' d\bar{\xi}' \, e^{(\bar{\xi}'-\bar{\xi})(\xi'-\xi)} \, \bar{A}(\xi, \bar{\xi}')\bar{B}(\xi', \bar{\xi}). \tag{95}$$

It is straightforward to generalize this to a product of an arbitrary number of operators. In that case a symbolic notation for the integration measure is used, of the form

$$\int d^n\xi d^n\bar{\xi} \equiv \int \prod_{\alpha=1}^{n} d\xi_\alpha d\bar{\xi}_\alpha. \tag{96}$$

Returning to the case of a spinning particle in 4-dimensional space-time we add two conjugate variables $(\bar{\xi}^1, \bar{\xi}^2)$ and introduce the symbols for the operators $(\hat{\gamma}^\mu, \hat{\gamma}_5)$, rescaled by factors $1/\sqrt{2m}$ for later convenience:

$$\psi^1 = \frac{1}{\sqrt{2m}} \left(\xi^1 + \bar{\xi}^1\right), \quad \psi^2 = \frac{-i}{\sqrt{2m}} \left(\xi^1 - \bar{\xi}^1\right),$$

$$\psi^3 = \frac{1}{\sqrt{2m}} \left(\xi^2 + \bar{\xi}^2\right), \quad \psi^0 = -i\psi^4 = \frac{1}{\sqrt{2m}} \left(\xi^2 - \bar{\xi}^2\right). \tag{97}$$

and

$$\psi_5 = \frac{1}{\sqrt{2m}} e^{2\bar{\xi}\cdot\xi} = \frac{1}{\sqrt{2m}} \left(2\xi^1\bar{\xi}^1 - 1\right)\left(2\xi^2\bar{\xi}^2 - 1\right). \tag{98}$$

Note again the non-standard feature that, whereas the ψ^μ are Grassmann-odd, the ψ_5 introduced here is Grassmann-even (bosonized).

In terms of these quantities, one can rewrite the Dirac equation in momentum space as

$$\int d^2\xi' d^2\bar{\xi} \, e^{\bar{\xi}\cdot(\xi'-\xi)} \, [p\cdot\psi + m\psi_5](\xi, \bar{\xi}) \, \Phi(\xi') = 0. \tag{99}$$

As in our representation the chirality equals the Grassmann parity, it follows that terms with different Grassmann parity in the wave function are mixed by a non-zero mass.

To obtain the propagator in momentum space we have to invert[3] the Grassmann integral operator in (99). As a first step, we observe that the product rule (95) implies the usual identity

$$\int d^2\xi' d^2\bar{\xi}' \, e^{(\bar{\xi}'-\bar{\xi})\cdot(\xi'-\xi)} \, [p\cdot\psi + m\psi_5]\,(\xi,\bar{\xi}')\,[p\cdot\psi + m\psi_5]\,(\xi',\bar{\xi}) \;=\; \frac{p^2 + m^2}{2m}.$$
(100)

The inverse of the Dirac operator in momentum space can then be written in an extension of the Schwinger representation as

$$
\begin{aligned}
[S_F \hat{\gamma}_5]\,(p_\mu;\xi,\bar{\xi}) &= \frac{-i}{\sqrt{2m}} \int_0^\infty dT \int d\sigma \, e^{-\frac{iT}{2m}\left(p^2+m^2-i\varepsilon\right)-\sigma(p\cdot\psi+m\psi_5)} \\
&= \sqrt{2m}\,\frac{[p\cdot\psi + m\psi_5]\,(\xi,\bar{\xi})}{p^2 + m^2 - i\varepsilon},
\end{aligned}
$$
(101)

where T is the usual Schwinger proper-time parameter, and σ is a Grassmann-odd counterpart. Eq.(100) then implies that

$$\sqrt{2m} \int d^2\xi' d^2\bar{\xi}' \, e^{(\bar{\xi}'-\bar{\xi})\cdot(\xi'-\xi)} \, [p\cdot\psi + m\psi_5]\,(\xi,\bar{\xi}')\,[S_F\hat{\gamma}_5]\,(p_\mu;\xi',\bar{\xi}) \;=\; 1.$$
(102)

In order to get the expression in the co-ordinate representation, we have to compute the Fourier transform. We also redefine the Grassmann variable, making it proportional to proper time T: $\sigma = T\chi$. Now introduce an integral kernel:

$$
\begin{aligned}
K_\chi(x-x';\xi,\bar{\xi}|T) &= \int \frac{d^4p}{(2\pi)^4}\, e^{ip\cdot(x-x')}\, e^{-\frac{iT}{2m}\left(p^2+m^2-i\varepsilon\right)-T\chi(p\cdot\psi+m\psi_5)} \\
&= -i\left(\frac{m}{2\pi T}\right)^2 e^{\frac{im}{2T}(x-x')^2-\frac{iT}{2}(m-i\varepsilon)-m\chi\psi\cdot(x-x')-mT\chi\psi_5}.
\end{aligned}
$$
(103)

Then the following results hold:
(i) The Feynman propagator for a Dirac fermion can be written as

$$[S_F\hat{\gamma}_5]\,(x-y;\xi,\bar{\xi}) \;=\; \frac{-i}{\sqrt{2m}} \int_0^\infty \frac{dT}{T} \int d\chi \, K_\chi(x-y;\xi,\bar{\xi}|T).$$
(104)

(ii) Huygens' composition principle holds in the form

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[3]Of course, this inverse is to be interpreted in the usual sense of distributions to deal with singularities.

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$$\int d^4 x' \int d^2 \xi' d^2 \bar{\xi}' \, e^{\left(\bar{\xi}'-\bar{\xi}\right)\cdot(\xi'-\xi)} \, K_\chi(x-x';\xi,\bar{\xi}'|T') \, K_\chi(x'-x'';\xi',\bar{\xi}|T'')$$

$$= K_\chi(x-x'';\xi,\bar{\xi};|T'+T''). \tag{105}$$

(iii) The integral kernel satisfies the boundary condition

$$K_\chi(x-y;\xi,\bar{\xi}|0) = \delta^4(x-y). \tag{106}$$

Apart from giving the explicit expression (101) for the Dirac-Feynman propagator, eq.(104) can be used to construct a path-integral representation of the propagator by iteration of eq.(105). Indeed, repeated use of this equation leads to the result

$$K_\chi(x-y;\xi,\bar{\xi}|T) = \int \prod_{k=1}^{N} \left[d^4 x_k d^2 \xi_k d^2 \bar{\xi}_k \right] e^{\frac{1}{2}\sum_{j=1}^{N}\left[\left(\bar{\xi}_j - \bar{\xi}_{j-1}\right)\cdot\xi_j - \bar{\xi}_j\cdot(\xi_{j+1}-\xi_j)\right]}$$

$$\times e^{+\frac{1}{2}\left(\bar{\xi}_0 - \bar{\xi}_N\right)\cdot\xi_{N+1} - \frac{1}{2}\bar{\xi}_0\cdot(\xi_1 - \xi_{N+1})} \prod_{s=0}^{N} K_\chi(x_{s+1}-x_s;\xi_{s+1},\bar{\xi}_s|\Delta T), \tag{107}$$

where $x_0 = y$, $x_{N+1} = x$, $\bar{\xi}_0 = \bar{\xi}$, $\xi_{N+1} = \xi$ and $\Delta T = T/(N+1)$. Furthermore

$$\prod_{s=0}^{N} K_\chi(x_{s+1}-x_s;\xi_{s+1},\bar{\xi}_s|\Delta T) =$$

$$= \left[\frac{1}{i}\left(\frac{m}{2\pi\Delta T}\right)^2\right]^{N+1} e^{\frac{i}{2}\sum_{s=0}^{N}\Delta T\left[m\left(\frac{x_{s+1}-x_s}{\Delta T}\right)^2 + 2im\chi\psi_s\cdot\left(\frac{x_{s+1}-x_s}{\Delta T}\right) + 2im\chi\psi_s^5 - m + i\varepsilon\right]} \tag{108}$$

In agreement with our earlier definitions, the symbols ψ_k^μ here are a short-hand notation for

$$\psi_k^1 = \frac{1}{\sqrt{2m}}\left(\xi_{k+1}^1 + \bar{\xi}_k^1\right), \qquad \psi_k^2 = \frac{-i}{\sqrt{2m}}\left(\xi_{k+1}^1 - \bar{\xi}_k^1\right), \tag{109}$$

with similar espressions for (ψ_k^3, ψ_k^0) in terms of $(\xi_{k+1}^2, \bar{\xi}_k^2)$, and the bosonized ψ_k^5 given by

$$\psi_k^5 = \frac{1}{\sqrt{2m}} e^{2\bar{\xi}_k\cdot\xi_{k+1}}. \tag{110}$$

In the continuum limit ($N \to \infty, T$ fixed) the free fermion propagator can now be written as a path integral

$$[S_F \hat{\gamma}_5] (x - y; \xi, \bar{\xi}) = \frac{-i}{\sqrt{2m}} \int_0^\infty \frac{dT}{T} \int d\chi \, \mathcal{D}x(\tau) \mathcal{D}\xi(\tau) \mathcal{D}\bar{\xi}(\tau) \, e^{iS_{ferm}\left[x(\tau), \xi(\tau), \bar{\xi}(\tau)\right]},$$
(111)

where up to boundary terms

$$S_{ferm}\left[x(\tau), \xi(\tau), \bar{\xi}(\tau)\right] =$$

$$\int_0^T d\tau \left[\frac{m}{2} \dot{x}_\mu^2 - \frac{i}{2} \left(\dot{\bar{\xi}} \cdot \xi - \bar{\xi} \cdot \dot{\xi} \right) + im\dot{x} \cdot \chi \, \psi(\xi, \bar{\xi}) + im\chi \, \psi_5(\xi, \bar{\xi}) - \frac{m}{2} \right].$$
(112)

Finally it can be cast into a manifestly Lorentz-invariant form[4] by replacing $(\xi^i(\tau), \bar{\xi}^i(\tau))$ by the vector-like variables $\psi^\mu(\tau)$ defined in eq.(97):

$$S_{ferm}\left[x(\tau), \psi^\mu(\tau)\right] = \frac{m}{2} \int_0^T d\tau \left[\dot{x}_\mu^2 + i\psi \cdot \dot{\psi} + 2i\dot{x} \cdot \chi \, \psi + 2i\chi \, \psi_5 - 1 \right]. \quad (113)$$

This expression resembles closely the one usually encountered in the literature [13, 14], [26]-[29], which is based on the realization of a (gauge-fixed) local world line supersymmetry. However, in contrast to the standard approach the present construction uses a Grassmann-*even* ψ_5, and as a result inclusion of the mass-term violates explicit supersymmetry at the classical level, even though it is realized on the operator level in the quantum theory, as shown by eq.(100). In fact, for non-vanishing m the pseudo-classical action S_{ferm} does not even have a well-defined Grassmann parity; eq.(84) and the discussion following it makes clear that this is a direct consequence of the non-conservation of chirality for massive fermions.

It is shown below, that the discrepancy between the result (112) and the manifestly supersymmetric spinning particle model has its origin in a doubling of the number of degrees of freedom in the supersymmetric case, which is regularly overlooked. More precisely, if one preserves supersymmetry of the classical action by taking ψ_5 to be Grassmann-odd, and if the classical algebra of Poisson-Dirac brackets is mapped in a straightforward way to the quantum (anti-)commutation relations, then one does not obtain the representation of the quantum operator $\hat{\psi}_5$ by the 4-dimensional Dirac matrix $\gamma_5 = -i/4! \, \varepsilon_{\mu\nu\kappa\lambda} \gamma^\mu \gamma^\nu \gamma^\kappa \gamma^\lambda$. Rather, a Grassmann-odd (fermionic) representation of the operator $\hat{\psi}_5$ is to be used, anti-commuting with the other $\hat{\psi}^\mu$. Then the equivalent matrix representation of this

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[4]There is a subtlety concerning the Lorentz-invariance of the continuum limit, as it seems to require that the main contribution to the continuum path integral comes from paths which are smooth in the sense that for $\Delta T \to 0$ one has $\|\xi_{k+1} - \xi_k\| = \mathcal{O}((\Delta T)^{\frac{1}{2}+p})$ with $p > 0$; from the calculation of explicit examples directly in the continuum limit this seems to be correct. Some arguments for a consistent continuum path-integral for simple spin systems, related to the Duistermaat-Heckman theorem, have been advanced in [31].

operator is a γ_5 taken from the Dirac algebra in six dimensions. This implies a doubling of the number of degrees of freedom of an irreducible spinor.

On the other hand, for free massless particles manifest supersymmetry is not violated even in the present theory; this is because chirality is conserved, and so only terms of even Grassmann parity occur in the action. One therefore sees that the apparent violation of manifest world line supersymmetry is a result of mass generation, and has the same dynamical origin.

8 Worldline supersymmetry

In this section we turn to the point-particle model with full classical super-reparametrization invariance on the world line, to compare it with our treatment of Dirac fermions in sect.(7). To realize complete off-shell supersymmetry[5] it is necessary to introduce three different types of super-multiplets (superfields):

1. The gauge multiplet (e, χ) consisting of the einbein e and its superpartner χ, also refered to as gravitino; under local worldline supersymmetry with parameter ε the multiplet transforms as

$$\delta e \ = \ -2i\varepsilon\chi, \qquad \delta\chi \ = \ \frac{d\varepsilon}{d\tau}. \tag{114}$$

2. Matter multiplets (x^μ, ψ^μ) describing the position and spin co-ordinates of the particle. The transformation rules are

$$\delta x^\mu \ = \ -i\varepsilon\psi^\mu, \qquad \delta\psi^\mu \ = \ \varepsilon\frac{1}{e}\frac{Dx^\mu}{D\tau}, \tag{115}$$

where the super-covariant derivative is constructed with the gravitino as the connection:

$$\frac{Dx^\mu}{D\tau} \ = \ \frac{dx^\mu}{d\tau} \ + \ i\chi\psi^\mu. \tag{116}$$

3. One or more fermionic multiplets (η, f) with Grassmann-odd η and even f; it is used in the following as an auxiliary multiplet and its transformation properties under local supersymmetry are

$$\delta\eta \ = \ \varepsilon f, \qquad \delta f \ = \ -i\varepsilon\frac{1}{e}\frac{D\eta}{D\tau}. \tag{117}$$

---

[5]The term *off shell* implies that the supersymmetry algebra is realized without using dynamical constraints like the equations of motion.

The super-covariant derivative is formed as before:

$$\frac{D\eta}{D\tau} = \frac{d\eta}{d\tau} - \chi f. \tag{118}$$

It may be checked that in all cases the commutator of two supersymmetry transformations gives a local translation (reparametrization) with parameter $a = (2i/e)\varepsilon_1\varepsilon_2$, plus a local supersymmetry transformation with parameter $\varepsilon' = -(2i/e)\varepsilon_1\varepsilon_2\chi$.

The minimal free particle action invariant under these one-dimensional local supersymmetry transformations is

$$S_{susy} = \frac{m}{2} \int d\tau \, \left(\frac{1}{e}\dot{x}_\mu^2 + i\psi \cdot \dot{\psi} + \frac{2i}{e}\chi\psi \cdot \dot{x} + i\eta\dot{\eta} + 2i\chi\eta + ef^2 - 2ef\right) \tag{119}$$

Obviously, the bosonic variable f appearing quadratically without derivatives does not represent a dynamical degree of freedom and may be eliminated using its algebraic equation of motion $f = 1$ (which is equivalent to completing the square). Then the classical action becomes

$$S_{susy} = \frac{m}{2} \int d\tau \, \left(\frac{1}{e}\dot{x}_\mu^2 + i\psi \cdot \dot{\psi} + +i\eta\dot{\eta} + \frac{2i}{e}\chi\psi \cdot \dot{x} + 2i\chi\eta - e\right). \tag{120}$$

In addition local super-reparametrization invariance may be used to fix the gauge multiplet (e, χ) to constant values. In particular, if the proper time τ is rescaled by e, so as to be measured in the same units as the laboratory time (effectively setting $e = 1$), and the constant value of χ is rescaled by the same amount, then comparison with the fermionic action (112) shows that S_{ferm} is formally related to S_{susy} by $\dot{\eta} \to 0$, $\eta \to \psi_5$. However, this last relation is frustrated by the mismatch in the Grassmann parity of the two quantities; moreover, contrary to f the variable η is a dynamical degree of freedom, associated with an additional two-valued parameter labeling the wave-functions of the particle. As it couples to the gravitino, and therefore appears in the first-class constraints of the theory, its dynamics cannot be taken into account by simply equating it to a constant in the action. At this point we observe however, that one could add more fermionic multiplets (η_k, f_k) whose auxiliary scalars have vanishing classical value:

$$\Delta S = \frac{m}{2} \int d\tau \, \left(i\eta_k\dot{\eta}_k + ef_k^2\right). \tag{121}$$

Eliminating the f_k by their algebraic Euler-Lagrange equation $f_k = 0$, there remain only free fermionic degrees of freedom η_k which decouple from the dynamics because they do not interact with the other physical or the gauge degrees of freedom. In fact these additional fermions define a $d = 1$ topological field theory, in

the sense that their action is reparametrization invariant without involving the metric (represented by the einbein e). It is also easy to see, that they do not contribute to the hamiltonian of the theory.

To construct the canonical quantum theory corresponding to the supersymmetric action S_{susy}, we employ the BRST procedure used before. Besides the reparametrization ghosts (b, c) and the lagrange multiplier λ we introduce commuting super-symmetry ghosts (α, β) and an anti-commuting multiplier s. The nilpotent BRST variations of the full set of variables read:

1. For the gauge multiplet:

$$\delta_\Lambda e = \frac{d(ce)}{d\tau} + 2\alpha\chi, \qquad \delta_\Lambda \chi = \frac{d(c\chi)}{d\tau} + i\dot{\alpha}. \qquad (122)$$

2. For the matter multiplets:

$$\delta_\Lambda x^\mu = c\dot{x}^\mu + \alpha\psi^\mu, \qquad \delta_\Lambda \psi^\mu = c\dot{\psi}^\mu + \frac{i\alpha}{e}\frac{Dx^\mu}{D\tau}. \qquad (123)$$

3. For the fermionic multiplet:

$$\delta_\Lambda \eta = c\dot{\eta} + i\alpha f, \qquad \delta_\Lambda f = c\dot{f} + \frac{\alpha}{e}\frac{D\eta}{D\tau}. \qquad (124)$$

4. For the ghost variables:

$$\delta_\Lambda c = c\dot{c} - \frac{i\alpha^2}{e}, \quad \delta_\Lambda \alpha = c\dot{\alpha} + \frac{\alpha^2}{e}\chi,$$

$$\delta_\Lambda b = -i\lambda, \qquad \delta_\Lambda \beta = s, \qquad (125)$$

$$\delta_\Lambda \lambda = 0, \qquad \delta_\Lambda s = 0.$$

In the classical theory defined by S_{susy} we wish to impose the gauge conditions

$$\dot{e} = 0, \qquad \dot{\chi} = 0. \qquad (126)$$

Two write down a relatively simple BRST-invariant gauge-fixed action, it is convenient to make the redefinitions:

$$ec \to c, \qquad \alpha - ic\chi \to \alpha. \tag{127}$$

Then the BRST variations are covariantized and simplified; for example

$$\delta_\Lambda \chi = i\dot{\alpha}, \qquad\qquad \delta_\Lambda c = -i\alpha^2, \qquad \delta_\Lambda \alpha = 0,$$

$$\delta_\Lambda x^\mu = \frac{c}{e}\frac{Dx^\mu}{D\tau} + \alpha\psi^\mu, \quad \delta_\Lambda \psi^\mu = \frac{c}{e}\frac{D\psi^\mu}{D\tau} + \frac{i\alpha}{e}\frac{Dx^\mu}{D\tau}, \tag{128}$$

$$\delta_\Lambda \eta = \frac{c}{e}\frac{D\eta}{D\tau} + i\alpha f, \qquad \delta_\Lambda f = \frac{c}{e}\frac{Df}{D\tau} + \frac{\alpha}{e}\frac{D\eta}{D\tau}.$$

In terms of these new variables the BRST-invariant gauge-fixed action reads

$$S_{gf} = S_{susy} + \int_0^T d\tau \left(\lambda\dot{e} + is\dot{\chi} + ib\dot{c} + \beta\dot{\alpha} + 2i\alpha b\chi \right), \tag{129}$$

where we take S_{susy} as in eq.(120), in which $f = 1$ has been inserted. This is consistent with the BRST variations (128) and their nilpotency, provided the equation of motion $\dot{\eta} = \chi$ is used.

The canonical momenta for the physical and gauge variables in this theory are

$$p^\mu = \frac{m}{e}\frac{Dx^\mu}{D\tau},$$

$$\pi^\mu = -\frac{im}{2}\psi^\mu, \quad \pi_\eta = -\frac{im}{2}\eta. \tag{130}$$

$$p_e = \lambda, \qquad \pi_\chi = -is,$$

This is to be supplemented with the ghost momenta

$$\pi_c = -i\dot{b}, \quad \pi_b = i\left(\dot{c} + 2\alpha\chi\right), \tag{131}$$

$$p_\alpha = \dot{\beta}, \qquad p_\beta = \dot{\alpha}.$$

A minor complication is the appearance of the second-class constraints for the fermionic momenta (π^μ, π_η). These are resolved in the standard way by Dirac's procedure, and one ends up with the gauge-fixed, unconstrained hamiltonian

$$H_{gf} = \frac{e}{2m}\left(p_\mu^2 + m^2\right) - i\chi\left(\psi \cdot p + m\eta - 2i\alpha\pi_c\right) - i\pi_b\pi_c + p_\alpha p_\beta, \tag{132}$$

plus the following Dirac-Poisson bracket for functions (F, G) on the unconstrained phase-space spanned by $(x^\mu, p_\mu; \psi^\mu; \eta; e, p_e; \chi, \pi_\chi; c, \pi_c; b, \pi_b; \alpha, p_\alpha; \beta, p_\beta)$:

$$\{F,G\} \;=\; \frac{\partial F}{\partial x^\mu}\frac{\partial G}{\partial p_\mu} - \frac{\partial F}{\partial p_\mu}\frac{\partial G}{\partial x^\mu} + \frac{\partial F}{\partial e}\frac{\partial G}{\partial p_e} - \frac{\partial F}{\partial p_e}\frac{\partial G}{\partial e}$$

$$\frac{\partial F}{\partial \alpha}\frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha}\frac{\partial G}{\partial \alpha} + \frac{\partial F}{\partial \beta}\frac{\partial G}{\partial p_\beta} - \frac{\partial F}{\partial p_\beta}\frac{\partial G}{\partial \beta}$$

$$+ (-1)^{a_F}\left(\frac{i}{m}\frac{\partial F}{\partial \psi^\mu}\frac{\partial G}{\partial \psi_\mu} + \frac{i}{m}\frac{\partial F}{\partial \eta}\frac{\partial G}{\partial \eta} + \frac{\partial F}{\partial \chi}\frac{\partial G}{\partial \pi_\chi} + \frac{\partial F}{\partial \pi_\chi}\frac{\partial G}{\partial \chi} \right)$$

$$+ (-1)^{a_F}\left(\frac{\partial F}{\partial c}\frac{\partial G}{\partial \pi_c} + \frac{\partial F}{\partial \pi_c}\frac{\partial G}{\partial c} + \frac{\partial F}{\partial b}\frac{\partial G}{\partial \pi_b} + \frac{\partial F}{\partial \pi_b}\frac{\partial G}{\partial b} \right).$$

$$(133)$$

The equations of motion then take the canonical form

$$\frac{dF}{d\tau} \;=\; \{F, H_{gf}\}. \tag{134}$$

Similarly, the BRST variation of a phase-space function F can now be obtained from

$$\delta_\Lambda F \;=\; (-1)^{a_F}\{F, \Omega\}, \tag{135}$$

where Ω is the nilpotent, conserved BRST charge

$$\Omega \;=\; \frac{c}{2m}\left(p_\mu^2 + m^2\right) + \alpha\left(p\cdot\psi + m\eta\right) - i\pi_b p_e + ip_\beta \pi_\chi - i\alpha^2 \pi_c, \tag{136}$$

$$\{\Omega, H_{gf}\} \;=\; 0, \qquad \{\Omega, \Omega\} \;=\; 0.$$

Like for the scalar particle, the full hamiltonian H_{gf} is BRST-exact:

$$H_{gf} \;=\; \{-e\pi_c + i\chi p_\alpha, \Omega\}. \tag{137}$$

In the BRST cohomology the hamiltonian is therefore equivalent to zero and the physical particle motions are on the mass-shell.

9 Quantum supersymmetry

The BRST-invariant classical action for the supersymmetric theory can now be taken as the starting point for the construction of a canonical quantum theory for a free point particle. Later this quantum theory is then used to construct a propagator, and it is shown that it can be expressed in path-integral form using precisely the classical action S_{gf}. Thus the correspondence between the

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classical and quantum theory is established in both directions. From the canonical formulation it will then be entirely clear, that for $m \neq 0$ this theory describes a degenerate pair of Dirac fermions, and that this is completely independend of the BRST quantization procedure; the origin of the doubling of the degrees of freedom can in fact be traced to the fermionic dynamical variable η.

The first step in defining the quantum theory is to postulate a set of (anti) commutation relations in correspondence with the classical Dirac-Poisson brackets (133). In addition to the operator (anti) commutators (43), we introduce the operator algebra

$$\left\{\hat{\psi}_\mu, \hat{\psi}_\nu\right\} = \frac{1}{m}\eta_{\mu\nu}, \quad \left\{\hat{\psi}_6, \hat{\psi}_6\right\} = \frac{1}{m},$$

$$\{\hat{\chi}, \hat{\pi}_\chi\} = -i, \tag{138}$$

$$[\hat{\alpha}, \hat{p}_\alpha] = i, \qquad \left[\hat{\beta}, \hat{p}_\beta\right] = i.$$

The notation $\hat{\psi}_6$ (rather than the more usual $\hat{\psi}_5$) has been introduced for the operator corresponding to η, to emphasize the Clifford algebra structure of the fermion anti-commutators, but also to avoid thinking of this operator as the product of the other $\hat{\psi}^\mu$ as in the case of the Dirac fermion in sect.(7). The above commutation relations can be satisfied by choosing the following *linear* representation of the operators: the $\hat{\psi}^\mu$ are realized in terms of two Grassmann variables (ξ^1, ξ^2) and their derivatives, as in (75) by

$$\hat{\psi}_\mu = \frac{1}{\sqrt{2m}} \hat{\gamma}_\mu(\xi^1, \xi^2). \tag{139}$$

As before, the operators $\hat{\psi}_0$ are hermitean only w.r.t. to the physical indefinite-metric inner-product involving Pauli-conjugate spinors.

Furthermore we introduce Grassmann variables ξ^3 and χ, as well as ordinary real variables (α, β) and define

$$\hat{\psi}_6 = \frac{1}{\sqrt{2m}} \left(\xi^3 + \frac{\partial}{\partial\xi^3}\right), \quad \hat{\pi}_\chi = -i\frac{\partial}{\partial\chi},$$

$$\hat{p}_\alpha = -i\frac{\partial}{\partial\alpha}, \qquad \hat{p}_\beta = -i\frac{\partial}{\partial\beta}. \tag{140}$$

Note that we have a fermionic (Grassmann-odd) representation of $\hat{\psi}_6$. As a result, having at our disposition the variable ξ^3, one can define at no expense the additional operator

$$\hat{\psi}_5 = -\frac{i}{\sqrt{2m}} \left(\xi^3 - \frac{\partial}{\partial\xi^3}\right), \tag{141}$$

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which anti-commutes with the other $\hat{\psi}_M$. This operator does not appear in the hamiltonian of the theory, which we define in correspondence with the classical hamiltonian (132):

$$
\begin{aligned}
\hat{H}_{gf} &= \frac{e}{2m}\left(-\Box + m^2\right) - i\chi\left(-i\hat{\psi}\cdot\partial + m\hat{\psi}_6\right) + 2i\alpha\chi\frac{\partial}{\partial c} + i\frac{\partial^2}{\partial b\partial c} - \frac{\partial^2}{\partial\alpha\partial\beta} \\
&\equiv e\hat{H}_0 - i\chi\hat{Q}_+ + 2i\alpha\chi\frac{\partial}{\partial c} + i\frac{\partial^2}{\partial b\partial c} - \frac{\partial^2}{\partial\alpha\partial\beta},
\end{aligned}
\tag{142}
$$

and a nilpotent BRST operator

$$
\begin{aligned}
\hat{\Omega} &= \frac{c}{2m}\left(-\Box + m^2\right) + \alpha\left(-i\hat{\psi}\cdot\partial + m\hat{\psi}_6\right) + i\frac{\partial^2}{\partial e\partial b} - i\frac{\partial^2}{\partial\beta\partial\chi} - \alpha^2\frac{\partial}{\partial c} \\
&= c\hat{H}_0 + \alpha\hat{Q}_+ + i\frac{\partial^2}{\partial e\partial b} - i\frac{\partial^2}{\partial\beta\partial\chi} - \alpha^2\frac{\partial}{\partial c}.
\end{aligned}
\tag{143}
$$

The short-hand notation in the second line of these equations has been introduced with a view to the supersymmetry algebra

$$
\hat{Q}_+^2 = \hat{H}_0.
\tag{144}
$$

At this point a remark of caution is appropriate: for $m \neq 0$ the operator $\hat{H}_0$ has zero-modes only in Minkowski space; but for lorentzian metrics the operator $\hat{Q}_+$ (essentially the Dirac operator) is not hermitean w.r.t. the positive-definite inner-product $\Psi^\dagger\Psi$ (only w.r.t. the Lorentz-invariant indefinite inner product $\overline{\Psi}\Psi$). Therefore the zero-modes of $\hat{H}_0$ (the physical states) are not necessarily zero-modes of $\hat{Q}_+$: one also finds back the negative-energy states associated with the vanishing of $\hat{Q}_+^\dagger$. Moreover, there is a similar relation for the Lorentz-invariant operator $\hat{Q}_-$ obtained by replacing $m \to -m$:

$$
\hat{Q}_-^2 = \hat{H}_0, \qquad \hat{Q}_- = \left(-i\hat{\psi}\cdot\partial - m\hat{\psi}_6\right).
\tag{145}
$$

The resolution of these difficulties lies of course in the full quantum-field theoretical treatment; it is of interest to see how the BRST procedure is implemented in this context, and as a first step we construct in this paper the free propagator, in sect.(10).

At this point it is however sufficient to note, that the BRST-cohomology can still be used to characterize the physical states, in the following way: since the hamiltonian H_{gf} is a BRST-exact operator:

$$
H_{gf} = \left\{ e\frac{\partial}{\partial c} - i\chi\frac{\partial}{\partial\alpha}, \hat{\Omega} \right\},
\tag{146}
$$

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one can pick a state from each equivalence class of solutions of the BRST condition

$$\hat{\Omega}\Psi_{phys} = 0, \tag{147}$$

by requiring the subsidiary condition

$$\left(e\frac{\partial}{\partial c} - i\chi\frac{\partial}{\partial\alpha}\right)\Psi_{phys} = 0, \tag{148}$$

for *arbitrary* values of e and χ, as suggested by super-reparametrization invariance. This amounts actually to two subsidiary conditions:

$$\frac{\partial}{\partial c}\Psi_{phys} = 0, \qquad \frac{\partial}{\partial\alpha}\Psi_{phys} = 0. \tag{149}$$

One may think of these conditions as defining the ghost vacuum. It now follows automatically from (146) that

$$H_{gf}\,\Psi_{phys} = 0. \tag{150}$$

Combined with the BRST invariance of the physical states expressed by (147), this gives the physical states precisely as solutions of the Dirac (and consequently Klein-Gordon) equation:

$$\left(-i\hat{\psi}\cdot\partial + m\hat{\psi}_6\right)\Psi_{phys} = 0. \tag{151}$$

The wave functions Ψ_{phys} can be decomposed as

$$\Psi_{phys}(\xi^1,\xi^2,\xi^3) = \Phi_1(\xi^1,\xi^2) + \xi^3\Phi_2(\xi^1,\xi^2), \tag{152}$$

where each of the terms $\Phi_{1,2}(\xi^1,\xi^2)$ is a 4-component spinor of the type (79). Therefore the physical states have eight spinor components rather than four. Working out the action of the Dirac operator on these wave functions, eq.(151), it can be written explicitly in matrix notation, in terms of ordinary four-dimensional Dirac matrices, as

$$\left(\begin{array}{cc} i\gamma_5\gamma\cdot p & m \\ m & -i\gamma_5\gamma\cdot p \end{array}\right)\left[\begin{array}{c} \Phi_1 \\ \Phi_2 \end{array}\right] = \left(\sum_{\mu=0}^{3}\Gamma_\mu p^\mu + m\Gamma_6\right)\Psi = 0, \tag{153}$$

where the Γ_M are an 8-dimensional representation of the Dirac matrices in 6-dimensional space-time. Thus the supersymmetric spinning particle can be thought of as a reduction of a 6-dimensional massless spinor to 4 space-time dimensions, by compactification on a circle in the 6th dimension ($p^6 = m$) and trivial in x^5 ($p^5 = 0$). It follows, that the states of this theory represent a degenerate pair of 4-dimensional massive fermions.

10   Supersymmetric propagator

In the co-ordinate representation, the quantum states of the supersymmetric particle in the full extended state space (including the ghost degrees of freedom) are represented by wave functions depending on the variables $Z = (x^\mu, \xi^k, e, \chi, c, b, \alpha, \beta)$, with $\mu = 0, ..., 3$ and $k = 1, 2, 3$. To obtain the second quantized propagator for this theory, we first construct the evolution operator associated with the hamiltonian $\hat{H}_{gf}$, eq.(142).

It is actually convenient to do this in two steps: first we construct the expression for the matrix elements of $\hat{K}(T) = e^{-iT\hat{H}_{gf}}$ restricted to the space of gauge- and ghost degrees of freedom $z = (e, \chi; c, b; \alpha, \beta)$, leaving the 'matter' content (x^μ, ξ^k) unspecified; only then we specify the precise model for the physical degrees of freedom. The advantage of this procedure is, that the results are easily applied to other models, for example particles interacting with background fields.

Our starting point is the general expression (142) for $\hat{H}_{gf}$:

$$\hat{H}_{gf} = e\hat{H}_0 - i\chi\hat{Q}_+ + 2i\alpha\chi\frac{\partial}{\partial c} + i\frac{\partial^2}{\partial b\partial c} - \frac{\partial^2}{\partial\alpha\partial\beta},$$

where $\hat{H}_0 = \hat{Q}_+^2$. For any such $\hat{H}_0$ and $\hat{Q}_+$ not depending on the gauge- and ghost degrees of freedom the following results hold:

1. The equation

$$i\frac{\partial}{\partial T}\hat{K}(z, z'|T) = \hat{H}_{gf}\hat{K}(z, z'|T), \tag{154}$$

   with the initial condition

$$\hat{K}(z, z'|0) = \delta(e - e')\delta(\chi - \chi')\delta(c - c')\delta(b - b')\delta(\alpha - \alpha')\delta(\beta - \beta')\hat{I}, \tag{155}$$

   has the solution

$$\hat{K}(z, z'|T) = \frac{1}{2\pi}\delta(e - e')\delta(\chi - \chi')e^{\frac{i}{T}(\alpha-\alpha')(\beta-\beta')-(\alpha+\alpha')(b-b')\chi-\frac{1}{T}(b-b')(c-c')}$$

$$\times e^{-iT(e\hat{H}_0 - i\chi\hat{Q}_+)}. \tag{156}$$

2. This solution satisfies the composition rule

$$\int dz'\, \hat{K}(z, z'|T')\,\hat{K}(z', z''|T'') = \hat{K}(z, z''|T' + T''). \tag{157}$$

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We observe that the gauge $(e, \chi) = const.$ is manifestly realized in the expression (156). Let us now first derive an operator expression for the propagator in the general case (before specifying $\hat{H}_0$ and $\hat{Q}_+$):

$$\hat{\Delta}_{gf}(z, z') = \frac{ie}{2m} \int_0^\infty dT \, \hat{K}(z, z'|T), \tag{158}$$

which is a solution of the generalized Klein-Gordon-Dirac equation

$$\hat{H}_{gf} \, \hat{\Delta}_{gf}(z, z') = \frac{e}{2m} \delta(z - z')\hat{1}. \tag{159}$$

To get an explicit expression for $\hat{\Delta}_{gf}$ we use the following intermediate results:

$$e^{\frac{i}{T}(\alpha - \alpha')(\beta - \beta')} = \frac{T}{2\pi} \int dp_\alpha dp_\beta \, e^{ip_\alpha(\alpha - \alpha') + ip_\beta(\beta - \beta') - iTp_\alpha p_\beta}, \tag{160}$$

and

$$e^{-\frac{1}{T}(b - b')(c - c')} = \frac{1}{T} \int d\pi_b d\pi_c \, e^{(b - b')\pi_b + \pi_c(c - c') - T\pi_b \pi_c}. \tag{161}$$

Note that the factors of T in front of these ghost determinants cancel in the expression for $\hat{K}(T)$, as expected from supersymmetry. It is now straightforward to obtain the operator expression for the propagator in the ghost-momentum space:

$$\hat{\Delta}_{gf}(z, z') = \frac{e}{8\pi^2 m} \delta(e - e')\delta(\chi - \chi') \, e^{-(\alpha + \alpha')(b - b')\chi} \times$$

$$\int dp_\alpha dp_\beta \int d\pi_b d\pi_c \, e^{ip_\alpha(\alpha - \alpha') + ip_\beta(\beta - \beta') + (b - b')\pi_b + \pi_c(c - c')} \left[e\hat{H}_0 - i\chi\hat{Q}_+ + p_\alpha p_\beta - i\pi_b \pi_c \right]^{-1}. \tag{162}$$

The matrix elements of the inverse operator inside the square brackets now have to be computed. Again, we first consider the evolution operator. For the free particle we use the elementary result

$$\langle x | e^{-iT(e\hat{H}_0 - i\chi\hat{Q}_+)} | x' \rangle = \int \frac{d^4p}{(2\pi)^2} e^{ip \cdot (x - x')} e^{-\frac{ieT}{2m}(p_\mu^2 + m^2) - T\chi(\hat{\psi} \cdot p + m\hat{\psi}_6)}. \tag{163}$$

In addition we replace the fermionic operators $(\hat{\psi}^\mu, \hat{\psi}_6)$ by their symbols as in eq.(97), supplemented by

$$\eta = \frac{1}{\sqrt{2m}} \left(\xi^3 + \bar{\xi}^3 \right), \tag{164}$$

where we have returned to the original notation for the additional fermionic degree of freedom. Similarly, for later convenience, we also introduce the symbol of the presently redundant operator $\hat{\psi}_5$, denoting it by η_1:

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$$\eta_1 = -\frac{i}{\sqrt{2m}}\left(\xi^3 - \bar{\xi}^3\right). \tag{165}$$

Carrying out the integration over momentum p_μ then gives the expression for the matrix element $K(Z, Z'|T)$:

$$K_\varepsilon(Z, Z'|T) = \frac{1}{2\pi}\,\delta(e - e')\delta(\chi - \chi')\,e^{\frac{i}{T}(\alpha - \alpha')(\beta - \beta') - (\alpha + \alpha')(b - b')\chi - \frac{1}{T}(b - b')(c - c')}$$

$$\times \left[-i\left(\frac{m}{2\pi eT}\right)^2 e^{\frac{im}{2eT}(x - x')^2 - \frac{ieT}{2}(m - i\varepsilon) - \frac{m}{e}\chi\psi\cdot(x - x') - mT\chi\eta}\right]. \tag{166}$$

Note that after rescaling T and χ by a factor e (or equivalently, taking $e = 1$), the expression in brackets involving the physical degrees of freedom is almost identical with the expression (103) for the kernel $K_\chi(T)$ of the single Dirac fermion, except for the replacement of the Grassmann-even $\psi_5(\xi^i, \bar{\xi}^i)$, $(i = 1, 2)$, by the Grassmann-odd $\eta(\xi^3, \bar{\xi}^3)$.

The expression for the propagator $\Delta_{gf}(Z, Z')$ in the full ghost-extended state space is similarly obtained by taking the matrix element of $\hat{\Delta}(z, z')$; this amounts to making the replacement

$$\frac{e}{2m}\left[e\hat{H}_0 - i\chi\hat{Q}_+ + p_\alpha p_\beta - i\pi_b\pi_c\right]^{-1} = \frac{ie}{2m}\int_0^\infty dT\, e^{-iT\left[e\hat{H}_0 - i\chi\hat{Q}_+ + p_\alpha p_\beta - i\pi_b\pi_c\right]}$$

$$\to \int \frac{d^4p}{(2\pi)^2}\,\frac{e^{ip\cdot(x - x')}}{p^2 + m^2 - i\varepsilon + \frac{2m}{e}\left[-i\chi(p\cdot\psi + m\eta) - i\pi_b\pi_c + p_\alpha p_\beta\right]}, \tag{167}$$

in the r.h.s. of eq.(162), and interpreting (ψ^μ, η) as the above symbols.

The construction of the path-integral formula for the propagator now repeats the steps for the Dirac fermion, with the difference that instead of the symbol of the physical part depending on two canonical pairs of Grassmann variables $(\xi^k, \bar{\xi}^k)$, there are now three such pairs. Then of course there is also the difference that we have included here additional ghost degrees of freedom. However, these do not cause any difficulties. As a result we obtain

$$K_\varepsilon(Z, Z'|T) = \int \left[\prod_{k=1}^N dZ_k\right]\, e^{\frac{1}{2}\sum_{j=1}^N\left[\left(\bar{\xi}_j - \bar{\xi}_{j-1}\right)\cdot\xi_j - \bar{\xi}_j\cdot\left(\xi_{j+1} - \xi_j\right)\right]}$$

$$\times e^{+\frac{1}{2}\left(\bar{\xi}_0 - \bar{\xi}_N\right)\cdot\xi_{N+1} - \frac{1}{2}\bar{\xi}_0\cdot(\xi_1 - \xi_{N+1})}\,\prod_{s=0}^N K_\varepsilon(Z_{s+1}, Z_s|\Delta T), \tag{168}$$

where the exponent involves 3 types of $(\xi^k, \bar{\xi}^k)$ and as before $\Delta T = T/(N + 1)$.

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Representing once more the delta-functions by their Fourier decomposition, and thereby including integrations over lagrange multipliers (λ_k, s_k) in the measure $\prod_k dZ_k$, we can use eq.(166) to evaluate the product of $K_\varepsilon(Z_{s+1}, Z_s|\Delta T)$-factors on the right:

$$\prod_{s=0}^{N} K_\varepsilon(Z_{s+1}, Z_s|\Delta T) = \left[\frac{1}{i}\left(\frac{m}{4\pi^2 e_0 \Delta T}\right)^2\right]^{N+1} \times$$

$$\times e^{i\sum_{s=0}^{N} \Delta T \left[\frac{m}{2e_s}\left(\frac{x_{s+1}-x_s}{\Delta T}\right)^2 - \frac{e_s}{2}(m-i\varepsilon) + i\frac{m}{e_s}\chi_{s+1}\psi_s \cdot \left(\frac{x_{s+1}-x_s}{\Delta T}\right) + im\chi_{s+1}\eta_s + \lambda_{s+1}\left(\frac{e_{s+1}-e_s}{\Delta T}\right) + is_{s+1}\left(\frac{\chi_{s+1}-\chi_s}{\Delta T}\right)\right]}$$

$$\times e^{i\sum_{s=0}^{N} \Delta T \left[i\left(\frac{b_{s+1}-b_s}{\Delta T}\right)\left(\frac{c_{s+1}-c_s}{\Delta T}\right) + \left(\frac{\beta_{s+1}-\beta_s}{\Delta T}\right)\left(\frac{\alpha_{s+1}-\alpha_s}{\Delta T}\right) + i(\alpha_{s+1}+\alpha_s)\left(\frac{b_{s+1}-b_s}{\Delta T}\right)\chi_{s+1}\right]}$$

$$(169)$$

Here the ψ_k^μ are defined as in eq.(109), and we take similarly

$$\eta_k = \frac{1}{\sqrt{2m}}\left(\xi_{k+1}^3 + \bar{\xi}_k^3\right), \qquad \eta_{1k} = -\frac{i}{\sqrt{2m}}\left(\xi_{k+1}^3 - \bar{\xi}_k^3\right). \qquad (170)$$

Also, in the front factor on the right-hand side of (169) we have included a contribution $(e_0)^{2(N+1)}$ in the denominator, instead of $\prod_{k=0}^{N}(e_k)^2$, by making use of the fact that e_k is constant: in the integral its value remains the same from time step to time step.

Finally, the exponent of the finite differences in $(\xi_k, \bar{\xi}_k)$ in eq.(168) can in the continuum limit be rewritten in terms of the ψ^μ, η and η_1 and their derivatives:

$$\sum_{j=1}^{N} \left[\left(\bar{\xi}_j - \bar{\xi}_{j-1}\right) \cdot \xi_j - \bar{\xi}_j \cdot \left(\xi_{j+1} - \xi_j\right)\right] \rightarrow -m\left(\psi \cdot \dot{\psi} + \eta\dot{\eta} + \eta_1\dot{\eta}_1\right). \qquad (171)$$

As a result we can now construct the propagator of the theory in the full ghost-extended state space as

$$\Delta_{gf}(Z, Z') = \frac{ie}{2m}\int_0^\infty dT \int_\Gamma DZ(\tau)\, e^{iS_{gf}[Z(\tau)]}, \qquad (172)$$

where integration over the lagrange multipliers $(\lambda(\tau), s(\tau))$ is to be included in the measure $\mathcal{D}Z(\tau)$, and modulo fermionic boundary terms the action S_{gf} is that of eq.(129) with the addition of a single topological fermion of the kind (121). As has been discussed before, at the classical level this additional fermion is completely harmless, whilst in the quantum theory it signifies the doubling of the number of components of the spinor wave-functions, or equivalently the doubling of the number of degrees of freedom in the propagator.

11 Conclusions

In this paper we have studied the propagators of free spin-0 and spin-1/2 parti-
cles and connected them to classical relativistic particle-mechanics through the
path-integral formalism. It has been established that starting from the known
field-theorietical expressions is advantageous, as it can specify the representation
of certain operators to be used, something which is usually not possible from the
canonical 'Poisson-bracket to commutator' quantization prescription. In the case
of Dirac fermions, this has been used to argue in favour of the bosonic represen-
tation of the operator $\hat{\psi}_5$, implying that non-manifestly supersymmetric models
are prefered to avoid doubling of the spectrum of states.

Our results can be generalized to the case of particles moving in certain back-
ground fields: scalar, vector (e.g., electro-magnetic) or gravitational fields can be
included in the path-integral expressions for the propagators. The constructions
we have presented have been designed in such a way that only minimal additional
work is needed to cover these more general cases. An example we would like to
mention is the inclusion of scalar fields, which by Yukawa couplings may give rise
to mass generation through the mechanism of spontaneous symmetry breaking.
The contribution of the scalar interactions to the classical action (119) is obtained
by replacing the terms for the fermionic multiplet (η, f), which for free particles
read $i\eta\dot{\eta} + 2i\chi\eta + ef^2 - 2ef$, by

$$\Delta L_{sc} = i\eta\dot{\eta} + ef^2 - 2gef\varphi(x) - 2ige\eta\psi^\mu\partial_\mu\varphi(x) + 2ig\chi\eta\varphi(x). \tag{173}$$

Here $\varphi(x)$ is the scalar field, and g is the Yukawa coupling constant. Eliminating
the auxiliary variable f by splitting off a square, this becomes

$$\Delta L_{sc} = i\eta\dot{\eta} + eg^2\varphi^2 - 2ige\eta\psi^\mu\partial_\mu\varphi + 2ig\chi\eta\varphi. \tag{174}$$

This result, here obtained through multilplet calculus, agrees with the result
derived by dimensional reduction in [32]. Taking $\varphi(x) = const. \neq 0$ returns
us to the original action for a free massive particle, and shows how masses are
generated by spontaneous symmetry breaking. It is quite straightforward to
apply these results to the other actions for scalar and Dirac particles, by either
removing all fermionic degrees of freedom, or by keeping them whilst replacing
the fermionic η by the bosonized ψ_5.

Interactions with the electro-magnetic or other vector fields can be introduced,
e.g. through minimal coupling, whilst gravitational interactions result from co-
variantizing the expressions with respect to general co-ordinate transformations.
A more general discussion of the inclusion of background fields will be presented
in a separate paper.

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