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Worldline Path Integrals for Fermions with Scalar, Pseudoscalar and Vector Couplings †

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Abstract

A systematic derivation is given of the worldline path integrals for the effective action of a multiplet of Dirac fermions interacting with general matrix-valued classical background scalar, pseudoscalar, and vector gauge fields. The first path integral involves worldline fermions with antiperiodic boundary conditions on the worldline loop and generates the real part of the one loop (Euclidean) effective action. The second path integral involves worldline fermions with periodic boundary conditions and generates the imaginary part of the (Euclidean) effective action, i.e. the phase of the fermion functional determinant. Here we also introduce a new regularization for the phase of functional determinants resembling a heat-kernel regularization. Compared to the known special cases, our worldline Lagrangians have a number of new interaction terms; the validity of some of these terms is checked in perturbation theory. In particular, we obtain the leading order contribution (in the heavy mass expansion) to the Wess-Zumino-Witten term, which generates the chiral anomaly.

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1. Introduction

The worldline formulation of quantum field theory (see [1]) provides a powerful alternative method for the calculation of Feynman diagrams and effective actions to first order in the loop expansion [2]. When implemented with the background field method, simplified Feynman rules may be derived and large numbers of Feynman diagrams may be combined into simple expressions [3]. This situation is reminiscent of string perturbation theory, where only a single diagram arises to each order in perturbation theory [4]. Indeed, first quantized worldline path integrals closely resemble string perturbation theory and interestingly, it is in the point particle limit of string theory that its power was rediscovered.

Surprisingly however, the worldline path integral formulation has not yet been presented for the quantization of fields with the most general interactions. While the case of scalar charged particles coupled to Abelian gauge fields was obtained long ago, the extensions to include charged fermions (see [2]) or non-Abelian gauge fields has been obtained only recently [5]. The worldline formulation for a single Dirac fermion in the presence of a scalar and a pseudoscalar field was obtained in [6] (although limited to diagrams with an even number of pseudoscalar vertices).

In the present paper, we shall extend the worldline path integral formulation to the case of a multiplet of Dirac fermions, coupled to matrix-valued scalar and pseudoscalar as well as to non-Abelian vector gauge fields, and obtain explicit formulas for the one loop effective action. The most general case would include a further axial vector and anti-symmetric tensor field, but this problem will be treated in a future publication. The methods presented here are expected to have a suitable extension to the general case.

The starting point of our method is the expression for the fermion effective action – coupled to a scalar, pseudoscalar and vector gauge field – in terms of the logarithm of a functional determinant of the Dirac operator $\mathcal{O}$, suitably continued to Euclidean spacetime. Due to the presence of the pseudoscalar field, the Euclidean effective action can have both a real and an imaginary part, respectively even and odd in its dependence on the pseudoscalar field. This discussion is presented in Section 2.

The real part of the effective action admits a manifestly global chiral invariant and gauge invariant regularized heat-kernel representation. To show this, we double the number of fermions by adding their charge conjugates. The real part of the effective action is then naturally (and apparently uniquely) given as the logarithm of the functional determinant of the operator $\mathcal{O}^\dagger\mathcal{O}$. Since the operator $\mathcal{O}^\dagger\mathcal{O}$ is now positive, the determinant admits a standard regularized heat-kernel representation, which is chiral invariant.

Using the coherent states formalism for fermions (and the coordinate-momentum representation for bosons), we derive a manifestly gauge invariant and globally chiral invariant worldline path integral representation for the real part of the effective action. The corresponding worldline action that enters the path integral involves worldline fermions obeying *antiperiodic boundary conditions* on the worldline loop and is even in the Grassmann variables. (This is in opposition to [5], where a worldline action was proposed with both even and odd terms in the Grassmann variables.) The real part of the effective action is discussed in Section 3, one of its subtleties is handled in Appendix A and a comparison with a Feynman diagram is given in Appendix B.

The imaginary part of the effective action is odd in the pseudoscalar field, proportional to the Levi-Civita, or totally anti-symmetric ε-tensor and may be expressed as the phase of the functional determinant of the Dirac operator $\mathcal{O}$. A new regularization by a heat-kernel-like expression of this phase is presented and is more subtle than the heat-kernel for the real part of the effective action. Several different equally natural choices may be made for the representation, none of which preserve global chiral symmetry in a manifest way (while gauge invariance remains manifest), in keeping with the appearance of the chiral anomaly. One choice is presented in the main body of the paper in Section 4, while another is discussed in Appendix C.

Using the coherent states formalism, we derive a simple worldline path integral representation for the imaginary part of the effective action. Gauge invariance is manifest while global chiral symmetry is explicitly violated as expected. It is shown that the corresponding worldline action involves worldline fermions obeying *periodic boundary conditions* on the worldline loop and is even in Grassmann variables. The zero modes of the periodic worldline fermions are responsible for producing the Levi-Civita tensor, which always appears in these amplitudes. These issues are discussed in Section 4, (with an alternative formulation in Appendix C), while an explicit calculation of the leading order contribution (in the heavy mass limit) to the Wess-Zumino-Witten term is given as an example of a contribution to the imaginary part of the effective action in Appendix D.

2. Effective Action

The field theory studied in this paper is that of a multiplet of Dirac fermions coupled to a background scalar $\Phi(x)$, pseudoscalar $\Pi(x)$, and vector gauge field $A(x)$. The most general (CPT invariant) classical action for the theory is given (in Minkowski spacetime) by

$$S_M[\bar\Psi,\Phi,\Pi,A,\Psi] = \int d^4x_M\ \bar\Psi^I\,[i\gamma_M^\mu\partial_\mu - \Phi + i\gamma_M^5\Pi + \gamma_M^\mu A_\mu]^{IJ}\Psi^J\ . \tag{2.1}$$

Here, the spacetime metric has signature $\eta = (+\ -\ -\ -)$ and we make use of the standard conventions

$$\begin{gathered}\{\gamma_M^\mu,\gamma_M^\nu\} = 2\eta^{\mu\nu}\ , \qquad \{\gamma_M^\mu,\gamma_M^5\} = 0\ , \qquad (\gamma_M^5)^2 = 1\\ (\gamma_M^0\gamma_M^\mu)^\dagger = \gamma_M^0\gamma_M^\mu\ \ ,\ \ {\gamma_M^5}^\dagger = \gamma_M^5\ \ ,\ \ \Phi^\dagger = \Phi\ \ ,\ \ \Pi^\dagger = \Pi\ \ ,\ \ A^\dagger = A\ .\end{gathered} \tag{2.2}$$

The superscripts, I and J, in the classical action refer to the internal quantum numbers of the fermion multiplet serving as reminders that the background fields are matrix-valued. Coupling constants have been absorbed into the definition of the background fields for ease of notation.

Action S_M is sufficiently general to describe, for example, the gauge coupling of Dirac fermions in an arbitrary representation $\mathcal{R}$ (possibly reducible) of a compact non-Abelian gauge group G, as well as the scalar and pseudoscalar couplings. Under these G-transformations, S_M is gauge invariant, provided Φ and Π transform under the representation $\mathcal{R}\otimes\mathcal{R}^*$ of G. The action S_M may also have a global chiral symmetry group

$H_L \times H_R$ whose generators commute with G. In this case, the chiral fermions ψ_L and ψ_R transform under some representations T_L and T_R respectively and $\Phi + i\Pi$ transforms under $T_L \otimes T_R^*$ by left and right multiplication respectively. In the special case of QCD-like theories with N_c colors and N_f flavors for example, we have $G = SU(N_c)$, $H_L = U(N_f)_L$ and $H_R = U(N_f)_R$. Of course, axial fermion number is in general anomalous with respect to color at the quantum level. Lastly, notice that the action S_M is parity conserving.

Also, it is understood that the scalar field Φ may assume a vacuum expectation value, rendering the Dirac fermion massive; in which case it is convenient to single out the mass by decomposing Φ as follows

$$\Phi = m + \varphi \tag{2.3}$$

and where φ has zero vacuum expectation value. For $\varphi = 0$, we recover the ordinary Dirac mass term.

The effective action, $W_M[\Phi, \Pi, A]$ for the fermions in the presence of the background fields may be defined by the functional integral :

$$e^{iW_M[\Phi,\Pi,A]} = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi e^{iS_M[\bar{\Psi},\Phi,\Pi,A,\Psi]} \tag{2.4a}$$

or by the functional determinant

$$iW_M[\Phi, \Pi, A] = \log \mathrm{Det}\, i[i\partial\!\!\!/_M - \Phi(x) + i\gamma_M^5\Pi(x) + A\!\!\!/_M(x)] \ , \tag{2.4b}$$

Here, $A\!\!\!/_M$ stands for $\gamma_M^\mu A_\mu$.

As it stands, this functional determinant makes sense only perturbatively in weak fields, e.g. with the help of dimensional regularization. However, it may be defined also non-perturbatively as a product of eigenvalues after suitable continuation to Euclidean spacetime. This definition requires an infrared regularization that renders the spectrum discrete, as well as an ultraviolet regularization on the infinite number of possibly large eigenvalues. Both regularizations can be naturally achieved using heat-kernel methods.

Analytic continuation is performed by Wick-rotating $t_M \to -it_E$ such that

$$\partial_{t_M} \to i\partial_{t_E} \quad , \quad (p_M)_0 \to i(p_E)_4 \quad \text{and} \quad A_0 \to iA_4 \ . \tag{2.5a}$$

The gamma matrices are unaffected by the continuation. However, it is useful to change notation and to define five generators of a Euclidean Clifford algebra, $(\gamma_E)_j \equiv i(\gamma_M)_j$, $(\gamma_E)_4 \equiv (\gamma_M)_0$, and $(\gamma_E)_5 \equiv \gamma_M^5$, satisfying

$$\{(\gamma_E)_a, (\gamma_E)_b\} = 2\delta_{ab} \ \text{and} \ (\gamma_E)_a^\dagger = (\gamma_E)_a \ \text{with} \ a, b = \mu, 5 \ . \tag{2.5b}$$

Thus, under analytic continuation we have also the following changes

$$\partial\!\!\!/_M \to i\partial\!\!\!/_E \qquad \text{and} \qquad A\!\!\!/_M \to iA\!\!\!/_E \ . \tag{2.5c}$$

The hermiticity property of the background fields remains unchanged by the analytic continuation. Under the continuation, the expression (2.4) for the effective action transforms into Euclidean space as follows (Henceforth, spacetime is taken to be Euclidean and the subscript E is dropped.):

$$-W[\Phi,\Pi,A] = \log \mathrm{Det}\,[\mathcal{O}]\ , \tag{2.6}$$

where the operator $\mathcal{O}$ is defined by

$$\mathcal{O} \equiv \not{p} - i\Phi(x) - \gamma_5\Pi(x) - \not{A}(x)\ . \tag{2.7}$$

Here, $p = -i\partial$ is the momentum operator and x should properly be viewed as the position operator, both of which are hermitian. The continuation procedure of the effective action given here, is free of the inconsistencies mention in ref. [7]: it was achieved without continuing the classical action directly – avoiding the more delicate Fermi fields.

As Eq. (2.7) involves the position and momentum operator, the reformulation of the effective action in terms of path integrals has now been posed as a problem in elementary quantum mechanics. It will be assumed that the fields dampen sufficiently fast at infinity so that spacetime can be effectively compactified. Also, zero modes of $\mathcal{O}$ – which would make the effective action blow up – may always be lifted by suitable perturbations of the external fields. Now, the operator $\mathcal{O}$ has an unbounded spectrum since its signature is linear in the momentum. To make use of the heat-kernel regularization, an operator with a positive real part is required. Bounded operators may be obtained most naturally by splitting the effective action into its real and imaginary parts:

$$-W_{\Re}[\Phi,\Pi,A] - iW_{\Im}[\Phi,\Pi,A] = \ln(|\mathrm{Det}\,[\mathcal{O}]|) + i\arg(\mathrm{Det}\,[\mathcal{O}])\ . \tag{2.8}$$

Both are parity conserving, since the original fermion action is parity conserving. In order to understand which diagrams they separately generate, we use the expansion of the determinant formula (2.6) in weak field perturbations, which we shall here give in dimensionally regularized momentum space (with the vacuum graph deleted)

$$\begin{aligned} W \ &= \sum_{n=1}^{\infty}\frac{1}{n}\int\frac{d^Dp_1}{(2\pi)^D}\cdots\frac{d^Dp_n}{(2\pi)^D}(2\pi)^D\delta^{(D)}(p_1+\cdots+p_n)\int\frac{d^Dq}{(2\pi)^D}\ \mathrm{tr}\frac{1}{\not{q}-im} \\ &\quad\times\left(i\tilde{\varphi}_1+\gamma_5\tilde{\Pi}_1+\tilde{\not{A}}_1\right)\cdots\frac{1}{\not{q}-\not{p}_1-\cdots-\not{p}_{n-1}-im}\left(i\tilde{\varphi}_n+\gamma_5\tilde{\Pi}_n+\tilde{\not{A}}_n\right)\ . \end{aligned} \tag{2.9}$$

From this perturbation expansion, it is easy to see that graphs with an even number of Π legs are real and contribute to $W_{\Re}$ while those with an odd number of Π legs are imaginary and contribute to $W_{\Im}$. Graphs with an odd number of pseudoscalar legs involve a single γ_5 and result in contributions proportional to the anti-symmetric tensor $\varepsilon_{\mu\nu\rho\sigma}$.

3. Worldline Path Integral for the Real Part of the Effective Action

The real part of the effective action defined by (2.8),

$$W_{\Re}[\Phi,\Pi,A] = -\frac{1}{2}\ln(\mathrm{Det}\,[\mathcal{O}^{\dagger}\mathcal{O}])\ , \tag{3.1}$$

automatically accommodates a positive operator, namely $\mathcal{O}^\dagger\mathcal{O}$ and so heat-kernel regularization may be applied here. However, its expression contains terms linear in the γ-matrices. Reformulation of this effective action in terms of a worldline path integral leads to a worldline action with terms linear in Grassmann variables. While there is nothing wrong with such actions from a mathematical point of view [5], they are usually regarded as physically unacceptable. This difficulty can be handled by doubling the number of fermions without altering the value of the effective action in the following way :

$$W_{\Re}[\Phi,\Pi,A] = -\frac{1}{2}\ln\mathrm{Det}\,[\Sigma]\ , \tag{3.2}$$

where the operator Σ is defined by

$$\Sigma \equiv \begin{pmatrix} 0 & \mathcal{O} \\ \mathcal{O}^\dagger & 0 \end{pmatrix}\ . \tag{3.3}$$

It is natural to introduce the six 8×8 gamma matrices [7]

$$\Gamma_\mu \equiv \begin{pmatrix} 0 & \gamma_\mu \\ \gamma_\mu & 0 \end{pmatrix}\ ,\ \Gamma_5 \equiv \begin{pmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{pmatrix}\ ,\ \Gamma_6 \equiv \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}\ , \tag{3.4a}$$

which are all hermitian and satisfy

$$\{\Gamma_A,\Gamma_B\} = 2\delta_{AB}\,\mathrm{I}_{8\times 8}\ . \tag{3.4b}$$

In terms of these matrices, Σ may now be recast as

$$\Sigma = \Gamma_\mu(p_\mu - A_\mu) - \Gamma_6\Phi - \Gamma_5\Pi\ . \tag{3.5}$$

Notice that this operator may be viewed as the six spacetime dimensional Dirac operator for a gauge field in which the four first components are those of the gauge field A_μ, while the fifth and sixth components are Π and Φ respectively, and for a Dirac fermion independent of the fifth and sixth coordinates.

The operator Σ still has an unbounded spectrum but since it is hermitian, its square Σ^2 is positive and the heat-kernel regularization may be applied to it. As a result, we have

$$W_{\Re}[\Phi,\Pi,A] = -\frac{1}{2}\ln\mathrm{Det}\,[\Sigma] = -\frac{1}{4}\ln\mathrm{Det}\,[\Sigma^2] = -\frac{1}{4}\mathrm{Tr}\ln[\Sigma^2]\ . \tag{3.6}$$

The trace is over spacetime, spin, as well as internal degrees of freedom. Now, the identity, valid for any positive real number $\mathcal{E}$ and for any σ with $\Re(\sigma) > 0$

$$\ln\sigma = \int_1^\sigma \frac{dt}{t} = -\int_0^\infty \frac{dT}{T}[e^{-\frac{\mathcal{E}}{2}T\sigma} - e^{-\frac{\mathcal{E}}{2}T}] \tag{3.7}$$

can be duly extended to the case where σ is an operator with positive real eigenvalues, such as Σ^2. At last then, the heat-kernel regularization is obtained:

$$W_{\Re}[\Phi,\Pi,A] = \frac{1}{4}\int_0^\infty \frac{dT}{T}\mathrm{Tr}\,e^{-\frac{\mathcal{E}}{2}T\Sigma^2}\ . \tag{3.8}$$

The constant term has been dropped since its role of subtracting out ultraviolet divergences will be replaced by dimensional regularization. $W_{\Re}$ has been expressed as the trace of the imaginary-time quantum mechanical evolution operator with Hamiltonian Σ^2. The T-integration serves as the sum over all inequivalent loops [2]. Using $[p_\mu, G(x)] = -i\partial_\mu G(x)$ as well as the anticommutation relations satisfied by the matrices Γ_A, the Hamiltonian may easily be worked out to be

$$\Sigma^2 = (p-A)^2 + \frac{i}{2}\Gamma_\mu F_{\mu\nu}\Gamma_\nu + i\Gamma_\mu\Gamma_6 D_\mu\Phi + i\Gamma_\mu\Gamma_5 D_\mu\Pi + \Phi^2 + \Pi^2 + \Gamma_6\Gamma_5[\Phi,\Pi]\ ,$$
$$D_\mu\Phi = \partial_\mu\Phi - i[A_\mu,\Phi]\ , \qquad D_\mu\Pi = \partial_\mu\Pi - i[A_\mu,\Pi]\ , \qquad F_{\mu\nu} = i[D_\mu, D_\nu]\ . \tag{3.9}$$

Notice that the operator Σ^2 is manifestly covariant under vector gauge transformations, as well as under global chiral rotations that transform the Higgs field $\Phi + i\Pi$. Also, this operator is hermitian and positive by construction and involves Γ-matrices only to even powers. Thus, it is suited for the construction of a worldline path integral representation of the real part of the fermion effective action, which we shall now derive.

Coherent State Formalism

The standard states of the coordinate-momentum representation of elementary quantum mechanics, shall be used to convert the bosonic sector of the trace in (3.1) into a bosonic path integral. To handle the fermionic sector analogously, the coherent states developed in ref. [8] shall be employed. The key observation here is that the matrices a_r^+ and a_r^-, $r = 1,2,3$ defined by $a_r^\pm \equiv \frac{1}{2}(\Gamma_r \pm i\Gamma_{r+3})$ satisfy standard Fermi-Dirac anticommutation relations, given by

$$\{a_r^+, a_s^-\} = \delta_{rs} \qquad\qquad \{a_r^+, a_s^+\} = \{a_r^-, a_s^-\} = 0\ . \tag{3.10}$$

Thus, a_r^+ and a_r^- are creation and annihilation operators, respectively, for a Hilbert space with a vacuum defined in the usual way: $a_r^-|0\rangle = \langle 0|a_r^+ = 0$. To construct coherent states, we introduce six independent Grassmann variables, θ_r and $\bar\theta_r$, $r = 1,2,3$, which anticommute with themselves, and with the Fermi operators a_r^+ and a_r^-. The differentials $d\theta_r$ and $d\bar\theta_r$ commute with one another and anticommute with all Grassmann variables and Fermi operators. Also, θ_r, $\bar\theta_r$, $d\theta_r$ and $d\bar\theta_r$ commute with the vacuum $|0\rangle$. The coherent states are defined as follows

$$\langle\theta| \equiv i\langle 0|\prod_{r=1}^{3}(\theta_r - a_r^-) \qquad |\theta\rangle \equiv \exp\big(-\sum_{r=1}^{3}\theta_r a_r^+\big)|0\rangle$$
$$\langle\bar\theta| \equiv \langle 0|\exp\big(-\sum_{r=1}^{3} a_r^-\bar\theta_r\big) \qquad |\bar\theta\rangle \equiv i\prod_{r=1}^{3}(\bar\theta_r - a_r^+)|0\rangle\ . \tag{3.11}$$

The product symbols are understood to order r in the sequence 123. These states satisfy the defining equations for coherent states

$$
\begin{aligned}
&\langle\theta|a_r^- = \langle\theta|\theta_r \qquad a_r^-|\theta\rangle = \theta_r|\theta\rangle \qquad \langle\theta|\bar\theta\rangle = \exp(\sum_{r=1}^{3}\theta_r\bar\theta_r)\\
&\langle\bar\theta|a_r^+ = \langle\bar\theta|\bar\theta_r \qquad a_r^+|\bar\theta\rangle = \bar\theta_r|\bar\theta\rangle \qquad \langle\bar\theta|\theta\rangle = \exp(\sum_{r=1}^{3}\bar\theta_r\theta_r)\ .
\end{aligned}
\tag{3.12}
$$

Further useful expressions may be obtained with the help of the Grassmann integration, defined by

$$
\int \theta_1 d\theta_1 = \int \bar\theta_1 d\bar\theta_1 = i\ . \tag{3.13}
$$

In particular, we have a *completeness relation*

$$
1 = \int |\theta\rangle\langle\theta|\, d^3\theta = \int d^3\bar\theta\, |\bar\theta\rangle\langle\bar\theta| \ \text{, with }\ ,\quad d^3\theta = d\theta_3 d\theta_2 d\theta_1 \quad ,\quad d^3\bar\theta = d\bar\theta_1 d\bar\theta_2 d\bar\theta_3\quad , \tag{3.14}
$$

and it is proven in [8] that traces may be evaluated with these coherent states as well :

$$
\mathrm{Tr}(U) = \int d^3\theta\, \langle -\theta|U|\theta\rangle\ . \tag{3.15}
$$

Notice that special care is needed in checking these formulas, since θ_r, $\bar\theta_r$, $d\theta_r$ and $d\bar\theta_r$ *anticommute* with two of the coherent states, namely $\langle\theta|$ and $|\bar\theta\rangle$, but *commute* with the other two, namely $|\theta\rangle$ and $\langle\bar\theta|$.

The Worldline Path Integral

Inserting complete sets of coordinate states and fermionic coherent states, the trace of the evolution operator in Eq. (3.8) may be written as

$$
\begin{aligned}
\mathrm{Tr}\, e^{-\frac{\mathcal{E}}{2}T\Sigma^2} &= \mathrm{Tr_c}\int d^4x d^3\theta\, \langle x,-\theta|e^{-\frac{\mathcal{E}}{2}T\Sigma^2}|x,\theta\rangle\\
&= \mathrm{Tr_c}\int_{\mathrm{BC}} \prod_{i=1}^{N}\Big(-d^4x^i d^3\theta^i\, \langle x^i,\theta^i|\exp[-\frac{\mathcal{E}}{2}\frac{T}{N}\Sigma^2]|x^{i+1},\theta^{i+1}\rangle\Big)\ .
\end{aligned}
\tag{3.16}
$$

Here $\mathrm{Tr_c}$ denotes the trace only over the internal degrees of freedom and shall not be explicitly evaluated here. The subscript BC denotes the boundary condition $(x^{N+1},\theta^{N+1}) = (x^1,-\theta^1)$ on the x and θ integrations. Before evaluating the matrix element of the evolution operator, it is useful to evaluate the matrix elements of the Dirac matrices :

$$
\langle\theta^i|\Gamma_A\Gamma_B|\theta^{i+1}\rangle = -\int d^3\bar\theta^{i,i+1}\, \langle\theta^i|\bar\theta^{i,i+1}\rangle\langle\bar\theta^{i,i+1}|\theta^{i+1}\rangle\, 2\, {}^i\psi_A\psi_B^{i+1}\ , \tag{3.17}
$$

where

$$\psi_r^{i+1} \equiv \frac{1}{\sqrt{2}}(\theta_r^{i+1} + \bar{\theta}_r^{i,i+1}) \quad , \quad \psi_{r+3}^{i+1} \equiv \frac{i}{\sqrt{2}}(\theta_r^{i+1} - \bar{\theta}_r^{i,i+1})$$
$${}^i\psi_r \equiv \frac{1}{\sqrt{2}}(\theta_r^{i} + \bar{\theta}_r^{i,i+1}) \quad , \quad {}^i\psi_{r+3} \equiv \frac{i}{\sqrt{2}}(\theta_r^{i} - \bar{\theta}_r^{i,i+1}) \ . \tag{3.18}$$

This is easily verified by writing the Γ-matrices in terms of the Fermi operators and inserting complete sets of the coherent states $|\bar{\theta}^{i,i+1}\rangle$ in the appropriate spots. Denoting the dependency of the Hamiltonian on the momentum operator, background fields and Γ-matrices by $\Sigma^2(p,\Phi,\Pi,A,\Gamma_A\Gamma_B)$, the matrix element of the evolution operator may now be readily computed:

$$\begin{aligned}&\langle x^i,\theta^i|\exp\big[-\frac{\mathcal{E}}{2}\frac{T}{N}\Sigma^2\big]|x^{i+1},\theta^{i+1}\rangle\\ &= -\int\frac{d^4p^{i,i+1}d^3\bar{\theta}^{i,i+1}}{(2\pi)^4}e^{i(x^i-x^{i+1})\cdot p^{i+1}+(\theta^i-\theta^{i+1})_r\bar{\theta}_r^{i,i+1}}\Big(1-\frac{\mathcal{E}}{2}\frac{T}{N}\Sigma_i^2+\mathrm{O}\big[(\frac{T}{N})^2\big]\Big) \ .\end{aligned}\tag{3.19}$$

Here, we have

$$\Sigma_i^2 \equiv \Sigma^2(p^{i,i+1},\Phi^{i,i+1},\Pi^{i,i+1},A^{i,i+1},2\,{}^i\psi_A\psi_B^{i+1}) \ , \tag{3.20}$$

and the fields $\Phi^{i,i+1}$, $\Pi^{i,i+1}$ and $A^{i,i+1}$ denote the averages of the corresponding fields with superscripts i and $i+1$. Substituting the matrix element of the evolution operator back into the expression for the trace and symmetrizing the positions of the Grassmann variables in the exponentials gives

$$\begin{aligned}&\mathrm{Tr}\, e^{-\frac{\mathcal{E}}{2}T\Sigma^2}\\ &= \mathrm{Tr_c}\int\prod_{i=1}^{N}\Big[\frac{d^4x^id^4p^{i,i+1}d^3\theta^id^3\bar{\theta}^{i,i+1}}{(2\pi)^4}\Big]\prod_{i=1}^{N}\Big(1-\frac{\mathcal{E}}{2}(\tau^i-\tau^{i+1})\Sigma_i^2+O\big[\frac{T^2}{N^2}\big]\Big)\\ &\quad\times\exp\Big(\sum_{i=1}^{N}[i(x^i-x^{i+1})\cdot p^{i,i+1}+\frac{1}{2}(\theta_r^i-\theta_r^{i+1})\bar{\theta}_r^{i,i+1}-\frac{1}{2}\theta_r^i(\bar{\theta}_r^{i-1,i}-\bar{\theta}_r^{i,i+1})]\Big) \ .\end{aligned}\tag{3.21}$$

Here, the boundary condition for the integrals on x, θ and $\bar{\theta}$ is given by $(x^{N+1},\theta^{N+1},\bar{\theta}^{0,1}) = (x^1,-\theta^1,-\bar{\theta}^{N,N+1})$. The interpolating propertime, τ, has been introduced such that $\tau^1 = T$, $\tau^{N+1}=0$ and $\tau^i-\tau^{i+1}=T/N$. In the limit $N\to\infty$, the product in the integrand of (3.21) is simply the standard path ordered exponential. In this limit, the worldline path integral is obtained:

$$\begin{aligned}\mathrm{Tr}\, e^{-\frac{\mathcal{E}}{2}T\Sigma^2} = \mathrm{Tr_c}\int\mathcal{D}p\int_{\mathrm{PBC}}\mathcal{D}x\int_{\mathrm{APBC}}\mathcal{D}\theta\mathcal{D}\bar{\theta}\ \mathcal{P}\exp\Big\{\int_0^T d\tau\Big[i\dot{x}\cdot p\\ +\frac{1}{2}\dot{\theta}_r\bar{\theta}_r-\frac{1}{2}\theta_r\dot{\bar{\theta}}_r-\frac{\mathcal{E}}{2}\Sigma^2(p,\Phi,\Pi,A,2\psi_A\psi_B)\Big]\Big\} \ ,\end{aligned}\tag{3.22}$$

where $\mathcal{P}$ is the path ordering operator which acts on the interaction part of the exponential only. The boundary conditions are periodic (PBC) on x, while antiperiodic (APBC) on

θ and $\bar{\theta}$: $(x(T), \theta(T), \bar{\theta}(T)) = (x(0), -\theta(0), -\bar{\theta}(0))$. Also, we have introduced the new variables $\psi_A(\tau)$, defined as

$$\psi_r(\tau) \equiv \frac{1}{\sqrt{2}}[\theta_r(\tau) + \bar{\theta}_r(\tau)] \quad , \quad \psi_{r+3}(\tau) \equiv \frac{1}{\sqrt{2}} i[\theta_r(\tau) - \bar{\theta}_r(\tau)] \,. \tag{3.23}$$

To complete this variable change, the fermionic kinetic term is re-expressed in terms of the ψ_A variables as well

$$\frac{1}{2}\dot{\theta} \cdot \bar{\theta} - \frac{1}{2}\theta \cdot \dot{\bar{\theta}} = -\frac{1}{2}(\psi \cdot \dot{\psi} + \psi_5 \dot{\psi}_5 + \psi_6 \dot{\psi}_6) = -\frac{1}{2}\psi_A \dot{\psi}_A \,, \tag{3.24}$$

and the associated boundary condition becomes $(x(T), \psi_A(T)) = (x(0), -\psi_A(0))$. The Jacobian of the transformation is simply absorbed into the normalization of correlation functions of the ψ_A fields and so need not be written down explicitly in the path integral expression.

The terms in the path integral which involve the momentum may be rearranged as follows

$$i\dot{x} \cdot p - \frac{\mathcal{E}}{2}(p - A)^2 = -\frac{\mathcal{E}}{2}(p - A - \frac{i\dot{x}}{\mathcal{E}})^2 - \frac{\dot{x}^2}{2\mathcal{E}} + i\dot{x} \cdot A \,. \tag{3.25}$$

The term $\frac{i\dot{x}}{\mathcal{E}}$ may clearly be shifted away by the p-integration. Showing then that $(p - A)$ may be replaced by p without changing the value of the path integral is more subtle but proven in Appendix A. The following normalization factor is left over

$$\mathcal{N} \equiv \int \mathcal{D}p \, e^{-\frac{\mathcal{E}}{2}\int_0^T d\tau p^2(\tau)} \,, \tag{3.26}$$

satisfying

$$\mathcal{N} \int\limits_{\mathrm{PBC}} \mathcal{D}x e^{-\int_0^T d\tau \frac{\dot{x}^2}{2\mathcal{E}}} = (2\pi\mathcal{E}T)^{-2} \int d^D x \,.$$

The real part of the one-loop effective action can now be recast into its final form, which is the main result of this Section. It is given by a worldline path integral with an action that is even in Grassmann fields ψ obeying antiperiodic boundary conditions

$$W_{\Re}[\Phi, \Pi, A] = \frac{1}{4}\int_0^\infty \frac{dT}{T} \mathcal{N} \int\limits_{\mathrm{PBC}} \mathcal{D}x \int\limits_{\mathrm{APBC}} \mathcal{D}\psi \, \mathrm{Tr_c} \, \mathcal{P} e^{-\int_0^T d\tau \mathcal{L}} \,. \tag{3.27a}$$

The worldline Lagrangian, $\mathcal{L}$, is given by

$$\begin{aligned} \mathcal{L}(\tau) = \frac{\dot{x}^2}{2\mathcal{E}} + \frac{1}{2}\psi_A \dot{\psi}_A - i\dot{x} \cdot A + \frac{i}{2}\mathcal{E}\psi_\mu F_{\mu\nu}\psi_\nu + \frac{1}{2}\mathcal{E}\Phi^2 + \frac{1}{2}\mathcal{E}\Pi^2 \\ + i\mathcal{E}\psi_\mu \psi_6 D_\mu \Phi + i\mathcal{E}\psi_\mu \psi_5 D_\mu \Pi + \mathcal{E}\psi_6 \psi_5 [\Phi, \Pi] \,. \end{aligned} \tag{3.27b}$$

Clearly this path integral preserves manifestly gauge symmetry. While global chiral symmetry was manifest at the operator level in terms of Σ^2 in Eqs. (3.8-9), this symmetry is less clearly seen once the Γ-matrices have been represented by a path integral over Grassmann variables ψ. Nonetheless, after Wick-contraction of the fields ψ, global chiral symmetry naturally transpires and we shall still denote it as manifest.

The worldline path integral of (3.27) reduces to that in [6] in the limit where the background fields all commute and the gauge field vanishes (their notation incidentally interchanges $\psi_5 \leftrightarrow \psi_6$). However, we stress that from our construction, it is clear that this worldline path integral can only generate one-loop diagrams with an even number of pseudoscalar vertices (which are real valued in Euclidean space).

The path integral in (3.27) is a one dimensional quantum field theory. Since $x(\tau)$ has a constant zero mode, the Green's function of the corresponding free field operator for the bosonic sector, $\frac{d^2}{d\tau^2}$, must be defined on the subspace orthogonal to the zero mode:

$$\frac{1}{\mathcal{E}}\frac{d^2}{d\tau^2} g_b(\tau-\tau') = \delta(\tau-\tau') - \frac{1}{T}\,, \tag{3.28}$$

where the solution satisfying $g_b(T-\tau') = g_b(0-\tau')$ is

$$g_b(\tau-\tau') = -\mathcal{E}\frac{(\tau-\tau')^2}{2T} + \frac{\mathcal{E}}{2}|\tau-\tau'| + \ \text{constant}\,. \tag{3.29}$$

Decomposing $x(\tau)$ on the constant zero mode and all the other modes orthogonal to the zero mode, that is, $x(\tau) = y(\tau) + x$, where $\frac{d}{d\tau}x = 0$ and $\int_0^T d\tau y(\tau) = 0$, the fundamental correlation function in the bosonic sector is

$$\langle\, e^{\int_0^T d\tau y(\tau)\cdot J(\tau)}\,\rangle = \frac{\int d^D x}{(2\pi\mathcal{E}T)^{2-\epsilon}} \exp\Big[-\frac{1}{2}\int_0^T d\tau_1 d\tau_2 J(\tau_1) g_b(\tau_1-\tau_2) J(\tau_2)\Big]\,, \tag{3.30}$$

where the dimension of spacetime has been taken to be $D = 4-2\epsilon$ in preparation for dimensional regularization. For the fermionic sector, the Green's function is defined as

$$\frac{d}{d\tau} g_f(\tau-\tau') = \delta(\tau-\tau')\,, \tag{3.31}$$

and must satisfy $g_f(T-\tau') = -g_f(0-\tau')$ due to the antiperiodicity of ψ_A; demanding further that $g_f(\tau-\tau') = -g_f(\tau'-\tau)$ leads to the solution

$$g_f(\tau-\tau') = \frac{1}{2}\text{sign}(\tau-\tau')\,. \tag{3.32}$$

The fundamental correlation function in the fermionic sector is

$$\begin{aligned}&\langle \psi_{A_1}\cdots\psi_{A_N}\rangle \\ &= 8\sum_{\text{all }\ j_i=1}^{N} \varepsilon_{j_1\ldots j_N}\delta_{A_{j_1}A_{j_2}}\cdots\delta_{A_{j_{N-1}}A_{j_N}} \frac{\text{sign}(\tau_{j_1}-\tau_{j_2})}{4}\cdots\frac{\text{sign}(\tau_{j_{N-1}}-\tau_{j_N})}{4}\,.\end{aligned} \tag{3.33}$$

The normalization $\langle 1\rangle_{\text{fermionic sector}} = \mathrm{Tr}(\mathrm{I}_{8\times 8}) = 8$ is to be contrasted with the one found in ref. [5] where 4×4 gamma matrices are used instead.

4. Worldline Path Integral for the Imaginary Part of the Effective Action

In this Section, we present the evaluation of the imaginary part of the effective action, i.e. the phase of the fermion functional determinant [9]. To this end, we first derive a new general regulator for the phase of functional determinants which resembles the heat-kernel regulator in that it involves an evolution operator and a positive Hamiltonian. By analogy with the evaluation of the real part of the effective action, we shall again double the number of fermions so that the heat-kernel-like representation is even in powers of the Γ-matrices and so hence the related worldline action is even in the Grassmann variables. First, the doubling of fermions is carried out as follows

$$-iW_{\Im}[\Phi,\Pi,A] = i\mathrm{argDet}[\mathcal{O}] = \frac{i}{2}\mathrm{argDet}[\mathcal{O}^2] = \frac{i}{2}\mathrm{argDet}[\Omega] \tag{4.1a}$$

where we define

$$\Omega \equiv \begin{pmatrix} 0 & \mathcal{O} \\ \mathcal{O} & 0 \end{pmatrix} \; . \tag{4.1b}$$

Using the six Γ-matrices defined in Section 3, Ω and $\Omega^\dagger$ (which we shall need shortly as well) may be recast in the following form :

$$\begin{aligned} \Omega &= \Gamma_\mu(p_\mu - A_\mu) - \Gamma_5\Pi - \Gamma_7\Gamma_6\Phi \\ \Omega^\dagger &= \Gamma_\mu(p_\mu - A_\mu) - \Gamma_5\Pi + \Gamma_7\Gamma_6\Phi \; , \end{aligned} \tag{4.2}$$

where the six dimensional analogue of γ_5 is defined by

$$\Gamma_7 \equiv -i\prod_{A=1}^{6}\Gamma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; . \tag{4.3}$$

As in Section 3, the goal of introducing the 8×8 Γ-matrices is to obtain a final worldline action even in the Grassmann variables. Thus, the starting point for the imaginary part of the effective action will be the following expression

$$-iW_{\Im}[\Phi,\Pi,A] = \frac{1}{4}\mathrm{Tr}\log\Omega - \frac{1}{4}\mathrm{Tr}\log\Omega^\dagger \; . \tag{4.4}$$

At this point, there are different choices for passing from this expression to a heat-kernel formulation. These different choices correspond to different regularizations carried out on the effective action and none of these will exhibit manifest chiral symmetry, in keeping with the appearance of the chiral anomaly. In this Section, we shall pursue one particular choice which appears to us most advantageous. An alternative natural choice will be presented in Appendix C.

We begin by recasting Eq. (4.4) in terms of a single trace with the help of an auxiliary integration over a parameter α as follows

$$\begin{aligned}-iW_{\Im}[\Phi,\Pi,A] &= \frac{1}{4}\int_{-1}^{1} d\alpha\ \frac{\partial}{\partial\alpha}\mathrm{Tr}\ln\left[\frac{1}{2}(\Omega+\Omega^\dagger)+\alpha\frac{1}{2}(\Omega-\Omega^\dagger)\right] \\ &= \frac{1}{4}\int_{-1}^{1} d\alpha\ \mathrm{Tr}(\Omega-\Omega^\dagger)\big[(\Omega+\Omega^\dagger)+\alpha(\Omega-\Omega^\dagger)\big]^{-1} .\end{aligned} \tag{4.5}$$

Contributions that are odd in α in the integrand vanish in the integral, so only the even part may be retained. This allows us to take the average of the integrands for α and $-\alpha$, which leads to the following expression

$$-iW_{\Im}[\Phi,\Pi,A] = \frac{1}{4}\int_{-1}^{1} d\alpha\ \mathrm{Tr}\big(\Omega^2-\Omega^{\dagger 2}\big)\Big[(\Omega+\Omega^\dagger)^2+2\alpha[\Omega,\Omega^\dagger]-\alpha^2(\Omega-\Omega^\dagger)^2\Big]^{-1} . \tag{4.6}$$

The distinctive advantage of this expression for the imaginary part of the effective action is that we have produced a formula where the denominator is a positive operator, as can be seen from the identity

$$(\Omega+\Omega^\dagger)^2+2\alpha[\Omega,\Omega^\dagger]-\alpha^2(\Omega-\Omega^\dagger)^2 = \big[(\Omega+\Omega^\dagger)+\alpha(\Omega-\Omega^\dagger)\big]\big[(\Omega+\Omega^\dagger)+\alpha(\Omega-\Omega^\dagger)\big]^\dagger . \tag{4.7}$$

As a result, we can represent this positive denominator in terms of a heat-kernel formula :

$$-iW_{\Im}[\Phi,\Pi,A] = \frac{\mathcal{E}}{32}\int_{-1}^{1} d\alpha\ \int_0^\infty dT\ \mathrm{Tr}\Big\{(\Omega^2-\Omega^{\dagger 2})e^{-\frac{\mathcal{E}}{2}T\left[\frac{1}{4}(\Omega+\Omega^\dagger)^2+\frac{1}{2}\alpha[\Omega,\Omega^\dagger]-\frac{1}{4}\alpha^2(\Omega-\Omega^\dagger)^2\right]}\Big\} . \tag{4.8}$$

The imaginary part of the effective action, namely the phase of $\mathrm{Det}[\Omega]$, has now been written as a completely well defined new heat-kernel expression, and we shall now evaluate the various ingredients in preparation for a path integral reformulation of it. The insertion works out to be

$$\Omega^2-\Omega^{\dagger 2} = 2i\Gamma_7\omega\ , \qquad \omega = i\Gamma_5\Gamma_6\{\Pi,\Phi\} - i\Gamma_\mu\Gamma_6\{(p_\mu-A_\mu),\Phi\}\ , \tag{4.9}$$

while the argument of the exponential involves

$$\begin{aligned}\mathcal{H}_0 &= \frac{1}{4}(\Omega+\Omega^\dagger)^2 - \frac{1}{4}\alpha^2(\Omega-\Omega^\dagger)^2 \\ &= (p-A)^2+\frac{i}{2}\Gamma_\mu F_{\mu\nu}\Gamma_\nu+\Pi^2+i\Gamma_\mu\Gamma_5 D_\mu\Pi+\alpha^2\Phi^2 \\ \mathcal{H}_1 &= \frac{1}{2}\Gamma_7[\Omega,\Omega^\dagger] = -i\Gamma_6\Gamma_\mu D_\mu\Phi+\Gamma_5\Gamma_6[\Pi,\Phi]\ .\end{aligned} \tag{4.10}$$

Putting all these contributions together, we obtain the following heat-kernel representation

$$W_{\Im}[\Phi,\Pi,A] = -\frac{\mathcal{E}}{16}\int_{-1}^{1} d\alpha\int_0^\infty dT\ \mathrm{Tr}\Big\{\Gamma_7\omega\ e^{-\frac{\mathcal{E}}{2}T(\mathcal{H}_0+\alpha\Gamma_7\mathcal{H}_1)}\Big\} . \tag{4.11}$$

The expansion of the exponential in powers of α produces non-zero contributions to the integral only for even powers. Since Γ_7 always occurs multiplied by α and commutes with $\mathcal{H}_0$ and $\mathcal{H}_1$, the Γ_7 in the exponential only occurs to even powers, which just reduce to the identity. Thus, the presence of the Γ_7 factor in the argument of the exponential is immaterial, and henceforth, we shall omit it. This leads to the following simplified expression

$$W_{\Im}[\Phi,\Pi,A] = -\frac{\mathcal{E}}{16}\int_{-1}^{1} d\alpha \int_0^\infty dT \ \mathrm{Tr}\Big\{\Gamma_7 \omega \ e^{-\frac{\mathcal{E}}{2}T(\mathcal{H}_0+\alpha\mathcal{H}_1)}\Big\} \ . \tag{4.12}$$

Notice that in (4.12), the exponential involves terms without Γ-matrices and terms quadratic in Γ-matrices only, exactly as was the case for the real part of the effective action. Furthermore, the Hamiltonian $\mathcal{H} \equiv \mathcal{H}_0 + \alpha\mathcal{H}_1$ coincides with the Hamiltonian expression (3.9) of the real case when $\alpha = 1$. It is clear from (4.9), (4.10) and (4.12) that the new regulator is manifestly gauge invariant while global chiral invariance is not realized in a manifest way due to the presence of the α-parameter.

Now, the conversion of (4.12) into a path integral is almost identical to the case of the real part in Section 3. The difference is that the presence of Γ_7 in the insertion causes the worldline fermions to have *periodic boundary conditions*. This can be seen by defining the worldline fermion number operator, $\mathcal{F}$, in terms of the Fermi creation and annihilation operators of Section 3:

$$\mathcal{F} \equiv \sum_{r=1}^{3} \mathcal{F}_r, \qquad \text{with} \qquad \mathcal{F}_r \equiv a_r^+ a_r^- \ . \tag{4.13}$$

Then, Γ_7 is identical to the fermion number counter $(-1)^{\mathcal{F}}$, also called the "G-parity operator" [4], and may be expressed as

$$(-1)^{\mathcal{F}} = \prod_{r=1}^{3}(1-2\mathcal{F}_r) = \Gamma_7 \ . \tag{4.14}$$

The presence of $(-1)^{\mathcal{F}}$ under a trace changes the boundary conditions on the worldline fermions, since at the level of coherent states, it operates as

$$\langle -\theta|(-1)^{\mathcal{F}} = i\langle 0| \prod_{r=1}^{3}(-\theta_r - a_r^-)(1-2a_r^+a_r^-) = i\langle 0|\prod_{r=1}^{3}(-\theta_r + a_r^-) = -\langle\theta| \ . \tag{4.15}$$

This identity makes it clear that while the trace of (3.15) involves antiperiodic boundary conditions on the worldline fermions, the presence of Γ_7 modifies these into periodic boundary conditions.

With this preparation, the trace in (4.12) may be readily converted into a path integral, using (3.22) together with the change to periodic boundary conditions worked out above. We find

$$\mathrm{Tr}\,\Gamma_7\,\omega\, e^{-\frac{\mathcal{E}}{2}T\mathcal{H}} = \mathrm{Tr}_{\mathrm{c}} \int d^4x d^3\theta \ \langle x,-\theta|\,(-1)^{\mathcal{F}}\ \omega\, e^{-\frac{\mathcal{E}}{2}T\mathcal{H}}\,|x,\theta\rangle \tag{4.16}$$

$$
\begin{aligned}
&= \mathrm{Tr_c} \int \mathcal{D}p \int\limits_{\mathrm{PBC}} \mathcal{D}x\mathcal{D}\theta\mathcal{D}\bar{\theta}\ \omega(p,\Phi,\Pi,A,2\psi_A\psi_B)(\tau=0) \\
&\times \mathcal{P}\exp\left\{\int_0^T d\tau \left[i\dot{x}\cdot p + \frac{1}{2}\dot{\theta}_r\bar{\theta}_r - \frac{1}{2}\theta_r\dot{\bar{\theta}}_r - \frac{\mathcal{E}}{2}\mathcal{H}(p,\Phi,\Pi,A,2\psi_A\psi_B)\right]\right\} ,
\end{aligned}
$$

where the periodic boundary conditions (PBC) are $(x(T),\theta(T),\bar{\theta}(T)) = (x(0),\theta(0),\bar{\theta}(0))$.

Being periodic on the loop, each worldline fermion, θ_r, $\bar{\theta}_r$ and ψ_A, can be decomposed as the sum of its zero mode and modes orthogonal to the zero mode (denoted with a prime). So under the change of variables from θ_r, $\bar{\theta}_r$ to ψ_A, there is a Jacobian, J, for the zero mode and a Jacobian, J', for the orthogonal modes. However, J' will be absorbed in the normalization of the correlation functions of the ψ'_A so it need not be exhibited explicitly whereas the zero modes are to be integrated out and so we must explicitly exhibit J. Thus the measure $\mathcal{D}\theta\mathcal{D}\bar{\theta} \equiv d\theta_3 d\theta_2 d\theta_1 d\bar{\theta}_1 d\bar{\theta}_2 d\bar{\theta}_3 \mathcal{D}\theta'\mathcal{D}\bar{\theta}'$ becomes

$$
\frac{1}{J} d\psi_1 d\psi_2 d\psi_3 d\psi_4 d\psi_5 d\psi_6 \mathcal{D}\psi' \equiv \frac{1}{J} d^6\psi \mathcal{D}\psi' = \frac{1}{J}\mathcal{D}\psi , \tag{4.17}
$$

which defines the ordering in the variable change, and so J is easily worked out as

$$
J = \det\left(\frac{\partial\theta,\bar{\theta}}{\partial\psi}\right) = i . \tag{4.18}
$$

This identity, $\mathcal{D}\theta\mathcal{D}\bar{\theta} = -i\mathcal{D}\psi$, may also be easily checked by direct comparison with the Γ-matrix algebra.

As in the previous Section, the terms in the exponential of the path integral in (4.16) involving the momentum may be arranged as in (3.25) and again the term $\frac{i\dot{x}}{\mathcal{E}}$ may be shifted away by the p-integration (which introduces $\frac{i\dot{x}}{\mathcal{E}}$ into the insertion). Showing then that $p - A$ may again be replaced by p, even in the insertion, is proven in Appendix A. Of course then after this replacement, the term in the insertion linear in the momentum can not contribute to the path integral. Hence the path integral of (4.16) acquires the same normalization factor as in (3.26). We therefore obtain the final form for the imaginary part of the effective action, which is the principal result in this Section.

$$
W_{\Im}[\Phi,\Pi,A] = \frac{1}{8}\int_{-1}^{1} d\alpha \int_0^\infty dT\, \mathcal{N} \int\limits_{\mathrm{PBC}} \mathcal{D}x\mathcal{D}\psi\ \mathrm{Tr_c}\, \mathcal{J}(0)\, \mathcal{P} e^{-\int_0^T d\tau \mathcal{L}_\alpha} . \tag{4.19}
$$

The worldline insertion, $\mathcal{J}(\tau)$, is given by

$$
\mathcal{J}(\tau) = [2i\psi_\mu\psi_6\dot{x}_\mu\Phi - \mathcal{E}\psi_5\psi_6\{\Pi,\Phi\}](\tau) \tag{4.20}
$$

and the worldline Lagrangian, $\mathcal{L}_\alpha$, is given by

$$
\begin{aligned}
\mathcal{L}_\alpha = \frac{\dot{x}^2}{2\mathcal{E}} + \frac{1}{2}\psi_A\dot{\psi}_A - i\dot{x}\cdot A + \frac{i}{2}\mathcal{E}\psi_\mu F_{\mu\nu}\psi_\nu + \frac{1}{2}\mathcal{E}\alpha^2\Phi^2 + \frac{1}{2}\mathcal{E}\Pi^2 \\
+ i\alpha\mathcal{E}\psi_\mu\psi_6 D_\mu\Phi + i\mathcal{E}\psi_\mu\psi_5 D_\mu\Pi + \alpha\mathcal{E}\psi_5\psi_6[\Pi,\Phi] .
\end{aligned} \tag{4.21}
$$

The final PBC is $(x(T), \psi_A(T)) = (x(0), \psi_A(0))$. Notice that the terms in this new worldline Lagrangian and insertion bear close resemblance to the structure of superstring perturbation theory to one loop order for odd spin structure [4]. Furthermore, notice that $\mathcal{L}_\alpha$ at $\alpha = 1$ is exactly the same worldline Lagrangian used to describe the real part of the effective action. The new worldline Lagrangian is again manifestly gauge invariant but now, due to the explicit presence of the α-parameter, global chiral symmetry is not realized in a manifest way.

The Green function and perturbative rules for the bosonic sector of the one dimensional field theory described by (4.19) are the same as those given for the real case. In contrast with the real case, the presence of the zero mode of $\psi_A(\tau)$ requires that the Green function, G_f, of the corresponding free field operator for the fermionic sector, $\frac{d}{d\tau}$, be defined on the subspace orthogonal to the zero mode:

$$\frac{d}{d\tau} G_f(\tau - \tau') = \delta(\tau - \tau') - \frac{1}{T} \tag{4.22}$$

and must satisfy $G_f(T - \tau') = G_f(0 - \tau')$; demanding furthermore that $G_f(\tau - \tau') = -G_f(\tau' - \tau)$ leads to the solution

$$G_f(\tau - \tau') = \frac{1}{2}\mathrm{sign}(\tau - \tau') - \frac{(\tau - \tau')}{T} \; . \tag{4.23}$$

The fundamental correlation function in the fermionic sector is the same as (3.33) except for the replacements $\psi_{A_j} \to \psi'_{A_j}$, $\frac{1}{4}\mathrm{sign}(\tau_{j_i} - \tau_{j_{i+1}}) \to \frac{1}{2} G_f(\tau_{j_i} - \tau_{j_{i+1}})$ and the normalization $8 \to N'$. The new normalization for the $\psi'(\tau)$ path integral, defined as $N' \equiv \int_{\mathrm{PBC}} \mathcal{D}\psi' \exp\left[-\frac{1}{2}\int_0^T d\tau \psi'_A \dot{\psi}'_A\right]$, can be easily worked out by introducing two Grassmann variables, η_1 and η_2:

$$\begin{aligned} N &= \frac{1}{T}\int d\eta_1 d\eta_2 \int\limits_{\mathrm{PBC}} d^6\psi \mathcal{D}\psi' \exp\Big\{-\int_0^T \big[\frac{1}{2}\psi_A\dot{\psi}_A - \eta_1\eta_2\psi_1(\tau)\cdots\psi_6(\tau)\big]\Big\} \\ &= \frac{1}{T}\int d\eta_1 d\eta_2 \, i \,\mathrm{Tr}\,\Gamma_7 \exp\Big(\frac{1}{8}\,\alpha\beta\,\Gamma_1\cdots\Gamma_6\, T\Big) = -1 \; . \end{aligned} \tag{4.24}$$

As an application of this worldline path integral formalism for one loop diagrams with an odd number of pseudoscalar couplings, we recover the Wess-Zumino-Witten term in Appendix D.

5. Conclusion

Worldline path integral formulations for the real and imaginary parts of the one-loop effective action for a multiplet of Dirac fermions coupled to matrix-valued scalar, pseudoscalar and vector gauge fields have been systematically derived.

For the real case, this was naturally achieved by a standard heat-kernel representation of a positive operator, which was then converted into a worldline path integral using a simple coherent state formalism for worldline fermions. We proved that worldline fermions must obey antiperiodic boundary conditions here. Gauge invariance and global chiral symmetry are manifest in this formulation.

For the imaginary case, no simple heat-kernel representation is directly available. Nonetheless, we obtained a new modified heat-kernel representation with the help of an additional integration parameter and an insertion operator, reminiscent of string perturbation theory to one loop for odd spin structure. We proved that worldline fermions must obey periodic boundary conditions here. While manifestly gauge invariant, this formulation does not exhibit chiral symmetry in a manifest way, and indeed, the Wess-Zumino-Witten term is recovered as an example of imaginary contributions to the effective action.

It is now noted that our two path integral formulations could be combined into a single path integral formulation (for the generation of one-loop diagrams both even and odd in the number of pseudoscalar couplings) through a summation over the two spin structures. This is in accordance with superstring perturbation theory [4].

The results of ref. [5] and [6] are special cases of our results and are in agreement within the respective restrictions of the fields. The dimensional reduction in [6] of a six dimensional gauge coupling to four dimensions is naturally understood in our reformulation, from the way the fermions are doubled, both for the case of the real as well as the imaginary part of the effective action. This doubling was utilized here in order to achieve worldline actions which are even in the Grassmann variables, showing that the worldline actions with mixed powers of Grassmann variables, obtained in [5] can naturally be avoided.

It appears that the coupling to an axial vector and anti-symmetric tensor field can be handled in an analogous way to the case treated here. These cases will be treated in a future publication.

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Appendix A : Proof of the Momentum Shift

In this Appendix we prove the momentum shifts used in the path integral reformulations of the real and imaginary parts of the effective action.

For the real case, it is useful to define

$$M(\lambda,T) \equiv \int \mathcal{D}p \; \mathcal{P} e^{-\int_0^T d\tau [(p-A)^2+\lambda B]} \;, \qquad (A.1)$$

where A and B are $N \times N$, τ-dependent fields which do not commute. We wish to prove that $M(\lambda, T)$ (as an analytic function of λ) is independent of A. To show this, we first diagonalize A by $A = U^\dagger D U$, where D is a diagonal matrix with entries $D^1, ..., D^N$, and $U^\dagger U = 1$. Next we note that for $\lambda = 0$ we get

$$M(0,T) = \lim_{K\to\infty} \prod_{k=0}^{K} \left[U_k^\dagger \,\mathrm{Diag}\left(\int_{p_k} \exp\left[-(p_k - D_k^1)^2 \frac{T}{K}\right], ..., \int_{p_k} \exp\left[-(p_k - D_k^N)^2 \frac{T}{K}\right] \right) U_k \right] . \tag{A.2}$$

Since each diagonal element $D^1, ..., D^N$ may be independently shifted away, we establish that $M(0,T)$ is independent of A. Now we calculate the first derivative

$$\frac{\partial M}{\partial \lambda}(\lambda, T) = -\int_0^T d\tau \int \mathcal{D}p \; \mathcal{P} e^{-\int_0^\tau d\tau' [(p-A)^2 + \lambda B]} \; B(\tau) \, \mathcal{P} e^{-\int_\tau^T d\tau' [(p-A)^2 + \lambda B]} \tag{A.3}$$

$$= -\int_0^T d\tau M(\lambda, \tau) B(\tau) M^{-1}(\lambda, \tau) M(\lambda, T) \; . \tag{A.4}$$

Thus $\frac{\partial M}{\partial \lambda}(0,T)$ is independent of A since each M in (A.4) is independent of A when $\lambda = 0$. It is easy to see that all higher derivatives of M will also be expressible as linear combinations of products of B, M and M^{-1} as in (A.4) so that at $\lambda = 0$ all higher derivatives of M are also independent of A. Therefore, $M(\lambda, T)$ is independent of A.

For the imaginary case, it is useful to define

$$M'(\lambda, T) \equiv \int \mathcal{D}p \; (p-A)_0 \; \mathcal{P} e^{-\int_0^T d\tau [(p-A)^2 + \lambda B]} \; . \tag{A.5}$$

We shall prove in analogy to the real case that $M'(\lambda, T)$ is independent of A and hence vanishes. The analogue of (A.2) is

$$\begin{aligned} & M'(0,T) \\ &= U_0^\dagger \,\mathrm{Diag}\left(\int_{p_0} (p-A)_0 \, e^{\left[-(p_0 - D_0^1)^2 \frac{T}{K}\right]}, ..., \int_{p_0} (p-A)_0 \, e^{\left[-(p_0 - D_0^N)^2 \frac{T}{K}\right]} \right) U_0 \; M(0,T) \\ &= \int_{p_0} p_0 \, e^{-\frac{T}{K} p_0^2} \; M(0,T) = 0 \; . \end{aligned} \tag{A.6}$$

Now, the fromula for the first derivative, $\frac{\partial M'}{\partial \lambda}$, is exactly the same as (A.4) except with M replaced by M'. Thus, $\frac{\partial M'}{\partial \lambda}(0,T) = 0$ since $M'(0,T) = 0$. Similarly, since all higher derivatives are easily expressible as local products of B, M' and $(M')^{-1}$, we see that all higher derivatives vanish at $\lambda = 0$. Therefore $M'(\lambda, T) = 0$.

Appendix B : Comparison with Perturbation Theory, Real Part

In this Appendix, the validity of the new term $\psi_6 \psi_5 [\Phi, \Pi]$ in the path integral of Eq. (3.27) will be checked by calculating the path integral at order $\Phi^2 \Pi^2 A^0$ and then comparing the result with that obtained (at the same order) using ordinary quantum field theory, namely using Eq. (2.9). This order is in the domain of applicability of the path integral (having an even number of pseudoscalar legs) and is the lowest order diagram which takes into account the noncommuting nature of Φ and Π. For simplicity, Φ is given a vanishing vacuum expectation value (keeping the fermion massless) and only the pole part of the graph will be calculated. Expanding the path integral in Eq. (3.27) to order $\Phi^2\Pi^2 A^0$, denoting it by $W_{\Re}[\Phi^2,\Pi^2]$, employing dimensional regularization and keeping only those terms which have poles leads to

$$W_{\Re}[\Phi^2,\Pi^2] = \frac{1}{4}\int\limits_{p_1,\dots,p_4}\int_0^\infty \frac{dT}{T}(-\frac{\mathcal{E}}{2})^2\int_0^T d\tau_1\int_0^{\tau_1} d\tau_2 \mathcal{X}_{p,T}(\tau_1,\tau_2)\ , \qquad (B.1)$$

where explicitly,

$$\begin{aligned}\mathcal{X}_{p,T}(\tau_1,\tau_2) &= 2\mathrm{Tr_c}(\tilde\Phi_1\tilde\Phi_3\tilde\Pi_2\tilde\Pi_4)\langle 1\rangle_F\langle e^{ix_1\cdot(p_1+p_3)+ix_2\cdot(p_2+p_4)}\rangle_B \\ &+ \mathrm{Tr_c}([\tilde\Phi_1,\tilde\Pi_2][\tilde\Phi_3,\tilde\Pi_4])4\langle\psi_{6,1}\psi_{5,1}\psi_{6,2}\psi_{5,2}\rangle_F\langle e^{ix_1\cdot(p_1+p_2)+ix_2\cdot(p_3+p_4)}\rangle_B\ .\end{aligned} \qquad (B.2)$$

The background fields have been written in terms of their fourier transforms:

$$\Phi(x_i) = \int\limits_{p_i}\tilde\Phi_i e^{ip_i\cdot x_i}\ \text{, where}\ \int\limits_{p_i} = \int\frac{d^Dp_i}{(2\pi)^D}\ ,\tilde\Phi_i = \tilde\Phi(p_i)\ \text{ and }\ x_i = x(\tau_i)\ , \qquad (B.3)$$

and likewise for Π. The notation $\psi_{A,i} = \psi_A(\tau_i)$ is also used.

The zero modes in the bosonic correlation functions integrate out as usual [4, 6] to give overall momentum conservation. Using (3.30) and introducing the scaled propertimes $u_{1,2} \equiv \tau_{1,2}/T$, the T-integration of the bosonic correlation function in either of the two terms in (B.2) becomes

$$\int_0^\infty \frac{dT}{T}T^2(2\pi\mathcal{E}T)^{\epsilon-2}e^{-\frac{\mathcal{E}}{2}TK(u_i,p_i)} = (2\pi\mathcal{E})^{\epsilon-2}\Gamma(\epsilon)[\frac{\mathcal{E}}{2}K(u_i,p_i)]^{-\epsilon} = \frac{1}{(2\pi\mathcal{E})^2}\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0)\ , \qquad (B.4)$$

where $K(u_i,p_i)$ is some function of the scaled propertimes and the momenta.

The fermionic correlation function in (B.2) is evaluated using (3.33):

$$\langle\psi_{6,1}\psi_{5,1}\psi_{6,2}\psi_{5,2}\rangle_F = -8\cdot 2\frac{\mathrm{sign}(\tau_1-\tau_2)}{4}\cdot 2\frac{\mathrm{sign}(\tau_1-\tau_2)}{4} = -2\ . \qquad (B.5)$$

The integration over the scaled propertimes is trivial when only the pole part of (B.2) is considered: $\int_0^1 du_1\int_0^{u_1} du_2 = \frac{1}{2!}$.

Putting everything together, (B.1) becomes

$$\begin{aligned}&W_{\Re}[\Phi^2,\Pi^2,\frac{1}{\epsilon}] \\ &= \frac{1}{4}\frac{1}{2!}(-\frac{\mathcal{E}}{2})^2\int\limits_{p_1,\cdots,p_4}(2\pi)^D\delta^{(\mathrm{D})}(p_1+\cdots+p_4)\cdot\frac{1}{(2\pi\mathcal{E})^2}\frac{1}{\epsilon} \\ &\times\Big\{2\mathrm{Tr_c}(\tilde\Phi_1\tilde\Phi_3\tilde\Pi_2\tilde\Pi_4)\cdot 8 + \mathrm{Tr_c}([\tilde\Phi_1,\tilde\Pi_2][\tilde\Phi_3,\tilde\Pi_4])\cdot 4\cdot(-2)\Big\}\ .\end{aligned} \qquad (B.6)$$

Observing that

$$
\begin{aligned}
&\int\limits_{p_1,\cdots,p_4} (2\pi)^D \delta^{(\mathrm{D})}(p_1+\cdots+p_4)\mathrm{Tr_c}([\tilde{\Phi}_1,\tilde{\Pi}_2][\tilde{\Phi}_3,\tilde{\Pi}_4]) \\
&= \int\limits_{p_1,\cdots,p_4} (2\pi)^D \delta^{(\mathrm{D})}(p_1+\cdots+p_4)[2\mathrm{Tr_c}(\tilde{\Phi}_1\tilde{\Pi}_2\tilde{\Phi}_3\tilde{\Pi}_4) - 2\mathrm{Tr_c}(\tilde{\Phi}_1\tilde{\Phi}_3\tilde{\Pi}_2\tilde{\Pi}_4)] \ ,
\end{aligned} \tag{B.7}
$$

the final result may be written as

$$
\begin{aligned}
&W_{\Re}[\Phi^2,\Pi^2,\frac{1}{\epsilon}] \\
&= (\frac{1}{8\pi^2})\frac{1}{\epsilon}\int\limits_{p_1,\cdots,p_4} (2\pi)^D \delta^{(\mathrm{D})}(p_1+\cdots+p_4)[2\mathrm{Tr_c}(\tilde{\Phi}_1\tilde{\Phi}_3\tilde{\Pi}_2\tilde{\Pi}_4) - \mathrm{Tr_c}(\tilde{\Phi}_1\tilde{\Pi}_2\tilde{\Phi}_3\tilde{\Pi}_4)] \ .
\end{aligned} \tag{B.8}
$$

Of course, the einbein, $\mathcal{E}$, canceled out as it should since it was introduced here as an arbitrary constant. Furthermore, in the limit that the fields commute, Eq. (B.8) becomes

$$
W_{\Re}[\Phi^2,\Pi^2,\frac{1}{\epsilon}] = (\frac{1}{8\pi^2})\frac{1}{\epsilon}\int\limits_{p_1,\cdots,p_4} (2\pi)^D \delta^{(\mathrm{D})}(p_1+\cdots+p_4)\mathrm{Tr_c}(\tilde{\Phi}_1\tilde{\Phi}_3\tilde{\Pi}_2\tilde{\Pi}_4) \ . \tag{B.9}
$$

This last expression agrees with ref. [6] modulo a factor of four, namely, the $\frac{1}{8}$ here is replaced by $\frac{1}{2}$ there. The reason is simple. In ref. [6], the external fields have been taken to be a superposition of two plane waves (thus yielding a cross-term factor of 2 for each particle type) while here only a single plane wave is used per external field.

Now, using (2.9) at order $\Phi^2\Pi^2A^0$ gives

$$
\begin{aligned}
W_{\Re}[\Phi^2,\Pi^2] = \frac{i^2}{4}&\int\limits_{p_1,\cdots,p_4} (2\pi)^D \delta^{(\mathrm{D})}(p_1+\cdots+p_4)\mathrm{Tr}[I(\not{p}_1,...,\not{p}_4)] \\
&\times \mathrm{Tr_c}\Big[-(\tilde{\Phi}_1\tilde{\Phi}_3\tilde{\Pi}_2\tilde{\Pi}_4 + 3\text{ cyclic perm.}) + (\tilde{\Phi}_1\tilde{\Pi}_2\tilde{\Phi}_3\tilde{\Pi}_4 + 1\text{ cyclic perm.})\Big] \ .
\end{aligned} \tag{B.10}
$$

The function $I(\not{p}_1,...,\not{p}_4)$ is a standard loop integral whose pole part works out to be $(\frac{1}{4\pi^2})\frac{1}{\epsilon}$. Plugging this pole contribution back in reproduces (B.8).

Appendix C : Alternative Formulation for the Imaginary Part of the Effective Action

In this Appendix, we shall briefly discuss a heat-kernel and worldline path integral reformulation for the imaginary part of the effective action that is different from the one developed in Section 4. The reason different formulations arise for the imaginary part of the effective action is that there does not seem to be a naturally unique way of obtaining a heat-kernel formulation for the imaginary part. To us, it seems that the formulation given in Section 4 is perhaps the most natural one. Yet, the different version given below may have also certain advantages, while at the same time being less attractive from other points of view.

The starting point is again relations (4.1) and (4.4). Now, we notice that

$$\mathrm{Det}(\Gamma_5 \Omega \Gamma_5) = \mathrm{Det}(\Omega) \tag{B.1}$$

and thus, we may define a candidate for the imaginary part of the effective action, which is different from that derived in Section 4, in the following way

$$-iW_{\Im} = \frac{1}{8} \mathrm{Tr} \ln\{-\Omega\Gamma_5\Omega\Gamma_5\} - \frac{1}{8} \mathrm{Tr} \ln\{-\Omega^\dagger \Gamma_5 \Omega^\dagger \Gamma_5\} \ . \tag{B.2}$$

The operators

$$\begin{aligned} -\Omega\Gamma_5\Omega\Gamma_5 &= H + \Gamma_7 \omega' \\ -\Omega^\dagger\Gamma_5\Omega^\dagger\Gamma_5 &= H - \Gamma_7 \omega' \end{aligned} \tag{B.3}$$

with

$$\begin{aligned} H &= (p-A)^2 + \frac{i}{2}\Gamma_\mu F_{\mu\nu}\Gamma_\nu + \Phi^2 - \Pi^2 - \Gamma_5\Gamma_\mu\{(p_\mu - A_\mu), \Pi\} \\ \omega' &= -i\Gamma_6\Gamma_\mu D_\mu \Phi - \Gamma_6\Gamma_5\{\Phi, \Pi\} \end{aligned} \tag{B.4}$$

are not hermitian, let alone positive. Nonetheless, for $\Pi = 0$ it is clear that $H \pm \Gamma_7\omega'$ is positive as each may be written as the product of an operator with its adjoint and therefore, a heat-kernel representation can be written which is exact in the fields Φ and A while correct to all orders in a weak field expansion of Π.

We now have the following expression for the imaginary part of the effective action in terms of a heat-kernel

$$\begin{aligned} -iW_{\Im} &= \frac{1}{8}\mathrm{Tr}\ln(H + \Gamma_7\omega') - \frac{1}{8}\mathrm{Tr}\ln(H - \Gamma_7\omega') \\ &= \frac{1}{8}\int_{-1}^{1} d\alpha \int_0^\infty dT \ \mathrm{Tr}\Big\{\Gamma_7 \omega' e^{-T(H+\alpha\Gamma_7\omega')}\Big\} \ . \end{aligned} \tag{B.5}$$

We use the same argument as was given in Section 4, to show that the Γ_7 factor in the argument of the exponential can be dropped since it is immaterial. Thus, we are left with the simplified expression

$$-iW_{\Im} = \frac{1}{8}\int_{-1}^{1} d\alpha \int_0^\infty dT \ \mathrm{Tr}\Big\{\Gamma_7 \omega' e^{-T(H+\alpha\omega')}\Big\} \ . \tag{B.6}$$

Using the coherent state formalism, we immediately can translate this expression into a worldline path integral formulation, given by

$$W_{\Im} = \frac{\mathcal{E}}{8}\int_{-1}^{1} d\alpha \int_0^\infty dT \mathcal{N} \int\limits_{\mathrm{PBC}} \mathcal{D}x\mathcal{D}\psi \ \mathrm{Tr_c}\, \mathcal{K}(0) \ \mathcal{P} e^{-\int_0^T d\tau \mathcal{L}'} \ . \tag{B.7}$$

Here, we have the following expressions for the worldline Lagrangian

$$\mathcal{L}' = \frac{\dot{x}^2}{2\mathcal{E}} + \frac{1}{2}\psi_A\dot{\psi}_A - i\dot{x}\cdot A + \frac{i}{2}\mathcal{E}\psi_\mu F_{\mu\nu}\psi_\nu + \frac{1}{2}\mathcal{E}\Phi^2 - \frac{1}{2}\mathcal{E}\Pi^2 + 2i\mathcal{E}\psi_\mu\psi_5\dot{x}_\mu\Pi + \alpha\mathcal{E}\mathcal{K} \quad (B.8)$$

and the insertion operator

$$\mathcal{K} = -i\psi_6\psi_\mu D_\mu\Phi + \psi_5\psi_6\{\Pi,\Phi\}\ . \quad (B.9)$$

While this formulation continues to be gauge invariant, it appears asymmetrical in the way Π and Φ enter and thus chiral symmetry is not manifest.

Appendix D : Comparison with Perturbation Theory, the Wess-Zumino-Witten term

We make the fermion massive by giving the scalar field a vacuum expectation value, $\Phi = \varphi + m$. We shall now compute the path integral in (4.19) to order $\varphi^0\Pi^5A^0$ in the background fields. Only the leading order in the heavy mass limit is considered. Expanding the path integral gives

$$W_\Im[\Pi^5] = \frac{1}{8}\int_{p^1,\ldots,p^5}\int_{-1}^{1}d\alpha\int_0^\infty dT\ (-2\mathcal{E}m\times\mathcal{E}^4\times T^4)\int_0^1 du_1\cdots\int_0^{u_3}du_4\,\mathcal{Y}_{p,T}(\alpha,u_1,...,u_4)\ , \quad (D.1)$$

where explicitly,

$$\mathcal{Y}_{p,T}(\alpha,u_1,...,u_4) = p^1_{\mu_1}\cdots p^4_{\mu_4}\mathrm{Tr_c}(\tilde{\Pi}_1\cdots\tilde{\Pi}_5)\langle e^{ix_1\cdot p^1+\cdots+ix_4\cdot p^4+ix_0\cdot p^5}\rangle_B\ e^{-\frac{\mathcal{E}}{2}T\alpha^2m^2} \quad (D.2)$$
$$\times\int_{\mathrm{PBC}} d^6\psi\mathcal{D}\psi'\,\psi_{5,0}\psi_{6,0}\psi_{\mu_1,1}\psi_{5,1}\cdots\psi_{\mu_4,4}\psi_{5,4}\exp\Big(-\frac{1}{2}\int_0^T d\tau\psi_A\dot{\psi}_A\Big)\ .$$

The result of the evaluation of the bosonic correlation function followed by the T-integration is standard. Performing the new α-integration on this result gives

$$\frac{1}{(2\pi\mathcal{E})^2}\Gamma(3)\int_{-1}^{1}d\alpha\Big[\frac{\mathcal{E}}{2}\alpha^2m^2+\frac{\mathcal{E}}{2}\kappa(u_i,p^i)\Big]^{-3} = -2\times\frac{1}{5}\times\frac{2!}{(2\pi\mathcal{E})^2}\times(\frac{2}{\mathcal{E}m^2})^3+\mathrm{O}(\frac{1}{m^8})\ , \quad (D.3)$$

where $\kappa(u_i,p^i)$ is some function of the scaled propertimes and the momenta.

The fermionic zero mode integration vanishes unless the integrand contains a factor of $\psi_1\cdots\psi_6$, the integration of which is $+1$. To obtain such a factor requires that the Lorentz indeces of the momenta must all be different. For four different momenta, there are then 4! possible momentum combinations which can be expressed succinctly by the factor $\varepsilon_{\mu_1\mu_2\mu_3\mu_4}p^1_{\mu_1}p^2_{\mu_2}p^3_{\mu_3}p^4_{\mu_4}$. The remaining scaled propertime integrations of the fermionic correlation function becomes

$$\int_0^1 du_1\cdots\int_0^{u_3}du_4\ \langle\psi'_{5,2}\psi'_{5,3}\psi'_{5,4}\psi'_{5,0} \quad (D.4)$$
$$-\psi'_{5,1}\psi'_{5,3}\psi'_{5,4}\psi'_{5,0}+\psi'_{5,1}\psi'_{5,2}\psi'_{5,4}\psi'_{5,0}-\psi'_{5,1}\psi'_{5,2}\psi'_{5,3}\psi'_{5,0}+\psi'_{5,1}\psi'_{5,2}\psi'_{5,3}\psi'_{5,4}\rangle_F\ .$$

Using the appropriate version of (3.33), the first, third and fifth (odd) correlation functions each easily work out to be $(1/120 - 1/96)$ while the second and fourth (even) correlation functions similarly work out to be $(1/80 - 1/96)$. Subtracting the even correlation functions from the odd ones yields $-\frac{1}{96}$.

Putting everything together, (C.1) becomes

$$W_{\Im}[\Pi^5] = -\frac{1}{240\pi^2}\frac{1}{m^5}\varepsilon_{\mu_1\mu_2\mu_3\mu_4}\int\limits_{p^1,\ldots,p^5}(2\pi)^4\delta^{(4)}(p^1+\cdots+p^5)p^1_{\mu_1}p^2_{\mu_2}p^3_{\mu_3}p^4_{\mu_4}\,\mathrm{Tr_c}(\tilde{\Pi}_1\cdots\tilde{\Pi}_5)\;. \tag{D.5}$$

Now using (2.9) at order $\varphi^0\Pi^5 A^0$ gives (after much algebra)

$$W_{\Im}[\Pi^5] = -\frac{i}{5}\int\limits_{p^1,\ldots,p^5}(2\pi)^4\delta^{(4)}(p^1+\cdots+p^5)\,(-4im)\,\mathrm{Tr_c}(\tilde{\Pi}_1\cdots\tilde{\Pi}_5)\varepsilon_{\mu_1\cdots\mu_4}p^1_{\mu_1}\cdots p^4_{\mu_4}I'(p^i)\;. \tag{D.6}$$

The loop integral works out to be $\frac{2!}{(4\pi)^2 4!}\cdot\frac{1}{m^6}+\mathrm{O}(\frac{1}{m^8})$. Plugging the leading order contribution in the heavy mass limit back in reproduces (D.5).

The contribution to the imaginary part of the effective action evaluated here is the Wess-Zumino-Witten action [10] to lowest order in the fields. The coefficient is found to agree with an explicit Feynman diagram calculation in the large fermion mass limit, as given also in [11].

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