

What Thermodynamics tells about QCD Plasma near Phase Transition

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Abstract

Due to a rapid change of the entropy density $s(T)$ across the critical temperature T_c of the QCD phase transition, the pressure $P(T)$ and the energy density $e(T)$ above T_c generally deviate from their Stefan-Boltzmann values. We shall demonstrate this both analytically and numerically for a general class of $s(T)$ consistent with thermodynamical constraints and make a qualitative comparison of the result with the lattice QCD data. Quantities related to $ds(T)/dT$ such as the specific heat and sound velocity are also discussed.

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Revealing the precise nature of the quark-hadron phase transition is one of the central issues in recent lattice and analytical studies of hot QCD (see the reviews [1].) Furthermore, the behavior of the bulk quantities such as the entropy density $s(T)$, energy density $e(T)$ and pressure $P(T)$ as a function of temperature T is relevant to the formation and evolution of the quark-gluon plasma in the ultrarelativistic heavy ion collisions planned at BNL and CERN.

Lattice QCD calculations have been providing interesting data on the plasma properties above the critical temperature T_c [1]. Among them are, however, some unexpected features: (a) P/T^4 approaches the Stefan-Boltzmann limit very slowly as T increases, (b) e/T^4 has a peak just above T_c and approaches an asymptotic value from above as T increase, and (c) $e - 3P \neq 0$ above T_c i.e. a large deviation from the ideal gas behavior is seen. They are sometimes identified as indications of the non-perturbative nature of the quark-gluon plasma.

In this letter, under generic assumptions on $s(T)$ consistent with the thermodynamic inequality, we examine what is the expected behavior of $e(T)$, $P(T)$ and other quantities such as the specific heat $C_V(T)$ and the sound velocity $c_s(T)$. We show that most of the “unexpected” behaviors measured in the lattice QCD simulations [1–3] can be explained at least qualitatively as a simple consequence of the rapid increase of $s(T)$ around T_c .

Our basic observation is that all the above quantities are simply parametrized by the entropy density $s(T)$ which depends only on the active degrees of freedom and is free from the complexities of the vacuum structure of QCD. This is seen by the thermodynamic relations at zero chemical potential such as

$$P(T) = \int_0^T s(t)dt, \quad (1)$$

$$e(T) = Ts(T) - P(T), \quad (2)$$

and

$$C_V = T \frac{\partial s(T)}{\partial T}, \quad (3)$$

$$c_s^2 = \frac{\partial P}{\partial e} = \left[\frac{\partial \ln s(T)}{\partial \ln T} \right]^{-1}. \quad (4)$$

In eq.(1), $P(T)$ is normalized by $P(T = 0) = 0$, which also leads to $e(T = 0) = 0$ from eq.(2). The same normalization is adopted also in lattice calculations.

Since we are interested only in the gross behavior of the thermodynamic quantities and not in the precise order of the phase transition [4], we make a smooth interpolation of $s(T)$ between the hadronic gas at low T and the quark-gluon plasma at high T . Note, however, that one cannot make arbitrary parametrization of $s(T)$ since it is constrained by the thermodynamic inequality and the Nernst’s theorem [5]

$$\frac{\partial s(T)}{\partial T} > 0, \quad s(0) = 0. \quad (5)$$

A simplest possible parametrization satisfying eq.(5) reads

$$s(T) = s_h(T)w_h(T) + s_q(T)w_q(T), \quad (6)$$

where $w_h(T) = 1 - w_q(T)$ and $w_q(T)$ is an increasing function. In this letter, we take the following form for $w_q(T)$, but our conclusion is not limited to this specific choice of $w_{h,q}(T)$:

$$w_q(T) = \frac{n \left(1 + \tanh\left(\frac{T-T_c}{\Gamma}\right)\right)}{m \left(1 - \tanh\left(\frac{T-T_c}{\Gamma}\right)\right) + n \left(1 + \tanh\left(\frac{T-T_c}{\Gamma}\right)\right)}. \quad (7)$$

Here $s_h(T) \equiv 12(\pi^2/90)T^3$ and $s_q(T) \equiv 148(\pi^2/90)T^3$ are the entropy densities of massless free gas with two flavors in the hadronic phase (pion gas) and the quark-gluon phase, respectively. The interaction between particles and the quark masses are neglected just for simplicity. 2Γ sets the width of the transition region. For T satisfying $|T - T_c| \gg \Gamma$, $s(T)$ approaches asymptotically to $s_h(T)$ below T_c and to $s_q(T)$ above T_c . m and n are introduced to consider the asymmetric superposition of the two phases around T_c , but we shall take $m = n$ in the following.

The equation of state obtained from eq.(6) is more general than that of the bag model [6] in the sense that (a) we need not refer to phenomenological parameters such as the bag constant, and (b) we can treat not only the strong first order phase transition but also the crossover by changing Γ in a thermodynamically consistent way. When $\Gamma = 0$, our equation of state is equivalent to the bag model one with a bag constant $4B = (s_q(T)/T^3 - s_h(T)/T^3)T_c^4$.

In Fig.1, $s(T)$ with $m = n = 1$ is shown for $\Gamma/T_c = 0, 0.05$ and 0.25 . Lattice measurements show a rapid variation of $s(T)$ in a narrow region of T (with a width ~ 10 MeV) [7], which is similar to Fig.1 with $\Gamma/T_c = 0.05$. Inclusion of the perturbative α_s corrections above T_c and the interactions of hadrons below T_c modifies the absolute value of $s(T)$ as well as its T dependence, but it does not change our conclusions qualitatively.

Once $s(T)$ is given, it is straightforward to calculate other quantities from eqs.(1,2,3,4). In Fig.2, e/T^4 , P/T^4 and $(e-3P)/T^4$ are shown as a function of temperature for $\Gamma/T_c = 0.05$. They can be directly compared with the lattice results [1-3] at least qualitatively. In fact, they look quite similar:

(i) P/T^4 increases rather slowly above T_c both on the lattice [2,3] and in Fig.2. Since $P(T)$ is given as an integral of $s(T)$, it is quite natural to expect such a continuous and slow rise. When $\Gamma = 0$, $P(T > T_c)$ has an analytic form

$$P(T)/P_{SB}(T) \sim 1 - (T_c/T)^4, \quad (8)$$

where $P_{SB}(T)$ denotes the Stefan-Boltzmann value. Eq.(8) tells us that $P(T)/P_{SB}(T) = 50\%$ (90%) for $T/T_c = 1.2$ (1.8) independent of the details of the dynamics. If Γ is increased, P/T^4 has even weaker T dependence. Thus the major part of the deviation of $P(T)$ from $P_{SB}(T)$ can be accounted for without introducing non-perturbative interactions of quarks and gluons above T_c .

(ii) e/T^4 has a peak just above T_c in Fig. 2 (note that $e(T)$ itself is a monotonically increasing function of T), which is also seen in lattice simulations [2]. In our case, this peak is a simple consequence of the rapid increase of $s(T)$ and the slow rise of $P(T)$ above T_c (see the definition eq.(2)). The width of the peak is correlated with the slow rise of P/T^4 .

One can even prove analytically that there must exist a peak around T_c from the following relation satisfied by arbitrary weight functions $w_{h,q}(T)$,

$$T^5 \left(\frac{e}{T^4} \right)' = T^2 (s_h(T)w_h'(T) + s_q(T)w_q'(T)) - \int_0^T t (s_h(t)w_h'(t) + s_q(t)w_q'(t))dt, \quad (9)$$

where the prime denotes a derivative with respect to T and we have used a fact $s_{h,q}(T) = (\text{constant}) \times T^3$. Under the conditions that $w_h(t) + w_q(t) = 1$, $0 \leq w_q(0) < 1$, $w_q'(t) > 0$ and $w_q'(t \rightarrow \infty) = o(1/t^4)$, $(e/T^4)'$ is positive (negative) for low (high) T and has a zero near T_c . These conditions are all satisfied in our case, and as a result e/T^4 has a peak. One should also note that, for high enough T , $(e/T^4)' \sim -\Delta(T)/T < 0$ with $\Delta \equiv (e - 3P)/T^4$.

(iii) $\Delta(T) = (e - 3P)/T^4$ is so called the ‘‘interaction measure’’ and has a peak near T_c both on the lattice [2,3] and in Fig.2. Again, the rapid increase of $s(T)$ (i.e. the liberation of quarks and gluons) is the reason for this peak, which is also seen from the general formula

$$\Delta(T) = \frac{1}{T^4} \int_0^T t (s_h(t)w_h'(t) + s_q(t)w_q'(t))dt. \quad (10)$$

Δ vanishes for $T = 0$ and ∞ and has a peak around T_c independent of the details of the dynamics. Also the peak becomes sharper and its height becomes higher as Γ decreases. When $\Gamma = 0$, one can rewrite Δ as $\Delta(T > T_c) = (s_q/T^3 - s_h/T^3)(T_c/T)^4$ which is equivalent to the bag model formula without α_s corrections $\Delta(T > T_c) = 4B/T^4$ [6]. Note here that, if we normalize the pressure by $P(\infty) = P_{SB}$ instead of $P(0) = 0$, the peak of $\Delta(T)$ does not arise and the naive Stefan-Boltzmann law is realized above T_c . In this sense, the $1/T^4$ tail of Δ at high T in Fig.2 can be interpreted as an artifact of the normalization and has nothing to do with non-perturbative interactions of quarks and gluons in the quark-gluon plasma.

In Fig.3, c_s^2 is shown for $\Gamma/T_c = 0.05$. Since the equation of state becomes soft near the critical region, the sound velocity slows down. This is the reason why c_s^2 has a sudden drop in the narrow region $T_c - \Gamma < T < T_c + \Gamma$. This sudden change is in contrast to the broad peak or slow rise of the quantities in Fig.2. The difference comes from the fact that c_s^2 is related to the derivative of $s(T)$ while $e(T)$ and $P(T)$ are related to the integral of $s(T)$. To see the effect of the finite pion mass on c_s^2 below T_c , c_s^2 using $s_h(T)$ with $m_\pi = 140\text{MeV}$ (and $T_c = 180\text{MeV}$) is shown in Fig.3 by the dashed line. The effect of finite m_π on the other quantities is small since they are small in any way at low T . The heat capacity C_V has also a sharp peak at T_c , since one needs to supply a large amount of heat to increase T across T_c to liberate quark-gluon degrees of freedom.

Although we need not refer to the vacuum parameters such as the bag constant in our approach, it might be instructive to introduce an effective bag constant above T_c defined as a deviation of the pressure from its Stefan-Boltzmann value,

$$B(T > T_c) = -(P(T) - P_{SB}(T))/T^4, \quad (11)$$

where we have neglected all the α_s corrections. $B(T)/T_c^4$ is shown in Fig.4. The asymptotic value $B(\infty)$ is the one usually used in the bag model. In our case, $B(\infty)$ depends on how one parametrizes $s(T)$: as Γ is increased, $B(\infty)$ also increases.

The main conclusion of this letter is that, whenever the entropy density has a rapid change near T_c , e/T^4 , P/T^4 and Δ behave as we know from the lattice simulations. Furthermore, we need not refer to the vacuum condensate or the bag constant to see this fact. We believe that our approach based on the parametrization of $s(T)$ provides a transparent and thermodynamically consistent way to study the qualitative feature of the bulk plasma quantities. Although we have taken a simplest possible form of $s(T)$, inclusion of the quark masses, hadronic interactions, α_s corrections and chemical potentials is straightforward. Possible non-perturbative effects at $T_c < T < 3T_c$ [3,8] appear as a deviation from the basic curves in Fig.2 and could be included by adjusting the functional form of $s(T)$. However, it is not an easy task to identify true non-perturbative effects from the lattice data, since Fig.2 has already similar behavior with the lattice data, and furthermore the finite volume effect and perturbative corrections are still large in the current lattice simulations [1].

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Figure Captions

- Fig.1: s/T^4 as a function of T for three different values of the width parameter Γ .
- Fig.2: The solid line, the dashed line and the dash-dotted line correspond to e/T^4 , P/T^4 and Δ , respectively. $\Gamma/T_c = 0.05$ is chosen.
- Fig.3: Squared sound velocity c_s^2 as a function of T . Solid (dashed) line corresponds to $m_\pi = 0$ (140) MeV.
- Fig.4: “Effective” bag constant $B(T)/T_c^4$ for $T > T_c$. Solid, dashed and dash-dotted lines correspond to $\Gamma/T_c = 0, 0.05$ and 0.25 , respectively.