Gonihedric 3D Ising actions

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August 30, 1995

Abstract

We investigate the generalized Ising actions containing nearest neighbour, next to nearest neighbour and plaquette terms that have been suggested as potential string worldsheet discretizations on cubic lattices by Savvidy and Wegner. We use both mean field techniques and Monte-Carlo simulations to sketch out the phase diagram. In particular, we look at the effect of varying the parameter κ that quantifies the effects of self-avoidance and note some differences in behaviour between the $\kappa = 0$ and $\kappa \neq 0$ transitions.

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1 Introduction

In a series of recent papers [1] Savvidy and his co-workers suggested a "Gonihedric" random surface action which could be written as

$$S = \frac{1}{2} \sum_{<ij>} |\vec{X}_i - \vec{X}_j| \theta(\alpha_{ij}), \tag{1}$$

on triangulated surfaces, where $\theta(\alpha_{ij}) = |\pi - \alpha_{ij}|$ and α_{ij} is the angle between the embedded neighbouring triangles with common link $< ij >$. This was intended as an alternative to gaussian plus extrinsic curvature lattice actions of the form

$$S = \sum_{<ij>} (\vec{X}_i - \vec{X}_j)^2 + \lambda \sum_{\Delta_i, \Delta_j} (1 - \vec{n}_i \cdot \vec{n}_j) \tag{2}$$

which have been much explored [2] as discretizations of rigid membranes and strings [3]. Although a simulation showed that the action of equ.(1) produced flat surfaces [4], potential problems arising from the failure to suppress the wanderings of vertices in the plane were pointed out in [5]. One possible way to cure this is to add additional Gaussian or linear terms to the action [6], but a study of the scaling of the string tension and mass gap in such a theory produced at best inconclusive results [7].

Another possibility for regularizing the Gonihedric action is to put the surfaces generated on a (hyper)cubic lattice. This approach has been pursued in some detail analytically in [8, 9, 10] and one numerical simulation carried out in three dimensions [9]. The crucial observation in this work is that the surface theory on a cubic or hypercubic lattice can be written equivalently as a one parameter family of Ising actions, where boundaries between the spin clusters are the original surfaces. The free parameter κ arises from the choice of a self-intersection coupling. The net result of these considerations is that the Hamiltonian of the system in three dimension has the form

$$H = 2\kappa \sum_{<x,y>} \sigma_x \sigma_y - \frac{\kappa}{2} \sum_{<<x,y>>} \sigma_x \sigma_y + \frac{1-\kappa}{2} \sum_{[x,y,z,t]} \sigma_x \sigma_y \sigma_z \sigma_t \tag{3}$$

where the generalized Ising action contains nearest neighbour ($< x, y >$), next to nearest neigbour ($<< x, y >>$) and round a plaquette ($[x, y, z, t]$) terms. Such actions are not new, having been investigated in some detail using both mean field methods and simulations in [11], but the particular combination of coefficients arising in equ.(3) was not considered explicitly there. Related surface models have also been simulated directly in [12], but again the particular Gonihedric set of coefficients was not of interest for this work. A very rich phase structure was observed in [11], in common with various other Ising models with extended interactions [13] of various sorts which display first and second order phase boundaries as well as incommensurate phases. Given this, the action of equ.(3) merits investigation from purely statistical mechanical considerations as well as from the point of view of finding potential continuum string theories.

A variation on the theme was explored in [10] where the action

$$H = (12\kappa + 4) \sum_{<x,y>} \sigma_x \sigma_y - (3\kappa + 4) \sum_{<<x,y>>} \sigma_x \sigma_y - (3\kappa + 4) \sum_{[x,y,z,t]} \sigma_x \sigma_y \sigma_z \sigma_t \tag{4}$$

was suggested as a lattice discretization for three dimensional gravity. This is effectively one of the classes of coefficients already considered in [11], so we discuss it only briefly in what follows.

In the context of string theory one is looking for a continuous transition (or transitions) at which a sensible continuum surface theory may be defined. It is perhaps worth recalling that even this does not guarantee a "good" surface theory. The interfaces in the standard nearest neighbour Ising model in three dimensions, which has a continuous phase transition, have been investigated in some detail recently and found to be very porous objects, decorated with lots of handles at the scale of the lattice cutoff [14]. Ideally one might hope that the surfaces generated by the Gonihedric action were smoother, given that it is derived from a sort of stiffness term.

Our motivation in this paper is to investigate the action of equ.(3) in order to sketch out a map of its phase structure for various values of κ. In [9] one particular value, $\kappa = 1$, was investigated in a simulation and the similarity to the *two* dimensional Ising model transition temperature and critical behaviour remarked on. A few cautionary words are in order before we go on to discuss the mean field approach and simulations. It has been emphasized in [8, 9] that the ground state of the action in equ.(3) is very degenerate as parallel planes of spins can be flipped at no energy cost, particularly for the case $\kappa = 0$. The ability to flip arbitrary spin planes makes defining a magnetic order parameter rather problematic. Even the staggered local order parameters defined in [11] would miss the lamellar phases with arbitrary intersheet spacings that could be generated at no cost by flips of spin planes. We have not attempted to measure the exhaustive global order parameters suggested in [9]

$$M^{\mu} = < \sum_{\vec{r}} \sigma^{\mu}_{\vec{r}}(vac)\, \sigma_{\vec{r}} > \tag{5}$$

(with $\mu = 1, 2 \ldots 2^{3L}$ for $\kappa = 0$ on a lattice of size L) in our simulations as this is prohibitively slow on even moderately sized lattices, but simply contented ourselves with the standard magnetization. This is sufficient to verify the absence of simple ferromagnetic order.

2 Zero Temperature and Mean Field

As the Gonihedric model is a special case of the general action considered in [11] we can apply the methods used there for both the zero temperature phase diagram and mean field theory. For the zero temperature case this involves writing the full lattice Hamiltonian as a sum over individual cube Hamiltonians

$$h_c = \frac{\kappa}{2} \sum_{<x,y>} \sigma_x \sigma_y - \frac{\kappa}{4} \sum_{<<x,y>>} \sigma_x \sigma_y + \frac{1-\kappa}{4} \sum_{[x,y,z,t]} \sigma_x \sigma_y \sigma_z \sigma_t \tag{6}$$

and observing that if the lattice can be tiled by a cube configuration minimizing the individual h_c then the ground state energy density is $\epsilon_0 = min\ h_c$.

The inequivalent spin configurations on a single cube and their multiplicities are listed in [11] for general coefficients. We repeat these in Table.1 using the same notation with our choice of couplings to highlight the degeneracies that appear with the Gonihedric action. In the list of spins the first column represents one face of the cube and the second the other. In the table two configurations are considered equivalent if one can be transformed into the other by reflections and rotations or if they are related by a global spin flip. The antiferromagnetic image of a configuration is obtained by flipping the three nearest neighbours and the spin at the other end of the cube diagonal from a given spin and is denoted by an overbar. With the Gonihedric values of the couplings the freedom to flip spin planes is clear even at this level as ψ_0, which would represent a large single surface when used to tile the lattice, and ψ_6 which would represent flipped spin layers, have the same energy for any value of κ. The higher energy configurations ψ_4 and $\psi_{\bar{4}}$ are also identical. The degeneracies increase when $\kappa = 0$, as the arguments of [9] indicate they should, the club of states of energy $-3/2$ is now composed of $\psi_0, \psi_{\bar{0}}, \psi_6, \psi_{\bar{6}}$ and various extra degeneracies appear for higher energy states. From these results the ground state is clearly highly degenerate whatever the value of κ.

In the mean field approximation the spins are in effect replaced by the average site magnetizations. The calculation of the mean field free energy is an elaboration of the method used above to investigate the ground states in which the energy is decomposed into a sum of individual cube terms. The next to nearest neighbour and plaquette interactions in the Gonihedric model give the total mean field free energy as the sum of elementary cube free energies $\phi(m_c)$, given by

$$\phi(m_C) = -\frac{\kappa}{2} \sum_{<x,y>\subset C} m_x m_y + \frac{\kappa}{4} \sum_{<<x,y>>\subset C} m_x m_y$$

$$-\frac{1-\kappa}{4} \sum_{[x,y,z,t]\subset C} m_x m_y m_z m_t + \frac{1}{16} \sum_{x \subset C} [(1+m_x) ln(1+m_x) + (1-m_x) ln(1-m_x)] \tag{7}$$

where m_C is the set of the eight magnetizations of the elementary cube. This gives a set of eight mean-field equations

$$\frac{\partial \phi(m_C)}{\partial m_i}_{(i=1\ldots8)} = 0 \tag{8}$$

(one for each corner of the cube) rather than the familiar single equation for the standard nearest neighbour Ising action. More explicitly, we have

$$\begin{aligned} m_1 &= tanh[4\beta\kappa(m_2+m_4+m_5) - 2\beta\kappa(m_3+m_6+m_8) \\ &\quad +2\beta(1-\kappa)(m_2m_3m_4+m_2m_5m_6+m_4m_5m_8)] \\ &\vdots \\ m_8 &= tanh[4\beta\kappa(m_2+m_5+m_7) - 2\beta\kappa(m_1+m_3+m_6) \\ &\quad +2\beta(1-\kappa)(m_3m_4m_7+m_1m_4m_5+m_5m_6m_7)] \end{aligned} \tag{9}$$

where we have labelled the magnetizations on a face of the cube counterclockwise $1\ldots4$ and similarly for the opposing face $5\ldots8$ as shown in Figure.1. If we solve these equations iteratively we arrive at zeroes for a paramagnetic phase or various combinations of ±1 for the magnetized phases on the eight cube vertices, and the mean field ground state is then give by gluing together the elementary cubes consistently to tile the complete lattice, in the manner of the ground state discussion.

Turning loose a numerical solver on the mean field equs.(9) gives generically a single transition to one of the phases listed in Table.1 from the high temperature paramagnetic phase. The transition temperatures and the resulting low temperature phase are listed in Table.2. We have taken the liberty of carrying out global flips where necessary to tidy up the table. Rather remarkably, we see that apart from $\kappa = 0$ the transition appears to be to the simple ferromagnetic phase, ψ_0. However, remembering that ψ_0 and ψ_6 have the same energy the best we can say is that we end up in a layered phase with arbitrary interlayer spacing in all directions. Although the $\kappa = 0$ case appears to be superficially different, the $\psi_{\bar{0}}$ phase that is found at low temperature here is one of the phases that is degenerate with ψ_0 and ψ_6 when $\kappa = 0$. Although $\kappa = 1$ fits the pattern as far as a transition to $\psi_{0,6}$ at decreasing β is concerned it appears to be rather atypical in that further transitions are observed at larger β. However, this is a numerical instability that is peculiar to this particular value of κ. It was observed in [11] that an iterative solution of the mean field equations written in the form

$$m_i^{(n+1)} = f[E_{,i}(m^n)] \tag{10}$$

where E is the individual cube Hamiltonian could fail to converge if an eigenvalue of $\partial m_i^{(n+1)}/\partial m_j^n$ was less than -1. Modifying the equations to

$$m_i^{(n+1)} = \frac{(f[E_{,i}(m^n)] + \alpha m_i^n)}{1+\alpha} \tag{11}$$

for suitable α cures this. This is precisely what happens for $\kappa = 1$, where introducing a non-zero α suppresses the extra "transitions".

The three dimensional gravity action of equ.(4) has zero temperature ground states of $\psi_{4,\bar{4}}, \psi_{7,\bar{7}}$ and the Monte Carlo simulations of [11] give a transition from a high temperature paramagnetic phase to a $\psi_{\bar{4}}$ or ψ_7 low temperature phase for any (positive) value of κ. This action thus appears to generate the lattice version of minimal surfaces, rather than the freely movable spin planes of equ.(3).

In summary, the mean field theory suggests a rather simple phase diagram for the Gonihedric model with action equ.(3), with a single transition that is pushed down to $\beta = 0$ at large κ. The low temperature phase is generically of the $\psi_{0,6}$ type, apart from $\kappa = 0$ where we see a $\psi_{\bar{0}}$ phase that is degenerate with these. Although there still appears to be only a single transition for the three dimensional gravity action of equ.(4), the ground states are different. We now go on to see how the results for the Gonihedric action tally with a direct Monte-Carlo simulation.

3 Simulations

We carried out simulations with rather modest statistics for $\kappa = 0.0, 0.5, 1.0, 2.0$ on lattices of size $5^3, 10^3, 15^3, 20^3$ and 25^3 and for $\kappa = 5.0, 10.0$ on the four smaller lattice sizes only. In all cases periodic boundary conditions were imposed in the three directions. We carried out 50K sweeps for each β value with a measurement every sweep after allowing sufficient time (typically 20K sweeps) for thermalization. A simple metropolis update was used because of the difficulty in concocting a cluster algorithm for a

Hamiltonian with such complicated interaction terms. The program was tested on the standard nearest neighbour Ising model and the parameters used in [11] to ensure it was working.

We measured the usual thermodynamic quantities for the model: the energy E, specific heat C, (standard) magnetization M, susceptibility χ and various cumulants. Bearing the earlier comments on the possibility of flipping spin planes in mind it is clear that the magnetization and susceptibility are unlikely to be particularly informative. This makes a serious scaling analysis rather difficult as the option of using Binder's cumulant for some staggered variant of the magnetization to determine the critical point and ν independently of the specific heat or susceptibility peaks is lost if the ground states are of the degenerate form suggested by the zero temperature and mean field analyses. In these circumstances the strongest evidence for phase transitions is likely to come from the specific heat and the energy.

The absolute value of the energy for various κ on lattices of size $L = 20$ is plotted against β in Figure.2, where we can see that the zero temperature prediction of $3/2(1+\kappa)$ is satisfied with good accuracy for sufficiently large β. We can therefore observe that the zero-temperature/mean-field analysis has correctly identified the ground state(s) of the theory: $\psi_{0,6}$ for $\kappa \neq 0$; or $\psi_{0,\bar{0},6,\bar{6}}$ for $\kappa = 0$ as these are the only states with the observed energies. The possibility of the simple ferromagnetic ordered state ψ_0 can be excluded by looking at the standard magnetization, which is plotted in Figure.3 for various κ, again on lattices of size $L = 20$. In all the cases it is either zero or fluctuates wildly as β is changed, showing that the standard M is not a good order parameter for the large β phase.

One feature of the full Monte-Carlo simulations, however, does not agree with the mean field analysis, namely the behaviour of the transition temperature itself. In the mean field theory β_c drops quite sharply with increasing β. Visual inspection of Figs. 2,3 shows that although β_c, taken as the crossover in the energy or spin values, drops from $\beta_c \simeq 0.6$ at $\kappa = 0$ to $\beta_c \simeq 0.44$ at $\kappa = 1$ (in agreement with the value measured in [9]) it appears to remain pegged at this value for $\kappa > 1$ rather than decreasing further.

Further evidence for this can be garnered by looking at the specific heats for various κ. In Figures 4,5,6 we show the specific heat curves for $\kappa = 0, 0.5$ and 1 respectively. The curves for $\kappa > 1$ are essentially identical in shape to $\kappa = 1$, with maxima around $\beta_c \simeq 0.44$ although the maximum height of the peaks rapidly drops to a value that is constant for increasing κ after reaching a maximum at $\kappa = 1$. The quality of our data for the specific heat peaks is not really sufficient to reliably extract the exponent α, especially given the presence of an additional constant in the scaling form for C and the absence of an independent estimate of β_c. We can, however, say with some degree of confidence that the transitions for all $\kappa > 0$ appear to be of second order.

We can confirm the continuous nature of the transition by examining the energy cumulants for various values of κ

$$U_E = 1 - \frac{< E^4 >}{3 < E^2 >^2} \tag{12}$$

which we would expect to scale trivially at a continuous transition to 2/3 with this choice of normalization. At a first order transition a non-trivial value would be observed. A glance at Figure.7 reveals that the continuous scaling is satisfied at $\beta_c \simeq 0.44$ for $\kappa \neq 0$ and $\beta_c \simeq 0.6$ for $\kappa = 0$, although the error bars are considerable in the latter case. In spite of this the specific heat peak suggests that the transition that is observed at $\kappa = 0$ is, if anything, weaker than the generic case and may even be of higher than second order.

The difference in behaviour at $\kappa = 0$ is further emphasized by looking at the standard susceptibility χ. As one might expect from the behaviour of the magnetization this is not particularly informative for general κ. Although it rises sharply around $\beta_c \simeq 0.44$, for $\beta > \beta_c$ it is essentially noise. $\kappa = 0$, however, throws up a surprise as we can see in Figure.8. The susceptibility is equal to one for $\beta < \beta_c$ and zero for $\beta > \beta_c$. Given that the standard magnetization is close to zero in both phases and that

$$\chi = < M^2 > - < M >^2 \tag{13}$$

a suitable order parameter for the $\kappa = 0$ theory would thus appear to be $< M^2 >$. This is curiously reminiscent of spin glass order, but in this case it is the *high*-temperature (small β) phase that is displaying a non-zero $< M^2 >$. It thus perhaps better to think of the system in the dual language discussed in several of the papers in [9]. The dual Hamiltonian can be written as

$$H = \sum_{i \neq j \neq k} \Lambda_j(\xi)\Lambda_k(\xi)\Lambda_j(\xi + e_i)\Lambda_k(\xi + e_i) \tag{14}$$

where each vertex now has three Ising spins and the e_i are the unit lattice vectors in the various directions. Both the original action and its dual above are of the general form suggested for models of self-induced disorder [15], so it is conceivable that such behaviour is playing a role here too. It would be an interesting exercise to simulate the dual model directly to examine the nature of its low temperature phase more carefully. The magnetization cumulant

$$U_M = 1 - \frac{< M^4 >}{3 < M^2 >^2} \tag{15}$$

is of no use in the general case, as we have indicated above, but its behaviour is smoother for $\kappa = 0$, increasing sharply from zero to 2/3 at $\beta \simeq 0.6$.

4 Conclusions

We have conducted zero-temperature, mean-field and Monte-Carlo investigations of the generalized Ising model action suggested in [8, 9, 10] as a cubic lattice discretization of the Gonihedric string action [1] using essentially the methods of [11]. Although the phase structure of such generalized Ising models is generically very rich [13], the one parameter family of models examined here seems to be a fairly simple "slice" of the phase diagram, with one transition, apparently of second order, to a layered ground state. The zero-temperature/mean-field analyses are in agreement with the Monte-Carlo simulations on the nature of the ground state and its energy, but intriguingly the full simulations indicate a transition temperature that changes little, if at all, from its value at $\kappa = 1$ ($\beta_c \simeq 0.44$), which is close to that of the two-dimensional Ising model. The mean field theory on the other hand indicates a fairly sharp decline in β_c as κ is increased.

We were unable to extract reliable values for the specific heat scaling exponent from our data essentially because of the absence of an independent estimate of β_c from a cumulant analysis and the extra adjustable constant that appears in the finite size scaling form for the specific heat. The degenerate nature of the ground state for any κ means that not even the staggered magnetizations considered in [11] would be useful for the models here.

As the increased symmetry that is present in the Hamiltonian at $\kappa = 0$ might indicate, this is a special case. The transition appears weaker, it is at $\beta_c \simeq 0.6$ and there are indications that $< M^2 >$ functions as an order parameter. It is somewhat surprising that the case with the highest degree of symmetry lends itself to such a simple characterization of phases. The form of the $\kappa = 0$ Hamiltonian suggests links with models of self-induced disorder which might be more transparent in the dual formalism, given that $< M^2 >= 1$ in the *high* temperature phase of the direct model.

The most obvious extension of the work reported here is to carry out a much higher statistics simulation near the transition point in order to extract accurate values for the specific heat exponent and β_c. Finding the magnetic exponents for these models requires first attacking the conceptual problem of defining a magnetic order parameter for general κ that is easier to handle than the suggested global magnetizations of [10] in equ.(5). Another obvious path is to investigate the various higher dimensional generalizations that were formulated in [8, 9, 10]. The links between the $\kappa = 0$ model and self-induced disorder are also intriguing and deserve pursuing.

If we return to our original stringy motivation another useful extension would be to characterize the surfaces generated by the gonihedric action in the style of [14] to see whether they were any less spongy than those in the standard 3D Ising model. As a playground for exploring plaquette discretizations of string and gravity inspired models the generalized Ising models clearly have some interesting quirks that are worthy of further exploration.

5 Acknowledgements

R.P.K.C. Malmini was supported by Commonwealth Scholarship SR0014.

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Figure Captions

Fig.1 The labelling of the cube vertices for the mean field equations.

Fig.2 The energies for various κ, all on lattices of size $L = 20$.

Fig.3 The standard (unstaggered) magnetization for various κ, all on lattices of size $L = 20$.

Fig.4 The specific heat for $\kappa = 0$ on various lattice sizes.

Fig.5 The specific heat for $\kappa = 0.5$.

Fig.6 The specific heat for $\kappa = 1$.

Fig.7 The Energy cumulant for various κ.

Fig.8 The susceptibility for $\kappa = 0$, showing the sharp transition to a value of one for $\beta < \beta_c$.

TABLES

Table.1

State	Top	Bottom	Energy	Multiplicity
ψ_0	+ + + +	+ + + +	$-3/2 - 3\kappa/2$	2
$\psi_{\bar{0}}$	- + + -	+ - - +	$-3/2 + 21\kappa/2$	2
ψ_1	+ + - +	+ + + +	$-3\kappa/2$	16
$\psi_{\bar{1}}$	- + + -	+ + - +	$9\kappa/2$	16
$\psi_{2,\bar{2}}$	- + + -	+ + + +	$1/2 + \kappa/2$	24
ψ_3	+ + - -	+ + + +	$-1/2 - 3\kappa/2$	24
$\psi_{\bar{3}}$	- - - +	+ + + -	$-1/2 + 5\kappa/2$	24
ψ_4	- + + +	+ + + -	$3/2 - 3\kappa/2$	8
$\psi_{\bar{4}}$	- - - +	- + + +	$3/2 - 3\kappa/2$	8
ψ_5	- - - +	+ + + +	$-3\kappa/2$	48
$\psi_{\bar{5}}$	- + + -	- + + +	$\kappa/2$	48
ψ_6	- - - -	+ + + +	$-3/2 - 3\kappa/2$	6
$\psi_{\bar{6}}$	+ + - -	- - + +	$-3/2 + 5\kappa/2$	6
$\psi_{7,\bar{7}}$	- - - +	+ - + +	$1/2 - 3\kappa/2$	24

Table.1: The inequivalent spin configurations of a single cube and the associated energies and degeneracies

Table.2

κ	β_c		
0.0	0.325	+ - - +	- + + -
0.25	0.31	+ + + +	+ + + +
0.5	0.278	+ + + +	+ + + +
1.0	0.167	+ + + +	+ + + +
2.0	0.09	+ + + +	+ + + +
5.0	0.0335	+ + + +	+ + + +
10.0	0.02	+ + + +	+ + + +
15.0	$<$0.02	+ + + +	+ + + +

Table.2: The ground state configurations and transition temperatures for various κ. The states shown appear above the quoted temperature.