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Transverse momentum dependence of Hanbury-Brown/Twiss correlation radii

Urs Achim Wiedemann, Pierre Scotto and Ulrich Heinz

Institut für Theoretische Physik, Universität Regensburg,

D-93040 Regensburg, Germany

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Abstract

The transverse momentum dependence of Hanbury-Brown/Twiss (HBT) interferometry radii for 2-body correlation functions provides experimental access to the collective dynamics in heavy-ion collisions. We present an analytical approximation scheme for these HBT radii which combines the recently derived model-independent expressions with an approximate determination of the saddle point of the emission function. The method is illustrated for a longitudinally boost-invariant hydrodynamical model of a heavy ion collision with freeze-out on a sharp hypersurface. The analytical approximation converges rapidly to the width of the numerically computed correlation function and reproduces correctly its exact transverse momentum dependence. However, higher order corrections within our approximation scheme are essential, and the previously published lowest order results with simple $m_\perp$ scaling behaviour are quantitatively and qualitatively unreliable.

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I. INTRODUCTION

While the total energy involved in a heavy ion collision can be measured directly by particle calorimeters, an analogous direct measurement of the locally reached energy density does not exist. An indirect determination of the size of the interaction region is possible through Hanbury-Brown/Twiss (HBT) intensity interferometry [1]. However, the interpretation of the measured correlations between the produced particles is in general model-dependent, and a considerable amount of theoretical effort has recently been spent on the question to what extent this intrinsic model dependence can be reduced by a refined analysis [2–7].

Compared to HBT interferometry on stars, the situation in heavy-ion collisions is complicated by the finite lifetime and the strong dynamical evolution of the particle emitting source [8–11]. As a result the 2-body correlation function is in general characterized by different width parameters ("HBT radii") if the relative momentum between the two identical particles points in different directions [9]. Furthermore, the dynamical expansion of the source leads to correlations between the momenta of the emitted particles and their emission point (so-called $x - K$ correlations) which in turn generate a characteristic dependence of the HBT radii on the total pair momentum [9].

In this paper we discuss this last issue in some detail. Our work was motivated by the simple scaling laws for the HBT radii as a function of the transverse mass $m_\perp$ of the particle pair which were proposed in Refs. [10,3,6,7,13]. These predictions are based on simple models for the emission function, and it is natural to ask which of the predicted features are independent of the model and which are not. For example, the model of Ref. [7] makes definite assumptions about the shape of the longitudinal and transverse expansion flow profiles whose influence on the result for the correlation function, in particular on the functional form of the $m_\perp$-dependence of the correlation radii, is not known. The work reported here provides an answer to this question within a certain class of models. More importantly, however, we discovered on our way that the simple scaling laws of Refs. [10,3,6,7] are based on approximations which are quantitatively unreliable and in some typical cases give even qualitatively misleading results. We will show, for example, that a fit of the exact correlation function with the simple analytical expression from Ref. [10] for the longitudinal HBT radius R_l gives an estimate for the decoupling time which is too large by a factor $\simeq 2$. (This was recently also pointed out in [13].) The effect of the finite duration of particle emission on the difference between the HBT radii in "out" and "side" directions, $R_o^2 - R_s^2$ which, as we will show, is also present in models with sharp freeze-out along a proper-time hypersurface, is completely missed by the approximations used in Ref. [7], giving rise to a qualitatively wrong $m_\perp$ dependence of the "out"-correlator. We conclude from our findings that a quantitative analysis of the $m_\perp$-dependence of HBT radii necessitates a numerical evaluation of the theoretical expressions for the correlation radii, or requires at least a much more sophisticated analytical approximation scheme than so far employed.

We begin by shortly reviewing the general approach and defining our notation. We restrict our discussion to two-particle correlations. We start from the usual abstraction of the collision region as an assembly of classical boson emitting sources in a certain space-time region [1,14]. Their Wigner transform [15–17] defines the emission function $S(x,p)$ which describes the probability that a particle with on-shell momentum p ($p^2 = m^2$) is emitted from the space-time point x. The emission function determines the single particle momentum spectrum $P_1(\mathbf{p}) = E_p\, dN/d^3p = \int d^4x\, S(x,p)$ as well as the HBT two-particle correlation function $C(\mathbf{K},\mathbf{q})$ where $\mathbf{K} = \frac{1}{2}\left(\mathbf{p}_1 + \mathbf{p}_2\right) = (\mathbf{K}_\perp, K_L)$ is the average momentum of the two particles and $\mathbf{q} = \mathbf{p}_1 - \mathbf{p}_2$ their relative momentum. In the plane wave approximation the latter is given in terms of the 1- and 2-particle distributions $P_1(\mathbf{p})$, $P_2(\mathbf{p}_1, \mathbf{p}_2)$ and the average number $\langle N \rangle$ ($\langle N(N-1) \rangle$) of particles (particle pairs) produced in the reaction as [14,15,17]

$$C(\mathbf{p}_1, \mathbf{p}_2) = \frac{\langle N \rangle^2}{\langle N(N-1) \rangle} \frac{P_2(\mathbf{p}_1, \mathbf{p}_2)}{P_1(\mathbf{p}_1) P_1(\mathbf{p}_2)} = 1 + \frac{\left| \int d^4x\, S(x,K)\, e^{iq\cdot x} \right|^2}{\int d^4x\, S(x,p_1)\ \int d^4x\, S(x,p_2)}\,. \tag{1.1}$$

In the numerator of the last expression we have introduced off-shell momentum 4-vectors K and q for the total and relative momentum of the particles in the pair by defining $K^0 = \frac{1}{2}(E_1 + E_2)$ and $q^0 = E_1 - E_2$ where $E_i = \sqrt{m^2 + \mathbf{p}_i^2}$. A popular approximation whose accuracy was studied quantitatively in [4] consists of putting K on-shell, $K^0 \simeq E_K = \sqrt{m^2+\mathbf{K}^2}$, and setting $p_1{=}p_2{=}K$ in the denominator:

$$C(\mathbf{K},\mathbf{q}) \simeq 1 + \frac{|\int d^4x\, S(x,K)\, e^{iq\cdot x}|^2}{|\int d^4x\, S(x,K)|^2}\,. \tag{1.2}$$

The aim of HBT interferometry is to obtain from the measured correlation function $C(\mathbf{K},\mathbf{q})$ information about $S(x,K)$ which in turn should characterize the size of the interaction region. One usually compares $C(\mathbf{K},\mathbf{q})$ to experimental data by taking recourse to the Gaussian approximation [4]

$$C(\mathbf{K},\mathbf{q}) \simeq 1 + \exp\left[-q_s^2 R_s^2(\mathbf{K}) - q_o^2 R_o^2(\mathbf{K}) - q_l^2 R_l^2(\mathbf{K}) - 2q_l q_o R_{lo}^2(\mathbf{K})\right]\,, \tag{1.3}$$

which is valid for azimuthally symmetric sources. Here q_l, q_o, q_s denote the spatial components of $\mathbf{q}$ in the beam direction ("longitudinal" or z-direction), parallel to the transverse components $\mathbf{K}_\perp$ of $\mathbf{K}$ ("out" or x-direction), and in the remaining third cartesian direction ("side" or y-direction), respectively. In practice the last term in (1.3) which mixes the components q_o and q_l has usually been neglected because its existence was only recently pointed out [4] and confirmed experimentally [18].

The main purpose of the present work is to investigate the transverse momentum dependence of the HBT correlation radii $R_i^2(\mathbf{K})$ in (1.3). As already mentioned it reflects the $x-K$-correlations in the emission function $S(x,K)$ generated by collective expansion of the source, information that can not be obtained from $\mathbf{K}$-averaged correlation radii. Our analysis will be based on a specific model for the emission function, but the method is general and can later be combined with a comprehensive investigation of the model dependence of 2-particle correlations. In the present paper we will compare two different methods to calculate the $R_i^2(\mathbf{K})$: the first consists of fitting the numerically determined HBT correlation function $C(\mathbf{K},\mathbf{q})$ to the form (1.3) in such a way that its half width is correctly reproduced, while in the second approach we will use the following model-independent expressions [13,4,19]:

$$R_s^2 = \langle y^2 \rangle,$$

$$R_o^2 = \langle (x-\beta_\perp t)^2 \rangle - \langle (x-\beta_\perp t) \rangle^2,$$

$$R_l^2 = \langle (z-\beta_L t)^2\rangle - \langle (z-\beta_L t)\rangle^2,$$

$$R_{lo}^2 = \langle (x-\beta_\perp t)(z-\beta_L t)\rangle - \langle (x-\beta_\perp t)\rangle \langle (z-\beta_L t)\rangle\,, \tag{1.4}$$

which are guaranteed to yield the correct Gaussian curvature of the correlation function $C(\mathbf{K},\mathbf{q})$ near $\mathbf{q}=0$ [4,5,13]. In Eqs. (1.4) we defined $\beta_i = 2K_i/(E_1+E_2) \approx K_i/E_K$ and introduced the notation

$$\langle \xi \rangle = \langle \xi \rangle(K) = \frac{\int d^4x\, \xi\, S(x,K)}{\int d^4x\, S(x,K)}\,. \tag{1.5}$$

For Gaussian sources, the set of equations (1.4) can be alternatively derived by expanding (1.1) for small relative momenta $\mathbf{q}$ up to second order and re-exponentiating [4,13], or by making a Gaussian saddle point approximation of the emission function around its maximum [5]. A more general justification of Eqs. (1.4), which (as we will show) remain rather accurate even for non-Gaussian sources, is based on the following parametrization of the emission function:

$$S(x,K) = S(\bar{x},K)\, e^{-\frac{1}{2}(x-\bar{x})^\mu (x-\bar{x})^\nu B_{\mu\nu}(\mathbf{K})} + \delta S(x,K)\,, \tag{1.6}$$

where the saddle point $\bar{x}(\mathbf{K})$ is defined as the point where all first derivatives of $S(x,K)$ vanish. If one identifies $B_{\mu\nu}$ with the tensor of second derivatives of $\ln S(\bar{x},K)$, the first term on the r.h.s. of Eq. (1.6) amounts to the Gaussian saddle point approximation around $\bar{x}$ which was used in [5–7]. For non-Gaussian forms of $S(x,K)$, the typical corrections to Eqs. (1.4), (1.5) from $\delta S(x,K)$ can be minimized by instead defining $B_{\mu\nu}(\mathbf{K})$ in terms of the variance of $S(x,K)$,

$$(B^{-1})_{\mu\nu} = \langle x_\mu x_\nu \rangle - \langle x_\mu \rangle \langle x_\nu \rangle\,. \tag{1.7}$$

which measures the width of $S(x,K)$. With this definition the Fourier transform (1.2) of the first term in (1.6) can be done analytically and leads directly to the expressions (1.3) and (1.4), with the expectation value (1.5) defined in terms of the *full* emission function $S(x,K)$. We will see that, in contrast to other approximation schemes (see for example [6,7]), the determination of the HBT radii via (1.4) yields an accurate representation of the width of the exact correlation function, independent of the validity of the Gaussian saddle point approximation for $S(x, K)$.

Except for the few special cases for which the four-dimensional Fourier transform of $S(x, K)$ is known exactly, an analytical investigation of equation (1.1) or (1.4) must involve a suitable approximation scheme. The existing analytical calculations [3,4,6,7] of (the $\mathbf{K}$-dependence of) the HBT radii for particular emission functions $S(x, K)$ cover certain limiting cases of parameter space only, and their validity has not been checked numerically. In the present work, we investigate the complete $\mathbf{K}$-dependence of the HBT radii by both analytical and numerical methods without any restriction to limiting cases.

Our paper is organized as follows: In section II, we review a recently introduced simple model for the emission function [6,7] which we will also study here. It may not be the most realistic model for heavy-ion collisions, but it possesses a number of essential physical features and allows for a controlled investigation of our analytical approximation scheme and a direct comparison with previously published approximations. In section III we develop a new analytical approximation scheme for the HBT radii (1.4) of this model. In Section IV we provide some intermediate results for the numerical evaluation of the correlation function. In Section V we compare the analytical and numerical values for the HBT radii. We find that the analytical approach works very well for a linear transverse flow profile, if the approximation scheme (which involves an expansion in powers of $\gamma_t m_\perp/T$, γ_t being the Lorentz factor associated with the transverse flow) is carried to third non-leading order. For the "out" and "longitudinal" radii the leading terms in this approximation scheme are found to be insufficient. For parabolic transverse flow profiles our approximation scheme and, *a fortiori*, all previously suggested simpler approximations are found to fail completely. The physical consequences of our findings are discussed in Section VI.

II. A SIMPLE MODEL FOR THE EMISSION FUNCTION

For ease of comparison with published results in the literature, we consider the emission function [6,7]

$$S(x,K) = \frac{K\cdot n(x)}{(2\pi)^3}\, e^{-K\cdot u/T}\, e^{-r^2/(2R^2)}\,, \tag{2.1}$$

where the Boltzmann factor $e^{-K\cdot u/T}$ reflects the assumed local thermal equilibration of a source with local temperature $T(x)$ moving with hydrodynamic 4-velocity $u_\mu(x)$. We will take T to be constant. We assume sharp freeze-out of the particles from the thermalized fluid along a hypersurface $\Sigma(x)$, and the 4-vector $n_\mu(x) = \int_\Sigma d^3\sigma_\mu(x')\,\delta^{(4)}(x-x')$ denotes the normal-pointing freeze-out hypersurface element. The factor $\exp\left[-r^2/(2R^2)\right]$ imposes a Gaussian transverse density profile with geometric radius R; the exponent can be interpreted in terms of an effective chemical potential $\mu(r)\!=-T\,r^2/(2R^2)$ which depends only on the transverse coordinate $r\!=\!\sqrt{x^2+y^2}$.

For the flow velocity profile $u^\nu(x)$ we assume longitudinal boost invariance by setting $v_L\!=\!z/t$, i.e. identifying the flow rapidity $\eta_{\rm flow}\!=\!\frac{1}{2}\ln[(1+v_L)/(1-v_L)]$ with the space-time rapidity $\eta\!=\!\frac{1}{2}\ln[(t+z)/(t-z)]$. We can thus parametrize $u^\nu(x)$ in the form

$$u_\nu(x) = \left(\mathrm{ch}\,\eta\,\mathrm{ch}\,\eta_t,\ \mathrm{sh}\,\eta_t\,\frac{x}{r},\ \mathrm{sh}\,\eta_t\,\frac{y}{r},\ \mathrm{sh}\,\eta\,\mathrm{ch}\,\eta_t\right), \tag{2.2}$$

where $\eta_t(r)$ is the transverse flow rapidity. The momentum K is parametrized in the usual way through the transverse mass $m_\perp\!=\!\sqrt{m^2+K_\perp^2}$ and the rapidity Y as $K_\nu\!=\!(m_\perp\mathrm{ch}\,Y, K_\perp, 0, m_\perp\mathrm{sh}\,Y)$.

Assuming sharp freeze-out at constant proper time τ_0, the freeze-out hypersurface is parametrized as $\Sigma(x)\!=\!(\tau_0\mathrm{ch}\,\eta, x, y, \tau_0\mathrm{sh}\,\eta)$, resulting in $K\!\cdot\! n(x)\!=\!m_\perp\mathrm{ch}\,(\eta-Y)\,\delta(\tau-\tau_0)$. The emission function can thus be rewritten as

$$S(x,K) = \frac{m_\perp\mathrm{ch}\,(\eta-Y)}{(2\pi)^3}\, e^{-[m_\perp\mathrm{ch}\,(\eta-Y)\,\mathrm{ch}\,\eta_t - K_\perp(x/r)\,\mathrm{sh}\,\eta_t]/T}\, e^{-r^2/(2R^2)}\,\delta(\tau-\tau_0)\,. \tag{2.3}$$

The δ-function renders the τ-integration in $d^4x\!=\!\tau\,d\tau\,d\eta\,dx\,dy$ trivial, and only the integrations over space-time rapidity η and the transverse coordinates x,y remain to be done.

Exploiting the boost invariance and infinite longitudinal extension of our source we can simplify the structure of Eqs. (1.4) by going to the longitudinally comoving system (LCMS). In this frame $Y{=}0{=}\beta_L$.

III. ANALYTICAL EVALUATION OF THE HBT RADII

In this section we will discuss the model-independent expressions (1.4) for the HBT radii. Our analysis of these expressions proceeds in two steps: First, the η-integration is done analytically. We then find the approximate saddle point of the resulting integrand in the x- and y-directions and carry out those integrations by saddle point integration.

A. Integration over η

To begin with we note that by the symmetry of our infinite boost-invariant emission function (2.3) the expectation values of all odd functions of η vanish. Therefore, in this model the cross term R_{lo} is zero [4].

Writing $t{=}\tau \mathrm{ch}\,\eta$, $z{=}\tau \mathrm{sh}\,\eta$ and including the $\mathrm{ch}\,\eta$ prefactor in Eq. (2.3) (recall that we use the LCMS frame where $\beta_L{=}Y{=}0$), the η-integrations in the expectation values (1.4) can be done analytically and expressed in terms of modified Bessel functions $K_\nu(a) = \int_0^\infty d\eta\, e^{-a\,\mathrm{ch}\,\eta}\,\mathrm{ch}\,(\nu\eta)$. Defining

$$G^{mn}(r) = \int_{-\infty}^{\infty} d\eta\, \mathrm{sh}^m\eta\, \mathrm{ch}^{n+1}\eta\, e^{-a(r)\,\mathrm{ch}\,\eta} \Bigg/ \int_{-\infty}^{\infty} d\eta\, \mathrm{ch}\,\eta\, e^{-a(r)\,\mathrm{ch}\,\eta} \tag{3.1}$$

with

$$a(r) = \frac{m_\perp}{T}\,\mathrm{ch}\,\eta_t(r)\,, \tag{3.2}$$

the expressions (1.4) for the HBT radii can be written as sums of terms of the general form

$$\langle f(x,y)\, G^{mn}(r)\rangle_* \equiv \frac{\int dx\, dy\, f(x,y)\, G^{mn}(r)\, F(x,y)}{\int dxdy\, F(x,y)}\,, \tag{3.3}$$

where $f(x,y)$ is a simple polynomial in x or y and

$$F(x,y) = K_1(a(r))\, e^{K_\perp(x/r)\,\mathrm{sh}\,\eta_t(r)/T}\, e^{-r^2/(2R^2)} \tag{3.4}$$

is the transverse weight function for the $\langle \dots \rangle_*$ average in (3.3). The functions G^{mn} needed in (1.4) are given explicitly by

$$\begin{aligned}
G^{01}(r) &= \frac{1}{a} + \frac{K_0(a)}{K_1(a)}\,, \\
G^{02}(r) &= 1 + \frac{K_2(a)}{a\,K_1(a)}\,, \\
G^{20}(r) &= \frac{K_2(a)}{a\,K_1(a)}\,,
\end{aligned} \tag{3.5}$$

with $a = a(r)$ from (3.2). With these definitions the HBT radii (1.4) take the form

$$\begin{aligned}
R_l^2 &= \tau_0^2 \left\langle G^{20}(r) \right\rangle_*\,, \\
R_s^2 &= \left\langle y^2 \right\rangle_*\,, \\
R_o^2 &= \left\langle x^2 \right\rangle_* - \langle x \rangle_*^2 \\
&\quad - 2\beta_\perp \tau_0 \left(\left\langle x\, G^{01}(r) \right\rangle_* - \langle x \rangle_* \left\langle G^{01}(r) \right\rangle_* \right) \\
&\quad + \beta_\perp^2 \tau_0^2 \left(\left\langle G^{02}(r) \right\rangle_* - \left\langle G^{01}(r) \right\rangle_*^2 \right)\,, \\
R_{lo}^2 &= 0\,.
\end{aligned} \tag{3.6}$$

For the model (2.1) these expressions are exact.

In some previous papers [6,7,10] the η-integration was not done exactly, but by saddle point integration around $\bar\eta = 0$. To this end one expands the exponent of (2.3) as $\mathrm{ch}\,\eta \simeq 1 + \eta^2/2$, and replaces all $\mathrm{ch}\,\eta$ prefactors by their saddle point value $\mathrm{ch}\,\bar\eta = 1$. This amounts to replacing the modified Bessel functions by their leading term for large a, $K_\nu(a) \simeq \sqrt{\pi/(2a)}\, e^{-a}$. It is easy to see that in this approximation the τ_0-dependent terms in Eq. (3.6) for R_o^2 vanish exactly. One would thus conclude that for sharp freeze-out at $\tau = \tau_0$ there is no influence of the source lifetime on R_o^2 and on the difference $R_o^2 - R_s^2$.

The exact expressions (3.6) show, however, that this is not true: $R_o^2 - R_s^2$ is sensitive to the time structure of the source. The physical mechanism behind this is quite interesting: since particles from different points on the freeze-out surface contribute to the correlation function as long as they are separated in z-direction by less than the longitudinal region of homogeneity R_l, the correlation function indeed probes a finite range of coordinate times t along the proper-time hyperbola $\tau{=}\tau_0$: $\Delta t \simeq \sqrt{\tau_0^2{+}R_l^2/4} - \tau_0$ (assuming that only half of the longitudinal region of homogeneity contributes effectively). This shows up in the HBT radius for the out-direction as a contribution from a finite duration of the emission process and is formally reflected by the last two lines in the expression (3.6) for R_o^2. We will see the practical consequences of this in Sections V and VI.

B. Transverse saddle point integration

Since in most realistic situations we can assume $a(r) \geq \frac{m_\perp}{T} > 1$ (low-$K_\perp$ photons from a high-temperature source being the exception), it makes sense to use in the transverse integrals the asymptotic expansion of the Bessel functions for large arguments:

$$K_\nu(a) = \sqrt{\frac{\pi}{2a}} e^{-a} \left(\sum_{n=0}^{p-1} \frac{1}{(2a)^n} \frac{\Gamma(\nu+n+\frac{1}{2})}{n!\Gamma(\nu-n+\frac{1}{2})} + R(\nu,p,a) \right) . \tag{3.7}$$

Clearly, how many orders p must be taken into account to obtain a good approximation for (3.3) depends on both the order ν and the lower limit for a, as reflected by the upper bound [20]

$$|R(\nu,p,a)| < \frac{\Gamma(\nu+p+\frac{1}{2})}{(2a)^p p! |\Gamma(\nu-p+\frac{1}{2})|} , \qquad p \geq \nu - \frac{1}{2} , \tag{3.8}$$

for the remainder term in Eq. (3.7).

Substituting the expansion (3.7) back into (3.6), we are left with sums of integrands of the type $f(x,y)\, G_n(\mathbf{r})$ where

$$G_n(\mathbf{r}) = (\mathrm{ch}\,\eta_t)^{-n-\frac{1}{2}}\, e^{-(m_\perp \mathrm{ch}\,\eta_t - K_\perp (x/r)\,\mathrm{sh}\,\eta_t)/T}\, e^{-r^2/(2R^2)} . \tag{3.9}$$

The integer n labels the order in the expansion (3.7). The resulting transverse integrals can be done by a saddle point approximation of $G_n(\mathbf{r})$. To obtain accurate results it is, however, important to find a good approximation for the exact saddle point $\bar{x}_n$ of $G_n(\mathbf{r})$ which cannot be obtained analytically except for simple limiting cases [6]. In previous studies [4,6,7] a Gaussian approximation of the first exponential factor in (3.9) around its own saddle point was used; the resulting product of two Gaussian factors was integrated out analytically, without however including the effects from the $(\mathrm{ch}\,\eta_t)^{-\left(n+\frac{1}{2}\right)}$ prefactor in (3.9). Unfortunately, we found that in most cases the resulting analytical expressions for the HBT radii do not exhibit the correct $m_\perp$ dependence of the exact results obtained by evaluating the expressions (3.6) numerically. In Appendix A we derive an analytical approximation scheme for obtaining a better estimate of the true saddle point which yields much more accurate results. We derive there the following Gaussian approximation for the transverse weight function $G_n(\mathbf{r})$:

$$G_n(\mathbf{r}) \approx C_n \exp\left[-\frac{y^2}{2\,R_s^2(n)} - \frac{(x-\bar{x}_n)^2}{2\;R_o^2(n)}\right] , \tag{3.10}$$

with effective Gaussian size parameters

$$\begin{aligned} \frac{1}{R_s^2(n)} &= \frac{1}{R^2} + \frac{1}{\lambda_s^2(\bar{x}_n)} , \\ \frac{1}{R_o^2(n)} &= \frac{1}{R^2} + \frac{1}{\lambda_o^2(\bar{x}_n)} . \end{aligned} \tag{3.11}$$

Here $\bar{x}_n$ is the x-component of the transverse saddle point $\bar{\mathbf{r}}_n$ which should be determined from the condition

$$\left(\left(n+\frac{1}{2}+\frac{m_\perp}{T}\right)\mathrm{sh}\,\bar{\eta}_n - \frac{K_\perp}{T}\mathrm{ch}\,\bar{\eta}_n\right)\frac{\bar{\eta}'_n}{\bar{x}_n} = -\frac{1}{R^2} . \tag{3.12}$$

Barred quantities with an index n denote values at the saddle point, and primes indicate x-derivatives. Since this equation cannot be solved analytically for $\bar{x}_n$, one replaces in Eqs. (3.11) the true saddle point $\bar{x}_n$ by an analytically obtained approximate value $\tilde{x}_n$ and calculates the 'lengths of homogeneity' $\lambda_s^2(\bar{x}_n)$ and $\lambda_o^2(\bar{x}_n)$ by evaluating the expressions (A7,A8) at that point. The correct saddle point $\bar{x}_n$ is then related to the approximate value $\tilde{x}_n$ by Eqs. (A12,A13), while the constant C_n is determined by Eq. (A15). A good choice for the approximate saddle point $\tilde{x}_n$ is also derived in Appendix A and given analytically by Eqs. (A16), (A5) and (A10).

Please note that according to Eqs. (A7,A8) the homogeneity lengths in the 'side' and 'out' directions, λ_s and λ_o, are in general different; this was also recently found by Akkelin and Sinyukov [7]. Only for a linear transverse rapidity profile with a small slope (the nonrelativistic transverse expansion scenario studied in Refs. [4,5]), where the transverse saddle point can be taken as $\bar{x}_n \approx 0$, do the two expressions agree. We will see in Section V that in some situations this difference can be drastic, causing sizeable differences between R_o and R_s which have nothing to do with a finite duration of the emission process.

Using the expansion (3.7), the HBT radii (3.6) can be rewritten in terms of integrals over the auxiliary functions $G_n(\mathbf{r})$, Eqs. (3.9/3.10). Both the numerator and denominator of (3.3) reduce to integrals of the type

$$\sum_{n=0}^{p} c_n \left(\frac{T}{m_\perp}\right)^{n+\frac{1}{2}} \int dx\, dy\, f(x,y)\, G_n(\mathbf{r}) \,, \tag{3.13}$$

with certain coefficients c_n which depend on the particular HBT radius under consideration (see Appendix B). For the polynomials $f(x,y)$ occuring in (3.6) the necessary integrals are easily evaluated in terms of the size parameters (3.11):

$$\begin{aligned}
\int dx\, dy\, G_n(\mathbf{r}) &= 2\pi C_n\, R_o(n)\, R_s(n) \,, \\
\int dx\, dy\, y^2\, G_n(\mathbf{r}) &= 2\pi C_n\, R_o(n)\, R_s^3(n) \,, \\
\int dx\, dy\, x\, G_n(\mathbf{r}) &= 2\pi C_n\, R_o(n)\, R_s(n)\, \bar{x}_n \,, \\
\int dx\, dy\, x^2\, G_n(\mathbf{r}) &= 2\pi C_n\, R_o(n)\, R_s(n) \left(R_o^2(n) + \bar{x}_n^2\right) .
\end{aligned} \tag{3.14}$$

Before combining these results into analytical expressions for the HBT radii (see Appendix B and Section III C), let us shortly comment on the validity of the transverse saddle point approximation introduced in this subsection. We have checked three possible sources of deviations of the approximate analytical HBT radii from the results obtained by exact numerical integration of the expressions (3.6):

(i) The expansion (3.7) of the Bessel functions $K_\nu(a)$ is asymptotic, but not convergent. In our case this is not a problem since we found that in all cases where the approximation scheme worked (see below), convergence was reached at low orders $p \leq 3$, cf. Section V A.

(ii) The factor $(\mathrm{ch}\,\eta_t)^{-\left(n+\frac{1}{2}\right)}$ in $G_n(\mathbf{r})$ was substituted by $\exp\left[-\left(n+\frac{1}{2}\right)(\mathrm{ch}\,\eta_t - 1)\right]$, cf. Appendix A. Numerical checks showed that in all the cases considered in Section V the errors introduced by this approximation are negligible.

(iii) The crucial approximation in our method turns out to be the substitution of the transverse weight function $G_n(\mathbf{r})$ (in the form (A2)) by the Gaussian (3.10). This is good only as long as the quadratic term in the expansion of $\ln G_n(\mathbf{r})$ around $\bar{x}_n$ dominates the behaviour of the integrand. As we will discuss in Section V B, this places a strong mathematical constraint on the form of the transversal flow $\eta_t(r)$.

C. Expressions for the HBT radii

It is now only a matter of patience to combine Eqs. (3.3), (3.5), (3.7), (3.13), and (3.14) to obtain analytical expressions for all the HBT radii (3.6) in terms of the two effective Gaussian size parameters (3.11). This is done in Appendix B. Since the comparison in Section V with numerical results will show that we typically must keep several terms from the expansion of the Bessel functions, these analytical results are rather complicated and don't lend themselves easily to an intuitive interpretation. In particular, different terms of the expansion generate different types of $m_\perp$ dependences, and there is no simple $m_\perp$ scaling law for the HBT radii.

It is nevertheless instructive to compare our analytical results with simpler ones previously proposed by several other authors [3,4,6,7,10,13]. Those results were attractive because they implied simple scaling laws for the HBT radius parameters as a function of the average transverse momentum $K_\perp$ of the boson pair (resp. of its transverse mass $m_\perp$), and for this reason they are still popular with experimentalists (see, e.g., [21–23]). We will discuss here which limits have to be taken in order to recover these results from our expressions, and what these limits imply.

Let us begin with the longitudinal HBT radius R_l. Historically the first expression was derived by Makhlin and Sinyukov in a model without transverse flow, via saddle point approximation around $\bar{\eta} = \bar{\mathbf{r}} = 0$. They found

$$R_l^2 = \tau_0^2 \frac{T}{m_\perp} \qquad [\textit{Makhlin and Sinyukov}\ [10]]\,, \tag{3.15}$$

and this has been used by experimentalists to fit their data [21–23]. For boson emitting sources with a non-relativistic linear transverse flow $v(r/R)$, $v \ll c$, a next order correction to (3.15) was calculated by Chapman *et al.* who found in the boost-invariant limit[1] ($\Delta\eta \to \infty$ in their notation)

$$R_l^2 = \tau_0^2 \frac{T}{m_\perp} \left(1 + \left(\frac{1}{2} + \frac{1}{1 + \frac{m_\perp}{T} v^2} \right) \frac{T}{m_\perp} \right) \qquad [\textit{Chapman, Scotto and Heinz}\ [4]]\,. \tag{3.16}$$

In that paper the transverse shift of the saddle point away from $\bar{\mathbf{r}} = 0$ was neglected, which is justified for small transverse flow velocities. An analytical expression without any approximation was recently derived for the case of vanishing transverse flow:

$$R_l^2 = \tau_0^2 \frac{T}{m_\perp} \frac{K_2(\frac{m_\perp}{T})}{K_1(\frac{m_\perp}{T})} \qquad [\textit{Herrmann and Bertsch}\ [13]]\,. \tag{3.17}$$

In the limit $v \to 0$, (3.16) reduces to the first two terms from an expansion of (3.17) for $m_\perp \gg T$. Our expression (3.6) for the longitudinal radius in the case of arbitrary transverse flow reads (cf. Appendix B)

$$R_l^2 = \tau_0^2 \left\langle \frac{1}{a(r)} \frac{K_2(\frac{1}{a(r)})}{K_1(\frac{1}{a(r)})} \right\rangle_* = \tau_0^2 \frac{\sum_{n=0}^p \tilde{c}_n F_n}{\sum_{n=0}^p c_n F_n} \qquad [\textit{our general expression}]\,. \tag{3.18}$$

---

[1]For the sake of completeness, we mention another generalization of the Makhlin-Sinyukov formula,

$$R_l^2 = \tau_0^2 \frac{T}{m_\perp} \frac{1}{1 + \frac{T}{m_\perp (\Delta\eta)^2}} \qquad [\textit{Csörgő and Lørstad}\ [3]]\,,$$

which was found by saddle point approximation for a source with a finite longitudinal extension $\Delta\eta$ which breaks the boost-invariance. In the limit $\Delta\eta \to \infty$, this expression reduces to (3.15). The results of Ref. [4] generalize it by non-leading corrections to the saddle point approximation. Our investigation here is restricted to boost-invariant models.

From this result Eq. (3.17) follows immediately in the limit of vanishing transverse flow, $\lim_{\eta_t=0} a(r) = \frac{m_\perp}{T}$, and Eq. (3.15) is the leading term of an expansion of (3.17) for large $\frac{m_\perp}{T}$. It was pointed out before [13] that keeping only this lowest order term is an insufficient approximation for realistic values of T and $m_\perp$, and that therefore (3.15) should not be used to extract the freeze-out time τ_0 from data.

In the presence of transverse flow the $m_\perp$-dependence of R_l is much more complicated than either (3.15) or (3.17). The approximate result (3.16) for a weak linear transverse flow cannot, in this form, be derived from (3.18); one can show that for $v^2 \ll \frac{T}{m_\perp} \ll 1$ the leading term $\sim v^2$ agrees in both expressions, but differences occur in higher orders. These arise from the transverse shift of the saddle point $\bar{\mathbf{r}}$ which was neglected in Ref. [4], but must be taken into account for stronger transverse flows. This leads to a significant further modification of the $m_\perp$-dependence which will be studied in detail in the following Section.

Let us now turn to the "side"- and "out"-radii. Here we can again compare[2] with the expressions obtained by Chapman *et al.* [4] for a weak linear transverse flow, expanding around $\bar{\mathbf{r}} = 0$. (Earlier results by Csörgő and Lørstad [3] can be obtained from those of Ref. [4] by keeping only the leading terms of this expansion.) In the limit of a boost-invariant source ($\Delta\eta \to \infty$ in their notation) and sudden freeze-out at proper time τ_0 ($\Delta\tau \to 0$) they find in the LCMS frame ($\beta_L = 0$)

$$\left.\begin{aligned} R_o^2 &= \frac{R^2}{1+\frac{m_\perp}{T}v^2} + \frac{1}{2}\left(\frac{T}{m_\perp}\right)^2 \beta_\perp^2 \tau_0^2 \,, \\ R_s^2 &= \frac{R^2}{1+\frac{m_\perp}{T}v^2} \,. \end{aligned}\right\} \qquad [\textit{Chapman, Scotto and Heinz } [4]] \tag{3.19}$$

Let us compare this with the lowest order ($n = 0$) contributions from our approximation scheme (see Appendix B):

---

[2]The results of Akkelin and Sinyukov [7] for systems with strong transverse flow, obtained by saddle point approximation, are difficult to compare with since the authors have for mathematical convenience modified the geometrical part of the emission function (2.3).

$$
\left.\begin{aligned}
\frac{1}{R_s^2} &= \frac{1}{R_s^2(0)} = \frac{1}{R^2} + \frac{1}{\lambda_s^2(\bar{x}_0)}\,, \\
\frac{1}{R_o^2} &= \frac{1}{R_o^2(0)} = \frac{1}{R^2} + \frac{1}{\lambda_o^2(\bar{x}_0)}\,.
\end{aligned}\right\} \qquad [\textit{our lowest order contribution}] \tag{3.20}
$$

For a weak linear transverse flow $\eta = \eta_{\rm f}(r/R)$, $\eta_{\rm f} \ll 1$, one may approximate the transverse saddle point by $\bar{x}_0$=0. Then with a little algebra Eqs. (A7/A8) can be shown to yield $\frac{1}{\lambda_o^2(0)} = \frac{1}{\lambda_s^2(0)} = A_0 \frac{\eta_{\rm f}^2}{R^2}$. Inserting this into (3.20) one recovers the first term on the right hand sides of Eqs. (3.19) except that in the denominator $\frac{m_\perp}{T}$ is replaced by $A_0 = \frac{m_\perp}{T} + \frac{1}{2}$. The reason for this slight discrepancy is that in Ref. [4] (as in all other previous discussions of the emission function (2.3)) the contribution of the factor $(\mathrm{ch}\,\eta_t)^{-n-\frac{1}{2}}$ in (3.9) to the lengths of homogeneity was overlooked.

These first terms in Eqs. (3.19) arise from the "geometric" parts, $\langle x^2\rangle - \langle x\rangle^2$ and $\langle y^2\rangle$, respectively, in the model-independent expressions (1.4), which for weak linear transverse flow coincide to lowest order of our approximation scheme. The second term in the expression (3.19) for R_o is due to the finite duration of particle emission by the source (2.3) and comes from the term $\beta_\perp^2\left(\langle t^2\rangle - \langle t\rangle^2\right)$ in (1.4). It is identical with the lowest non-vanishing contribution from Eq. (B6). Thus the authors of [4] have correctly identified the lowest order contributions to $\langle x^2\rangle - \langle x\rangle^2$, $\langle y^2\rangle$, and $\beta_\perp^2\left(\langle t^2\rangle - \langle t\rangle^2\right)$. (The term $-2\beta_\perp\left(\langle xt\rangle - \langle x\rangle\langle t\rangle\right)$ in R_o^2 is much smaller and according to (B5) can only be seen if the effect of the transverse flow on the saddle point $\bar{x}_i$ is taken into account.) In the following Section we will show, however, that for realistic values of the transverse flow velocity these lowest order contributions are unreliable, and a correct description of the $K_\perp$-dependence of the HBT radii requires the inclusion of higher order corrections from our general expressions in Appendix B.

We already noted that for transverse flows which are not weak or have a non-linear dependence on r, the two transverse homogeneity lengths λ_o and λ_s are no longer equal (see also [7]). In that case it is no longer guaranteed that the finite duration of particle emission yields the dominant contribution to the difference $R_o^2 - R_s^2$ as in models with weak or vanishing transverse flow [4,5,24]. Since transverse flow is a generic feature of the sources generated in heavy-ion collisions [25], its effect must to be taken into account when trying to interpret measured differences between R_o and R_s in terms of the emission time.

IV. NUMERICAL EVALUATION OF THE CORRELATOR

The η-integration for the full 2-particle correlation function $C(\mathbf{K},\mathbf{q})$ as given by Eq. (1.2) can be done analytically by the same methods as used in Section III A. The resulting 2-dimensional transverse Fourier integrals will then be evaluated numerically. The half widths of the numerically computed exact correlator can then be compared with the analytical expressions for the HBT radii from the previous Section, allowing us to check our approximations.

The η-integral for the denominator of Eq. (1.2) was already evaluated in Section III A:

$$\int d^4x\, S(x,K) = \frac{2\tau_0 m_\perp}{(2\pi)^3} \int dx\, dy\, K_1(a(r))\, e^{K_\perp (x/r)\, \mathrm{sh}\, \eta_t/T}\, e^{-r^2/(2R^2)}\,. \tag{4.1}$$

For the numerator let us first study the limit $q_l = 0$. Writing $t{=}\tau\, \mathrm{sh}\, \eta$, $z{=}\tau\, \mathrm{ch}\, \eta$ and using $q^0{=}\beta_\perp q_o$ (which is true in the LCMS frame where $\beta_L = 0$), we find

$$\begin{aligned}
&\int d^4x\, S(x,K)\, e^{i(\beta_\perp q_o t - q_o x - q_s y)} \\
&= \frac{2\tau_0 m_\perp}{(2\pi)^3} \int dx\, dy\, e^{-i(q_o x + q_s y)}\, K_1\Big(a(r) - i\tau_0\beta_\perp q_o\Big)\, e^{K_\perp (x/r)\, \mathrm{sh}\, \eta_t(r)/T}\, e^{-r^2/(2R^2)}\,.
\end{aligned} \tag{4.2}$$

The q_l-dependence of $C(\mathbf{K},\mathbf{q})$ can be obtained by using

$$\int_{-\infty}^{\infty} d\eta\, \mathrm{ch}\, \eta\; e^{-a\, \mathrm{ch}\, \eta - i\, b\, \mathrm{sh}\, \eta} = \frac{2\, a}{\sqrt{a^2+b^2}}\, K_1\left(\sqrt{a^2+b^2}\right) \tag{4.3}$$

for real values of a and b. This leads to

$$\begin{aligned}
&\int d^4x\, S(x,K)\, e^{-i q_l z} \\
&= \frac{2\tau_0 m_\perp}{(2\pi)^3} \int dx\, dy\, \frac{a(r)}{\sqrt{a(r)^2 + q_l^2\tau_0^2}}\, K_1\left(\sqrt{a(r)^2 + q_l^2 \tau_0^2}\right)\, e^{K_\perp (x/r)\, \mathrm{sh}\, \eta_t(r)/T}\, e^{-r^2/(2R^2)}\,.
\end{aligned} \tag{4.4}$$

With these (exact) expressions the correlation functions $C(\mathbf{K}, q_o, q_s, q_l{=}0)$ and $C(\mathbf{K}, q_o{=}q_s{=}0, q_l)$ reduce to a ratio of 2-dimensional integrals. These have been evaluated numerically. From the numerical result for the correlator we determine new HBT radii $R_i^c(\mathbf{K})$ by requiring that the function

$$C(\mathbf{K}, q_i) = 1 + e^{-R_i^c(\mathbf{K})^2 q_i^2}\,, \qquad i = o,\, l,\, s, \tag{4.5}$$

reproduces exactly the half point, $C(\mathbf{K}, \mathbf{q}) = 1 + \frac{1}{2}$, of the numerically computed correlator (the other two components $q_{i \neq j}$ are set equal to zero). In the following Section these values $R_i^c(\mathbf{K})$ will be compared with HBT radii obtained from the expressions (3.6) by either numerical or analytical evaluation of the transverse integrals $\langle \ldots \rangle_*$.

V. RESULTS

In this section we give a detailed and quantitative analysis of the HBT radii for the source function (2.1/2.2), both for a linear and a quadratic transverse flow rapidity profile $\eta_t(r)$. We will compare the analytic methods developed in Section III with exact numerical results. We will show that for the linear transverse flow the analytical approximation scheme works very well, but only if higher order contributions (non-leading terms in the Bessel function expansion and an improved value for the transverse saddle point) are properly taken into account. For the quadratic flow the saddle point methods from Section III are found to fail. For a weak quadratic transverse flow we also find an unexpected effect: in the $x{-}y$-plane the emission region *increases* with rising transverse momentum of the pair. Although this case may be somewhat pathological, it provides a specific counter example to the folklore [9,3,4] that transverse collective flow leads to a reduction of the transverse HBT radii at finite transverse pair momentum. Another example with a different velocity profile, but similar behaviour for R_o was found in Ref. [7].

A. Linear transverse flow rapidity profile

All the sources studied in this Section have a transverse geometric (Gaussian) radius $R = 3$ fm and freeze out along a hyperbola of constant longitudinal proper time $\tau_0 = 3$ fm/c at temperature $T = 150$ MeV. The longitudinal flow is always boost-invariant, and we study the sensitivity to the transverse flow.

Let us begin with a linear transverse flow rapidity profile:

$$\eta_t(r) = \eta_{\rm f}\, \frac{r}{R}\,. \tag{5.1}$$

For this case the transverse momentum dependence of the HBT radii R_s, R_o, and R_l is shown in Fig. 1. In the figure we compare various approximations: The solid lines are obtained by numerical evaluation of the expectation values in the model-independent expressions (1.4) in the form (3.6) with our source (2.1). The long-dashed lines show the radii R_i^c from Eq. (4.5) which reproduce the half width of the exact correlation function in direction i. The short-dashed lines represent our analytical results (B2) - (B6), evaluated with the effective size parameters (3.11) obtained in Appendix A and including all terms up to order $p = 3$ from the expansion of the Bessel functions. The dash-dotted lines show the same analytic expressions but truncating the expansion (3.7) at lowest order $p = 0$.

Let us summarize the most important features of Fig. 1:

1. For R_s our analytical expression (B2) approximates very accurately, even for $p = 0$, the exact value obtained numerically from Eq. (3.6). Thus for R_s the leading order expression

$$\frac{1}{R_s^2} = \frac{1}{R^2} + \frac{1}{\lambda_s^2(\bar{x}_0)} \tag{5.2}$$

can be used with excellent accuracy. However, it is necessary that the homogeneity length λ_s is evaluated sufficiently closely to the exact transverse saddle point $\bar{x}_0$ as described in Appendix A. Our studies showed that, if used with the lowest order estimate (A9), Eq. (5.2) develops for large transverse flow rapidities $\eta_{\rm f} > 0.3$ a much stronger $K_\perp$-dependence than the exact "side"-radius. Clearly this renders an analytical determination of the transverse flow velocity from the $K_\perp$-dependence of R_s a somewhat subtle issue, to say the least.

The exact numerical value for R_s from Eq. (3.6) also coincides very accurately with R_s^c from (4.5). For the source (2.1) our model-independent expressions (1.4) thus correctly reproduce not only the curvature of the correlation function at small values of q_s (which is difficult to access experimentally), but also its half width. This should not be too surprising after one checks that in the "side"-direction the source (2.1) has a rather Gaussian shape for all relevant values of $K_\perp$.

2. In the "out"-direction the situation is more complicated: there the leading term from the Bessel function expansion is seen to yield qualitatively wrong results. After including higher order terms up to p=3, the agreement with the exact numerical results improves dramatically, although some deviations remain at large $K_\perp$ for strong transverse flows. This remaining discrepancy can be traced back to the fact that we use for the transverse integration only an approximate saddle point. For the "naive" saddle point x_n^d the disagreement is much worse. This shows that for the "out"-direction and strong transverse flow the usefulness of the analytical approach relies crucially on an accurate approximation for the transverse saddle point.

It is important to note that the lowest order approximation (dash-dotted line) completely misses the rise of R_o at small values of $K_\perp$. This is a lifetime effect and arises from the time variance given in (B3). The rise was also noted previously in correlation functions based on numerical hydrodynamic calculations [26], but without a detailed theoretical analysis of its origin. As seen from Eq. (B9), the time variance only begins to receive contributions at third order of the Bessel function expansion, but the coefficients of these higher order terms are larger than unity. The physics of this effect has been explained at the end of Section III A. For weak flow ($\eta_{\rm f} \leq 0.3$) it yields the dominant contribution to the difference $R_o^2 - R_s^2$ between the "out" and "side" radii.

For strong transverse flow, that difference remains appreciable even if the contribution from the time variance is neglected; it is then due to the difference between the homogeneity lengths in the two directions, see Eqs. (A7/A8). This difference can be rather accurately estimated by comparing the dash-dotted lines in Fig. 1a (our lowest order analytical results for R_o which give essentially $\sqrt{\langle x^2\rangle - \langle x\rangle^2)}$ with Fig. 1b. For $\eta_{\rm f} = 0.9$ one finds, for example, that in the region $K_\perp > 500$ MeV/c the expressions $\langle y^2\rangle$ and $\langle x^2\rangle - \langle x\rangle^2$ differ by more than 50%.

The agreement between the exact HBT radius R_o from the model-independent expression (1.4) and the half widths described by R_o^c is excellent for all values of $\eta_{\rm f}$ and $K_\perp$. The reason is again the rather Gaussian shape of the source function also in the "out"-direction.

3. For the longitudinal radius R_l, the leading term from the Bessel function expansion is insufficient again (this is even true for vanishing transverse flow, see [13]), but excellent agreement with the exact value from the model-independent expression (1.4) is reached for $p = 3$, for all values of $K_\perp$. For small transverse momenta, however, one sees a disagreement with the half width of the exact correlator as given by the parameter $R_l^c(K_\perp)$. On the other hand we checked numerically that $R_l^2(K_\perp)$ from Eq. (1.4) correctly reproduces the curvature of $C(K_\perp, q_l)$ at $q_l = 0$, as it should. In this case the disagreement arises from non-Gaussian features of our source (2.1) in the longitudinal (η) direction: for small $K_\perp$ the source decreases at large values of η much more steeply than a Gaussian. However, the discrepancy is small, and in spite of these non-Gaussian features of the source the model-independent radii (1.4), evaluated with the full source (2.1), reproduce the half width of the correlation function with an accuracy of a few percent.

4. All three HBT radii are strongly affected by the transverse flow, even at $K_\perp = 0$. The influence on R_l is relatively weaker than on R_s and R_o. As transverse flow increases, the transverse region of homogeneity decreases, and the HBT correlator receives contributions from a smaller and smaller fraction of the total source. At $K_\perp = 0$ the "out-" and "side-" radii are equal, $R_s(K_\perp{=}0){=}R_o(K_\perp{=}0)$; this must be true because for $K_\perp = 0$ there is nothing to distinguish between the two transverse directions [4].

An extraction of the geometric transverse radius R from the data requires to disentangle geometrical from dynamical effects. Based on Fig. 1 it appears to us that the most promising way to achieve this is as follows: one estimates the amount of transverse flow from the strength of the $K_\perp$-dependence of the "side"-radius R_s. One then extrapolates $R_s(K_\perp)$ to $K_\perp = 0$ and uses the estimate for the transverse flow rapidity to correct it for flow effects, thus extracting the geometric transverse radius R.

5. From Fig. 1a it is clear that the extraction of the duration of particle emission, which is related to the rate at which $R_o(K_\perp)$ rises for small $K_\perp$, requires an accurate measurement of the $K_\perp$-dependence of the "out"-radius in the domain $K_\perp < 100$ MeV/c. This is not easy, because for very small $K_\perp$ the detection efficiency tends to decrease (in a TPC the track density becomes high, in a single arm spectrometer one of the two particles tends to miss the detector). It therefore helps to know that R_o agrees with R_s at $K_\perp = 0$; the latter can be extracted much more easily by extrapolation, due to its smooth $K_\perp$-dependence.

Fig. 1 shows that the leading terms from a saddle point approximation produce quantitatively and qualitatively unreliable results for the $m_\perp$-dependence of the HBT radii R_l and R_o. Quantitative attempts to extract values for the freeze-out time τ_0, the transverse flow $\eta_{\rm f}$ and the transverse geometric size R from HBT data must thus be based on a numerical evaluation of the model-independent expressions (1.4) rather than on simple $m_\perp$-scaling laws extracted from insufficient analytical approximations.

B. Quadratic transverse flow rapidity profile

A linear transverse flow rapidity profile is perhaps the simplest, but not obviously the most realistic assumption. Dynamical studies based on a numerical solution of the hydrodynamic equations [27] produce transverse flow velocity profiles which for small values of r look parabolic and for large r saturate at the light velocity; the parabolic rise is zero initially, but builds up gradually as the hydrodynamic pressure generates transverse flow. On the other hand, when combining such hydrodynamic calculations with a dynamic model for the freeze-out kinetics, taking into account sequential transverse freeze-out of the matter from the dilute edge at early times to the center at later times, the resulting flow profile *along the freeze-out hypersurface* turns out [28] to be *linear* again for small r, reaching a maximum at intermediate r and dropping to zero again at the edge of the firetube (because that part of the matter freezes out immediately due to its diluteness, before transverse flow is able to build up). Numerical kinetic simulations of pion production based on the RQMD code, finally, produce a transverse flow profile at freeze-out [29] which rises quadratically at small r, reaches a maximum and slightly drops again at large r.

The shape of the transverse flow profile is thus not at all clear. Investigations of particle freeze-out based on hydrodynamic or kinetic models, which result in source functions which are only numerically known, and their influence on the shape of the HBT correlation function should thus be supplemented by systematic studies of simple model sources with various types of simple flow profiles. In this subsection we therefore supplement the results from the previous one by a discussion of quadratic transverse flow rapidity profiles:

$$\eta_t(r) = \eta_{\rm f} \, \frac{r^2}{R^2} \, . \tag{5.3}$$

The resulting transverse momentum dependence of the HBT radii is shown in Fig. 2. The labelling and meaning of the various curves is the same as in the previous subsection. However, the curves corresponding to analytical approximations are missing; the reason is that the scheme developed in Section III and Appendix A fails for a quadratic transverse flow rapidity profile. That scheme relied on a Gaussian saddle point approximation in the transverse direction; since for small $K_\perp$ the dominant contribution from the transverse flow arises from the factor $\mathrm{ch}\,\eta_t$ in the exponent which contributes only at order r^4, a Gaussian saddle point approximation misses the dominant flow effects completely. An improved analytical approximation scheme on the other hand would be much more complicated.

Comparing Figs. 1 and 2 we see that qualitatively the $K_\perp$-dependence of the HBT radii is similar for both types of transverse flow profile. In all cases the model-independent expressions (1.4) yield HBT radii which give an excellent representation of the half-width of the correlator. For a given flow scale $\eta_{\rm f}$, the flow effects on the HBT radii are stronger for the quadratic flow (5.3) than for the linear flow (5.1). This indicates that emission from regions $r > R$ in the source plays a significant role for the shape of the correlation function.

The only unusual feature in Fig. 2 is the rise of R_o with $K_\perp$ for a weak quadratic transverse flow with $\eta_{\rm f} = 0.1$. It can be traced to a rise of the variance in the "out"-direction, $\langle x^2 \rangle - \langle x \rangle^2$, with $K_\perp$. This is different from the generic decrease of the effective region of homogeneity with increasing $K_\perp$ which is seen in all other cases. It seems to be due to an accidental coincidence of parameters: In Fig. 3 we show a contour plot of the transverse weight function (3.4); only for small quadratic transverse flow it gets wider in the x direction with increasing $K_\perp$. Although exotic, this example should be kept in mind as a reminder that transverse flow not always leads to a decrease of the effective source size.

VI. SUMMARY AND CONCLUSIONS

The aim of HBT interferometry with hadrons produced in heavy ion collisions is to obtain information about the space-time structure of the emitter. Unfortunately, a unique reconstruction of the emission function $S(x, K)$ from the 2-particle correlation function $C(\mathbf{q}, \mathbf{K})$ is impossible, since the frequency q^0 and the spatial momenta $\mathbf{q}$ in the Fourier transformation relating the two functions are not independent, as the final state hadrons are on mass-shell. However, the class of possible models for the emission function can be strongly restricted by a careful multidimensional HBT analysis of the 2-body correlation function. In particular, a careful analysis of the dependence of the HBT radii (which describe the width of the correlator as a function of the relative momentum $\mathbf{q}$) on the pair momentum $\mathbf{K}$ provides crucial information on the correlations between emission point x and particle momentum K in the source. Such $x-K$ correlations are generated, for example, by the collective expansion of the sources generated in heavy ion collisions. Recently heavy ion experiments [21–23] have begun to provide first quantitative information on the **K**-dependence of the HBT correlation functions. The hope to obtain access to the collective behaviour in heavy ion collisions by analyzing these results has refuelled the theoretical interest in the **K**-dependence of the HBT radii.

In this paper we reanalyzed this issue with a combination of numerical and analytical methods. For a given source model, the numerical methods provide us with exact predictions for the transverse momentum dependence of the HBT radii. They thus establish a reliable link between various possible models for the emission function and the experimental data for the correlation function. The analytical approach provides, in the context of a specific source model studied in this paper, a mathematical understanding of the numerical results and a bridge between approximate analytical expressions published previously and the exact numerical results obtained here, and thus permits to test their reliability.

Our numerical studies showed that the model-independent expressions (1.4) for the HBT radii not only give the correct curvature of the correlation function $C(\mathbf{q},\mathbf{K})$ for small values of $\mathbf{q}$ (as was known before [4,13]), but also reproduce on most cases the width of the correlator quite accurately. As long as the measured correlation function has a roughly Gaussian shape in $\mathbf{q}$, these expressions provide an accurate and valuable link between theory and experiment and permit to relate the (**K**-dependent) width parameters of the correlation function to the (**K**-dependent) space-time width of the original source function. In this context it should be cautioned, however, that the effects of resonance decays, which are numerically known to give the correlation function a non-gaussian shape [26,28], remain to be studied in more detail. Furthermore, this result means that *only* the half-widths of the emission function in space-time can be reconstructed from the width of the correlation function. Finer details of the space-time structure of the emitter (like sharp edges or holes in the center) do not affect the width of the correlator, but can only be estimated by a very accurate study of the long-range decay of the correlator (e.g. by looking for side maxima at larger values of $\mathbf{q}$).

The numerical study also enables us to give a detailed assessment of the accuracy of previously published approximate analytical results for the HBT radius parameters. This is of particular interest since some of these results have gained great popularity because of their simple scaling behaviour as a function of $m_\perp$. Here our results cause disappointment: none of the so far suggested simple scaling laws is quantitatively reliable, except for very limiting situations which are not likely to be established in experiments. To make quantitative use of the measured **K**-dependence of the correlation function and to obtain reasonably accurate estimates for the lifetime, transverse geometric size and collective expansion rate of the source, a full-blown numerical evaluation of the expressions (1.4) is required, at least. The simple scaling laws should not be used – we showed that they yield badly misleading results.

For the specific source studied in this paper the duration of particle emission is very short, of order $\Delta t(K_\perp) \simeq \sqrt{\tau_0^2 + R_l^2(K_\perp)/4} - \tau_0$ (≈ 0.9 fm/c at $K_\perp = 0$). This leads to a rise of R_o at small $K_\perp$. The rise stops at $K_\perp \simeq m_\pi$ (when $\beta_\perp$ approaches unity), and beyond that point R_o and the difference between $R_o - R_s$ decreases again (because $\Delta t(K_\perp)$ decreases). The experimental verification of the effect of a finite duration of particle emission thus requires a high-statistics study of the correlation function at small $K_\perp$. One might argue that realistic models in which the sharp freeze-out along a proper time hyperbola is replaced by continuous freeze-out over a longer time period should lead to a larger effect which is more easily seen in the data. However, in such models usually the source shrinks as the particles evaporate, and particles emitted at later times come from smaller regions near the center of the fireball; the net effect on the difference $R_o - R_s$ seems to remain small and concentrated at small $K_\perp < 100$ MeV/c [28]. In the light of these new expectations the failure of the experiments so far [21–23] to provide positive evidence for a finite period of particle emission appears less puzzling. We no longer believe that these data show that in heavy-ion collisions pion emission occurs "in a flash".

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APPENDIX A: DETERMINATION OF THE SADDLE POINT

Here we give the technical details of our saddle point approximation of expression (3.9). We begin by writing the prefactor as $(\mathrm{ch}\,\eta_t)^{-(n+\frac{1}{2})} = \exp[-(n+\frac{1}{2})\ln(\mathrm{ch}\,\eta_t)]$ and expanding the logarithm around $\mathrm{ch}\,\eta_t = 1$:

$$(\mathrm{ch}\,\eta_t)^{-(n+\frac{1}{2})} = e^{-(n+\frac{1}{2})(\mathrm{ch}\,\eta_t - 1)} \left(1 + \frac{2n+1}{16}\eta_t^4 + O(\eta_t^6)\right) . \tag{A1}$$

We checked numerically that keeping only the first term in the brackets is sufficient in the following. This allows us to combine the exponent with the $\mathrm{ch}\,\eta_t$-term from the Boltzmann weight factor and to rewrite $G_n(\mathbf{r})$ in the form

$$G_n(\mathbf{r}) \approx \exp\Big(n + \frac{1}{2} + d_n(\mathbf{r}) - r^2/(2R^2)\Big) , \tag{A2}$$

where

$$d_n(\mathbf{r}) = -A_n\,\mathrm{ch}\,\eta_t(r) + B\,\frac{x}{r}\,\mathrm{sh}\,\eta_t(r) \tag{A3}$$

with

$$A_n = n + \frac{1}{2} + \frac{m_\perp}{T} , \qquad B = \frac{K_\perp}{T} . \tag{A4}$$

The saddle point $\mathbf{r}_n^d = (x_n^d, y_n^d)$ of the modified dynamical term $\exp(d_n(\mathbf{r}))$ alone is found to satisfy

$$y_n^d = 0 , \qquad \mathrm{th}\,\eta_n^d \equiv \mathrm{th}\,\eta_t(x_n^d, 0) = \frac{K_\perp}{m_\perp + (n+\frac{1}{2})T} = \frac{B}{A_n} . \tag{A5}$$

Omitting the term $\left(n+\frac{1}{2}\right)T$ in the denominator, this reduces to the expression given in [7] whose shortcomings were discussed in section V A. In this case the saddle point is given by that point of the source at which the transverse flow velocity $\mathrm{th}\,\eta_t$ agrees with the transverse velocity $\beta_\perp$=$(K_\perp/m_\perp)$ of the pion pair. Numerically a gaussian approximation around this point leads, however, to unsatisfactory results. Including the effects of the $(\mathrm{ch}\,\eta_t)^{-(n+\frac{1}{2})}$ prefactor both in the saddle point and in the gaussian curvature considerably improves the approximation. In addition, the saddle point $\bar{\mathbf{r}}_n$ of the full function $G_n(\mathbf{r})$ is further shifted away from $\mathbf{r}_n^d$ by the geometric factor $\exp(-r^2/(2R^2))$ in (A2). Due to the symmetry of the function $G_n(\mathbf{r})$ this shift will be only in x-direction, i.e. $\bar{y}_n$ will stay zero.

To estimate the position of the exact saddle point $\bar{\mathbf{r}}_n{=}(\bar{x}_n, 0)$ we proceed as follows: we expand $\exp(d_n(\mathbf{r}))$ to second order around an arbitrary point $\tilde{\mathbf{r}}_n{=}(\tilde{x}_n, 0)$. After combining the resulting gaussian with the geometric factor $\exp(-r^2/(2R^2))$ we try to adjust $\tilde{x}_n$ iteratively to obtain an analytical approximation to the true saddle point $\bar{x}_n$ of the full weight function.

The quadratic expansion of the dynamical term reads

$$\begin{aligned} d_n(\tilde{x}_n + \delta x, \delta y) = &-\Big(A_n \operatorname{ch}\tilde{\eta}_n - B \operatorname{sh}\tilde{\eta}_n\Big) - \Big(A_n \operatorname{sh}\tilde{\eta}_n - B \operatorname{ch}\tilde{\eta}_n\Big)\tilde{\eta}_n'\,\delta x \\ &-\frac{\delta x^2}{2\lambda_o^2(\tilde{x}_n)} - \frac{\delta y^2}{2\lambda_s^2(\tilde{x}_n)}\,, \end{aligned} \tag{A6}$$

where $\tilde{\eta}_n{\equiv}\eta_t(\tilde{\mathbf{r}}_n)$, $\tilde{\eta}_n'{\equiv}(\partial\eta_t/\partial x)(\tilde{x}_n, 0)$, etc., and

$$\begin{aligned} \frac{1}{\lambda_s^2(\tilde{x}_n)} &= \Big(A_n \operatorname{sh}\tilde{\eta}_n - B \operatorname{ch}\tilde{\eta}_n\Big)\frac{\partial^2}{\partial y^2}\eta_t(\tilde{\mathbf{r}}_n) \\ &\quad + \Big(A_n \operatorname{ch}\tilde{\eta}_n - B \operatorname{sh}\tilde{\eta}_n\Big)\left(\frac{\partial}{\partial y}\eta_t(\tilde{\mathbf{r}}_n)\right)^2 + B\,\frac{\operatorname{sh}\tilde{\eta}_n}{\tilde{x}_n^2}\,, \end{aligned} \tag{A7}$$

$$\frac{1}{\lambda_o^2(\tilde{x}_n)} = \left(A_n \operatorname{sh}\tilde{\eta}_n - B \operatorname{ch}\tilde{\eta}_n\right)\tilde{\eta}_n'' + \left(A_n \operatorname{ch}\tilde{\eta}_n - B \operatorname{sh}\tilde{\eta}_n\right)\tilde{\eta}_n'^2\,. \tag{A8}$$

Inserting for $\tilde{x}_n$ the saddle point x_n^d of the dynamical factor as determined by (A5), the linear term in (A6) as well as the first terms on the right hand sides of Eqs. (A7,A8) vanish as they should:

$$\frac{1}{\lambda_s^2(x_n^d)} = \frac{B^2}{\sqrt{A_n^2 - B^2}}\,\frac{1}{(x_n^d)^2}\,, \tag{A9}$$

$$\frac{1}{\lambda_o^2(x_n^d)} = \sqrt{A_n^2 - B^2}\,(\eta_n^d)'^2\,. \tag{A10}$$

At any other point $\tilde{x}_n$ this no longer true. Inserting the expansion (A6) into (A2) and completing the squares we end up with a single gaussian

$$\ln G_n(\mathbf{r}) \approx \ln C_n - \frac{\delta y^2}{2\,R_s^2(n)} - \frac{(x - \bar{x}_n)^2}{2\;R_o^2(n)}\,, \tag{A11}$$

whose saddle point $\bar{x}_n$ is related to the point $\tilde{x}_n$ around which the dynamical factor was expanded by

$$\bar{x}_n = \tilde{x}_n - \varepsilon_n \tag{A12}$$

with

$$\varepsilon_n = \frac{R_o^2(n)}{R^2}\tilde{x}_n + R_o^2(n)\Big(A_n \mathrm{sh}\,\tilde{\eta}_n - B\mathrm{ch}\,\tilde{\eta}_n\Big)\tilde{\eta}_n' \,. \tag{A13}$$

The radius parameters in (A11) are given in terms of the geometric radius R and the homogeneity lengths (A7,A8) by

$$\frac{1}{R_s^2(n)} = \frac{1}{R^2} + \frac{1}{\lambda_s^2(\tilde{x}_n)}\,, \qquad \frac{1}{R_o^2(n)} = \frac{1}{R^2} + \frac{1}{\lambda_o^2(\tilde{x}_n)}\,, \tag{A14}$$

while the constant term is given by

$$\ln C_n = n + \frac{1}{2} - \Big(A_n\,\mathrm{ch}\,\tilde{\eta}_n - B\,\mathrm{sh}\,\tilde{\eta}_n\Big) - \frac{\tilde{x}_n^2}{2R^2} + \frac{\varepsilon_n^2}{2R_o^2(n)}\,. \tag{A15}$$

Identifying the expansion point $\tilde{x}_n$ with the true saddle point $\bar{x}_n$ requires setting ε_n=0 and amounts to solving the condition (3.12). The constant J_n then simplifies accordingly. Instead of solving (3.12) numerically we can, however, solve (A5) analytically for x_n^d and then set (see Eq. (A10))

$$\tilde{x}_n = \frac{R^2}{R^2 + \lambda_o^2(x_n^d)}\,x_n^d\,. \tag{A16}$$

This is the saddle point of $G_n(\mathbf{r})$ which would be obtained by expanding $d_n(\mathbf{r})$ around $\mathbf{r}_n^d$. We want to stress that this procedure largely corrects for the fact that the radius parameters $R^2_{s,o}(n)$ should be calculated from the curvature at the full saddle point $\bar{x}_n$ instead of the saddle point x_n^d of the modified dynamical factor alone. As shown in Section V, it is found to yield a very good approximation for the HBT radii and their $m_\perp$-dependence.

APPENDIX B: CALCULATION OF HBT-RADII

In this appendix, we give expressions for the HBT-radii (3.5) with coefficients calculated from the series expansions (3.13/3.14) explicitly up to order $p = 3$. Using the notational shorthand

$$F_n = \left(\frac{T}{m_\perp}\right)^{n+\frac{1}{2}} R_o(n)\, R_s(n)\, C_n\,, \tag{B1}$$

and the results derived in Section III B, we obtain

$$R_s^2 = \frac{\sum_{n=0}^{p} c_n\, F_n\, R_s^2(n)}{\sum_{n=0}^{p} c_n\, F_n}\,, \qquad c_n = \left(1, \tfrac{3}{8}, -\tfrac{15}{128}, \tfrac{105}{1024}, \dots\right)\,, \tag{B2}$$

and

$$R_l^2 = \tau_0^2 \frac{\sum_{n=0}^{p} \tilde{c}_n\, F_n}{\sum_{n=0}^{p} c_n\, F_n}\,, \qquad \tilde{c}_n = \left(0, 1, \tfrac{15}{8}, \tfrac{105}{128}, \dots\right)\,. \tag{B3}$$

The expansion of the "out"-radius R_o^2 involves three contributions:

$$\begin{aligned}
\langle x^2\rangle - \langle x\rangle^2 &= \frac{\sum_{n=0}^{p} c_n\, F_n\, \left(R_o^2(n) + \bar{x}_n^2\right)}{\sum_{n=0}^{p} c_n\, F_n} - \frac{\left(\sum_{n=0}^{p} c_n\, F_n\, \bar{x}_n\right)^2}{\left(\sum_{n=0}^{p} c_n\, F_n\right)^2} \\
&= \frac{\sum_{i,j=0}^{p} d_{ij}^{xx}\, F_i\, F_j \left(R_o^2(i) + R_o^2(j) + (\bar{x}_i - \bar{x}_j)^2\right)}{\left(\sum_{n=0}^{p} c_n\, F_n\right)^2}\,,
\end{aligned} \tag{B4}$$

$$-2\beta_\perp \Big(\langle xt\rangle - \langle x\rangle\langle t\rangle\Big) = \beta_\perp \tau_0 \frac{\sum_{i,j=0}^{p} d_{ij}^{xt}\, F_i\, F_j\, (\bar{x}_i - \bar{x}_j)}{\left(\sum_{n=0}^{p} c_n\, F_n\right)^2}\,, \tag{B5}$$

$$\beta_\perp^2 \left(\langle t^2\rangle - \langle t\rangle^2\right) = \frac{\beta_\perp^2 \tau_0^2}{4} \frac{\sum_{i,j=0}^{p} d_{ij}^{tt}\, F_i\, F_j}{\left(\sum_{n=0}^{p} c_n\, F_n\right)^2}\,. \tag{B6}$$

Here, up to order $p = 3$ the coefficients are given by

$$d_{ij}^{xx} = \begin{pmatrix} \frac{1}{2} & \frac{3}{16} & -\frac{15}{256} & \frac{105}{2048} & \cdots \\ \frac{3}{16} & \frac{9}{128} & -\frac{45}{2048} & \cdots & \cdots \\ -\frac{15}{256} & -\frac{45}{2048} & \cdots & \cdots & \cdots \\ \frac{105}{2048} & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}\,, \tag{B7}$$

$$d_{ij}^{xt} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{9}{16} & -\frac{75}{256} & \cdots \\ \frac{1}{2} & 0 & \frac{69}{256} & \cdots & \cdots \\ \frac{9}{16} & \frac{69}{256} & \cdots & \cdots & \cdots \\ -\frac{75}{256} & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}\,, \tag{B8}$$

$$d_{ij}^{tt} = \begin{pmatrix} 0 & 0 & \frac{3}{2} & \frac{45}{16} & \dots \\ 0 & -1 & -\frac{9}{16} & \dots & \dots \\ \frac{3}{2} & -\frac{9}{16} & \dots & \dots & \dots \\ \frac{45}{16} & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} . \tag{B9}$$

It is interesting to note that only the expressions (B3) and (B6) contain coefficients larger than 1 (cf. our discussion in Section III C).

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FIGURES

FIG. 1. $K_\perp$-dependence of the HBT radii R_o (a), R_s (b), and R_l (c), for the emission function (2.3) with parameters T=150 MeV, τ_0=3 fm/c, R=3 fm. A linear transverse flow rapidity profile $\eta_t(r)$=$\eta_{\rm f}(r/R)$ was assumed. Curves for different values of $\eta_{\rm f}$ are shown. The solid lines are calculated numerically from Eq. (1.4); the long-dashed lines parametrize the widths R_i^c of the numerically computed correlator $C(\mathbf{K}_\perp, q)$ according to Eq. (4.5); the short-dashed lines represent our analytical results to order $p = 3$, while the dash-dotted lines denote the corresponding lowest order results.

FIG. 2. Same as Fig. 1, but for a quadratic transverse flow profile $\eta_t(r)$=$\eta_{\rm f}(r^2/R^2)$. Only the exact HBT radii from a numerical integration of the model-independent expressions (1.4) (solid lines) and the width parameters R_i^c of the numerically computed correlator $C(\mathbf{K}_\perp, q)$ according to (4.5) (dashed lines) are shown.

FIG. 3. Contour plots for the emission function (3.4) in the transverse $x-y$ plane, with parameters T=150 MeV, τ_0=3 fm/c, R=3 fm, for a quadratic transverse flow profile $\eta_t(r)$=$\eta_{\rm f}(r^2/R^2)$. From center to edge the lines correspond to 90% ... 10% of the peak value (in steps of 10%). The two left diagrams show that for a weak quadratic transverse flow with $\eta_{\rm f} = 0.1$) the emission region actually increases in the x-direction with increasing transverse momentum. The two right diagrams (for $\eta_{\rm f} = 0.3$) show the generic decrease of the source in both x and y directions with increasing $K_\perp$.