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Some Problems of Topology Change Description in the Theory of Space-Time

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Abstract

The problem of topology change description in gravitation theory is analized in detailes. It is pointed out that in standard four-dimensional theories the topology of space may be considered as a particular case of boundary conditions (or constraints). Therefore, the possible changes of space topology in (3+1)-dimensions do not admit dynamical description nor in classical nor in quantum theories and the statements about dynamical supressing of topology change have no sence. In the framework of multi-dimensional theories the space (and space-time) may be considered as the embedded manifolds. It give the real posibilities for the dynamical description of the topology of space or space-time.

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1 Introduction

The assumption that topology of 3-space may change dynamically or undergo quantum fluctuations was stated for the first time by Wheeler in connection with his geometrodynamics program [1]. According to Wheeler observable properties of matter and fields must be explained by geometrical and topological properties of space-time and the dynamics of matter and fields configuration are the result of the dynamics of space-time geometry and topology. This assumption and its motivations were discussed by many authors [2]-[17]. It was shown that the properties of matter and fields is indeed closely connected with the topology of space-time [18]-[20]. Some analogies between the topology of space-time and the properties of matter fields were also found recently [31]. The considerable progress was achieved in the investigation of in the investigation of the "existence" and properties of the topology change models [3], [4], [11]-[13], where the well known results of differential geometry and topology may be used. The most essential results in this areas are the theorems about global hyperbolicity and Cauchy problem in general relativity [21]-[24], the Geroch theorem [3] and it's generalization [4]. The theorems about global hyperbolicity state that globally hyperbolic space-time model (i.e. the space-time model which may be considered as a solution of the Cauchy problem) must have topology of the direct product $\mathbf{M}^3 \times \mathbf{R}^1$ of 3-dimensional space $\mathbf{M}^3$ and the real line $\mathbf{R}^1$. The theorem of Geroch [3], [4] may be considered as a supplement to the global hyperbolicity theorems. It states that space-time model with different topologies of space sections must be singular or contain closed time-like curves. These statements jointly establish the impossibility of dynamical consideration (i.e. as a solution of some Cauchy problem) of classical topology changes processes in the framework of four-dimensional theory but they do not forbid to consider topology change models as solutions of some boundary problem. As all experiments yields only local data, we have no *a priori* basis for excluding from consideration the space-time models which solve some boundary problem, but the physical meaning of such models is unclear.

Some additional restrictions on the topology change models were imposed also in a number of papers [11], [12], [13] but the most results of such type were obtained under different additional assumptions which concerns the global properties of both space-time itself and some additional structures. The realization of such assumptions in the real space-time do not follow from observations or from some fundamental physical principles which are essentially local. For instance, in the recent proposal of so called "selection rules for topology changes" by Gibbons and Hawking [13] the existence of spinor structure is supposed. However the existence of any globally nonzero field (spinor, vector or tensor field) does not follows directly from observations. If we omit the condition of existence of globally nonzero sections of spinor bundle than the more general class of four-manifolds than it claimed in [13] and hence more general topological changes models will be admitted.

The particular models of topology changes were also discussed by several authors [17], [26]-[28], [29], [30]. Unfortunately, all such models are made "by hand" and there is no topology change models ware obtained as a solution of some dynamical or boundary problem. Moreover there is almost no progress in the constructive description of topology change: up to now there is no real (or toy) theory which describe the possible changes of space topology. All existing attempts to describe topology change or topological fluctuation are phenomenological. It is often supposed that topology changes are pure quantum phenomena [2], [10], [9], [13], but there is no real progress in their quantum description also.

Such situation indicate that the difficulties which arise in connection with the problem of topology change description are of principal nature. It is indeed the case because to describe the manifold structure the following data must be given: (*i*) the finite or countable set of coordinate maps and the order of their junction; (*ii*) the set of functions which connect the coordinate systems of different maps in their intersections [33], [32]. These data must be given before the solution of any equations. They may be considered as constraints or as a part of boundary conditions which are given "by hand". Therefore to describe the changes of space or space-time topology the number of the coordinate maps, the order of their junction and the junction conditions must be converted from the class of constraints or boundary data into the class of dynamical variables. This problem does not trivial because any change of the number of coordinate maps and identification or re-identification of points of space-time must induce the redefining of the space of functions on it. However, the formalism of the current field theory does not contain any tools which make possible to change the number of coordinate maps and to perform identification of space or space-time points including the redefinition of the function space. By this reason, the Hawking's proposal to include degenerate metrics into path integrals for description of the topological fluctuations [9] does not solve the problem because the values of functions along degenerate space path do not identified automatically. The same is true for the so-called 'minimalist wormhole model' which was supposed recently by Smolin [31].

The main goal of this paper is to analyze formally several aspects of topological change description, consider several different attempts to the solution of this problem and discuss some perspectives. For this purpose in the next section we discuss the role of space-time topology in the field theory. In particular, using the results of our previous papers [35], [36] it will be formalized the above statement that in the current classical field theory topology of space-time is a constraint. Some possibilities of the topology change consideration in the framework of $(3+1)$-dimensional theory will be discussed in the section 3. The rest part of paper contains some discussion and speculations about different possibilities of the topology change description in multidimensional space-time theory.

2 Topology of space-time in current field theory

Consider the action integral of the some field theory (both classical and quantum in its path integral form) in four-dimensional space-time in the following general form

$$S = \int\limits_{\mathbf{M}} L(\Phi_A, \Phi_{A,\alpha}) d^4\sigma, \tag{1}$$

where $L(\Phi_A, \Phi_{A,\alpha})$ is the Lagrangian which is depend from the fields potentials Φ_A and their derivatives, A is the cumulative index and $d^4\sigma$ is the invariant volume element which in the local coordinates $\{x^\alpha, \alpha = 0, ..., 3\}$ has the form

$$d^4\sigma = \sqrt{-g} d^4x = \sqrt{-g} dx^0 \Lambda ... \Lambda dx^3 \tag{2}$$

where Λ is the exterior product of differential forms, $g = \det \|g_{\alpha\beta}\|$ and $g_{\alpha\beta}$ is the metric tensor of Lorentzian signature $diag(+,-,-,-)$ on $\mathbf{M}$.

In the equality (1) the integration are carried out over the full manifold $\mathbf{M}$, so both action S and corresponding Feynman amplitude $\exp\{iS/h\}$ are the functionals of both field variables Φ_A and the manifold itself.

To formalize the action dependence from the manifold structure, let us consider some atlas $\mathbf{U} = \{\mathbf{V}_k\}$ of $\mathbf{M}$, i.e. finite or countable covering of $\mathbf{M}$ by coordinate maps $\mathbf{V}_k$ which are diffeomorphic to unit cube $\mathbf{D}^4$ of the Euclidean space $\mathbf{R}^4 : \mathbf{M} = \bigcup_{k\in\mathbf{P}} \mathbf{V}_k$, where $\mathbf{V}_k \sim \mathbf{D}^4$ and $\mathbf{P} \subset \mathbf{N}$ is a subset of the set $\mathbf{N}$ of natural numbers, which numerate the elements of the covering $\mathbf{U}$. Let $\{x_k^\alpha\}$ are the local coordinates in the region $\mathbf{V}_k$ and $\left\{x_{i_0\ldots i_l}^\alpha\right\}$ are some local coordinates in the intersection $\mathbf{V}_{i_0} \cap \ldots \cap \mathbf{V}_{i_l}$ (which is also diffeomorphic to $\mathbf{D}^4$). In any intersection $\mathbf{V}_{i_0} \cap \ldots \cap \mathbf{V}_{i_l}$ the field potentials Φ_A must satisfy to the natural consistency conditions which may be considered as an additional constraints.

In the atlas $\mathbf{U}$ the integral (1) may be rewritten in the following form [35]

$$
\begin{aligned}
S = \sum_{k\in P} \int_{V_k} L(\Phi_A(x_k), \Phi_{A,\alpha}(x_k))\sqrt{-g}d^4x_k - \\
\sum_{k<l} \int_{V_k \cap V_l} L(\Phi_A(x_{kl}), \Phi_{A,\alpha}(x_{kl}))\sqrt{-g}d^4x_{kl} + \ldots \\
+(-1)^K \sum_{i_0<\ldots<i_K} \int_{V_{i_0}\cap\ldots\cap V_{i_K}} L(\Phi_A(x_{i_0\ldots i_K}), \Phi_{A,\alpha}(x_{i_0\ldots i_K}))\sqrt{-g}d^4x_{i_0\ldots i_K}
\end{aligned}
\tag{3}
$$

where $K < \infty$ because of standard supposition about paracompactness of space-time manifold $\mathbf{M}$ [21].

For the following formalization of the action integral (1) consider the set $\mathbf{\Lambda_U}$ of subsets of the set $\mathbf{P}$ such that $(i_0, \ldots, i_k) \in \mathbf{\Lambda_U}$ if and only if $\mathbf{V}_{i_0} \cap \ldots \cap \mathbf{V}_{i_k} \neq \emptyset$, where $\emptyset$ denotes the empty set. The set $\mathbf{\Lambda_U}$ which satisfies to such condition is called a nerve of the covering $\mathbf{U}$ [48] and its elements of type $I_k = (i_0, \ldots, i_k)$ are known as k-dimensional simplexes [48]. The zero-dimensional simplexes, i.e. elements of the type $I_0 = i_0$, are vertexes. It is follows from definition that if $I_k \in \mathbf{\Lambda_U}$ and $J_l \in I_k$, where $l < k$, then $J_l \in \mathbf{\Lambda_U}$. This property shows that the nerve $\mathbf{\Lambda_U}$ of the covering $\mathbf{U}$ is a particular case of the abstract simplicial complex [48]. Such constructions are widely used in the algebraic topology, in particular, in Cech cohomology theory whose connections with topological quantization was discussed in [49].

With the nerve $\mathbf{\Lambda_U}$ of the covering $\mathbf{U}$ we may associate the system of its characteristic functions which will be denoted as $\mathbf{F_\Lambda} = \left\{f_\mathbf{\Lambda}^0, \ldots, f_\mathbf{\Lambda}^K\right\}$ [35], [36] where the functions $f_\mathbf{\Lambda}^l = f_\mathbf{\Lambda}^l(I_l) \in \mathbf{F_\Lambda}$, $0 \leq l \leq K$, are defined on the set of natural numbers as follows

$$
f_\mathbf{\Lambda}^l = \sum_{i_0<\ldots<i_l} a_\mathbf{\Lambda}^{I^l} f_{i_0}^0 \wedge \ldots \wedge f_{i_l}^0 \tag{4}
$$

where $i_m \in \mathbf{N}$, $\mathbf{N}$ is the set of natural numbers, "$\wedge$" denote exterior multiplication,

$$
f_i^0 = f_i^0(j) = \delta_{ij} \tag{5}
$$

and

$$a_{\mathbf{\Lambda}}^{I_l}=1,\quad \text{if}\quad I_l\in\mathbf{\Lambda}_U,\quad \text{and}\quad a_{\mathbf{\Lambda}}^{I_l}=0,\quad \text{if}\quad I_l\notin\mathbf{\Lambda_U} \tag{6}$$

It is follows from definitions that the connection of the system $\mathbf{F_\Lambda}$ of characteristic function of nerve $\mathbf{\Lambda_U}$ with atlas $\mathbf{U}$ is one-to one and hence $\mathbf{F_\Lambda}$ defines the topology of the manifold $\mathbf{M}$ as well as corresponding atlas $\mathbf{U}$.

Using the definitions (4)-(6) we may rewrite equality (3) as follows [35], [36]

$$S=\sum_{k=0}^{K}(-1)^k\sum_{i_0<\ldots<i_k}S_k^{I_k}f_{\mathbf{\Lambda}}^k(I_k) \tag{7}$$

where

$$S_k^{I_k}=\int\limits_{\mathbf{V}_{i_0}\cap\ldots\cap\mathbf{V}_{i_k}}L(\Phi_A(x_{i_0\ldots i_k}),\Phi_{A,\alpha}(x_{i_0\ldots i_k}))\sqrt{-g}d^4x_{i_0\ldots i_k} \tag{8}$$

Equalities (7), (8) together with definitions (5)-(6) formalize the dependence of the action integral from the topology of space-time. They make possible to do several observations.

First, equalities (7) and (8) formalize the above statement that the topology of manifold play the role of the additional constraint. Second, the topology of arbitrary manifold may be coded by the system $\mathbf{F_\Lambda}$ which may be done finite for compact manifolds. Some another methods of the manifold structure coding are described in [47], but they are less suitable for our purpose. It is known also, that independently from the method of the manifold structure coding in three or more dimensions, the set of codes which describe all manifolds of the given dimensionality is infinite with infinite subset of codes which define the given manifold. Moreover, if dimension of manifold is three or more then there is no so simple classification of manifold structures as in two dimensions [47]. Third, any changes in the topology of manifold $\mathbf{M}$ may be represented as corresponding changes of the system $\mathbf{F_\Lambda}$. Really to change the topology of $\mathbf{M}$ it is necessary to change its atlas $\mathbf{U}$, i.e. the order in which the coordinate maps $\mathbf{V}_i$ are joined with each other and their number. The change of atlas $\mathbf{U}$ induce the change of its nerve $\mathbf{\Lambda_U}$ and hence the system $\mathbf{F_\Lambda}$ because the correspondences $\mathbf{U}\leftrightarrow\mathbf{\Lambda_U}\leftrightarrow\mathbf{F_\Lambda}$ are one-to-one. However the representation of the action functional S in the form (7) does not contain any sign of the $F_{\mathbf{\Lambda}}$ changes. Moreover, such representation does not contains any information about joining conditions in the intersections of coordinate maps. So, any changes of space or space-time topology may be done only "by hands" and does not follow from the general formalism. Therefore the standard methods of the current field theory (both classical and quantum in its path-integral form), which are based on the usage of the action functional S, does not permit to describe the dynamical change of space-time topology. To make possible such description it is necessary to use functionals which contain not only $\mathbf{F_\Lambda}$ but also some objects that may be called as "discrete derivatives" of $\mathbf{F_\Lambda}$ (as an example of such objects may be used operators $\rho_{I_k}^{\pm}$ [35], [36] which may be interpreted as creation and annihilation operators of the simplex I_k). The introduction of such objects is equivalent to introduction of some non local (topological) interaction which has no analogies in the current field theory therefore it is almost hopeless to solve this problem directly but it is possible to investigate some possibilities in the construction of the consecutive topology change theory and its main features in the scope of the standard theories.

In the above the general 4-dimensional form of the action integral was considered while in the context of the topology change description the usage of some parameterization would be more suitable. The introduction of such parameterization is straightforward and we do not consider it here. In particular, in parameterized form the action integral (1) reads

$$S = \int L dt \tag{9}$$

where the Lagrangian L is defined analogously to (7), (8):

$$L = \sum_{k=0}^{K} (-1)^k \sum_{i_0 < \ldots < i_k} L_k^{I_k} f_\Lambda^k(I_k, t) \tag{10}$$

where $f_\Lambda^k(I_k, t)$ and $L_k^{I_k}$ are straightforward analog of (4)-(6), (8).

It is easy to see that parameterization of the action integral does not change result: (i) both in general four-dimensional form and in the parameterized form the topological variables are the discrete valued functions, and (ii) the action functional S contains the topological variables only as parameters (or constraints).

3 Topology change in four-dimensional theory: application of Morse theory

To simplify the problem consider the particular case than the part of space-time is a compact four-manifold $\mathbf{M}^4$ whose boundary is a disjoint sum of three-dimensional space-like manifolds $\mathbf{M}_1^3$ and $\mathbf{M}_2^3$: i.e. $\partial\mathbf{M}^4 = \mathbf{M}_1^3 \cup \mathbf{M}_2^3$, and $\mathbf{M}_1^3 \cap \mathbf{M}_2^3 = \emptyset$. The manifold $\mathbf{M}^4$ is often called as interpolating manifold. Such models may be described in the framework of Morse theory [33], [34] which state that:

(i) there is a smooth function φ on the manifold $\mathbf{M}^4$, such that $0 \leq \varphi(p) \leq 1$ for all $p \in \mathbf{M}^4$, $\varphi(p_1) \equiv 0$ for all $p_1 \in \mathbf{M}_1^3$, $\varphi(p_2) \equiv 1$ for all $p_2 \in \mathbf{M}_2^3$, and moreover φ has a finite number of nondegenerate critical points on $\mathbf{M}^4$ (the point $p \in \mathbf{M}^4$ is called the nondegenerate critical point of smooth function φ if in arbitrary system of local coordinates x^α the following conditions are satisfied: $\varphi,_\alpha(p) = 0$ and $\det \|\varphi,_{\alpha\beta}(p)\| \neq 0$);

(ii) $\mathbf{M}_2^3$ may be obtained from $\mathbf{M}_1^3$ by a finite number of spherical modifications which correspond to the critical points of φ.

The correspondence between the non-degenerate critical points of function φ and the topology of the level surfaces of this function is the follows [33].

Let p_* is a non-degenerate critical point of φ, $\varphi(p_*) = c$ and let there is no other critical points on the level surface $\varphi = c$. In some neighborhood of p_* a system of local coordinates $\{x^\alpha\}$ exists such that φ is represented in the form

$$\varphi = \varphi(p_*) + \frac{1}{2} \sum_{\alpha=1}^{4} a_\alpha \cdot (x^\alpha)^2 \tag{11}$$

where coefficients a_α are equal to ± 1. Let $r + 1$ is the number of negative a_α in (11): $r + 1 = Ind_- \|\varphi,_{\alpha\beta}(p_*)\|$. Then, the level surface $\varphi = c + \epsilon$ may be obtained from the level surface $\varphi = c - \epsilon$, where $\epsilon = const > 0$, through the spherical modification of rank r [33], [34]. In the case of 3-manifolds such modification may be represented as a contraction of the sphere $\mathbf{S}^r$ into the critical point p_* along the trajectories of the vector field $l_\alpha = \varphi,_\alpha$ and the following inflation of the sphere $\mathbf{S}^{3-r-1}$ from the same point p_*. Within this the contraction of $\mathbf{S}^r$ is realized in the subspace with coordinates x^α which coincides with $a_\alpha = -1$ in (11), while the inflation of $\mathbf{S}^{3-r-1}$ is realized in the subspace with coordinates x^β with $a_\beta = +1$. More formally such modification is described by the equation

$$\mathbf{M}_2^3 = \left(\mathbf{M}_1^3 \setminus (\mathbf{E}^{3-r} \times \mathbf{S}^r\right) \cup \left(\mathbf{E}^{r+1} \times \mathbf{S}^{3-r-1}\right) \tag{12}$$

where $\mathbf{M}_1^3$ and $\mathbf{M}_2^3$ are the level surfaces of $\varphi = c-\epsilon$ and $\varphi = c+\epsilon$ respectively and $\mathbf{E}^{3-k} \times \mathbf{S}^k$ is the tubular neighborhood of a directly embedded sphere S^k. The generalization of such procedure on arbitrary-dimensional case is straightforward.

The 3-sphere creation ($\emptyset \to \mathbf{S}^3$) and its annihilation ($\mathbf{S}^3 \to \emptyset$) are described by spherical modifications of rank $r = -1$ and $r = 3$ respectively, while the wormhole creation is the spherical modification of the zero rank.

It is obvious that the spherical modifications theory may be applied to the description of 3-spaces topology changes on the given four-manifold $\mathbf{M}^4$ with given topological structure, because the topological structure of manifold must be given before introduction of smoothness and before definition of Morse function φ (or arbitrary smooth function). The main principles of such application are the follows (for details see [38] - [40]).

It is supposed that the level surfaces $\varphi =$const are space-like and the vector fields $l_\alpha = \varphi,_\alpha$ is time-like everywhere except the critical points of φ, where $l_\alpha = 0$. Outside the critical points of φ the metric tensor of space-time may be represented in the form

$$g_{\alpha\beta} = \frac{2l_\alpha l_\beta}{f} - \tilde{g}_{\alpha\beta} \tag{13}$$

where $f = g^{\rho\sigma} l_\rho l_\sigma = \tilde{g}^{\rho\sigma} l_\rho l_\sigma$ and $\tilde{g}_{\alpha\beta}$ is a positive definite metric on $\mathbf{M}^4$.

Using representation (13) we may investigate the asymptotic properties of space-time models in the vicinity of nondegenerate critical point of function φ (i.e. e. in the vicinity of the topology change points). In particular, the direct calculation of Ricci tensor and scalar curvature gives

$$R_{\alpha\beta} = \tilde{R}_{\alpha\beta} + \sum_{n=1}^{4} \left(\frac{1}{f}\right)^n \overset{n}{R}_{\alpha\beta} \tag{14}$$

and

$$R = -\tilde{R} + \left(\frac{1}{f}\right)\left\{4l^\alpha l^\beta \tilde{R}_{\alpha\beta} - 2\left((l^\alpha_{|\alpha})^2 - l^\alpha_{|\beta} l^\beta_{|\alpha}\right) - 2f^\alpha,_\alpha\right\} + \left(\frac{2}{f}\right)^2 \left\{2l^\alpha_{|\alpha} l^\sigma f,_\sigma + 2f,^\alpha f,_\alpha + l^\alpha l^\beta f,_{\alpha|\beta}\right\} - \frac{4}{f^3}\left(l^\alpha f,_\alpha\right)^2 \tag{15}$$

where tilde "~" denote quantities correspond to the metric $\tilde{g}_{\alpha\beta}$ and the $\overset{n}{R}_{\alpha\beta}$ are certain polynomials on l_α, $f,_\alpha$ and their covariant derivatives with respect to $\tilde{g}_{\alpha\beta}$ (The explicit form of $R^\alpha_{\beta\gamma\delta}$ and $R_{\alpha\beta}$ are given in [29]).

It is easy to see from (11), (13) that non-degenerate critical points of φ are essentially singular points of metric $g_{\alpha\beta}$: the limits of right-hand side of (13) at these points exists, but depends on the direction [38], [39]. Further, it is easy to see that near non-degenerate critical points of φ the curvature tensors of space-time have the following asymptotic

$$\overset{n}{R}{}^{\alpha}_{\beta\gamma\delta}, \quad \overset{n}{R}_{\alpha\beta} \sim f^{n-1} \tag{16}$$

and hence

$$R^{\alpha}_{\beta\gamma\delta}, \quad R_{\alpha\beta}, \quad R \sim \frac{1}{f} \tag{17}$$

Therefore the topology change points are the scalar curvature singularities of space-time. More detailed investigation of space-time properties near topology change points and some simple examples may be found elsewhere [39], [29].

The representation of the Lorentzian metric $g_{\alpha\beta}$ in the form (13) may be used not only for investigation of space-time near the critical points of the function φ (i.e. near the topology change points) but for the construction of some variants of the topology change theory on the given four manifold. For this purpose the scalar function φ and positive definite metric $\tilde{g}_{\alpha\beta}$ are used as new independent variables instead of the Lorentzian metric $g_{\alpha\beta}$ which defines the motion of the sources. Such program leads to a new class of scalar-tensor theories of gravity, which partially realized Hawking's idea about the Euclidean nature of space-time [10]. The outline of such model theories were discussed in [39]-[40]. In particular, the action integral in such theory in the approximation of minimal coupling with sources may be written in standard form

$$L = \int (L_g + \kappa L_m)\, d\sigma \tag{18}$$

where L_g is the gravitational Lagrangian depending on scalar field φ and metric tensor $\tilde{g}_{\alpha\beta}$ and their derivatives, L_m is the standard Lagrangian of the source fields θ_A (A is the cumulative index) and their covariant derivatives with respect to $g_{\alpha\beta}$, $d\sigma$ is invariant volume element. The stationarity condition gives

$$\delta L_g/\delta\tilde{g}^{\alpha\beta} - \kappa T_{\rho\sigma} D^{\rho\sigma}_{\alpha\beta} = 0, \tag{19}$$

$$\delta L_g/\delta\varphi - \kappa (T_{\rho\sigma} P^{\rho\sigma\alpha})_{|\alpha} = 0 \tag{20}$$

and

$$\delta L_m/\delta\theta_A = 0 \tag{21}$$

where $\delta/\delta\tilde{g}^{\alpha\beta}$, $\delta/\delta\varphi$ and $\delta/\delta\theta_A$ are variational derivatives, $T_{\alpha\beta}$ is the standard energy-momentum tensor of the source fields (the same as in classical general relativity) and tensors $D^{\rho\sigma}_{\alpha\beta}$ and $P^{\rho\sigma\alpha}$ are equal to

$$D^{\rho\sigma}_{\alpha\beta} = \partial g^{\rho\sigma}/\partial\tilde{g}^{\alpha\beta} = \frac{4}{f} l^{(\rho}\delta^{\sigma)}_{(\alpha} l_{\beta)} - \frac{4}{f^2} l^{\rho} l^{\sigma} l_{\alpha} l_{\beta} - \delta^{\rho}_{(\alpha}\delta^{\sigma}_{\beta)}, \tag{22}$$

$$P^{\alpha\beta\sigma} = \partial g^{\alpha\beta}/\partial\varphi_{,\sigma} = (4/f)\, l^{(\alpha} g^{\beta)\sigma} - \left(4/f^2\right) l^{\alpha} l^{\beta} l^{\sigma} \tag{23}$$

Equations (21) are the classical equations for the source fields and the equations (19) and (20) define the scalar field φ, the positive definite metric $\tilde{g}_{\alpha\beta}$ and the pseudo-Riemannian (Lorentzian) structure of space-time through (13).

Some problems of such approach, namely, the choice of the Lagrangian L_g and the singularities problem were discussed in [29], [30], [39]-[40]. Here we want to point out several main features of such approach.

First, the structure of four-manifold must be given. This condition is very restrictive, because it automatically excludes from consideration a big class of topological changes or a big class of histories (intermediate states).

Second, the equation (20) is an elliptic one. So, this approach does not lead to the dynamical description of the topological change, because minimum one independent variable is a subject of the boundary problem. In application to the Universe evolution it means that both initial and final states of the Universe are given. The physical meaning of such problem, in particular, the nature of such boundary conditions, is very unclear.

At last, the direct application of the Morse theory lead to the singular space-time models and singular theory as it is follows from the above consideration. Using this fact De Witt made conclusion about dynamical suppressing of topology changes in quantum gravity [6]. His conclusion were reanalyzed in [7]. Nevertheless, De Witt conclusion cannot be considered as a general theorem because both in [6] and [7] only particular topology change model were considered in the framework of standard general relativity without any references to some theory of topology change. To obtain nonsingular theory some type of regularization [40] or so-called Lorentz cobordism [5] may be used. In the first case the additional boundary conditions must be introduced but the possibility to include them into the general formalism is not obvious. In the Lorentz cobordism case some additional restrictions on the 4-manifold structure is necessary and vector field l_α in (13) become nonintegrable. The resulting model will describe transition from "initial" manifold $\mathbf{M}_1$ to the "final" manifold $\mathbf{M}_2$ and may be both singular and regular but non-causal [3]. To obtain nonsingular model some additional surgical operations may be necessary. The consideration of full set of histories between "initial" and "final" states is impossible in both methods.

4 Topology change in multidimensional gravity

The most of the current unified theories require space-time to be of more then four dimensions. Independently from the reasons which lead to the nonobservability of additional dimensions, the multidimensionality of space-time gives several possibilities for the dynamical description of topology change of 3-space. Here we shall discussed briefly two such possibilities.

4.1 Effective topology change via dynamical inflation-contraction

In the standard dynamical dimensional reduction paradigm multidimensional space-time supposed to have the topological structure of direct product of real line (global time coordinate) and several topological spaces, i.e.

$$\mathbf{M} = \mathbf{R} \times \mathbf{M}_1 \times \mathbf{M}_2 \times ... \times \mathbf{M}_k$$

such that one of space, for instance $\mathbf{M}_1$, has dimension 3, while dimension of other spaces may be arbitrary. It is supposed also, that all spaces $\mathbf{M}_i$, $i = 1,...,k$, are compact. The model of space-time is constructed by such a way that 3-manifold $\mathbf{M}_1$ (our universe) expands from initial singular or Plank state while other manifolds contract from the state with finite scales to the Plank scales.

It may be speculatively supposed that universe may pass through several stages such that at each stage one 3-manifold expands and one 3-manifold contracts while other manifolds remain in the state with Plank scales. As a result at different stages of evolution universe will have different effective topologies.

Unfortunately, this idea has two serious deficiencies. First, to describe the wide class of the topological changes the total dimension of multidimensional space-time in such scheme must be enormously big (infinite) while there are no quantum field theories with total dimension of space-time more then 26. Second, it is very difficult to find some natural mechanism which may control the transitions between different stages of the universe evolution in such scheme. Therefore, the paradigm of effective topology change of universe via inflation-contraction must be considered today as pure principal possibility.

4.2 Topology change in the embedding models

Analysis of the Morse theory gives some ideas about topology change description in multidimensional theories. Namely, it is easy to see that the Morse function φ defines embedding of space-time $(\mathbf{M}^4, g)$ in 5-dimensional topological space $\mathbf{V}^5 = \mathbf{M}^4 \times \mathbf{R}$. As the natural generalization, we may consider our space-time as submanifold of some multidimensional manifold with given topology, in particular, as a submanifold of the Euclidean space of the appropriate dimensionality N which depends from the type of embedding. For instance, for arbitrary four-dimensional manifold $N = 8$, 10 and (10+7) for smooth, isometrical Riemannian and isometrical pseudo-Riemannian (with arbitrary metric signature) embedding respectively [41], [42]. In difference with standard Kaluza-Klein-type theories we do not demand nor the existence of any dimensional reduction mechanism, nor the representation of the whole manifold, namely $\mathbf{R}^N$, as a direct product $\mathbf{M}^n \times \mathbf{V}^{N-n}$, where $\mathbf{M}^n$ and $\mathbf{V}^{N-n}$ are some smooth manifolds. Instead this the existence of two type of observes and fields is supposed: the external observers and fields which are defined in the whole $\mathbf{R}^N$, and the internal observers and fields which are defined on $\mathbf{M}^n$. The class of internal fields may include both the induced fields and the surface distributed fields. The full theory must contains two parts: the description of the embedding and the description of the interior dynamics.

It is necessary to note, that the possibility of the space-time consideration as a membrane in higher dimensional space is not new and were discussed by several authors [43]-[46], but only the local properties of embedding were considered. The most principle feature of our approach is the consideration of both global and local properties of embedding in the unified formalism.

Here we shall briefly discuss only the outline of such approach. The detailed consideration is the subject of separate papers and will be published elsewhere.

We begin our consideration from reproduction of some well-known facts about embedding of manifolds into Euclidean space.

Let $\mathbf{M}^n$ is a smooth $n-$dimensional submanifold of the Euclidean space $\mathbf{R}^N$ with Cartesian coordinates $\left\{X^P, P = 1, ..., N\right\}$, and the number $k = N - n$ is called the codimension of $\mathbf{M}^n$. The embedding $\mathbf{M}^n \longrightarrow \mathbf{R}^N$ is defined locally by the set of equations

$$X^A = X^A\left(x^1, ..., x^n\right) \tag{24}$$

where $(x^1, ..., x^n)$ are the local coordinates in $\mathbf{M}^n$ and the matrix $\left\|\partial X^P/\partial x^i\right\|$, $i = 1, ..., n$, has maximal rank n. Equations (24) may be obtained as a solution of the well known Gauss-Kodazzi equations [32], [44] which connect the intrinsic and extrinsic geometries of submanifolds. Unfortunately, such description does not admit to define the global structure of $\mathbf{M}^n$ or investigate its dynamics. Therefore we shell use the alternative description of $\mathbf{M}^n$ as the intersection of $N - n$ hypersurfaces in $\mathbf{R}^N$, i.e. by the system of equations

$$\Phi^A\left(X^P\right) = 0 \tag{25}$$

where $\Phi^A\left(X^P\right)$, $A = 1, ..., k$, are some smooth functions on $\mathbf{R}^N$, and the matrix $\left\|\Phi^A{}_{,P}\right\| = \left\|\partial\Phi^A/\partial X^P\right\|$ must have maximal rank, i.e.

$$rank\left\|\Phi^A{}_{,P}\right\| = \left\|\partial\Phi^A/\partial X^P\right\| = k = N - n \tag{26}$$

To introduce the atlas on $\mathbf{M}^n$ let's consider arbitrary point $p \in \mathbf{M}^n \subset \mathbf{R}^N$. By force of the implicit function theorem there is a neighborhood $\mathbf{U}_p \subset \mathbf{R}^N$ of arbitrary point $p \in \mathbf{M}^n \subset \mathbf{R}^N$ such that the system (25) may be solved in the form (24) with $x^i = X^{m_i}$, $i = 1, ..., n$ and the numbers $\{m_1, ..., m_n\}$ are a subset of the set $\{1, ..., N\}$. It means that there is a one-to-one correspondence between the points of $\mathbf{U}_p \cap \mathbf{M}^n$ and one of n -dimensional coordinate planes in $\mathbf{R}^N$. Taking different points of $p \in \mathbf{M}^n$ we may obtain the covering of $\mathbf{M}^n$ by the regions $\widetilde{\mathbf{U}}_p = \mathbf{U}_p \cap \mathbf{M}^n$ which define the atlas on $\mathbf{M}^n$.

Equations (25) are the algebraic constraints, but the smooth functions Φ_A may be arbitrary. In particular, these fields may be considered as usual scalar fields in $\mathbf{R}^N$. In the simplest model the action functional will have the form

$$S_N = \int d^N X\left(\eta^{PQ}\delta_{AB}\Phi^A{}_{,P}\,\Phi^B{}_{,Q} - V\left(\Phi^A\right)\right) \tag{27}$$

where $d^N X$ - is an invariant volume element of $\mathbf{R}^N$, δ_{AB} - is Kronecker symbols, $A, B = 1, ..., k$; $k = N - n$ is a codimension of $\mathbf{M}^n$, η_{PQ} is the Euclidean or pseudo-Euclidean metric on $\mathbf{R}^N$ with given signature, $\Phi^A{}_{,P} = \partial\Phi^A/\partial X^P$ and $V\left(\Phi^A\right)$ is the potential. Of cause, the action S_N may contain not only the scalar fields Φ^A but additional scalar, vector and tensor fields also. Moreover, instead of Kronecker symbols δ_{AB} an arbitrary nondegenerate $k \times k$ matrix may be used.

The set of equations

$$\delta S_N / \delta\Phi_A = 0 \tag{28}$$

together with the constraints (25), the consistency condition (26) and appropriate initial or boundary conditions give dynamical description of the topology of $\mathbf{M}^n$ by means of its embedding into $\mathbf{R}^N$. Such description of the topology of $\mathbf{M}^n$ does not unique because the same manifold may be embedded into $\mathbf{R}^N$ by different manners. The interior dynamics of fields on $\mathbf{M}^n$ does not described by these equations.

To describe the dynamics of fields on M^n the full action must contain additional surface term. For definiteness we shall write this term in the form

$$S_n = \int\limits_{\Phi_A=0} d^n\sigma \left(R_n + L_m\right) \tag{29}$$

where $d^n\sigma$ - is an invariant volume element on $\mathbf{M}^n$, R_n - is a scalar curvature and L_m - is the Lagrangian density of the matter fields on $\mathbf{M}^n$. In general, L_m may contain both induced fields, in particular, the fields $X^A = X^A\left(x^1, ..., x^n\right)$, and the fields which are distributed on $\mathbf{M}^n$ (surface-distributed fields). In the simplest case

$$L_m = \eta_{AB} g^{\mu\nu} X^A{}_{,\mu} X^B{}_{,\nu} \tag{30}$$

where $g_{\mu\nu}$ is some metric on $\mathbf{M}^n$, and the action S_n become the direct generalization of the Nambu-Goto string action. The full action

$$S = S_N + S_n \tag{31}$$

describes both the topology of $\mathbf{M}^n$ (by means of its embedding into $\mathbf{R}^N$) and the dynamics of fields on $\mathbf{M}^n$. Equations

$$\delta S_n / \delta g^{\mu\nu} = 0, \tag{32}$$

$$\delta S_n / \delta X^A = 0 \tag{33}$$

and other analogous equations describe the dynamics of fields on $\mathbf{M}^n$ and its internal geometry. It is easy to see that these equations give additional constraints for the equations (25) which define the topology of $\mathbf{M}^n$ and the embedding $\mathbf{M}^n \longrightarrow \mathbf{R}^N$.

Such scheme may be directly applied to the description of our space-time (in particular, to the Universe evolution) if we put in the above equalities $n = 4$ and $N \geq 17$ (so that $k = N - n \geq 13$). It is not contradict also to the standard Kaluza-Klein approach, if we suppose that the multidimensional space-time of the Kaluza-Kline theory is a submanifold of some $\mathbf{R}^N$. However we will not consider such possibility here.

More detailed consideration of the considered approach is the subject of the separate paper. Nevertheless some additional remarks are necessary. The existence theorems gives only necessary conditions for isometrical pseudo-Riemannian embedding. Unfortunately, the induced metric is not necessary Lorentzian even if the space $\mathbf{R}^N$ is Lorentzian. To obtain Lorentzian metric on $\mathbf{M}^n$ we may use several possibilities. First, we may omit the isometrical condition and suppose that $g_{\mu\nu}$ is an arbitrary metric with Lorentzian signature on M^n (simultaneously the necessary dimension of $\mathbf{R}^N$ will be reduced). Second we may demand that induced metric $g_{\mu\nu}$ on $\mathbf{M}^n$ is Lorentzian. Both ways are seems to be unsatisfactory because in the first case the connection between external and internal geometries is very weak and in the second case additional constraints on the functions $\Phi^A\left(X^P\right)$ must be introduced. It seems that the most appropriate choice is to introduce additional smooth function Ψ on $\mathbf{R}^N$ and take metric $g_{\mu\nu}$ on $\mathbf{M}^n$ in the form (13), i.e.

$$g_{\alpha\beta} = \frac{2 l_\alpha l_\beta}{f} - \tilde{g}_{\alpha\beta} \tag{34}$$

where l_α and f are the same as in (13), ϕ is the restriction of Ψ on $\mathbf{M}^n$, i.e. $\phi = \Psi\ |_M$, and $\tilde{g}_{\alpha\beta}$ is induced by the Euclidean metric of $\mathbf{R}^N$. As an example of the function Ψ the projection of $\mathbf{M}^n$ on T-axes may be used. It is easy to see, that if external space $\mathbf{R}^N$ has the only time-like direction then such projection will be defined by the elliptic equation and so the definition of global structure of space-time could not be reduced to the pure dynamical problem. Moreover, definition of the metric of $\mathbf{M}^n$ in the above form to the same singularity problem as in the case of direct application of Morse theory discussed in section 2. To avoid this problem the condition of integrability of vector field l_α may be omitted. In this case the Lorentz cobordism models will be included in the general formalism also.

5 Conclusion

We have consider some possible approaches to the description of the topology of space-time and the topology changes in the framework of both four-dimensional and multidimensional theories. Our results may be summarized as follows.

First, the simplicial approach make possible to formalize the statement that the topology of manifold play the role of the additional constraint. Namely, in this approach topology of space or space-time is represented by means of the system $\mathbf{F}_{\mathbf{\Lambda}}$ of characteristic function of the nerve of some atlas of space-time. However action functional S contains $\mathbf{F}_{\mathbf{\Lambda}}$ by linear manner and does not contain any sign of the $\mathbf{F}_{\mathbf{\Lambda}}$ changes. Therefore the standard methods of the current field theory (both classical and quantum in its path-integral form), which are based on the usage of the action functional S, does not permit to describe the dynamical change of space-time topology. To make possible such description it is necessary to use functionals which contain not only $\mathbf{F}_{\mathbf{\Lambda}}$ but also some objects that may be called as "discrete derivatives" of $\mathbf{F}_{\mathbf{\Lambda}}$ (as an example of such objects may be used operators $\rho^{\pm}_{I_k}$ [35], [36] which may be interpreted as creation and annihilation operators of the simplex I_k). The introduction of such objects is equivalent to introduction of some non local ("topological") interaction which has no analogies in the current field theory.

Second, the direct application of Morse theory or its nonintegrable generalization (Lorentz cobordism) make possible to describe topology change on the given four-dimensional manifold. This condition is very restrictive because they do not permit to consider all possible intermediate states or all topological histories. Moreover, this approach leads to the singular space-time models. To obtain nonsingular models some additional conditions must be imposed or additional topological transformations must be made. Both way are unsatisfactory because they could not be included in the general formalism. Furthermore, the equation (20), which defines the simultaneity hypersurface is an elliptic one. Therefore the space topology in such models is a subject of the boundary problem with boundary conditions on different space-like hypersurfaces. The physical meaning of such boundary conditions is unclear.

Third, the multidimensional theories give several possibilities for the topology change description. The most radical way is to describe space-time as a membrane in the Euclidean space of appropriate dimensionality. In difference with the existing embedding space-time theories, the full action in such approach must contain two terms: the term which define embedding of 4-dimensional space-time into Euclidean space of appropriate dimensionality and the surface term which describe the dynamics of fields. Such approach make possible to consider all topological histories including 4-manifolds with exotic smoothness, whose possible role in physics was discussed recently by Brans [50]. Of cause, this approach does not free from the number of difficulties most of which are the subject of separate paper. Here we pointed out only two of such problems. The first one is pure technical: it is also hopeless to obtain exact solution with non trivial topology of space-time in any variant of such theory while any known solution may be easily rewritten in such scheme. The second problem is of principle nature because the finite classifications of smooth manifolds whose dimension more or equal 3 does not exist. Moreover, the problem of identification of manifolds is algorithmically unsolvable in 4 or more dimensions and its algorithmic solvability in 3 dimensions is an open question [47]. Nevertheless, such approach is seems to be of considerable interest because it is consistent with multidimensional paradigm of the most current field theories and admit to consider all possible topological configurations of 4-dimensional space-time.

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