The Stückelberg-Kibble Model as an Example of Quantized Symplectic Reduction

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Abstract

Recently, it has been observed that a certain class of classical theories with constraints can be quantized by a mathematical procedure known as Rieffel induction. After a short exposition of this idea, we apply the new quantization theory to the Stückelberg-Kibble model. We explicitly construct the physical state space $\mathcal{H}_{phys}$, which carries a massive representation of the Poincaré group. The longitudinal one-particle component arises from a particular Bogoliubov-transformation of the five (unphysical) degrees of freedom one has started with. Our discussion exhibits the particular features of the proposed constrained quantization theory in great clarity.

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1 Introduction

Classical gauge field theories may be defined by a set of fields $\mathcal{A}$, subject to a set of constraints $\mathcal{B}$, which, in turn, generate gauge transformations. The quantization of such theories is not a unique procedure. Indeed, already in the two best-established quantization methods very different technical setups are chosen. On the one hand, one has the canonical operator formalism, originating with Heisenberg and Pauli [1], and now well-adapted to handle non-abelian gauge theories [2], whereas on the other hand Feynman's path integral formalism [3] allows the quantization of such theories through the Faddeev-Popov procedure [4]. Both methods lead to identical perturbative expansions, but even at a mathematically heuristic level their possible equivalence is only known in perturbation theory.

It is certainly of general interest to have as many conceptually and mathematically different quantization schemes as possible, and to examine the particular features of each of them. The hope of obtaining some hints on how to quantize gravity may provide further motivation for investigating new quantization schemes. Especially, the modern formulation of classical mechanics in terms of symplectic manifolds and Poisson algebras (see e.g. [5]) has suggested more refined quantization procedures, such as geometric quantization [6], and strict deformation quantization [7, 8].

A particular feature of classical gauge theories that should somehow be reflected in the quantization method is that the physical (reduced) phase space may be written as a so-called Marsden-Weinstein quotient [9, 10]. It was shown in [11] that this classical reduction procedure has a satisfactory quantum analogue in a procedure from operator algebra theory known as Rieffel induction [12]. The way we apply this technique is mainly operator-theoretic, but a certain aspect of the path integral formalism, viz. the integration over the gauge group, will play a rôle as well.

This work discusses certain features of the Rieffel induction procedure, as applied to the quantization of constrained systems, which provides a conceptually and technically new method for the quantization of certain gauge field theories. The method in question has already been successfully applied to certain finite-dimensional constrained systems [11], as well as to free quantum electrodynamics [13, 14].

The present work draws on these results. Its aim is two-fold. Firstly, we would like to present the strategy of this new quantization method in a form accessible to a wider scientific community. Therefore, in Chapter 2, we briefly review the main line of argument, leading to the quantization proposal. To keep our presentation reasonably short, we refer for some of the technicalities to the aforementioned papers. Subsequently, in Chapter 3 we apply the new quantization scheme to the Stückelberg-Kibble model. This toy model has often been used in the investigation of the Higgs mechanism and of spontaneous symmetry breaking, see e.g. [15]. Here, we have chosen it since it already shows many of the typical complications of spontaneously broken gauge theories without the need to restrict oneself to a perturbative discussion.

As we shall demonstrate explicitly for this model, the Rieffel induction procedure provides a scheme for the construction of the physical state space of a constrained quantum theory, starting from a larger (unphysical) state space on which the unconstrained theory is defined. Our discussion will focus on the particular properties of this Rieffel-induced physical Hilbert space $\mathcal{H}_{phys}$. Especially, we find that $\mathcal{H}_{phys}$ carries a trivial representation of the gauge group and a massive representation of the Poincaré group. Also, the positive spectrum condition turns out to be satisfied. As an important by-product, we are able to trace back how "would-be Goldstone bosons rearrange to a massive, longitudinal component" in a theory exhibiting the Higgs mechanism.

The context of our work is modern symplectic geometry and reduction theory on the classical side, and algebraic quantum field theory on the quantum side. We only use the 'soft' side of these theories. Good recent introductions are [5, 16, 17], respectively.

2 The quantization of gauge theories with Rieffel induction

After presenting schematically the strategy which leads to Rieffel induction in the quantization of theories with constraints, the remainder of this section briefly specifies some notational and technical prerequisites.

2.1 Quantization of Marsden-Weinstein reduction

The general symplectic reduction procedure, which is quantized by Rieffel induction in its full generality, is described in [11]. Here we are merely concerned with a special case, viz. Marsden-Weinstein reduction at the zero level of the moment map, cf. [5, 18]. To introduce our notation, let us consider free classical electrodynamics. For the functional-analytic and other details which are suppressed in what follows, we refer the interested reader to [13].

We start with the space M of four-component real-valued weak solutions A_μ of the wave equation whose Fourier-transformed Cauchy-data lie in $L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$. That is, $M = \{A_\mu | \Box A_\mu = 0\}$. The imaginary part

$$B(A, A') = 2\mathrm{Im}(A, A')_M = -i \int \frac{d^3\mathbf{p}}{(2\pi)^3} [A^\mu(\mathbf{p}) \overline{A'}_\mu(\mathbf{p}) - \overline{A}^\mu(\mathbf{p}) A'_\mu(\mathbf{p})] \tag{2.1}$$

of the indefinite covariant scalar product $(*, *)_M$ turns M into a symplectic space (M, B), which is the phase space of the unconstrained classical system. The set of constraints is given by the gauge group G, which acts on M via $A_\mu \to A_\mu + \partial_\mu g$, where

$$G = \{g \in \mathcal{S}'(\mathbb{R}^4) \mid \Box g = 0; dg \in M\}. \tag{2.2}$$

Here, the space of distributions $\mathcal{S}'(\mathbb{R}^4)$ is the dual of the usual Schwartz space of rapidly decreasing test functions. In the present example, the reduced phase space (M_c, B_c) of the corresponding constrained system may be obtained by a so-called Marsden-Weinstein reduction [18]. This involves the moment map J from M into the dual of the Lie algebra of G. As G is a vector space, we may identify it with its Lie algebra, so we simply write $J_g(A)$ for the value of $J(A)$ on $g \in G$. Explicitly, the moment map turns out to be $J_g(A) = \mathrm{Im}(\partial g, A)_M$, cf. [13]. The preimage of its zero level is

$$J^{-1}(0) = \{A_\mu \in M \mid \partial_\mu A^\mu = 0\}. \tag{2.3}$$

Then, M_c is given by the Marsden-Weinstein quotient

$$M_c = J^{-1}(0)/G, \tag{2.4}$$

and B_c inherits its structure from B. It is easy to see that (M_c, B_c) defined this way indeed describes the physical degrees of freedom of free electrodynamics: picking $J^{-1}(0)$ fixes the gauge (thus imposing the Gauss law constraint, which on elements of M becomes the Lorentz gauge condition), and quotienting by G removes the gauge degeneracy of the symplectic form B with respect to the action of G on $J^{-1}(0)$.

In principle, there are two possibilities to quantize a reduced phase space (M_c, B_c). Either, we directly quantize the Marsden-Weinstein reduced (i.e. constrained) classical system (M_c, B_c), or we quantize the unconstrained classical system (M, B) together with the set of constraints. In the latter case, a scheme has to be found which imposes constraints on the unconstrained quantized theory, thereby providing a quantum analogue of the classical Marsden-Weinstein reduction. Examples of such schemes are the Dirac or the BRST method. According to the proposal of [11], the so-called Rieffel induction procedure of operator algebra theory [12] (which we explain below) provides a rival scheme, which in all examples studied so far works as well as, or better than the methods mentioned above.

More precisely, let us consider schematically a quantization prescription $Q_\hbar$ which relates the symplectic space (M, B) (or rather the Poisson algebra of functions on it) to some algebra of field operators on a Hilbert space $\mathcal{A}$, G to some algebra $\mathcal{B}$ generated by G, and (M_c, B_c) to some (a priori unknown) algebra of observables (in the sense of gauge-invariant operators) $\mathcal{A}_{obs}$. Then, according to our quantization proposal, the following diagram commutes:

$$\begin{array}{ccc} (M,B);G & \xrightarrow{Q_\hbar} & \mathcal{A};\mathcal{B} \\ {\scriptstyle \text{Marsden–Weinstein Reduction}}\downarrow & & \downarrow{\scriptstyle \text{Rieffel Induction}} \\ (M_c,B_c) & \xrightarrow{Q_\hbar} & \mathcal{A}_{obs} \end{array} \tag{2.5}$$

Our program in this paper is to specify the entries of this diagram for the Stückelberg-Kibble model. To this end, we briefly recall how, for a linear field theory, a symplectic space (M, B) can be related to a field algebra $\mathcal{A}$ of canonical commutation relations, and we explain how Rieffel induction allows one to construct new Hilbert spaces for quantum field theories, thereby eventually specifying $\mathcal{A}_{obs}$.

2.2 Weyl algebras of canonical commutation relations

The general theory behind this subsection is explained in great detail and rigour in, e.g., [17], and the application to electromagnetism is from [19]. We merely mention some of the main points.

For $\phi, \phi' \in M$, the operators $W(\phi)$, $W(\phi')$, satisfying the Weyl form of the canonical commutation relation (CCR)

$$W(\phi)W(\phi') = W(\phi + \phi')e^{\frac{-i}{2}B(\phi,\phi')}, \tag{2.6}$$

specify a field algebra with C^*-structure which we denote by $\mathcal{A}(M, B)$. In most cases, one is primarily interested in the properties of the operator vector potential A_μ, for which we use the same notation as for its classical counterpart, as no confusion will arise. The A_μ satisfy the canonical commutation relations

$$[A_\mu(x), A_\nu(y)] = -ig_{\mu\nu}D(x - y), \tag{2.7}$$

where D denotes the commutator function satisfying $\Box D = 0$, with initial conditions $D(\mathbf{x}, 0) = 0$, $\frac{\partial}{\partial t}D(\mathbf{x}, t)|_{t=0} = -\delta^{(3)}(\mathbf{x})$. To see the connection between (2.6) and (2.7), we consider the vector potential $A(f) = \int d^4x A_\mu(x) f^\mu(x)$, smeared with real test functions f. Now, (2.7) reads $[A(f), A(g)] = i\sigma(f, g)$, where $\sigma(f, g) = -\int d^4x d^4y D(x - y) f^\mu(x) g_\mu(y)$. Formally, this allows for the introduction of the operators $U(f) = e^{[iA(f)]}$ which according to the Baker-Campbell-Haussdorff formula satisfy the Weyl form of the canonical commutation relations $U(f)U(g) = U(f + g)e^{[-\frac{i}{2}\sigma(f,g)]}$. Here, however, $U(f)$ and $U(f')$ have the same commutation relations as long as $\int d^4x D(x - y)(f^\mu(x) - f'^\mu(x)) = 0$ for almost all y. To remove this degeneracy and to obtain a one-to-one correspondence between Weyl operators and test functions, one uses the map $f \to \phi$, defined by the convolution $\phi_\mu = D * f_\mu$. Then, the space M of solutions of the wave equation $\Box\phi_\mu = 0$, $\phi_\mu(\mathbf{x}, t) = \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{2k_0} [\phi_\mu(\mathbf{k}) e^{-ikx} + \overline{\phi_\mu}(\mathbf{k}) e^{ikx}]$, is[1]

$$M = \overline{\{\phi = D * f\}} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^4. \tag{2.8}$$

Now, the operators $W(\phi) = U(f)$, $\phi \in M$ satisfy (2.6) with symplectic form B induced by σ and given in (2.1).

Having established the connection between Weyl operators and vector potentials, we can introduce formal annihilation and creation operators a_μ, a^*_μ. E.g. for the free electromagnetic field, $A_\mu(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0} [e^{-ikx} a_\mu(\mathbf{k}) + e^{ikx} a^*_\mu(\mathbf{k})]|_{k_0 = \mathbf{k}}$,

$$iA(f) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0} [a_\mu(\mathbf{k})\overline{\phi_\mu}(\mathbf{k}) - a^*_\mu(\mathbf{k})\phi_\mu(\mathbf{k})] =: a_\mu(\phi^\mu) - a_\mu(\phi^\mu)^*. \tag{2.9}$$

Clearly, in terms of the annihilation and creation operators, the Weyl operators read $W(\phi^\mu) = \exp[a_\mu(\phi^\mu) - a_\mu(\phi^\mu)^*]$, where $[a_\mu(\phi^\mu), a_\nu(\phi'^\nu)^*] = (\phi', \phi)_M$, $(.,.)_M$ denoting the indefinite Minkowski inner product. Heuristically, one has

$$\frac{d}{d\lambda} W(\lambda\phi)|_{\lambda=0} = iA(f). \tag{2.10}$$

It is well-known that this derivative does not exist in the operator norm but with respect to regular representations only, and thereby the a_μ, a^*_μ only exist in such representations, too. Nevertheless, in what follows we shall adopt the formal expressions (2.9) and (2.10), even when no explicit reference to a particular representation is made.

As a final preparatory step, we point out that subalgebras of $\mathcal{A}(M, B)$ can be specified by selecting subspaces of M. In particular, for free QED,

$$\begin{aligned} N &= \{\phi_\mu \in M | k^\mu \phi_\mu(\mathbf{k}) = 0\} = \{\phi_\mu \in M | \partial^\mu \phi_\mu(x) = 0\}, \\ T &= \{\phi_\mu \in M | \phi_\mu(\mathbf{k}) = ik^\mu g(\mathbf{k})\} = \{\phi_\mu \in M | \phi_\mu(x) = \partial^\mu g(x), \Box g(x) = 0\} \end{aligned} \tag{2.11}$$

[1] Our notation does not distinguish between functions ϕ and their Fourier transforms, since no confusion should arise.

define subalgebras $\mathcal{A}(N,B)$, $\mathcal{A}(T,B)$ of $\mathcal{A}(M,B)$. Note that $T \subset N$, so that $\mathcal{A}(T,B) \subset \mathcal{A}(N,B)$. These subalgebras are Poincaré-invariant, as may be seen by recalling that the action of elements (Λ, a) of the Poincaré group $\mathcal{P}$ on $\mathcal{A}(M,B)$ is defined via the algebraic automorphism $\alpha_{(\Lambda,a)}$,

$$\alpha_{(\Lambda,a)}(W(\phi^\mu)) = W(\gamma_{(\Lambda,a)}(\phi^\mu)) \qquad \text{with} \qquad (\gamma_{(\Lambda,a)}(\phi^\mu))(x) = \Lambda^\mu_\nu \phi^\nu(\Lambda^{-1}(x-a)). \tag{2.12}$$

2.3 Rieffel induction

This subsection gives a quick 'review by example' of some parts of the theory developed in [11] and [13].

In physics, induction methods are mainly known from Wigner's classification and construction of all irreducible unitary representations of the Poincaré group P. In general, the method of induced representations of (locally compact) groups allows one to construct a representation of the complete group from a representation of a subgroup, cf. e.g. [20].

Also, in the theory of operator algebras (particularly C^*-algebras) a method exists for constructing a representation of an algebra, given a representation of some other algebra [12]. The latter is not necessarily a subalgebra of the former; instead, the two algebras need to be connected by a bimodule with certain additional properties. Whatever the technical details, the main idea is that the representation one induces from should be straightforward, and yet capable of producing an appropriate representation of the algebra one is really interested in. This idea will be fully realized in our context, for the second algebra will be the algebra generated by the gauge group, and the representation induced from is the trivial one. With a suitable choice of bimodule, the induced representation of the algebra of observables comes out to be the vacuum representation on a Fock space of physical photon states.

To facilitate our presentation, we proceed by example, abstracting general features afterwards. For free QED, in the diagram (2.5) we choose the field algebra $\mathcal{A} = \mathcal{A}(M,B)$ and the 'algebra of constraints' $\mathcal{B} = \mathcal{A}(T,B)$ (cf. the previous subsection), where the choice of $\mathcal{B}$ is motivated by observing that the gauge group G equals

T, cf. (2.11).[2] Also, we introduce the 'algebra of weak observables' $\mathcal{A}_c := \mathcal{A}(N, B)$, which is the largest subalgebra of $\mathcal{A} = \mathcal{A}(M, B)$ commuting with $\mathcal{B} = \mathcal{A}(T, B)$.

The Rieffel induction procedure will produce a representation of $\mathcal{A}_c$ induced from a representation of $\mathcal{B}$. To this end, we need a bimodule for $\mathcal{A}_c$ and $\mathcal{B}$, that is, a linear space on which $\mathcal{A}_c$ acts from the left, and $\mathcal{B}$ acts from the right (that is, in an anti-representation), so that these two actions commute. In the case at hand, $\mathcal{A}_c$ and $\mathcal{B}$, which is abelian, are each other's commutant in the field algebra $\mathcal{A}$, so that a representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ automatically defines such a bimodule. Finally, we need a representation of $\mathcal{B}$ to induce from. This is the trivial one, defined on the Hilbert space $\mathcal{H}_{tr} = \mathbb{C}$. Schematically,

$$\mathcal{A}_c \longrightarrow \mathcal{H} \longleftarrow \mathcal{B} \longrightarrow \mathcal{H}_{tr}. \tag{2.13}$$

The restriction of the action π on $\mathcal{H}$ of $\mathcal{A}$ to its subalgebra $\mathcal{B}$ defines a representation U of the gauge group, that is, one has $U(\phi) = \pi(W(\phi))$.

As will be discussed in more detail below, this setup allows the construction of a positive semidefinite sesquilinear form $(.,.)_0$ on $L \otimes \mathcal{H}_{tr}$, where L is a suitable dense subspace of L. In the present case, this form is given by

$$(\psi \otimes v, \varphi \otimes w)_0 = v\overline{w} \int_G [\mathcal{D}\phi](U(\phi)\psi, \varphi). \tag{2.14}$$

Here $[\mathcal{D}\phi]$ denotes the non-existent 'Lebesgue' measure on the gauge group G. The point is, however, that this flat 'measure' combines with a factor in the integrand to define a mathematically well-defined path integral (cylindrical) measure on G [13]. Furthermore, ψ, φ are in $\mathcal{H}$, $v, w \in \mathcal{H}_{tr} = \mathbb{C}$, and $(.,.)$ is the inner product on $\mathcal{H}$.

Irrespective of the explicit form of $(.,.)_0$, the induced physical Hilbert space is then defined as the completion of the quotient of $L \otimes \mathcal{H}_{tr}$ by the null space of $(.,.)_0$, i.e.,

$$\mathcal{H}_{phys} = \overline{(L \otimes \mathcal{H}_{tr})/\mathcal{N}}, \tag{2.15}$$

where $\mathcal{N} \subset L \otimes \mathcal{H}_{tr}$ is the subset of vectors with vanishing $(.,.)_0$ norm. The collection of vectors in $\mathcal{H}_{phys}$ of the form $\psi \tilde{\otimes} v$, defined as the image of $\psi \otimes v \in L \otimes \mathcal{H}_{tr}$ under the

[2] For simplicity, we here ignore some mathematical difficulties in defining algebras $\mathcal{B}$ for groups G which are not locally compact. This greatly simplifies our presentation. For more details, we refer to [13, 14].

quotient projection from $L \otimes \mathcal{H}_{tr}$ to $\mathcal{H}_{phys}$, are clearly dense in $\mathcal{H}_{phys}$. The action of elements A of $\mathcal{A}_c$ on $\mathcal{H}_{phys}$ is then given on this dense set by $\pi_{phys}(A)\psi\tilde{\otimes}v = (\pi(A)\psi)\tilde{\otimes}v$. Under appropriate continuity conditions [12, 13] this action may be extended to all of $\mathcal{H}_{tr}$.

The reader should note that $\mathcal{H}_{phys}$ satisfies an essential requirement of a non-degenerate physical Hilbert space: the gauge degeneracy of elements of $\mathcal{A}_c$ is removed in $\pi_{phys}(\mathcal{A}_c)$. To see this, choose an arbitrary element $W(\phi) \in \mathcal{A}_c$. From equation (2.14), it is obvious that for $\phi_t \in T$ (which, we recall, coincides with the gauge group G), $\pi_{phys}(W(\phi))\psi\tilde{\otimes}v = \pi_{phys}(W(\phi+\phi_t))\psi\tilde{\otimes}v$ for all vectors $\psi\tilde{\otimes}v \in \mathcal{H}_{phys}$. Hence, $\pi_{phys}(W(\phi)) = \pi_{phys}(W(\phi+\phi_t))$. This removal of the gauge degeneracy of $\mathcal{A}_c$ is independent of the choice of $\mathcal{H}$, and hence we indentify $\pi_{phys}(\mathcal{A}_c)$ with the representation-independent algebra of observables $\mathcal{A}_{obs}$, cf. (2.5).

Let us now turn to the abstract setting which has led to the $(.,.)_0$-inner product (2.14). As stated, the aim of the Rieffel induction procedure is to obtain a representation π_{phys} of $\mathcal{A}_c$ induced from a representation of $\mathcal{B}$ on some Hilbert space $\mathcal{H}_\chi$. Our example, and all similar examples involving gauge theories, have the special feature that $\mathcal{H}_\chi = \mathcal{H}_{tr} = \mathbb{C}$, that is, one induces from the trivial representation of the gauge group. This will imply that the algebra of constraints $\mathcal{B}$ is represented trivially on the induced space $\mathcal{H}_{phys}$. Technically, the construction of π_{phys} proceeds according to the following three step method.

1. Given a bimodule L for $\mathcal{A}_c$ and $\mathcal{B}$, a $\mathcal{B}$-valued scalar product $\langle .,. \rangle_{\mathcal{B}}$ has to be found on L, that is, for $\psi, \varphi \in L \subset \mathcal{H}$, $\langle \psi, \varphi \rangle_{\mathcal{B}} \in \mathcal{B}$,[3]

[3]Mathematically, $\langle .,. \rangle_{\mathcal{B}}$ is a so-called rigging map which has to satisfy the following conditions for all ψ, $\varphi \in L$ [12]:

(a) $\langle \lambda\psi, \mu\varphi \rangle_{\mathcal{B}} = \overline{\lambda}\mu\langle \psi, \varphi \rangle_{\mathcal{B}}$ for all $\lambda, \mu \in \mathbb{C}$;

(b) $\langle \psi, \varphi \rangle_{\mathcal{B}}^* = \langle \varphi, \psi \rangle_{\mathcal{B}}$ (where the * denotes the hermitian conjugate in $\mathcal{B}$);

(c) $\langle \psi, \varphi B \rangle_{\mathcal{B}} = \langle \psi, \varphi \rangle_{\mathcal{B}} B$ for all $B \in \mathcal{B}$ (on the left-hand side, B acts in the given right-representation on the bimodule L, whereas on the right-hand side B acts by multiplication in the algebra $\mathcal{B}$);

(d) $\langle A\psi, \varphi \rangle_{\mathcal{B}} = \langle \psi, A^*\varphi \rangle_{\mathcal{B}}$ for all $A \in \mathcal{A}$.

(e) $\langle A\psi, A\psi \rangle \leq \| A \|^2 \langle \psi, \psi \rangle$ for all $\psi \in L$, $A \in \mathcal{A}$.

2. Given such an operator-valued scalar product, the tensor product $L \otimes \mathcal{H}_\chi$ is equipped with a sesquilinear form $(.,.)_0$,

$$(\psi \otimes v, \varphi \otimes w)_0 := (\pi_\chi(\langle \varphi, \psi \rangle_{\mathcal{B}})v, w)_\chi. \tag{2.16}$$

   Crucially, this form is positive-semidefinite if the postivity condition $\pi_\chi(\langle \psi, \psi \rangle_{\mathcal{B}}) \geq 0$ for all $\psi \in L$ is satisfied, which is the case in all our examples.

3. The subspace $\mathcal{N} \subset L \otimes \mathcal{H}_{tr}$ of vectors with vanishing $(.,.)_0$-norm is determined and the physical Hilbert space is defined as in (2.15).

The most difficult part of this procedure is to find $\langle .,. \rangle_{\mathcal{B}}$. Here, one is guided by mathematical examples [11]. One may consider e.g. $\mathcal{B} = C^*(G)$, the C^*-group algebra of a locally compact group G (cf. [17]; this is essentially the convolution algebra on the group w.r.t. the Haar measure). Then, it can be shown that a rigging map $\langle .,. \rangle_{\mathcal{B}}$ is defined as follows: $\langle \psi, \varphi \rangle_{\mathcal{B}}$ has to be some element of $C^*(G)$, i.e., a function on the group, and we prescribe that the value of this function at $g \in G$ is given by $\langle \psi, \varphi \rangle_{\mathcal{B}}(g) = (U(g)\varphi, \psi)$, where U is a continuous unitary representation of G on $\mathcal{H}$, commuting with $\pi(\mathcal{A}_c)$, $x \in G$. Inducing from the trivial representation $\mathcal{H}_{tr} = \mathbb{C}$, one obtains[4]

$$(\psi, \varphi)_0 = \int_G dx (U(x)\psi, \varphi), \tag{2.17}$$

of which (2.14) is a special case, at least in a heuristic sense.

In what follows, we shall take a suitable generalization of (2.17) as our starting point, thereby obviating the need for a discussion of the explicit form and a verification of the mathematical properties of $\langle .,. \rangle_{\mathcal{B}}$. In fact, our presentation of the Rieffel induction procedure for quantum field theories has been slightly oversimplified with respect to this point. While the existence of a so-called 'rigged' inner product $(.,.)_0$, defined in (2.16), is always sufficient for the quantization proposal to apply, it is not always possible to derive it from a mathematically well-defined rigging map $\langle .,. \rangle_{\mathcal{B}}$. We refer to [13] for a discussion of the technical points involved.

[4]In what follows, we use the shorthand $(\psi, \varphi)_0$ for $(\psi \otimes v, \varphi \otimes w)_0$, since $v, w \in \mathbb{C}$ are complex numbers which can be absorbed in the definition of ψ and φ.

To sum up: In this chapter, we have seen that Rieffel induction provides a well-defined scheme for the construction of a physical Hilbert space $\mathcal{H}_{phys}$, on which gauge transformations act trivially. In the corresponding algebra of observables $\pi_{phys}(\mathcal{A}_c)$, all gauge degeneracies are removed, i.e., Rieffel induction is a method to impose constraints on quantum field theories. The physical Hilbert space $\mathcal{H}_{phys}$ is obtained by forming the quotient of a larger Hilbert space $L \otimes \mathcal{H}_{tr}$ with respect to a null space.

This is somehwat reminiscent of the BRST (or, in case of QED, the Gupta-Bleuler) procedure, with the major difference that with Rieffel induction no negative-norm subspace exists, obviating the need to select a physical subspace of $\mathcal{H}$. Also, certain functional-analytic problems that appear in the BRST as well as in the Dirac method are absent with our present techniques [11, 13]. By definition of the inner product on the physcial Hilbert space $\mathcal{H}_{phys}$, calculations of correlation functions of operators in $\mathcal{A}_c$ (as represented on $\mathcal{H}_{phys}$) may be performed in $L \otimes \mathcal{H}_{tr}$, [14].

3 Application to the Stückelberg-Kibble model

In this chapter, we specify (2.13) and (2.14) for the Stückelberg-Kibble model, thereby constructing a physical Hilbert space $\mathcal{H}_{phys}$ for this model. The Stückelberg-Kibble model is an abelian Higgs model with the modulus η of the scalar field $\phi(x) = \eta(x)e^{\varphi(x)}$ frozen to unity, $\eta(x) = 1$. It is given by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\left(\partial_\mu\varphi + eA_\mu\right)\left(\partial^\mu\varphi + eA^\mu\right). \tag{3.18}$$

Despite its linearity, this model has non-trivial features, and has been used as testing ground for investigations of the Higgs mechanism before [15]. Its equations of motion can be written in terms of a gauge-invariant current $j^\mu = \partial^\mu\varphi + eA^\mu$, satisfying

$$\left(\Box + e^2\right) j^\mu = 0 \qquad ; \qquad \partial_\mu j^\mu = 0. \tag{3.19}$$

In fact, this is nothing but the Proca equation [21] of a massive gauge-invariant vector field. To make this model amenable to treatment by symplectic reduction and quantum induction methods, we now make a move that is analogous to rewriting the Maxwell equation for A_μ as a massless Klein-Gordon equation plus a subsidiary Lorentz condition. Thus we pass back to the gauge-dependent fields A_μ and φ, and choose what is essentially the 't Hooft gauge as the subsidiary condition:

$$\partial_\mu A^\mu = e\varphi. \tag{3.20}$$

With this constraint, the equations of motion read

$$\left(\Box + e^2\right) A^\mu = 0 \qquad , \qquad \left(\Box + e^2\right) \varphi = 0, \tag{3.21}$$

and the gauge group $G = \{g | \left(\Box + e^2\right) g = 0\}$ acts on A_μ, φ via

$$A_\mu \to A_\mu + \partial_\mu g \qquad , \qquad \varphi \to \varphi - eg. \tag{3.22}$$

3.1 Marsden-Weinstein reduction for the Stückelberg-Kibble model

A mathematically rigorous treatment of the following material, in the style of [13], is possible, but we leave the details to the interested reader; instead, readability commands us to give somewhat loose formulations.

Our investigation of the Stückelberg-Kibble model starts from the symplectic space (M_{sk}, B_{sk}), defined by

$$M_{sk} = \{(A_\mu, \varphi) \mid A_\mu \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^4, \varphi \in L^2(\mathbb{R}^3); \left(\Box + e^2\right) A_\mu = \left(\Box + e^2\right) \varphi = 0\},$$
$$B_{sk}(A_\mu, \varphi; A'_\mu, \varphi') = 2\mathrm{Im}(A_\mu, A'_\mu)_M - 2\mathrm{Im}(\varphi, \varphi'). \tag{3.23}$$

The gauge group G acts on this space by the gauge transformation (3.22). This action is strongly Hamiltonian, and hence, in particular, it is symplectic. We evidently may identify the gauge group with the following subspace of M_{sk}

$$T_{sk} = \{(A_\mu, \varphi) \in M_{sk} \mid A_\mu = \partial_\mu g, \varphi = -eg; g \in L^2(\mathbb{R}^3); \left(\Box + e^2\right) g = 0\}. \tag{3.24}$$

From this, the Marsden-Weinstein reduced space $(M_{c,sk}, B_{c,sk})$ is easily calculated. With similar notation as in subsection 2.1, the moment map reads

$$J_g(A_\mu, \varphi) = 2\mathrm{Im}(\partial_\mu g, A_\mu)_M - 2\mathrm{Im}(-eg, \varphi), \tag{3.25}$$

which leads to

$$J^{-1}(0) = \{(A_\mu, \varphi) \in M_{sk} | \partial_\mu A^\mu = e\varphi\}. \tag{3.26}$$

Hence in view of (3.20) the Marsden-Weinstein quotient reads

$$M_{c,sk} = J^{-1}(0)/G = \{j_\mu \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^4 | \left(\Box + e^2\right) j_\mu = 0; \partial_\mu j^\mu = 0\}. \tag{3.27}$$

The symplectic form $B_{c,sk}$ on $M_{c,sk}$ inherits its structure from B_{sk}, and is given by

$$B_{c,sk}(j, j') = \frac{2}{e^2} \mathrm{Im}(j_\mu, {j'}_\mu)_M. \tag{3.28}$$

Clearly, $(M_{c,sk}, B_{c,sk})$ is the phase space of a massive vector boson, which indeed represents the physical degrees of freedom of the Stückelberg-Kibble model. This completely specifies the left-hand side of the diagram (2.5).

3.2 Rieffel induction for the Stückelberg-Kibble model

3.2.1 Construction of the field algebra

Consider the canonical commutation relations of the operator fields A_μ and φ (denoted by the same symbol as their classical counterparts):

$$\begin{aligned} [\varphi(x), \varphi(y)] &= i\triangle(x-y), \\ [A_\mu(x), A_\nu(y)] &= -ig_{\mu\nu}\triangle(x-y), \end{aligned} \tag{3.29}$$

where the commutator function $\triangle$ satisfies $(\Box + e^2)\triangle(x) = 0$ with initial conditions $\triangle(\mathbf{x}, 0) = 0$, $\frac{\partial}{\partial t}\triangle(\mathbf{x}, t)|_{t=0} = -\delta^{(3)}(\mathbf{x})$. In analogy with our discussion of free QED, we specify the formal connection between the fields A_μ, φ and the corresponding Weyl operators, $W(\phi_\mu, \phi) = e^{iA_\mu(f^\mu) + i\varphi(f)}$, where $\phi_\mu = \triangle * f_\mu$, $\phi = \triangle * f$. Here, either as a consequence of (3.29), or imposed axiomatically, the operators $W(\phi_\mu, \phi)$, $W({\phi'}_\mu, \phi')$ satisfy the Weyl form of the canonical commutation relations

$$W(\phi_\mu, \phi)W(\phi'_\mu, \phi') = W(\phi_\mu + \phi'_\mu, \phi + \phi')e^{-\frac{i}{2}B_{sk}(\phi_\mu, \phi; {\phi'}_\mu, \phi')}. \tag{3.30}$$

The field algebra of the model is then defined as the Weyl algebra $\mathcal{A}(M_{sk}, B_{sk})$ generated by the W's subject to these commutation relations (cf. [17]).

Now, we want to construct the quantum counterpart of Marsden-Weinstein reduction, i.e., we want to complete the right hand side of the diagram (2.5). Therefore, we invoke the quantization prescription for symplectic spaces as discussed in Chapter 2. This leads to the field algebra $\mathcal{A} \equiv \mathcal{A}(M_{sk}, B_{sk})$ defined by (3.23). Also, in analogy with our discussion in Chapter 2, we choose the algebra of constraints $\mathcal{B} = \mathcal{A}(T_{sk}, B_{sk})$; once again, the motivation for this is that it is the (C^*) algebra generated by the gauge group. Consequently, the algebra of weak observables, which by definition is the largest subalgebra of $\mathcal{A}(M_{sk}, B_{sk})$ commuting with $\mathcal{B}(T_{sk}, B_{sk})$, is given by $\mathcal{A}_c = \mathcal{A}(N_{sk}, B_{sk})$, where

$$N_{sk} = \{(\phi_\mu, \phi) \mid \partial^\mu \phi_\mu = e\phi\} \subset M_{sk}; \tag{3.31}$$

compare this with (3.26). The subspaces N_{sk} and $T_{sk} \subset N_{sk}$ of M_{sk} are invariant under the action of symplectic transformations $\gamma_{\Lambda,a}$ associated with elements (Λ, a) of the Poincaré group $\mathcal{P}$, $(\gamma_{\Lambda,a}(\phi_\mu, \phi))(x) := (\Lambda^\nu_\mu \phi_\nu, \phi)(\Lambda^{-1}(x - a)))$, cf. (2.12). Consequently, the subalgebras $\mathcal{A}_c$ and $\mathcal{B}$ are Poincaré-invariant.

3.2.2 Representing the algebra of observables

Rieffel induction starts from the input data of diagram (2.13). So far, we have determined the algebra of weak observables $\mathcal{A}_c = \mathcal{A}(N_{sk}, B_{sk})$ and the algebra of constraints $\mathcal{B} = \mathcal{A}(T_{sk}, B_{sk})$ of the Stückelberg-Kibble model; note that $\mathcal{B} \subset \mathcal{A}_c$. What is needed is a representation of these algebras on some subspace L of a Hilbert space $\mathcal{H}$. In this subsection, we give such a representation on a bosonic Fock space (cf. the corresponding procedure for QED in [13]).

For simplicity, in a first step we introduce a representation for elements $W(\phi_\mu, \phi = 0) \in \mathcal{A}(M_{sk}, B_{sk})$ only. This will subsequently be generalized to the whole algebra. We start from the canonical commutation relations for the smeared annihilation and creation operators $\hat{a}_\mu$, $\hat{a}^*_\mu$,

$$\hat{a}(f) = \hat{a}_\mu(f^\mu) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}[\hat{a}_0(\mathbf{k})\overline{f}_0(\mathbf{k}) + \hat{a}_i(\mathbf{k})\overline{f}_i(\mathbf{k})], \tag{3.32}$$

namely

$$[\hat{a}(f), \hat{a}^*(g)] = (g, f)_E := \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0} g_\mu(\mathbf{k})\delta^{\mu\nu}\overline{f}_\nu(\mathbf{k}). \tag{3.33}$$

For reasons to become clear soon, we have employed the so-called Fermi trick [19] which consists in defining the creation and annihilation operators of a vector field such that their commutator is a Euclidean scalar product. Introducing a vacuum state $|0\rangle$ with the property $\hat{a}(f)|0\rangle = 0$ for all f, the creation and annihilation operators generate a bosonic Fock space $\mathcal{H}_1$ in the usual way. Mathematically $\mathcal{H}_1$ is, of course, the symmetric Hilbert space [22] over $L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$.

We can now represent the field algebra $\mathcal{A}$, and thence its subalgebras $\mathcal{A}_c$ and $\mathcal{B}$, on $\mathcal{H}_1$ as follows:

$$\pi(W(\phi^\mu, \phi = 0)) = e^{[\hat{a}_\mu(\tilde{\phi}_\mu) - \hat{a}_\mu(\tilde{\phi}_\mu)^*]}, \tag{3.34}$$

where $\tilde{\phi}_\mu = \left(-\overline{\phi}_0, \phi_i\right)$, and the symbol π for a representation has been introduced. The essential point is that the Euclidean commutation relations (3.33) are able to represent the Minkowski commutators (3.29) because of the special definition of $\tilde{\phi}_\mu$.

Now, we present a very economical notation for symmetric n-particle states by introducing 'exponential vectors' [22]. To this aim, we represent the algebra $\mathcal{A}_c$ on the dense subset L_1 of $\mathcal{H}_1$, which is the span of all exponential vectors

$$\begin{aligned} L_1 &= \{\sum_{i=1}^{N} \lambda_i e^{\psi^{(i)}} \mid \lambda_i \in \mathbb{C}, \psi^{(i)} \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^4, N < \infty\}; \\ e^\psi &:= 1 \oplus \psi \oplus \frac{1}{\sqrt{2}} \psi \otimes \psi \oplus \frac{1}{\sqrt{3!}} \psi \otimes \psi \otimes \psi \oplus \ldots, \end{aligned} \tag{3.35}$$

where the tensor products are understood to be symmetrized. Note that the prefactors $\frac{1}{\sqrt{n!}}$ of the n-particle contributions to e^ψ have been chosen differently from those of a Taylor expansion of e^x. This allows for a simple form of the scalar product on L_1,

$$(e^\psi, e^\varphi) = e^{(\psi,\varphi)_E}. \tag{3.36}$$

A useful remark is now that symmetric n-particle states can be obtained from suitably normalized derivatives of exponential vectors,

$$\psi_1 \otimes_s \ldots \otimes_s \psi_n = \frac{1}{\sqrt{n!}} \frac{d}{dr_1} \ldots \frac{d}{dr_n} e^{\sum_i r_i \psi_i}|_{r_i=0}. \tag{3.37}$$

The representation of $W(\phi_\mu, 0)$ takes a very simple form on L_1. From (3.35) we have $e^{\hat{a}_\mu(\phi^\mu)}e^\psi = e^{(\psi,\phi)_E}e^\psi$, $e^{\hat{a}_\mu(\phi^\mu)^*}e^\psi = e^{(\psi+\phi)}$ and hence[5]

$$\pi(W(\phi_\mu,0))e^\psi = e^{\frac{-1}{2}(\phi,\phi)_E+(\psi,\tilde{\phi})_E}e^{(\psi-\tilde{\phi})}. \tag{3.38}$$

The construction given above is easily generalized to the whole algebra $\mathcal{A}(M_{sk},B_{sk})$ acting on the dense subspace $L = L_1 \otimes L_2$ of $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, where $\mathcal{H}_2$ is the bosonic Fock space over $L^2(\mathbb{R}^3)$. With

$$L_2 = \{\sum_i^N \lambda_i e^{\psi^{(i)}} \mid \psi^{(i)} \in L^2(\mathbb{R}^3); \lambda_i \in \mathbb{C}, N < \infty\} \tag{3.39}$$

the scalar product of vectors in L, reads

$$(e^{\psi_\mu} \otimes e^\psi, e^{\chi_\mu} \otimes e^\chi) = e^{(\psi_\mu,\chi_\mu)_E+(\psi,\chi)}, \tag{3.40}$$

and the action of $\mathcal{A}(M_{sk},B_{sk})$ (denoted by π as well, with slight abuse of notation) is

$$\pi(W(\phi_\mu,\phi))e^{\psi_\mu} \otimes e^\psi = e^{\frac{-1}{2}(\phi_\mu,\phi_\mu)_E+(\psi_\mu,\tilde{\phi_\mu})_E}e^{\frac{-1}{2}(\phi,\phi)+(\psi,\phi)}e^{\psi_\mu-\tilde{\phi_\mu}} \otimes e^{\psi-\phi}. \tag{3.41}$$

It should be pointed out that L is only stable under finite linear combinations of the W's (which span a dense subalgebra of $\mathcal{A}$), and not under all elements of $\mathcal{A}$. Hence, strictly speaking, the induction process is performed relative to the corresponding dense subalgebras of $\mathcal{A}_c$ and $\mathcal{B}$.

3.2.3 Constructing the physical one-particle Hilbert space

With (3.41), we have specified the bimodule L for $\mathcal{A}_c$ and $\mathcal{B}$, which in this case is a subspace of an 'unphysical' Hilbert space $\mathcal{H}$. Our next step is to construct the corresponding physical Hilbert space, i.e., to carry out the discussion following (2.13). In this and the next subsection, we determine the null space $\mathcal{N}_{sk}$ for the Stückelberg-Kibble model, thereby eventually obtaining $\mathcal{H}_{phys}$.

We start from the inner product on elementary vectors in L

$$(e^{\psi_\mu} \otimes e^\psi, e^{\chi_\mu} \otimes e^\chi)_0 = \int_{T_{sk}} [\mathcal{D}g](\pi(W(\partial_\mu g, -eg))e^{\psi_\mu} \otimes e^\psi, e^{\chi_\mu} \otimes e^\chi), \tag{3.42}$$

[5]To see that this defines a representation, we check that

$$\pi(W(\phi_\mu,0))\pi(W(\varphi_\mu,0)) = e^{[i\mathrm{Im}[(\overline{\phi}_0,\overline{\varphi}_0)_E+(\phi_i,\varphi_i)_E]]}\pi(W(\phi_\mu+\varphi_\mu,0)),$$

where $\mathrm{Im}[(\overline{\phi}_0,\overline{\varphi}_0)_E + (\phi_i,\varphi_i)_E] = B_{sk}(\varphi_\mu,0;\phi_\mu,0)$.

which is a natural generalization of (2.14) (and can, at least heuristically, be derived from an appropriate rigging map defined by a unitary representation of the gauge group on $\mathcal{H}$). As in [13], the heuristic path integral (3.42) can be turned into a well-defined integral w.r.t. a certain cylindrical measure on $T_{sk} = G$, but here we shall proceed with the formal flat measure $\mathcal{D}g$, and certify that all manipulations below can be rigorously justified.

Using the representation (3.41) of $\mathcal{A}(N_{sk}, B_{sk})$, we obtain, with $k_0 = \sqrt{e^2 + \mathbf{k}^2}$, and $d\tilde{k} = \frac{d\mathbf{k}^3}{(2\pi)^3 2k_0}$,

$$\begin{aligned}(e^{\psi_\mu} \otimes e^{\psi} \quad , \quad e^{\chi_\mu} \otimes e^{\chi})_0 &\\ = e^{\int d\tilde{k} \frac{-1}{k_0^2}[(k_i\psi_i - ie\psi)k_0\psi_0 + (k_i\overline{\chi}_i + ie\overline{\chi})k_0\overline{\chi}_0]} &\\ \times e^{\int d\tilde{k}\psi_i\left(\delta_{ij} - \frac{k_ik_j}{\mathbf{k}^2}\right)\overline{\chi}_j + \left(\frac{e}{k_0}\psi_i + i\frac{k_i}{k_0}\psi\right)\frac{k_ik_j}{\mathbf{k}^2}\overline{\left(\frac{e}{k_0}\chi_j + i\frac{k_j}{k_0}\chi\right)}}&, \end{aligned} \tag{3.43}$$

where we have used $\left(\delta_{ij} - \frac{k_ik_j}{k_0^2}\right) = \left(\delta_{ij} - \frac{k_ik_j}{\mathbf{k}^2}\right) + \frac{e^2k_ik_j}{k_0^2\mathbf{k}^2}$ to write (3.43) in terms of projection operators.

To investigate the structure of the null space $\mathcal{N}_{sk}$, we derive the $(.,.)_0$-inner product for n-particle vectors in $\mathcal{H}$ from (3.43). For one-particle vectors in the (unphysical) space $\mathcal{H}$, we have

$$\frac{d}{dr}e^{r\psi_\mu} \otimes e^{r\psi}|_{r=0} = \psi_\mu \otimes \Omega' + \Omega'' \otimes \psi, \tag{3.44}$$

where $\Omega = \Omega'' \otimes \Omega'$ denotes the vacuum state in $\mathcal{H}$. Since such expressions become cumbersome for higher derivatives, for notational convenience we define

$$\psi_*^{(1)} \times \ldots \times \psi_*^{(n)} := \frac{1}{\sqrt{n!}}\frac{d}{dr_1}\ldots\frac{d}{dr_n}e^{\sum_i r_i\psi_\mu^{(i)}} \otimes e^{\sum_j r_j\psi^{(j)}}|_{r_i=0}. \tag{3.45}$$

Then, the $(.,.)_0$-inner product on one-particle vectors in $\mathcal{H}$ reads

$$(\psi_*, \chi_*)_0 = \int d\tilde{k}\psi_i\left(\delta_{ij} - \frac{k_ik_j}{\mathbf{k}^2}\right)\overline{\chi}_j + \left(\frac{e}{k_0}\psi_i + i\frac{k_i}{k_0}\psi\right)\frac{k_ik_j}{\mathbf{k}^2}\overline{\left(\frac{e}{k_0}\chi_j + i\frac{k_j}{k_0}\chi\right)}. \tag{3.46}$$

Clearly, the two transversal components $P_T\psi_* := \left(\delta_{ij} - \frac{k_ik_j}{\mathbf{k}^2}\right)\psi_j$ and a linear combination $P_L\psi_*$ of the longitudinal component $\frac{k_ik_j}{\mathbf{k}^2}\psi_j(\mathbf{k})$ with the scalar component $\psi(\mathbf{k})$ survive, while the remaining two components lie in $\mathcal{N}_{sk}$. To be more precise, we introduce for ψ_* the *Bogoliubov-transformed components* ψ_L, ψ_N,

$$\begin{aligned}\psi_{L,i}(\mathbf{k}) &:= \cos\theta\frac{k_i k_j \psi_j(\mathbf{k})}{\mathbf{k}^2} + i\sin\theta\frac{k_i\psi(\mathbf{k})}{|\mathbf{k}|},\\ \psi_{N,i}(\mathbf{k}) &:= -\sin\theta\frac{k_i k_j \psi_j(\mathbf{k})}{\mathbf{k}^2} + i\cos\theta\frac{k_i\psi(\mathbf{k})}{|\mathbf{k}|},\end{aligned} \tag{3.47}$$

where $\cos\theta = \frac{e}{k_0}$, $\sin\theta = \frac{|\mathbf{k}|}{k_0}$. With ψ_L, ψ_T and ψ_N, the five-component vector $\psi_*^{(i)}$ can be specified as

$$\psi_*(\mathbf{k}) := (P_T\psi_\mu(\mathbf{k}), \psi_L(\mathbf{k}), \psi_N(\mathbf{k}), \psi_0(\mathbf{k}))\,, \tag{3.48}$$

and the projection operator P_p onto the 'physical' one-particle components is given by

$$(P_p\psi_*)(\mathbf{k}) = (P_T\psi_\mu(\mathbf{k}), \psi_L(\mathbf{k}), 0, 0)\,. \tag{3.49}$$

This is exactly what one expects: the five 'unphysical' degrees of freedom have combined into three physical ones in such a way that the longitudinal component in $\mathcal{H}$ has mixed with the scalar component.

3.2.4 The physical Hilbert space $\mathcal{H}_{phys}$

To extend (3.49) to n-particle states, we rewrite (3.43), using

$$\begin{aligned}\exp(\sum_i r_i\psi_*^{(i)}) &:= \exp(\sum_i r_i\psi_\mu^{(i)}) \otimes \exp(\sum_i r_i\psi^{(i)}),\\ (e^{\psi_*}, e^{\chi_*})_0 &= (e^{\psi_*}, \Omega)_0(\Omega, e^{\chi_*})_0(e^{P_p\psi_*}, e^{P_p\chi_*}).\end{aligned} \tag{3.50}$$

Here we have used the remark following (3.46), which implies that

$$(\exp(P_p\psi_*), \exp(P_p\chi_*))_0 = (\exp(P_p\psi_*), \exp(P_p\chi_*)).$$

From (3.50) we obtain

$$\begin{aligned}\psi_*^{(1)}\times\ldots\times\psi_*^{(n)} &= \frac{d}{dr_1}\cdots\frac{d}{dr_n}(e^{\sum_i r_i\psi_*^{(i)}}, \Omega)_0 e^{\sum_i r_i\psi_*^{(i)}}|_{r_i=0}\\ &= \sum_{q=0}^{n}\sum_{(p_i)_1^q\in\mathcal{P}_{q,n}} \lambda_{(p_i)_1^q}(P_p\psi_*^{(p_1)})\times\ldots\times(P_p\psi_*^{(p_q)}) + \vec{n},\end{aligned} \tag{3.51}$$

where $\mathcal{P}_{q,n}$ contains all sets of q indices $\{(p_i)_1^q\}$ out of $\{1,...,n\}$, such that $\{(p_i)_1^q\} \cup \{(\hat{p}_i)_1^{n-q}\} = \{1,...,n\}$. Here,

$$\lambda_{(p_i)_1^q} = \sqrt{\frac{(q)!(n-q)!}{n!}}(\psi_*^{(\hat{p}_1)} \times ... \times \psi_*^{(\hat{p}_{n-q})}|_{p_q(I_{n,q})}, \Omega)_0 \tag{3.52}$$

are c-number coefficients and $\vec{n}$ denotes an element in $\mathcal{N}_{sk}$.

Vectors of the type (3.49) generate a Hilbert space of physical one-particle states. The bosonic Fock space over this one-particle space is evidently $\mathcal{F}_{phys} := \mathcal{S}(L^2(\mathbb{R}^3) \otimes \mathbb{C}^3)$, the symmetric Hilbert space over $(L^2(\mathbb{R}^3) \otimes \mathbb{C}^3)$. It should be clear from equation (3.51) that the induced space $\mathcal{H}_{phys}$ from the Rieffel induction procedure is naturally isomorphic to this physical Fock space.[6] To prove this, we define a map $V : L \to \mathcal{F}_{phys}$ by linear extension of $V \exp(\psi_*) = (\exp(\psi_*), \Omega)_0 \exp(P_p\psi_*)$. It follows from an argument similar to the one in section 3.3 of [13] that this map is well-defined (which is a nontrivial property, as the basis $\{\exp(\psi_*)\}$ is overcomplete). Eq. (3.50), and the fact that the inner product in $\mathcal{F}_{phys}$ is just the one in $\mathcal{H}$, restricted to the physcial states, then implies the crucial property

$$(V\Psi, V\Phi) = (\Psi, \Phi)_0 \tag{3.53}$$

for all $\Psi, \Phi \in L$, where the inner product on the l.h.s. is evidently the one in $\mathcal{F}_{phys}$. Hence the null space $\mathcal{N}_{sk}$ of $(.,.)_0$ is precisely the kernel of V, and the quotient map $\tilde{V} : L/\mathcal{N}_{sk} \to \mathcal{F}_{phys}$ can be extended to a unitary map (denoted by the same symbol) $\tilde{V} : \mathcal{H}_{phys} \to \mathcal{F}_{phys}$.

3.2.5 n-point correlation functions and gauge-invariance

Having specified the physical Hilbert space $\mathcal{H}_{phys}$, the next step is to determine the action of $\pi_{phys}(\mathcal{A}_c)$. To this end, we consider the generating functional ω_{vac} for vacuum expectation values,

$$\begin{aligned} \omega_{vac}(\phi^\mu, \phi) &:= (\pi(W(\phi_\mu, \phi))\Omega, \Omega)_0 \\ &= e^{\frac{1}{2}(\phi_\mu,\phi_\mu)_M} e^{-\frac{1}{2}(\phi,\phi)} e^{-\frac{1}{k_0^2}(k_0\overline{\phi}_0(k_\mu\phi_\mu + ie\phi))}, \end{aligned} \tag{3.54}$$

[6] Of course, all Hilbert spaces of the same dimension are unitarily equivalent, but to impose such equivalence one generally has to pick a basis. We use the term 'naturally isomorphic' to indicate that a unitary equivalence exists which doesn't require the choice of a basis. From the point of view of representation theory, this equivalence intertwines the actions of appropriate operator algebras, cf. the next subsection.

where $\Omega \in \mathcal{H}$ is the (unphysical) 'vacuum' state. By construction, only $\mathcal{A}_c = \mathcal{A}(N_{sk}, B_{sk})$ acts on $\mathcal{H}$ (cf. (3.31)), and for $(\phi_\mu, \phi) \in N_{sk}$, $k_0\phi_0 = k_i\phi_i - ie\phi$, we obtain

$$\begin{aligned}
\omega_{vac}(\phi^\mu, \phi) &= e^{-\frac{1}{2}(\phi_\mu, P_T\phi_\mu)_E} e^{-\frac{1}{2}(\frac{e}{k_0}\phi_i + i\frac{k_i}{k_0}\phi)\frac{k_ik_j}{\mathbf{k}^2}(\frac{e}{k_0}\overline{\phi}_j - i\frac{k_i}{k_0}\overline{\phi})} \\
&=: (\pi_{phys}(\tilde{W}(P_p\phi_*))\Omega_{phys}, \Omega_{phys})_{phys} \\
&= e^{-\frac{1}{2}\int d\tilde{k}[\overline{P_T\phi(\mathbf{k})}P_T\phi(\mathbf{k}) + \overline{\phi}_{L,i}(\mathbf{k})\phi_{L,i}(\mathbf{k})]}. \qquad (3.55)
\end{aligned}$$

Here, $\Omega_{phys} \in \mathcal{H}_{phys}$ is the physical vacuum state; it is just the projection of $\Omega \in L$ onto $L/\mathcal{N}_{sk} \subset \mathcal{H}_{phys}$.

We observe that for $(\phi_\mu, \phi) \in T_{sk}$, $\pi(W(\phi_\mu, \phi))$ equals the unit operator, cf. (3.24). This implies that the gauge group is represented trivially on $\mathcal{H}_{phys}$. Moreover, one infers that $\mathcal{A}_{obs} := \pi(\mathcal{A}_c) \simeq \mathcal{A}(N_{sk}/T_{sk}, B_{c,sk})$, since the image of a representation of a C^*-algebra is isomorphic to the algebra quotiented by the kernel of the representation. Now $N_{sk}/T_{sk} \simeq P_pN_{sk}$ as vector spaces (but not as carrier spaces of actions of the Poincaré group!), so that, equally well, $\mathcal{A}_{obs} \simeq \mathcal{A}(P_pN_{sk}, B_{sk})$.[7] Then, it is clear from section 3.1 that $\mathcal{A}_{obs}$ is precisely the Weyl algebra over de Marsden-Weinstein reduced space (i.e., the physical phase space) of the Stückelberg-Kibble model. Hence it describes three gauge-invariant, massive field components.

Thus $\tilde{W}(P_p\phi_*))$ can be viewed as a Weyl operator in $\mathcal{A}(P_pN_{sk}, B_{sk})$. In particular, the representation of $\mathcal{A}(P_pN_{sk}, B_{sk})$ on exponential vectors $e^\psi \in \mathcal{H}_{phys} = \mathcal{S}(L^2(\mathbb{R}^3) \otimes \mathbb{C}^3)$ is given by

$$\begin{aligned}
\pi_{phys}(\tilde{W}(P_p\phi_*))e^\psi &= e^{-\frac{1}{2}(P_p\phi_*, P_p\phi_*)_p + (\psi, P_p\phi_*)_p} e^{(\psi - P_p\phi_*)} \\
(\psi, P_p\phi_*)_p &= \int d\tilde{k}[\overline{P_T\phi(\mathbf{k})}P_T\phi(\mathbf{k}) + \overline{\phi}_{L,i}(\mathbf{k})\phi_{L,i}(\mathbf{k})]. \qquad (3.56)
\end{aligned}$$

From $\omega_{vac}(\phi_\mu, \phi)$, n-point correlation functions can be obtained as multiple derivatives of $\tilde{W}(P_p\phi_*) := e^{i\tilde{A}(f)}$, where $P_p\phi_* = \triangle * f \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^3$.

$$i^n \quad (\pi_{phys}(\tilde{A}(f_1)...\tilde{A}(f_n)\Omega_{phys}, \Omega_{phys})_{phys} = \frac{d}{dr_1}...\frac{d}{dr_n}\omega_{vac}(\sum_i r_i\phi_\mu^{(i)}, \sum_i r_i\phi^{(i)})|_{r_i=0}$$

[7]However, the isomorphism between $\mathcal{A}(P_pN_{sk}, B_{sk})$ and $\mathcal{A}(N_{sk}/T_{sk})$ does not preserve the (automorphic) action of the Poincaré group, which, indeed, acts on the latter but not on the former, cf. [19].

$$= \sum_{(p_i,q_i)_i^{\frac{n}{2}} \in \mathcal{S}_n} \prod_{i=1}^{\frac{n}{2}} (\pi_{phys}(\tilde{A}(f_{p_i})\tilde{A}(f_{q_i})\Omega_{phys}, \Omega_{phys})_{phys}(-1)^{\frac{n}{2}} \tag{3.57}$$

for n even and zero otherwise. Here, $\mathcal{S}_n$ denotes the set of all symmetric partitions of $\{1, ..., n\}$ into a set of unordered pairs (p_i, q_i). We conclude from (3.57) that the n-point correlation functions can be decomposed into products of 2-point correlation functions, i.e., Wick's theorem is satisfied. The reader should note, however, that this form of Wick's theorem is satisfied for elements in $\mathcal{A}(N_{sk}, B_{sk})$ only. The crucial point is that in general, the $(.,.)_0$-inner product preserves the adjoint for test functions in N_{sk} only. This can be seen by comparing, e.g., $\frac{d}{dr_1}\frac{d}{dr_2}(\pi(W(\sum_i r_i\phi_\mu^{(i)}, \sum_i r_i\phi^{(i)}))\Omega, \Omega)_0|_{r_i=0}$ with $\frac{d}{dr_1}\frac{d}{dr_2}(\pi(W(\phi_\mu^{(1)}, \phi^{(1)}))\Omega, \pi(W(\phi_\mu^{(2)}, \phi^{(2)}))\Omega)_0|_{r_i=0}$, cf. (3.54).

There is an interesting parallel between this restriction of the Rieffel induced expectation values to $\mathcal{A}(N_{sk}, B_{sk})$ and the general set-up of the Gupta-Bleuler indefinite metric formalism as presented in [23]. In the latter, one starts from an unphysical Hilbert space $\mathcal{H}_{GB}$ from which the physical one is obtained as a quotient $\mathcal{H}'/\mathcal{H}''$. Without reviewing this construction, we note that $\mathcal{H}$ has to be restricted to a suitable subspace $\mathcal{H}' \subset \mathcal{H}_{GB}$ before quotiening by a null space $\mathcal{H}'$. Obviously, in our setting, a similar restriction is needed on the level of the algebra, $\mathcal{A}(N_{sk}, B_{sk}) \subset \mathcal{A}(M_{sk}, B_{sk})$. This restriction emerges in a systematic way, for as we pointed out before, the subalgebra in question is the commutant of the algebra generated by the constraints (i.e., by the gauge group).

This observation is closely related to the result of Narnhofer and Thirring [24] that covariant formulations without indefinite inner metric are possible as long as the representation on the physical Hilbert space is restricted to a certain subalgebra of weak observables. In the example of Narnhofer and Thirring, non-regular states have to be introduced. This can be avoided in the Rieffel induction setting, cf. [13, 14] for further details.

3.2.6 Positivity of the Hamiltonian and action of the Poincaré group

On the algebra of weak observables of the Stückelberg-Kibble model $\mathcal{A}(N_{sk}, B_{sk})$, the time evolution is given as an automorphism group τ_t,

$$\tau_t[W(\phi_\mu, \phi)] = W(e^{it\sqrt{D+e^2}}\phi_\mu, e^{it\sqrt{D+e^2}}\phi), \tag{3.58}$$

where $(D\phi)_\mu = (-\triangle\phi_0, -\triangle\phi_1, -\triangle\phi_2, -\triangle\phi_3)$. We want to construct the Hamiltonian H, corresponding to τ_t on $\mathcal{H}$. H is a representation-dependent operator, implementing the time evolution τ_t in the representation π by

$$e^{itH}\pi(W(\phi_\mu, \phi))e^{-itH} = \pi(\tau_t[W(\phi_\mu, \phi)]). \tag{3.59}$$

Comparing this with the explicit form of the representation in terms of annihilation and creation operators $\hat{a}^*_\mu$, $\hat{a}_\mu$ for the vector field and $\hat{b}^*$, $\hat{b}$ for the scalar field, we obtain

$$H = -\int d\tilde{k}\sqrt{\mathbf{k}^2 + e^2}\hat{a}^*_\mu(\mathbf{k})g^{\mu\nu}\hat{a}_\nu(\mathbf{k}) + \int d\tilde{k}\sqrt{\mathbf{k}^2 + e^2}\hat{b}^*(\mathbf{k})\hat{b}(\mathbf{k}). \tag{3.60}$$

Regarded as an operator on $\mathcal{H}$ (with its Hilbert space inner product), this Hamiltonian clearly has the entire real axis as its spectrum. However, it is easy to see that

$$(\Psi, H\Psi)_0 \geq 0 \tag{3.61}$$

for all $\Psi \in \mathcal{H}$. The point is that arbitrary (normalized) components of the physical one-particle state space, $\left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right)\psi_j$ and $\frac{k_i}{\mathbf{k}}\psi_i \cos\theta + i\psi \sin\theta$ pick up (the same) positive energy contributions. For multi-particle states, this holds true due to their decomposition into such components. The elements of $\mathcal{H}$ carrying the negative energy spectrum have ended up in the null space. Hence the induced Hamiltonian H_{phys} on $\mathcal{H}_{phys}$ is positive.

Finally, we note that $\mathcal{H}_{phys}$ carries a massive representation of the Poincaré group $\mathcal{P}$. Indeed, ω_{vac} is Poincaré invariant on N_{sk} and hence [16, 17] there exists a Poincaré invariant vacuum state $\Omega_{phys} \in \mathcal{H}_{phys}$ and a representation U_p of the Poincaré group, such that

$$U_p(\Lambda, a)\pi_{phys}(W(\phi_\mu, \phi))\Omega_{phys} = \pi_{phys}(W(\gamma_{\Lambda,a}(\phi_\mu, \phi)))\Omega_{phys} \tag{3.62}$$

for all $(\phi_\mu, \phi) \in N_{sk}$. It is easily shown that H_{phys} is the generator of the time-translation part of the representation thus defined. Since the spectrum of the Hamiltonian H_{phys} shows a mass gap, we are dealing with a massive representation ($m^2 = e^2$) of the Poincaré group, i.e., the three components of the vector $P_p \psi_*^{(i)}$ transform as a massive one-particle state under the action of the little group $SO(3)$ [20].

We conclude that $\mathcal{H}_{phys}$ has the main properties required by a physical Hilbert space: it transforms trivially under the gauge group, satisfies the positive spectrum condition and carries a unitary representation of the Poincaré group.

4 Conclusion

The quantization proposal employed in this paper provides a detailed scheme for imposing constraints on gauge quantum field theories. As explained in Chapter 2, the main tool of this proposal is the Rieffel induction procedure, which provides a systematic scheme for the construction of representations of C^*-algebras. It may be viewed as the quantum counterpart of the symplectic reduction technique; as we have shown, this is particulary obvious for Weyl C^*-algebras. This leads to a new quantization method for gauge field theories.

In the present work, we have applied this method to the Stückelberg-Kibble model. To this end, we have defined a field algebra $\mathcal{A}$ corresponding to the field content of the Lagrangian, and an algebra of constraints $\mathcal{B}$ corresponding to the gauge group acting on $\mathcal{A}$. Also, we have specified a representation π of subalgebras of $\mathcal{A}$ on a (unphysical) Hilbert space $\mathcal{H}$. From these input data, we have constructed a representation of the physical, gauge-invariant fields on a new Hilbert space $\mathcal{H}_{phys}$.

The construction of $\mathcal{H}_{phys}$ shows some parallels to the Gupta-Bleuler indefinite metric formalism. In both settings, a degenerate inner product is defined on a (unphysical) Hilbert space $\mathcal{H}$, and $\mathcal{H}_{phys}$ is constructed by quotiening $\mathcal{H}$ by a null space with respect to this degenerate inner product. Yet, there are important differences. In contrast to the indefinite metric inner product $\langle .,. \rangle$, defined on $\mathcal{H}_{GB}$ in the Gupta-Bleuler formalism, the $(.,.)_0$-inner product is positive semidefinite. More importantly, it is a conceptual advantage of our quantization method that $(.,.)_0$ is derived from first principles (namely from the requirement to impose quantum constraints by a quantized version of the classical phase space reduction method), whereas the Gupta-Bleuler formalism takes $\langle .,. \rangle$ as starting point without further justification. A similar comment applies to the BRST technique: although a classical analogue of this procedure exists, the quantum BRST procedure is *not* in any satisfactory sense the quantization of the classical scheme.

Another remarkable difference between both formalisms is that the Gupta-Bleuler formalism restricts the unphysical Hilbert space before forming the quotient while the proposal of [11] restricts itself to a representation of the subalgebra $\mathcal{A}_c$ of weak observables on $\mathcal{H}$, before quotiening by the appropriate null space. As a consequence, the $(.,.)_0$-inner product preserves the adjoint for elements in $\mathcal{A}_c$ only. It remains to be seen how far this feature alters applications of usual perturbative techniques in more complicated models.

Most of our effort in Chapter 3 has gone into characterizing the particular features of the physical state space $\mathcal{H}_{phys}$. By construction, $\mathcal{H}_{phys}$ carries a trivial representation of the gauge group. Also, the states are physical in the sense that they obey a positive spectrum condition and that they carry a massive representation of the Poincaré group. Since the Stückelberg-Kibble model has been widely used in investigations of the Higgs mechanism, we emphasize again the result obtained for the one-particle subspace in $\mathcal{H}_{phys}$. The point is that in our proposal, the particular construction method of $\mathcal{H}_{phys}$ allows one to trace back how the (unphysical) components of $\mathcal{H}$ end up in the physical Hilbert space. In the present case, we have shown that the longitudinal physical one-particle component arises from a particular Bogoliubov-transformation of the unphysical longitudinal and the scalar component. As expected from general considerations, two of the five components in $\mathcal{H}$ have ended up in the one-particle null space.

We conclude our discussion of the Rieffel induction procedure by pointing out that our presentation has focused on a particular way of applying Rieffel induction to gauge quantum field theories. Conceptually, the scheme is much wider. It remains to be seen how far other choices for the inner product $(.,.)_0$ and the unphysical Hilbert space $\mathcal{H}$ allow for other realisations of the physical observables of gauge field theories.

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