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A RELATION BETWEEN GRAVITY IN $(3+1)$–DIMENSIONS AND PONTRJAGIN TOPOLOGICAL INVARIANT

by

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ABSTRACT

A relation between the MacDowell-Mansouri theory of gravity and the Pontrjagin toplogical invariant in $(3+1)$ dimensions is discussed. This relation may be of especial interest in the quest of finding a mechanism to go from non-dynamical to dynamical gravity.

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Topology has been a fascinating subject in mathematics. The surprise for many mathematicians is that in the last few years topology has become also a very fascinating subject in theoretical physics. The main motivation to become theoretical physicist interested in topological aspects arises mainly from a number of articles written by Witten [1]-[4]. In particular the search for higher symmetries in the string theory [5]-[9] has lead Witten [10] into deeper topological aspects. One of the central ideas in Witten's work is to find a topological action from which the string theory may be derived. This means, roughy speaking, that starting from of a theory (topological field theory) with non-dynamical metric a theory (string theory) with dynamical metric may be derived. In spite of many attempts, however, it seems that this idea has not been completely achieved [11]-[13].

Motivated by this idea we became intrigued that in the MacDowell-Mansouri theory of gravity [14]-[15] the relation between dynamical and non-dynamical gravity depends on the choice of the metric associated to the de Sitter (or anti-de Sitter) group. In fact, if such a metric corresponds to the Killing metric the action of the MacDowell-Mansouri formalism gives the Pontrjagin topological invariant, while if the metric is chosen to be proportional to the Levi-Civita tensor in four dimensions the action becomes Einstein-Hilbert gravitational action with cosmological constant and Gauss-Bonnet term included. So, a natural question is whether it is possible to find a mechanism to go from the Killing metric to the metric which is proportional to the Levi-Civita tensor in the MacDowell-Mansouri approach. If the answer to this question is affirmative we could in principle apply similar procedure to the case of the string theory.

In this article we analize a algebraic transformation which translates precisely the Killing metric of the anti de Sitter group to the metric which is proportional to the Levi-Civita tensor. Such a transformation, in fact, allows a relation between the MacDowell-Mansouri action for gravity and the Pontrjagin topological invariant.

Let us first briefly review the MacDowell-Mansouri theory of gravity. Consider the Einstein-Hilbert action written in the tetrad formalism:

$$S_1 = -\int d^4\xi \; \epsilon^{\mu\nu\alpha\beta} e^a_\mu e^b_\nu R^{cd}_{\alpha\beta} \epsilon_{abcd}, \tag{1}$$

where ξ^α are "spacetime" coordinates; $\epsilon^{\mu\nu\alpha\beta}$ and ϵ_{abcd} are Levi-Civita tensors; the tetrad e^a_μ is related to the metric $g_{\mu\nu}$ by the expression $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$ and

$$R^{ab}_{\mu\nu} = \partial_\mu w^{ab}_\nu - \partial_\nu w^{ab}_\mu + \frac{1}{2} C^{ab}_{efgh} w^{ef}_\mu w^{gh}_\nu \tag{2}$$

is the Riemann curvature tensor written in terms of the gauge connection w^{ab}_μ. Here C^{ab}_{efgh} are the structure constants of the Lorentz group $SO(1,3)$. Notice, that we are using appropiate unites in order to avoid writting in (1) the usual constant factor $\frac{1}{8\pi G}$, where G is the Newton gravitational constant.

It is well know that it is possible to add to S_1 the cosmological constant term:

$$S_2 = \int d^4\xi \; \epsilon^{\mu\nu\alpha\beta} e^a_\mu e^b_\nu e^c_\alpha e^d_\beta \epsilon_{abcd}. \tag{3}$$

The cosmological constant factor in (3) may arise by writting the tetrad e^a_μ as λe^a_μ and by rescaling the total action $S_1 + S_2$ as $\lambda^{-2}(S_1 + S_2)$ (see ref. [15]).

The central idea behind the MacDowell-Mansouri formalism is to add to the action $S_1 + S_2$ the Gauss-Bonnet topological invariant

$$S_0 = \frac{1}{4}\int d^4\xi \ \epsilon^{\mu\nu\alpha\beta} R^{ab}_{\mu\nu} R^{cd}_{\alpha\beta} \epsilon_{abcd}. \tag{4}$$

Properly combining the integrals S_0, S_1 and S_2 it is not difficult to show that the action $S = S_0 + S_1 + S_2$ may be written as

$$S = \frac{1}{4}\int d^4\xi \ \epsilon^{\mu\nu\alpha\beta} \Re^{ab}_{\mu\nu} \Re^{cd}_{\alpha\beta} \epsilon_{abcd}, \tag{5}$$

where

$$\Re^{ab}_{\mu\nu} = R^{ab}_{\mu\nu} - (e^a_\mu e^b_\nu - e^a_\nu e^b_\mu). \tag{6}$$

Let us now make the identification

$$W^{4a}_\mu = e^a_\mu, \quad W^{ab}_\mu = w^{ab}_\mu. \tag{7}$$

Using these relations the curvature $\Re^{ab}_{\mu\nu}$ may be written as

$$\Re^{ab}_{\mu\nu} = \partial_\mu W^{ab}_\nu - \partial_\nu W^{ab}_\mu + \frac{1}{2} C^{ab} W_\mu W_\nu, \tag{8}$$

where the indices ... etc run from 0 to 4, and the only nonvanishing structure constants C^{ab} are

$$C^{ab} = \begin{cases} C^{ab}_{efgh} & \text{-Lorentz structure constants,} \\ & \\ C^{ab}_{4f4h} = & -\frac{1}{2}(\delta^a_f \delta^b_h - \delta^a_h \delta^b_f). \end{cases} \tag{9}$$

This extension of the structure constants ($C^{ab}_{efgh} \longrightarrow C^{ab}$) suggests an extension of the curvature

$$\Re^{ab}_{\mu\nu} \longrightarrow \Re_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + \frac{1}{2} C W_\mu W_\nu, \tag{10}$$

with

$$\Re^{4a}_{\mu\nu} = \partial_\mu W^{4a}_\nu - \partial_\nu W^{4a}_\mu + C^{4a}_{4fgh} W^{4f}_\mu W^{gh}_\nu + C^{4a}_{ef4h} W^{ef}_\mu W^{4h}_\nu, \tag{11}$$

which in virtue of the relation $W^{4a}_\mu = e^a_\mu$ may be identified as the torsion

$$T^a_{\mu\nu} \equiv \Re^{4a}_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + w^a_{\mu b} e^b_\nu - w^a_{\nu b} e^b_\mu. \tag{12}$$

The quantities

$$C = \frac{1}{2}[\delta\delta\eta - \delta\delta\eta - \delta\delta\eta + \delta\delta\eta] - \frac{1}{2}[\leftrightarrow], \tag{13}$$

may be identified, now, with the structure constants of the anti-de Sitter group $S(2,3)$ (a similar result may be obtained in the case of the de Sitter group $SO(1,4)$).

Finally, let us introduce the quantity

$$g = \begin{cases} g_{abcd} = \epsilon_{abcd}, \\ \\ g_{4a4b} = -\eta_{ab}, \\ \\ 0 \ otherwise. \end{cases} \tag{14}$$

Notice that $g = -g = -g = +g$.

Making the torsion $T^a_{\mu\nu} = 0$ we find that the action (4) may be written as

$$S = \frac{1}{4}\int d^4\xi\ \epsilon^{\mu\nu\alpha\beta}\Re_{\mu\nu}\Re_{\alpha\beta}g. \tag{15}$$

An important feature of this action is that it is independent of the metric $g_{\mu\nu}$ and the Christoffel simbols $\Gamma^{\mu}_{\alpha\beta}$ and it depends only on the gauge field $W_\mu = -W_\mu$ associated to the anti-de Sitter group $SO(2,3)$ (or the de Sitter group $SO(1,4)$).

The action (15) is so similar to the Pontrjagin topological invariant action

$$= \frac{1}{8}\int d^4\xi\ \epsilon^{\mu\nu\alpha\beta}{}_{\mu\nu\alpha\beta}G, \tag{16}$$

where

$${}_{\mu\nu} = \partial_{\mu}{}_{\nu} - \partial_{\nu}{}_{\mu} + \frac{1}{2}f{}_{\mu\nu} \tag{17}$$

and

$$G = (\eta\eta - \eta\eta) \tag{18}$$

that we become intrigued if there is some kind of transformation

$$\Re_{\mu\nu} \leftrightarrow {}_{\mu\nu} \tag{19}$$

and

$$g \leftrightarrow G \tag{20}$$

such that

$$S \leftrightarrow . \tag{21}$$

In the expression (17) the quantities f are the structure constants related, of course, to the anti-de Sitter group (or the de Sitter group).

In order to find such a transformation we first expand (15) and (16) as follows:

$$S = \frac{1}{4}\int d^4\xi\ \epsilon^{\mu\nu\alpha\beta}\Re^{ab}_{\mu\nu}\Re^{cd}_{\alpha\beta}\epsilon_{abcd} - \int d^4\xi\ \epsilon^{\mu\nu\alpha\beta}T^a_{\mu\nu}T^b_{\alpha\beta}\eta_{ab} \tag{22}$$

and

$$= \frac{1}{8}\int d^4\xi\ \epsilon^{\mu\nu\alpha\beta}{}^{ab}_{\mu\nu}{}^{cd}_{\alpha\beta}(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}) - \frac{1}{2}\int d^4\xi\ \epsilon^{\mu\nu\alpha\beta}{}^{4a}_{\mu\nu}{}^{4b}_{\alpha\beta}\eta_{ab}. \tag{23}$$

Of course, in the MacDowell-Mansouri approach $T^a_{\mu\nu} = 0$, but for the moment let us consider that $T^a_{\mu\nu} \neq 0$. At this respect we should mention important aspects. Since we are considering here pure gravity the necessity to impose the constraint $T^a_{\mu\nu} = 0$ arises if we want the action (22) to be consistent with Einstein's gravitational therory. A remarkable feature of the action (22) is that by eliminating the second integral, the first integral precisely reproduce the constraint $T^a_{\mu\nu} = 0$ under variation. We note, however, that this compatibility holds at the classical level. This is an important observation which should be carefully considered at the quantum level (see ref. [14] for more details).

From the two expressions (22) and (23) we see that in principle $T^{a}_{\mu\nu}$ may be identified with ^{4a} and the first term in S may be identified with the first term in . So, let us first concentrate our attention in the integrals:

$$\hat{S} = \frac{1}{4}\int d^4\xi\ \epsilon^{\mu\nu\alpha\beta}\Re^{ab}_{\mu\nu}\Re^{cd}_{\alpha\beta}\epsilon_{abcd} \tag{24}$$

and

$$\hat{} = \frac{1}{8}\int d^4\xi\ \epsilon^{\mu\nu\alpha\beta}{}^{ab}_{\mu\nu}{}^{cd}_{\alpha\beta}(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}). \tag{25}$$

In order to find a relation between these two integrals it is convenient to introduce the quantity

$${}^{\pm}N^{ab}_{cd} = \frac{1}{2}(\delta^{ab}_{cd} \pm \epsilon^{ab}{}_{cd}), \tag{26}$$

where

$$\delta^{ab}_{cd} = \delta^{a}_{c}\delta^{b}_{d} - \delta^{a}_{d}\delta^{b}_{c}. \tag{27}$$

We can check that this quantity satisfies the relations

$$\frac{1}{2}{}^{\pm}N^{ab}_{ef}{}^{\pm}N^{cd}_{gh}(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}) = \pm\epsilon_{efgh}, \tag{28}$$

$${}^{+}N^{ab}_{cd}{}^{-}N^{cd}_{ef} = \delta^{ab}_{ef}. \tag{29}$$

Thus, using (28) we find that $\hat{S}$ becomes

$$\hat{S} = \frac{1}{8}\int d^4\xi\ \epsilon^{\mu\nu\alpha\beta}\Re^{ab}_{\mu\nu}\Re^{cd}_{\alpha\beta}{}^{+}N^{ef}_{ab}{}^{+}N^{gh}_{cd}(\eta_{eg}\eta_{fh} - \eta_{eh}\eta_{fg}). \tag{30}$$

Terefore, the problem to obtain $\hat{}$ from $\hat{S}$ may be accomplished if we consider the transformation

$${}^{ab}_{\mu\nu} = {}^{+}N^{ab}_{cd}\Re^{cd}_{\mu\nu}, \tag{31}$$

because then

$$\hat{S} = \hat{}. \tag{32}$$

However, the solution is not so simple because in order to have (31) we need to find the relations;

$$w^{ab}_{\mu} \mapsto {}^{ab}_{\mu},$$

$$\tag{33}$$

$$w^{4a}_{\mu} \mapsto {}^{4a}_{\mu},$$

and

$$C^{ab} \mapsto f^{ab}. \tag{34}$$

From the definitions of ${}^{ab}_{\mu\nu}$ and $R^{ab}_{\mu\nu}$ we find that (31) follows if the relations

$${}^{ab}_{\mu} = {}^{+}N^{ab}_{cd}w^{cd}_{\mu}, \tag{35}$$

$$f^{ab}_{efgh} = \frac{1}{4}{}^{+}N^{ab}_{cd}C^{cd}_{mnij}{}^{-}N^{mn}_{ef}{}^{-}N^{ij}_{gh}, \tag{36}$$

$$f^{ab}_{4f4h} = \frac{1}{2}{}^{+}N^{ab}_{cd}C^{cd}_{4f4h}, \tag{37}$$

and

$${}^{4a}_{\mu} = \sqrt{2}\ w^{4a}_{\mu} \tag{38}$$

are satisfied. The relation (35) follows directly from (31). The expressions (36), (37), and (38) , however, are more difficult to obtain and require special attention to numerical factors. We note that in order to obtain (36) we used (29). It is interesting to observe the important role played by the quantity (26) and the relations (28) and (29); without these relations to find the expressions (35)-(36) would be very much difficult.

It remains to clarify the meaning of the structure constants f. What we know is that C are the structure constant of the anti-de Sitter group $SO(3,2)$ whose generators S satisfy the algebra:

$$[S, S] = CS. \tag{39}$$

This algebra can be broken as follows

$$[S_{ef}, S_{gh}] = C^{ab}_{efgh}S_{ab}, \tag{40}$$

$$[S_{4f}, S_{gh}] = 2C^{4b}_{4fgh}S_{4b}, \tag{41}$$

$$[S_{4f}, S_{4h}] = C^{ab}_{4f4h}S_{ab}. \tag{42}$$

The first bracket (40) may be multiplied by $\frac{1}{4}{}^{-}N^{ef}_{ij}{}^{-}N^{gh}_{kl}$ in order to obtain

$$[{}^{-}S_{ij}, {}^{-}S_{kl}] = \frac{1}{4}{}^{+}N^{ab}_{rs}C^{rs}_{efgh}{}^{-}N^{ef}_{ij}{}^{-}N^{gh}_{kl}{}^{-}S_{ab}, \tag{43}$$

where we used (29) and the definition ${}^{-}S_{ij} = \frac{1}{2}{}^{-}N^{kl}_{ij}S_{kl}$. From this expression we see that the structure constants f^{ab}_{efgh} given in (36) precisely corresponds to the factor in front of ${}^{-}S_{ab}$ in (43). In fact, using (36) we get

$$[{}^{-}S_{ij}, {}^{-}S_{kl}] = f^{rs}_{ijkl}{}^{-}S_{rs}. \tag{44}$$

Similarly, multiplying (41) by $\frac{1}{2\sqrt{2}}{}^{-}N^{gh}_{ij}$ and defining ${}^{-}S_{4a} = \frac{1}{\sqrt{2}}S_{4a}$ we find

$$[{}^{-}S_{4f}, {}^{-}S_{ij}] = 2(\frac{1}{2}{}^{-}N^{gh}_{ij}C^{4b}_{4fgh}{}^{-}S_{4b}) \tag{45}$$

which suggests to write

$$f^{4b}_{4fij} = \frac{1}{2}{}^{-}N^{gh}_{ij}C^{4b}_{4fgh}. \tag{46}$$

Further, the bracket (42) can be written as

$$[{}^{-}S_{4f}, {}^{-}S_{4h}] = \frac{1}{2}C^{ab}_{4f4h}{}^{+}N^{ij}_{ab}{}^{-}S_{ij}, \tag{47}$$

with

$$f^{ij}_{4f4h} = \frac{1}{2}{}^{+}N^{ij}_{ab}C^{ab}_{4f4h}. \tag{48}$$

Therefore, the anti-de Sitter algebra (39) may be rewritten as

$$[{}^{-}S, {}^{-}S] = f\, {}^{-}S, \tag{49}$$

where ${}^{-}S_{ab} = \frac{1}{2} {}^{-}N_{ab}^{ef} S_{ef}$ and ${}^{-}S_{4a} = \frac{1}{\sqrt{2}} S_{4a}$. Consequently (49) implies that the structure constants f also correspond to the anti-de Sitter group.

With all these results we can see that, the connection between the integrals

$$\hat{S}_T = - \int d^4\xi \; \epsilon^{\mu\nu\alpha\beta} T^a_{\mu\nu} T^b_{\alpha\beta} \eta_{ab} \tag{50}$$

and

$$\hat{}_T = -\frac{1}{2} \int d^4\xi \; \epsilon^{\mu\nu\alpha\beta} {}^{4a}_{\mu\nu} {}^{4b}_{\alpha\beta} \eta_{ab} \tag{51}$$

is straightforward. In fact, we find $T^a_{\mu\nu} = \frac{1}{\sqrt{2}} {}^{4a}_{\mu\nu}$.

To conclude let us make the following comments. The message given by this work may be expressed as follows: Start with Einstein-Hilbert action, add a cosmological constant term and Gauss-Bonnet topological invariant form, transform each term according to (31) and finally add a torsion term breaking reflexion what we get it is the Pontrjagin topological invariant. Or inversely, starting with Pontrjagin topological invariant make the torsion vanishes then make the transformation (31) and eliminate the Gauss-Bonnet topological invariant and the cosmological constant what we get it is the Eistein-Hilbert action. In other words, the transition from a non-dynamical to dynamical gravity depends of making the torsion zero and making appropriate transformation using ${}^{+}N^{ab}_{cd}$. Finally, it is interesting that all this mechanism has been derived classically. Of course, It will be very interesting to find a similar dynamical mechanism and to exploite such a dynamical mechanism at the quantum level. But at the present time we do not see how to achieve this goal. Nevertheless, we think that the present work may be very useful in that direction. Further, since the Pontrjagin topological invariant may be related to Chern-Simons of gauge group $SO(2,3)$ which at the same time is related to $(1+1)$-conformal field theory [16]-[19] we think our work may be thought as bridge between gravity in $(3+1)$-dimensions and $(1+1)$-conformal field theory. Since $(1+1)$-conformal theory is closely related to string theory it seems that our work suggests that string theory may be obtained from gravity!

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