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Covariant Thermodynamics and "Realistic" Friedmann Model

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Abstract

We discuss a cosmological Friedmann model modified by inclusion of off-shell matter which has an equation of state $p, \rho \propto T^5$, $p = 1/4\rho$. Such matter is shown to have energy density comparable with that of non-interacting radiation at temperatures of the order of the Hagedorn temperature, $\sim 10^{12}$ K, indicating the possibility of a phase transition. It is argued that the T^5-phase, or an admixture, lies below the high-temperature T^4-phase.

Key words: relativistic mechanics, relativistic thermodynamics, off-shell matter, cosmology, Friedmann model

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1 Introduction

The cosmological equations (without cosmological constant) for a uniform universe with the space-time metric

$$-ds^2 = -dt^2 + R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{1.1}$$

are [1]

$$\left(\frac{dR}{dt}\right)^2 = \frac{8\pi G}{3}\rho R^2 - k, \tag{1.2}$$

$$R^3\frac{dp}{dt} = \frac{d}{dt}\left[R^3(\rho c^2 + p)\right], \tag{1.3}$$

where $R \equiv R(t)$ scales the comoving coordinates, $\rho(t)$ is the energy density, $p(t)$ the pressure, $k = 0, \pm 1$ the curvature constant and t is a universal cosmic time. The Friedmann models [2] have the properties: $p \simeq 0,\ k = 0, \pm 1$ ($k = 0$ is in fact the Einstein-de Sitter model [3]), and were improved and made more realistic by Lemaître [4] who included non-interacting radiation. If one denotes by subscripts m and r matter and radiation, respectively, one finds from (1.3), for non-interacting matter and radiation: $\rho_m \propto R^{-3},\ \rho_r \propto R^{-4}$, for $p_m << \rho_m c^2,\ p_r = \rho_r c^2/3$. Eq. (1.2) becomes

$$\left(\frac{dR}{dt}\right)^2 = \alpha_r R^{-2} + \alpha_m R^{-1} - k, \tag{1.4}$$

$$\alpha_r = 8\pi G\rho_r R^4/3, \quad \alpha_m = 8\pi G\rho_m R^3/3. \tag{1.5}$$

Eq. (1.4) is just the Lemaître equation. Its solution for $k = 1$ was given by de Sitter [5], Tolman [6] and Alpher and Herman [7], for $k = 0, \pm 1$ by Chernin [8] and Cohen [9], and for $k = 0$ by Jacobs [10]. Cosmological models containing both matter and radiation were also discussed by McIntosh [11] and Harrison [12].

For non-interacting matter and radiation, $\rho_r \propto T^4$ and hence $T \propto R^{-1}$; and $\rho_m \propto n \propto T^3$, where n is the mean matter number density. Therefore, the dimensionless quantity

$$\eta = n\left(\frac{\hbar c}{k_B T}\right)^3 = n^{(0)}\left(\frac{\hbar c}{k_B T^{(0)}}\right)^3 \tag{1.6}$$

is constant. Let us evaluate this quantity by assuming that we are dealing with baryonic matter.

There are about 10^{57} nucleons in a typical star. There are about 10^{11} galaxies in the universe, and each galaxy has about 10^{11} stars. Thus there are about 10^{79} baryons in the universe (in comparison with 10^{89} photons, a number which is obtained by thermodynamic arguments). The present size of the observable universe is 10^{28} cm. The baryon number density is therefore given by

$$n_b \sim \frac{10^{79}}{\frac{4\pi}{3}(10^{28})^3} \sim 10^{-6}\ \mathrm{cm}^{-3},$$

which we take for $n^{(0)}$. Taking $T^{(0)} \simeq 2.7$ K (the microwave background), one obtains $\eta \sim 10^{-9}$. Let $\beta = \rho_r^{(0)}/\rho_m^{(0)}$ (zero superscript denotes the present epoch); then because $\rho_r c^2 \sim k_B T (k_B T/\hbar c)^3$,

$$\beta \sim \frac{k_B T^{(0)}}{\eta m_n c^2} \sim 10^{-3}, \tag{1.7}$$

where m_n is the nucleon mass. Let the superscript (1) denotes the epoch where $\rho_r \simeq \rho_m$; then

$$\frac{\alpha_r}{\alpha_m} = R^{(1)} = \beta R^{(0)}, \tag{1.8}$$

and $\rho_r^{(1)} = \rho_m^{(0)}\beta^{-3} \sim 10^9 \rho_m^{(0)}$. Also, $T^{(1)} = T^{(0)}/\beta \simeq 3000$ K, and when $T > T^{(1)}$, the energy density of radiation exceeds that of matter.

The numerical values of the energy densities are [13]

$$\rho_m^{(0)} \sim 2 \cdot 10^{-31}\ \mathrm{g\ cm}^{-3}, \quad \rho_r^{(0)} \sim 4 \cdot 10^{-34}\ \mathrm{g\ cm}^{-3}, \tag{1.9}$$

$$\rho_r^{(1)} \sim 10^{-22}\ \mathrm{g\ cm}^{-3}. \tag{1.10}$$

It follows that the temperature dependence of the energy densities can consistently have the form

$$\rho_m(T) \simeq 10^{-31} \left(\frac{T}{3}\right)^3 \mathrm{g\ cm}^{-3}, \quad \rho_r(T) \simeq 10^{-22} \left(\frac{T}{3 \cdot 10^3}\right)^4 \mathrm{g\ cm}^{-3}. \tag{1.11}$$

As remarked by Harrison [12], when $T > T^{(2)} \sim m_e c^2/k_B \simeq 5 \cdot 10^9$ K (m_e is the electron mass), the Lemaître equation breaks down because of lepton and hadron pair production. At temperatures $T \gtrsim T^{(2)}$ the interaction of the hadrons is strong and has mainly a resonant character, the masses of the resonances being comparable with the temperature. Under these conditions hadronic matter is neither an ideal nor an ultra-relativistic gas, and can be well characterized by a resonance spectrum [14]

$$\tau(m) \sim m^a \exp{(m/T_0)}, \tag{1.12}$$

where a and T_0 are parameters. The statistical sum diverges when $T > T_0$, which indicates that the theory involves a limiting temperature (the so-called "hadronic boiling point" of Hagedorn [14]), whose numerical value is found to be

$$T_0 = m_\pi c^2/k_B \sim 10^{12}\ \mathrm{K} \tag{1.13}$$

(m_π is the pion mass), and in the vicinity of T_0 particle creation is so violent that it is impossible to exceed this temperature. In the temperature range $T^{(2)} \lesssim T \lesssim T_0$ the hadronic gas is thought of as being well described by the "realistic" equation of state suggested by Shuryak [15], $p, \rho \propto T^6$ (which gives the value of the velocity of sound $c_s^2 = dp/d\rho = 0.20$), in agreement with some of the relevant experimental data [16, 17, 18]. There are indications that c_s^2 may, however, take the values both more and less than 0.20, i.e., 0.25 [19, 20, 21] or 0.17 [18, 22, 23, 24].

The existence of the limiting temperature T_0 suggests that there is some phase transition. It also means that the Lemaître equation, although reasonably realistic, is a good approximation throughout the lifetime of the universe *except* for its earliest moments.

It is clear that, in fact, the known resonances form an essentially discrete set of states, and the well-known arguments applied above are based on an approximate idealization that considers the envelope of these resonances as a continuous mass spectrum. There is, however, a consistent (proper time) formulation of a manifestly covariant statistical mechanics [25, 26, 27], based on the ideas of Fock [28] and Stueckelberg [29], in which the four components of energy-momentum are considered as independent degrees of freedom, permitting fluctuations from the mass shell.

In the present paper we shall use this manifestly-covariant framework (which we review briefly in the next section) as a model for the description of phenomena which take place in hot hadronic matter. We argue that the phase transition at the Hagedorn limiting temperature can represent a phase transition from an off-shell sector of the theory, in which the relativistic ensemble is described within the framework of the manifestly covariant relativistic statistical mechanics mentioned above, to an ultrarelativistic independent particle phase. Recently we have studied thermodynamic properties of the off-shell phase [27, 30] and its possible consequences in hadronic physics [31], astrophysics [32], and cosmology [33, 34].

As shown in ref. [31], the behavior of hadronic matter below the Hagedorn limiting temperature coincides with that of a system which includes both particles and antiparticles, with the additional mass potential [25] $\mu_K \simeq 0$; such a system, within the covariant framework, is described by the equation of state corresponding to Shuryak's "realistic" one. In the present paper we shall show that off-shell matter, if included into the Friedmann-Lemaître model on an equal footing with baryonic matter and non-interacting radiation, has energy density comparable to that of non-interacting radiation at temperatures of the order of the Hagedorn temperature, indicating the possibility of a phase transition from strongly interacting (off-shell) phase to non-interacting one.

2 Relativistic N-body system

In the framework of a manifestly covariant relativistic statistical mechanics, the dynamical evolution of a system of N particles, for the classical case, is governed by equations of motion that are of the form of Hamilton equations for the motion of N *events* which generate the space-time trajectories (particle world lines) as functions of a continuous Poincaré-invariant parameter τ, called the historical time [29, 35]. These events are characterized by their positions $q^\mu = (t, \mathbf{q})$ and energy-momenta $p^\mu = (E, \mathbf{p})$ in an $8N$-dimensional phase-space. For the quantum case, the system is characterized by the wave function $\psi_\tau(q_1, q_2, \ldots, q_N) \in L^2(R^{4N})$, with the measure $d^4q_1 d^4q_2 \cdots d^4q_N \equiv d^{4N}q$ ($q_i \equiv q_i^\mu$; $\mu = 0, 1, 2, 3$; $i = 1, 2, \ldots, N$), describing the distribution of events, which evolves with a generalized Schrödinger equation [35]. The collection of events (called "concatenation" [36]) along each world line corresponds to a *particle,* and hence, the evolution of the state of the N-event system describes, *a posteriori,* the history in space and time of an N-particle system.

For a system of N interacting events (and hence, particles) one takes [35] (we use the metric $g^{\mu\nu} = (-,+,+,+)$)

$$K = \sum_i \frac{p_i^\mu p_{i\mu}}{2M} + V(q_1, q_2, \ldots, q_N), \tag{2.1}$$

where M is a given fixed parameter (an intrinsic property of the particles), with the dimension of mass, which we take to be the same for all the particles of the system. The Hamilton equations are

$$\frac{dq_i^\mu}{d\tau} = \frac{\partial K}{\partial p_{i\mu}} = \frac{p_i^\mu}{M},$$
$$\frac{dp_i^\mu}{d\tau} = -\frac{\partial K}{\partial q_{i\mu}} = -\frac{\partial V}{\partial q_{i\mu}}. \tag{2.2}$$

In the quantum theory, the generalized Schrödinger equation

$$i\frac{\partial}{\partial \tau}\psi_\tau(q_1, q_2, \ldots, q_N) = K\psi_\tau(q_1, q_2, \ldots, q_N) \tag{2.3}$$

describes the evolution of the N-body wave function $\psi_\tau(q_1, q_2, \ldots, q_N)$.

2.1 Covariant thermodynamics and cosmology

Thermodynamic functions for a many-body system can be derived from the grand partition function, which, for an ensemble of off-shell events at temperature T is defined by the following expression, modified, in comparison with the standard one, by the presence of the term $\mu_K K$ (henceforth we use the system of units in which $\hbar = c = k_B = 1$, unless the other specified):

$$Z = Tr\left[\exp\left\{-(\hat{E}^{(N)} - \mu\hat{N} - \mu_K\hat{K}^{(N)})/T\right\}\right], \tag{2.4}$$

where μ is the chemical potential and μ_K is the additional mass potential of the ensemble [25]. Here $\hat{E}^{(N)}$ is the operator of the total energy of the N-body system, $\hat{E}^{(N)} = \sum_{i=1}^N p_i^0$, $\hat{K}^{(N)}$ is the generalized N-body Hamiltonian defined by Eq. (2.1), and $\hat{N}$ is the operator of the number of events (and therefore, particles). As shown in ref. [33] (we shall review this point briefly below), the quantity $-\mu_K\kappa$, where κ is the generalized Hamiltonian density, $\kappa = \langle\hat{K}^{(N)}\rangle/V$, represents a (55)-component of a generalized energy-momentum tensor, which in the local rest frame takes on the form $T_{\alpha\beta} = \mathrm{diag}(-\rho, p, p, p, -\sigma\mu_K\kappa)$, $\quad \alpha,\beta = 0,1,2,3,5$. Here the sign of σ corresponds to the invariance group of the extended (x^μ,τ) manifold which could be $O(4,1)$ $(\sigma = 1)$ or $O(3,2)$ $(\sigma = -1.)$ In the present paper we shall restrict ourselves to the case $\sigma = 1$, i.e., the $O(4,1)$ invariance of the (x^μ,τ) manifold having the metric $g^{\alpha\beta} = (-,+,+,+,+)$.

Expressions for p and ρ, using the grand canonical ensemble obtained by Horwitz, Schieve and Piron [25] in their study of manifestly covariant statistical mechanics, were found in [30] in terms of confluent hypergeometric functions as

$$p = \frac{T_{\triangle V}}{4\pi^3}\frac{M^2}{\mu_K^3}T^3\sum_{s=1}^\infty \frac{(\pm 1)^{s+1}}{s^3}e^{s\mu/T}\Psi(3,3;\frac{sM}{2\mu_K T}), \tag{2.5}$$

$$p+\rho = \frac{3T_{\triangle V}}{4\pi^3}\frac{M^3}{\mu_K^4}T^2\sum_{s=1}^\infty \frac{(\pm 1)^{s+1}}{s^2}e^{s\mu/T}\Psi(4,4;\frac{sM}{2\mu_K T}), \tag{2.6}$$

where $T_{\triangle V}$ is a characteristic interval of τ for a trajectory to pass through a small representative four-volume. For T small, one finds from (2.5),(2.6) [30, 33] that $p,\rho \propto T^6$, $\rho \simeq 5p$, and, in fact, that $\mu_K\kappa \propto T^7$ is negligible in comparison with ρ. On the other hand, for T large, it follows from these expressions that $p,\rho,\mu_K\kappa \propto T^5$, $p \simeq 1/4\rho \simeq -\mu_K\kappa$. Now, we remark that the energy-momentum tensor

$$T^{\mu\nu} = (p+\rho)u^\mu u^\nu - pg^{\mu\nu} \tag{2.7}$$

can be extended to a five-dimensional form

$${}^{(5)}T_{\alpha\beta} = \left({}^{(5)}T_{\mu\nu},\ {}^{(5)}T_{55}\right); \tag{2.8}$$

the requirement that the limiting case of the corresponding gravitational theory (for zero curvature in the τ direction) coincide with the Einstein equations results in the identification ${}^{(5)}T_{\mu\nu} = {}^{(4)}T_{\mu\nu}$ and ${}^{(5)}T_{55} = -\mu_K\kappa$. For high temperature, it therefore follows that (as discussed in [33])

$$T^{\alpha\beta} = (p+\rho)u^\alpha u^\beta - pg^{\alpha\beta}, \quad u^\lambda u_\lambda = -1, \tag{2.9}$$

so that, in the local rest frame, $T_{\alpha\beta} = \mathrm{diag}\,(-\rho,p,p,p,p)$.

The first law of thermodynamics reads

$$dE = TdS - pdV + \mu_i dN_i, \tag{2.10}$$

where S is the entropy, $V = V(t)$ (the universal cosmic time t corresponds to the τ of (2.2) and (2.3)) is a comoving volume element of an ideally uniform universe, and N_i is the number of the ith kind of particles having the chemical potential μ_i, which implies that the energy E is a thermodynamic function of S, V, and $N_i : E = E(S, V, N_i)$. In the case of varying N_i it is convenient to consider the set of variables (T, V, μ_i) instead of (S, V, N_i). The change of variables is obtained with the help of a Legendre transformation

$$\Omega(T, V, \mu_i) = E - TS - \mu_i N_i, \tag{2.11}$$

which introduces the thermodynamic potential Ω. The use of (2.10) in the differential of (2.11) leads to

$$d\Omega = -SdT - pdV - N_i d\mu_i. \tag{2.12}$$

The coefficients S, p and N_i are given by the partial derivatives

$$S = -\left(\frac{\partial\Omega}{\partial T}\right)_{V,\mu_i} \quad ; \quad p = -\left(\frac{\partial\Omega}{\partial V}\right)_{T,\mu_i} \quad ; \quad N_i = -\left(\frac{\partial\Omega}{\partial \mu_i}\right)_{T,V}. \tag{2.13}$$

A fundamental result of statistical mechanics relates the thermodynamic potential to the grand partition function:

$$\Omega(T, V, \mu_i) = -T \ln Z. \tag{2.14}$$

By combining (2.13) with (2.14), the entropy and the pressure become

$$S = \frac{\partial(T \ln Z)}{\partial T}, \tag{2.15}$$

$$p = \frac{\partial(T \ln Z)}{\partial V}. \tag{2.16}$$

In the case of no interactions the latter relation reduces to

$$p = \frac{T}{V} \ln Z = -\frac{\Omega}{V}, \tag{2.17}$$

which is the equation of state of a free relativistic ensemble [25]

$$\frac{pV}{T} = \ln Z. \tag{2.18}$$

Substitution of $\Omega = -pV$ into Eq. (2.11) yields the thermodynamic relation

$$E = -pV + TS + \mu_i N_i, \tag{2.19}$$

which represents the formula for the entropy [37]

$$S = \frac{E + pV - \mu_i N_i}{T} = \frac{(\rho + p - \mu_i N_{0i})V}{T}, \tag{2.20}$$

where $\rho \equiv E/V$, $N_{0i} \equiv N_i/V$ are the corresponding energy and particle number densities.

Since the lepton and baryon numbers are relatively small and therefore negligible in the early universe, equilibrium conditions correspond to

$$\mu_i = 0. \tag{2.21}$$

Then the first law reads

$$d(\rho V) = TdS - pdV, \tag{2.22}$$

and Eq. (2.20) takes on the form

$$S = \frac{(\rho + p)V}{T}. \tag{2.23}$$

Since $V \sim R^3$, it follows from (1.3) and (2.22) that

$$\frac{dS}{dt} = 0, \tag{2.24}$$

and the entropy is conserved. Note that Eq. (1.3) represents the local energy conservation. Indeed, for the energy-momentum tensor of an ideal cosmological fluid,

$$T^{\mu\nu} = (p + \rho)u^\mu u^\nu - pg^{\mu\nu}, \quad u^\rho u_\rho = -1,$$

the local energy conservation, $\nabla_\mu T^{\mu\nu} = 0$, takes on the form

$$R\frac{d\rho}{dt} + 3(\rho + p)\frac{dR}{dt} = 0, \tag{2.25}$$

which reduces to (1.3).

By rewriting (2.22) in the form

$$TdS = d\left[(\rho + p)V\right] - Vdp \tag{2.26}$$

and using the integrability condition [37]

$$\frac{\partial^2 S}{\partial T \partial V} = \frac{\partial^2 S}{\partial V \partial T}, \tag{2.27}$$

the energy density and pressure are related as follows,

$$\rho + p = T\frac{dp}{dT}. \tag{2.28}$$

Given an equation of state, the temperature dependence of p and ρ can be derived with the help of (2.28). Assuming that in the early universe the pressure is proportional to the energy density and using Zeldovich's equation [38]

$$p = (\Gamma - 1)\rho, \tag{2.29}$$

we find

$$p, \rho \propto T^{\Gamma/(\Gamma-1)}. \tag{2.30}$$

In the standard framework one can obtain from (2.25),(2.29),

$$p, \rho \propto R^{-3\Gamma}. \tag{2.31}$$

For the radiation-dominated universe $\Gamma = 4/3$, so that

$$p, \rho \propto T^4 \propto R^{-4}. \tag{2.32}$$

In the covariant framework Eq. (2.28), and therefore (2.30), hold, while Eq. (2.25) should be modified. For the generalized energy-momentum tensor (2.8) the local energy conservation, $\nabla_\alpha T^{\alpha\beta} = 0$, yields

$$R\frac{d\rho}{dt} + 4(\rho + p)\frac{dR}{dt} = 0, \tag{2.33}$$

resulting, via (2.29), in

$$p, \rho \propto R^{-4\Gamma}. \tag{2.34}$$

Since in this case [33] $\Gamma = 5/4$, we obtain from (2.30),(2.34)

$$p, \rho \propto T^5 \propto R^{-5}. \tag{2.35}$$

3 "Realistic" Friedmann model

Let us modify the Friedmann-Lemaître model by inclusion of off-shell matter with the equation of state[1] $\rho_{m'} \propto T^5 \propto R^{-5}$, $p_{m'} = \rho_{m'}c^2/4$. The modified equation reads

$$\left(\frac{dR}{dt}\right)^2 = \alpha_{m'}R^{-3} + \alpha_r R^{-2} + \alpha_m R^{-1} - k, \tag{3.1}$$

$$\alpha_{m'} = 8\pi G \rho_{m'} R^5/3, \tag{3.2}$$

α_r, α_m being defined in (1.5).

Let $\beta' = \rho_{m'}^{(1)}/\rho_r^{(1)}$, where, as previously, the superscript (1) denotes the epoch of $T^{(1)} \simeq 3000$ K. Because $\rho_{m'}c^2 \sim k_B T(k_B T/\hbar c)^4$,

$$\beta' \sim \frac{k_B T^{(1)}}{\rho_r^{(1)}c^2} = \frac{k_B T^{(0)}/\beta}{\rho_m^{(0)}c^2/\beta^3} = \beta^2 \frac{k_B T^{(0)}}{\rho_m^{(0)}c^2} = \beta^3 \sim 10^{-9}. \tag{3.3}$$

[1] The equation of state of relativistic off-shell matter $p = 1/4\rho$ was obtained also by Hakim [39] within a different framework.

If superscript (3) denotes the epoch where $\rho_{m'} \simeq \rho_r$ (implying that the pressure of radiation and off-shell matter are equal; from (1.11) one sees that the contribution of baryonic matter is negligible at high temperature), then

$$\frac{\alpha_{m'}}{\alpha_r} = R^{(3)} = \beta' R^{(1)}, \tag{3.4}$$

and

$$\rho_{m'}^{(3)} = \rho_r^{(1)}(\beta')^{-4} \sim 10^{36}\rho_r^{(1)} \sim 10^{14} \text{ g cm}^{-3}. \tag{3.5}$$

Also

$$T^{(3)} = \frac{T^{(1)}}{\beta'} \simeq 3 \cdot 10^{12} \text{ K}. \tag{3.6}$$

We see that this temperature is just of the order of the Hagedorn one. At $T \simeq T^{(3)}$, off-shell matter and radiation have comparable energy densities, indicating the possibility of a phase transition. In general,

$$\rho_{m'}(T) \simeq 10^{14} \left(\frac{T}{3 \cdot 10^{12}}\right)^5 \text{ g cm}^{-3}. \tag{3.7}$$

Since the cosmological phase transition at $T_c \sim 10^{12}$ K is normally associated with the transition from a strongly interacting hadronic phase to a weakly interacting quark-gluon plasma phase [24, 40], we associate the T^4-phase above the transition temperature with a phase of of weakly interacting quarks and gluons, and T^5-phase below the transition temperature with a phase of strongly interacting hadrons (as we show in [41], T^4 behavior indeed results from the very high temperature asymptotic behavior of the distribution function, for small μ_K.) In this phase, particles undergo continual mutual interaction and are necessarily off-shell. Therefore, the effect of strong interaction in such a system may be represented by the off-shellness of its particles, justifying the use of the off-shell framework for the description of this phase.

If the phase transition at temperature $\sim T^{(3)}$ is not sufficiently sharp, there exists some region of the "mixed" phase, in which both (or the three, including baryonic matter) phases coexist (a similar situation may occur for both hadronic and the quark-gluon plasma phases, according to the "realistic" scenario suggested by Shuryak [15]). If the off-shell phase extended well below $T^{(3)}$, it would have energy density comparable with that of baryonic matter at $T \sim 10^8$ K, as follows from (1.11) and (3.7), so that the temperature range of the mixed phase would be $10^8 \text{ K} \lesssim T \lesssim 10^{12}$ K, where $\rho_m \lesssim \rho_{m'} \lesssim \rho_r$. In fact, the mixed phase, as we explain below, may exist in the temperature range

$$10^{10} \text{ K} \lesssim T \lesssim 10^{12} \text{ K}, \tag{3.8}$$

where 10^{10} K is the electron-positron threshold for hadron production in the e^+e^- annihilation process, and at temperatures well below 10^{10} K the off-shell phase is exponentially suppressed. For T from the range (3.8), we obtain, with the help of the relations

$$c^2 = \frac{dp}{d\rho} = \frac{dp/dT}{d\rho/dT},$$

$$p = p_m + p_r + p_{m'}, \quad \rho = \rho_m + \rho_r + \rho_{m'},$$

the following expression for the velocity of sound:

$$\begin{aligned} c^2 &= \frac{n_m p_m + n_r p_r + n_{m'} p_{m'}}{n_m \rho_m + n_r \rho_r + n_{m'} \rho_{m'}} \\ &= \frac{n_m c_m^2 + n_r c_r^2 \rho_r/\rho_m + n_{m'} c_{m'}^2 \rho_{m'}/\rho_m}{n_m + n_r \rho_r/\rho_m + n_{m'} \rho_{m'}/\rho_m}, \end{aligned} \tag{3.9}$$

where n_m, n_r, $n_{m'}$ are the powers of temperature in the corresponding formulas for the pressure and the energy densities, and $c_m^2 = dp_m/d\rho_m = p_m/\rho_m$, etc. are the sound velocities in the corresponding phases. Using the relations

$$c_m^2 << 1, \quad c_r^2 = \frac{1}{3}, \quad c_{m'}^2 = \frac{1}{4},$$

we finally obtain

$$c^2 \simeq \frac{4/3\ \rho_r/\rho_m + 5/4\ \rho_{m'}/\rho_m}{3 + 4\rho_r/\rho_m + 5\rho_{m'}/\rho_m}. \tag{3.10}$$

If the energy densities of the corresponding phases were of the same order, $\rho_m \sim \rho_r \sim \rho_{m'}$, one would obtain

$$c^2 \simeq \frac{31}{144} \approx 0.21. \tag{3.11}$$

In fact, the use of Eqs. (1.11),(3.7) in the formula (3.10) provides for the sound velocity a numerical value which is practically 1/3, up to temperature $\sim 2 \cdot 10^{12}$ K. At $T = T^{(3)}$ ($\simeq 3 \cdot 10^{12}$ K), Eq. (3.10) yields $c^2 \approx 0.28$. One sees that for the model of the mixed phase, a dip in the sound velocity as a function of temperature in the vicinity of the transition temperature is predicted, in agreement with both the phenomenological models [15, 42] and the recent data on QCD lattice simulations [43]. Let us also note that the value of the sound velocity $c^2 = 0.28$ at $T \sim 10^{12}$ K was reported in ref. [44].

The value of the sound velocity $c^2 \approx 0.25$, in agreement with the relevant experimental data [19, 20, 21], is obtained only in the assumption of a sufficiently sharp transition, when the phase below the transition temperature is that of strongly interacting (off-shell) matter alone, with temperature dependence T^5.

Thus, the "realistic" Friedmann model studied here describes qualitatively the decrease of the sound velocity in relativistic gas at values of T in the vicinity of 10^{12} K, which reaches the numerical value $c^2 \approx 0.25$, consistent with at least some of the experimental results [19, 20, 21], and significantly less than the ultra-relativistic

Stefan-Boltzmann gas value 0.33. It may be thought that the progressive population of states of higher mass creates an admixture of heavy particles, thus decreasing the factor relating pressure to energy density.

The above estimates have been made in a classical framework. Now we shall show how similar results can be obtained within the framework of the conventional (on-shell) quantum theory. As mentioned above, at $T^{(2)} \sim m_e \simeq 5 \cdot 10^9$ K, lepton and hadron pair production starts; in the temperature range $T^{(2)} \lesssim T \lesssim T^0$, where T_0 is the temperature of the Hagedorn phase transition, $T_0 \sim T^{(3)}$, the system is characterized by a resonance spectrum of the Hagedorn form (1.12), for which there is some experimental basis [14, 15]. At $T \sim 10^{10} - 10^{11}$ K $<< T_0$, one can neglect the exponential in (1.12) and use, as done by Shuryak [15, 45], a resonance spectrum of the form

$$\tau(m) \sim m^a, \tag{3.12}$$

where one chooses on phenomenological grounds $a = 1$ [15].

The expressions for the pressure and energy density are then written as [15, 45] (we neglect the difference in properties of Bose and Fermi particles at high temperature)

$$\begin{aligned} p &= \frac{g}{3}\int_\triangle^\infty dm\ \tau(m)\int\frac{d^3k}{(2\pi)^3}\frac{k^2}{\sqrt{k^2+m^2}}e^{-\sqrt{k^2+m^2}/T} &= g\int_\triangle^\infty dm\ \tau(m)p_m, \quad (3.13)\\ \rho &= g\int_\triangle^\infty dm\ \tau(m)\int\frac{d^3k}{(2\pi)^3}\sqrt{k^2+m^2}e^{-\sqrt{k^2+m^2}/T} &= g\int_\triangle^\infty dm\ \tau(m)\rho_m, \quad (3.14) \end{aligned}$$

where g is the number of hadronic degrees of freedom, p_m and ρ_m are expressions for the pressure and energy density of relativistic gas of particles with given mass m, and we may choose, sufficient for our present purposes, $\triangle \cong 2m_e \simeq 10^{10}$ K. Since at temperatures $T \gtrsim \triangle$ one should take into account electron-positron annihilation, leading, via quark loops, to the formation of hadronic jets [46],

$$e^+e^- \to q\bar{q} \to \text{hadrons}, \tag{3.15}$$

which can be thought of as resonances on the excitation background of virtual leptonic pair states, the e^+e^- threshold should be taken as the lowest possible mass for the resonance spectrum $\tau(m)$, justifying the introduction of the corresponding cut-off in the formulas (3.13),(3.14).

The use of the standard expression [47]

$$p_m = \frac{m^2T^2}{2\pi^2}K_2\left(\frac{m}{T}\right) \tag{3.16}$$

in Eq. (3.13), in which we take $\tau(m) = Cm,\ C = \text{const}$, yields [48]

$$p = \frac{gCT^2}{2\pi^2}\int_\triangle^\infty dm\ m^3K_2\left(\frac{m}{T}\right) = \frac{gCT^3}{2\pi^2}\triangle^3K_3\left(\frac{\triangle}{T}\right). \tag{3.17}$$

At temperatures $T \sim 10^{11}$ K where $T >> \triangle$ (but still $T << T_0$, so that the use of (3.12) instead of (1.12) is justified), one uses the asymptotic formula [49]

$$K_\nu(z) \sim \frac{1}{2}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu}, \quad z << 1 \tag{3.18}$$

and obtains

$$p = \frac{4gC}{\pi^2}T^6. \tag{3.19}$$

Similarly, one can obtain for such temperatures [15],

$$\rho = \frac{20gC}{\pi^2}T^6 = 5p, \tag{3.20}$$

so that

$$c^2 = \frac{dp}{d\rho} = 0.20. \tag{3.21}$$

Equations (3.19)-(3.21) represents the "realistic" equation of state suggested by Shuryak for hot hadronic matter [15].

For $T << \triangle$, one uses another asymptotic formula [49]:

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z}, \quad z >> 1 \tag{3.22}$$

and obtains

$$p = \frac{gCT^{7/2}\triangle^{5/2}}{2^{5/2}\pi^{3/2}}e^{-\triangle/T}, \tag{3.23}$$

and similar expression for ρ. Thus, at temperatures well below $\triangle \simeq 10^{10}$ K, the off-shell phase is suppressed by the exponential.

The use of $\tau(m) \sim m^a$ in the formulas (3.13),(3.14) will analogously lead to the equation of state at $T \sim 10^{11}$ K [15, 45]

$$p, \rho \sim T^{a+5}, \quad p = \rho/(a+4), \tag{3.24}$$

which coincides with (3.19)-(3.21) for $a = 1$. The values for the sound velocity in the hadronic phase 0.25 and 0.17, in agreement with some of the experimental results [18]-[24], is achieved, in view of (3.24), for $a = 0$ and $a = 2$, respectively.

4 Solutions to cosmological equation

The solutions of Eq. (3.1) for $t > t^{(3)},\ R > R^{(3)}$ are well known [12],

$$
\begin{aligned}
k = 0 :& \\
R &= \sqrt{\alpha_r}\tau + \frac{1}{4}\alpha_m\tau^2, \\
t &= \frac{1}{2}\sqrt{\alpha_r}\tau^2 + \frac{1}{12}\alpha_m\tau^3; \\
k = 1 :& \\
R &= \sqrt{\alpha_r}\sin\tau + \alpha_m\sin^2\frac{\tau}{2}, \\
t &= 2\sqrt{\alpha_r}\sin^2\frac{\tau}{2} + \frac{1}{2}\alpha_m(\tau - \sin\tau); \\
k = -1 :& \\
R &= \sqrt{\alpha_r}\sinh\tau + \alpha_m\sinh^2\frac{\tau}{2}, \\
t &= 2\sqrt{\alpha_r}\sinh^2\frac{\tau}{2} + \frac{1}{2}\alpha_m(\sinh\tau - \tau).
\end{aligned}
$$

For $0 < t < t^{(3)},\ 0 < R < R^{(3)}$, since the curvature term is negligible in the early universe, we have the equation of the general form

$$\left(\frac{dR}{dt}\right)^2 = \alpha_n R^{2-n}, \quad \alpha_n = 8\pi G\rho R^n/3, \tag{4.1}$$

with $n = 5$. Eq. (4.1) has the solution

$$R = \alpha_n^{1/(n-2)}\left(\frac{n-2}{2}\tau\right)^{2/(n-2)}, \tag{4.2}$$

$$t = \frac{2}{n}\alpha_n^{1/(n-2)}\left(\frac{n-2}{2}\tau\right)^{n/(n-2)}; \tag{4.3}$$

so that

$$R = \left(\frac{n}{2}\right)^{2/n}\alpha_n^{1/n}t^{2/n}, \tag{4.4}$$

and

$$t = \frac{2}{n}\left(\frac{8\pi G}{3}\rho\right)^{-1/2}. \tag{4.5}$$

For the energy density at $T = T^{(3)}$, $\rho_{m'}^{(3)} \sim 10^{14}$ g cm^{-3}, we obtain from (4.5), with $n = 5$,

$$t^{(3)} \sim 10^{-5}\text{c}. \tag{4.6}$$

Equation (4.1) corresponds to that of Zeldovich and Novikov [50] for the early charged symmetric universe, whereas the Lemaître equation (1.4) is for the subsequent charge asymmetric universe in which the asymmetry is either local or global.

5 Concluding remarks

We have studied the cosmological Friedmann model modified by introduction of off-shell matter with the equation of state $\rho \propto T^5 \propto R^{-5},\ p = \rho/4$. The energy density of such a matter is comparable with that of the non-interacting radiation at temperature of the order of the Hagedorn limiting temperature, $\sim 10^{12}$ K, indicating the possibility of a phase transition.

A cosmological phase transition at $T_c \sim 10^{12}$ K is normally associated with the transition from a strongly interacting hadronic phase to a weakly interacting quark-gluon plasma phase [24, 40]. The simplest classical model considered in the present paper implies the possibility of a first order phase transition from the T^5-phase of strongly interacting (off-shell) matter to the T^4-phase of non-interacting radiation-like matter. Cosmological consequences of such a phase transition are discussed in ref. [34], where it is suggested that the transition may be sufficiently smooth (second order) to preserve the expansion rate. Although a first order phase transition might be preferable for some cosmological implications, due to the fluctuations which are generated at the transition and could produce planetary mass black holes [51] which, in turn, could provide a possible explanation for the dark matter of the universe and even be seeds in galaxy formation [52, 53], experimental indications on the order of this phase transition are still absent. Indeed, presently available lattice data on $SU(N)$ pure gauge theory lattice simulations indicate that a phase transition to a weakly interacting phase is of apparently first order for $SU(3)$ and second order for $SU(2)$ theory [54]. In ref. [55], however, the apparent first order nature of the transition in the case of $SU(3)$ pure gauge theory has been called in question. Moreover, there are indications from lattice QCD calculations that when fermions are included, the phase transition may be of second or higher order [56]. In this case, as remarked by Ornik and Weiner [24], the phase transition would be hardly distinguishable from a situation in which no phase transition would have taken place (radiation-dominated universe alone).

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References

[1] S. Weinberg, *Gravitation and Cosmology,* (Wiley, New York, 1972), p. 472

[2] A. Friedmann, Z. Phys. **10** (1922) 377, **21** (1924) 326

[3] A. Einstein and W. de Sitter, Proc. Nat. Acad. Sci. **18** (1932) 213

[4] G. Lemaître, Ann. Soc. Sci. Brux. **47 A** (1927) 49; Bull. Astr. Insts. Neth. **5** (1930) 200; Mon. Not. R. Astr. Soc. **95** (1931) 483

[5] W. de Sitter, Bull. Astr. Insts. Neth. **5** (1930) 193

[6] R.C. Tolman, *Relativity, Thermodynamics and Cosmology,* (Dover, New York, 1987), pp. 407-411

[7] R.A. Alpher and R.C. Herman, Nature Lond. **142** (1948) 419

[8] A.D. Chernin, Soviet Astr. **9** (1966) 871

[9] J.M. Cohen, Nature Lond. **216** (1967) 249

[10] K.C. Jacobs, Nature Lond. **215** (1967) 1157

[11] C.B.G. McIntosh, Mon. Not. R. Astr. Soc. **138** (1968) 423

[12] E.R. Harrison, Mon. Not. R. Astr. Soc. **140** (1968) 281

[13] Ref. [1], Chapter 15, Section 6

[14] R. Hagedorn, Nuovo Cimento **35** (1965) 216; Suppl. Nuovo Cimento **3** (1965) 147, **6** (1968) 311

[15] E.V. Shuryak, *The QCD Vacuum, Hadrons and the Superdense Matter,* (World Scientific, Singapore, 1988)

[16] O.V. Zhirov and E.V. Shuryak, Sov. J. Nucl. Phys. **21** (1975) 443

[17] M.I. Gorenstein *et al.,* Phys. Lett. B **60** (1976) 283

[18] B. Andersson *et al.,* Nucl. Phys. B **112** (1976) 413

[19] E.V. Shuryak, Sov. J. Nucl. Phys. **24** (1976) 330

[20] E.V. Shuryak, Phys. Lett. B **78** (1978) 150

[21] O.V. Zhirov, Sov. J. Nucl. Phys. **30** (1979) 571

[22] S. Sohlo and G. Wilk, Lett. Nuovo Cimento **13** (1975) 375

[23] K. Wehrberger and R.M. Weiner, Phys. Rev. D **31** (1985) 222

[24] U. Ornik and R.M. Weiner, Phys. Rev. D **36** (1987) 1263

[25] L.P. Horwitz, W.C. Schieve and C. Piron, Ann. Phys. **137** (1981) 306

[26] L.P. Horwitz, S. Shashoua and W.C. Schieve, Physica A **161** (1989) 300

[27] L. Burakovsky and L.P. Horwitz, Physica A **201** (1993) 666

[28] V.A. Fock, Phys. Z. Sowjetunion **12** (1937) 404

[29] E.C.G. Stueckelberg, Helv. Phys. Acta **14** (1941) 372, 588; **15** (1942) 23

[30] L. Burakovsky and L.P. Horwitz, Found. Phys. **25** (1995) 785

[31] L. Burakovsky and L.P. Horwitz, Found. Phys., *in press*

[32] L. Burakovsky, L.P. Horwitz and W.C. Schieve, Statistical Mechanics of Relativistic Degenerate Fermi Gas I.Cold Adiabatic Equation of State, Preprint TAUP-2124-93

[33] L. Burakovsky and L.P. Horwitz, Manifestly Covariant Relativistic Thermodynamics and Avoidance of Gravitational Singularities, Preprint TAUP-2173-94

[34] L. Burakovsky and L.P. Horwitz, 5D Generalized Inflationary Cosmology, to be published in Gen. Rel. Grav.

[35] L.P. Horwitz and C. Piron, Helv. Phys. Acta **46** (1973) 316

[36] R. Arshansky, L.P. Horwitz and Y. Lavie, Found Phys. **13** (1983) 1167

[37] E.W. Kolb and M.S. Turner, *The Early Universe,* (Addison-Wesley, Redwood, CA, 1990), p. 66

[38] Y.B. Zeldovich, Sov. Phys. JETP **14** (1962) 1143

[39] R. Hakim, J. Math. Phys. **15** (1974) 1310

[40] T. DeGrand and K. Kajantie, Phys. Lett. B **147** (1984) 273

[41] L. Burakovsky, L.P. Horwitz and W.C. Schieve, Towards a Realistic Equation of State of Strongly Interacting Matter, Preprint TAUP-2239-95

[42] E.V. Shuryak and O.V. Zhirov, Phys. Lett. B **89** (1980) 253

[43] T. Çelik, J. Engels and H. Satz, Phys. Lett. B **129** (1983) 323;
K. Redlich and H. Satz, Phys. Rev. D **33** (1986) 3747

[44] E. Suhonen *et al.,* Phys. Rev. Lett. **31** (1973) 1567

[45] E.V. Shuryak, Sov. J. Nucl. Phys. **16** (1973) 220

[46] D.H. Perkins, *Introduction to High Energy Physics,* (Addison-Wesley, Reading, MA, 1982), p. 305

[47] Ref. [37], p. 61

[48] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series, and Products,* (Academic Press, New York, 1980), subsection 6.561

[49] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions,* (Dover, New York, 1972), p. 375

[50] Y.B. Zeldovich and I.D. Novikov, Sov. Phys. JETP Lett. **4** (1966) 80

[51] M. Crawford and D.N. Schramm, Nature **298** (1982) 538

[52] K. Freese, R. Price and D.N. Schramm, Astrophys. J. **275** (1983) 405

[53] E. Witten, Phys. Rev. D **30** (1984) 272

[54] B. Müller, *The Physics of the Quark-Gluon Plasma,* (Springer-Verlag, Berlin, 1985), p. 66

[55] F.R. Brown *et al.,* Phys. Rev. Lett. **61** (1988) 2058

[56] B. Svetitsky, Nucl. Phys. A **461** (1987) 71c