

# Irrotational dust with $\text{div } H = 0$

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## Abstract

For irrotational dust the shear tensor is consistently diagonalizable with its covariant time derivative:  $\sigma_{ab} = 0 = \dot{\sigma}_{ab}$ ,  $a \neq b$ , if and only if the divergence of the magnetic part of the Weyl tensor vanishes:  $\text{div } H = 0$ . We show here that in that case, the consistency of the Ricci constraints requires that the magnetic part of the Weyl tensor itself vanishes:  $H_{ab} = 0$ .

*Subject headings:*

cosmology - galaxies: clustering,  
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## 1 Introduction

The local non-linear dynamics of irrotational dust with a purely “electric” Weyl tensor has been investigated by Matarresse *et al.* [7, 8, 9]. These authors characterized such solutions as ‘silent universes’, because the time-evolution equations form a set of ordinary differential equations. In [5] it was shown that all the integrability conditions for this case are satisfied and are consistent, provided the initial constraint equations are satisfied.

Then the development along each world line is independent of those along neighbouring world lines; once the evolution is underway, no information is exchanged between neighbouring particles.

The dynamics of such universes have been studied mostly by employing an orthonormal tetrad, which is a simultaneous eigentetrad of the shear  $\sigma_{ab}$  and the “electric” part of the Weyl tensor  $E_{ab}$ . Variables defined in terms of these tetrads allow development of useful phase-plane portraits [2], which show *inter alia* that the generic collapse configuration is a cigar or spindle rather than a pancake.

In the linear theory, the magnetic part of the Weyl tensor  $H_{ab}$  only contains vector and tensor modes [1]; also if vorticity is zero, no vector modes are present, and  $H_{ab}$  at most contains tensor modes with vanishing divergence, which describe gravitational waves. In the non-linear theory the physical content of  $H_{ab}$  is however less clear. At the core of the dynamics of silent universe is the assumption that in the absence of  $H_{ab}$ , no gravitational waves occur.

For irrotational dust the shear tensor is consistently diagonalizable with its covariant time derivative:  $\sigma_{ab} = 0 = \dot{\sigma}_{ab}$ ,  $a \neq b$ , if and only if the divergence of the magnetic Weyl tensor vanishes. Furthermore if the electric Weyl tensor vanishes then the magnetic Weyl tensor vanishes [6]. So for irrotational dust there are no solutions with a purely magnetic Weyl tensor.

The aim of this paper is to prove that for irrotational dust there are no consistent solutions with a non-vanishing magnetic part of the Weyl tensor that has a vanishing divergence. Thus the only consistent solutions with vanishing divergence of  $H_{ab}$  are those described by the silent universes [2,7-9]. Mathematically, we start from the requirement that the shear tensor  $\sigma_{ab}$  is consistently diagonalizable with  $\dot{\sigma}_{ab}$ . We then prove that this is possible iff  $div H = 0$  ([6]). This is a dynamical restriction, for it is a constraint on the gravitational tidal field. The implication of this is seen in the time derivative of the Ricci constraints (which prescribes spatial restrictions on both variables  $\sigma_{ab}$ ,  $H_{ab}$ ), and this yields  $H_{ab} = 0$ .

*Notation:* Latin indices run from 0 to 3, and Greek indices from 1 to 3; semicolons denote covariant derivatives. Covariant differentiation along the velocity vector  $u^a$  is denoted by  $(\cdot)$  e.g., the acceleration vector is  $a_a \equiv \dot{u}_a := u_{a;b}u^b$ . Kinematic and dynamic quantities are the same as in the other papers of this series (they are comprehensively defined in [4]).

## 2 Equations for Kinematic and Dynamic Quantities

### 2.1 Conservation Equations

For a perfect fluid, the fluid acceleration is only determined by pressure gradients so the restriction of vanishing pressure implies that  $a^a = 0$ . This means that each fluid element moves along a geodesic. The conservation of energy and momentum  $T^{ab}{}_{;b} = 0$  leads to only one equation, the continuity equation:

$$\dot{\rho} = -\rho\Theta . \quad (1)$$

### 2.2 The Ricci identities

With the second restriction of vanishing vorticity  $\omega_{ab} = 0$ , the equations for the kinematic quantities follow from the Ricci identity:  $u_{a;d;c} - u_{a;c;d} = R_{abcd}u^b$ .

#### 2.2.1 Propagation equations

The expansion scalar  $\Theta$  obeys the Raychaudhuri equation:

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + 2\sigma^2 + \frac{1}{2}\kappa\rho = 0 , \quad (2)$$

where  $\sigma^2 \equiv \frac{1}{2}\sigma^{ab}\sigma_{ab}$  is the shear scalar and  $\kappa = 8\pi G$  is the gravitational constant. The remaining kinematic evolution equation is for the shear and is given by:

$$\dot{\sigma}_{ab} + \sigma_{ac}\sigma^c{}_b - \frac{2}{3}\sigma^2 h_{ab} + \frac{2}{3}\Theta\sigma_{ab} + E_{ab} = 0 , \quad (3)$$

where  $E_{ac} = E_{(ac)} \equiv C_{abcd}u^b u^d$  is the “electric” part of the Weyl tensor  $C_{abcd}$  (satisfying  $E_{ac}u^c = 0$ ,  $E^a{}_a = 0$ ).  $E_{ab}$  is that part of the gravitational field which describes tidal interactions. The Weyl tensor can be decomposed into  $E_{ab}$  and another tensor called the “magnetic” part:  $H_{ac} = H_{(ac)} \equiv \frac{1}{2}\eta_{ab}{}^{gh}C_{ghcd}u^b u^d$  (satisfying  $H_{ac}u^c = 0$ ,  $H^a{}_a = 0$ ). This is the part of the gravitational field that describes gravo-magnetic effects, and gravitational waves.

### 2.2.2 Constraint equations

Besides the evolution equations for the kinematical equations, there are several constraints that our variables must satisfy. On setting  $p = \omega_{ab} = 0$  we obtain as non-trivial constraints,

$$h^e{}_b \left( \frac{2}{3} \Theta^{;b} - h^d{}_c \sigma^{bc}{}_{;d} \right) = 0, \quad (4)$$

the ‘(0,  $\nu$ )’ field equations, and

$$H_{ad} = -h^t{}_a h^s{}_d \sigma_{(t}{}^{b;c} \eta_{s) fbc} u^f. \quad (5)$$

which we refer to as the ‘ $H_{ab}$ ’ constraint, also (4) and (5) are both referred to as Ricci constraints. As we are considering the zero-vorticity case, we can also write down the Gauss-Codacci equations for the 3-spaces orthogonal to  $u^a$  (see [4]); however we do not need them for what follows.

### 2.3 Bianchi identities

Additionally, the Bianchi identities must be satisfied, as they are the integrability conditions for the other equations.

With the restrictions we have put on so far, they take the form:

$$h^t{}_a h^d{}_s E^{as}{}_{;d} - \eta^{tbpq} u_b \sigma^d{}_q H_{dq} = \frac{1}{3} h^t{}_b \rho^{;b}, \quad (6)$$

$$h^m{}_a h^t{}_c \dot{E}^{ac} + h^{mt} \sigma^{ab} E_{ab} + \Theta E^{mt} + J^{mt} - 3 E_s^{(m} \sigma^{t)s} = -\frac{1}{2} \rho \sigma^{tm}, \quad (7)$$

which we refer to as the “div  $E$ ” and “ $\dot{E}$ ” equations respectively, and

$$h^t{}_a H^{as}{}_{;d} h^d{}_s + \eta^{tbpq} u_b \sigma^d{}_p E_{qd} = 0, \quad (8)$$

$$h^m{}_a h^t{}_c \dot{H}^{ac} + h^{mt} \sigma^{ab} H_{ab} + \Theta H^{mt} - I^{mt} - 3 H_s^{(m} \sigma^{t)s} = 0, \quad (9)$$

which are the “div  $H$ ” and “ $\dot{H}$ ” equations respectively, where we have defined the curls of  $E$  and  $H$  respectively as

$$\text{‘Curl } E\text{’} : \quad I^{mt} = -h_a^{(m} \eta^{t)rsd} u_r E^a{}_{s;d} \quad (10)$$

$$\text{‘Curl } H\text{’} : \quad J^{mt} = h_a^{(m} \eta^{t)rsd} u_r H^a{}_{s;d}. \quad (11)$$

The Bianchi identities are analogous to Maxwell’s electromagnetic equation [4]. The gradient  $\frac{1}{3} h_a^b \rho_{,b}$  acts as a source of the divergence of the  $E_{ab}$  field

in the “div E” constraint. For zero vorticity the  $H_{ab}$  field is source-free in the “div H” constraint. In the case of the time-development equations the “curl E” term  $I^{mt}$  acts as a source of  $\dot{H}$ . On the other hand  $\frac{1}{2}\rho\sigma^{tm}$  acts as a source of  $\dot{E}$ , as well as the “curl H” source term  $J^{mt}$ .

### 3 Tetrad Description

We now show that the shear tensor  $\sigma_{ab}$  is diagonalizable in the same principal frame as  $\dot{\sigma}_{ab}$  iff  $div H_{ab} = 0$ .

We first introduce an orthonormal tetrad that diagonalizes the shear tensor i.e.,

$$\sigma_{ab} = 0 \quad (a \neq b) . \quad (12)$$

This immediately implies  $\partial\sigma_{ab}/\partial\tau = 0$  for  $(a \neq b)$ , where  $\mathbf{u} = \partial/\partial\tau$ , but this gives no direct restrictions on  $\dot{\sigma}_{ab}$  because of the rotation coefficient terms in those quantities. However from (3) the tetrad satisfying (12) also satisfies  $\dot{\sigma}_{ab} = 0$  for  $a \neq b$  provided

$$E_{ab} = 0 \quad (a \neq b) \quad (13)$$

in this tetrad. From the “div H” equation (8), equations (12) and (13) imply that

$$div H = h^t{}_a H^{as}{}_{;d} h^d{}_s = 0 . \quad (14)$$

Conversely if  $div H = 0$  then  $\sigma_{ab}$  and  $E_{ab}$  are simultaneously diagonalizable from (8), as shown in [3]; furthermore then  $\dot{\sigma}_{ab} = 0$  for  $a \neq b$  from (3). Hence, For irrotational dust the shear tensor is consistently diagonalizable with its covariant time derivative:  $\sigma_{ab} = 0 = \dot{\sigma}_{ab}$  for  $a \neq b$ , if and only if the divergence of the magnetic Weyl tensor vanishes.

We now adopt the assumption that  $div H = 0$ , as defined by equation (14); and choose the preferred frame that diagonalizes  $\sigma_{ab}$  and  $\dot{\sigma}_{ab}$ . For simplicity we denote  $\sigma_{aa}$  (no sum) by  $\sigma_a$ , and  $E_{aa}$  by  $E_a$ . Both the shear tensor  $\sigma_{ab}$  and the ‘electric’ part of the Weyl tensor  $E_{ab}$  satisfy the trace-free property, i.e.,

$$\begin{aligned} E_1 + E_2 + E_3 &= 0 , \\ \sigma_1 + \sigma_2 + \sigma_3 &= 0 . \end{aligned} \quad (15)$$

If we now write out  $0 = \dot{\sigma}_{ab} = \sigma_{ab;c}u^c$  for ( $a \neq b$ ), we get the following tetrad relation:

$$(\sigma_a - \sigma_b)\Gamma^a{}_{ob} = 0; \quad a \neq b \quad (16)$$

where  $u^c = \delta^c_0$  and  $u^a u_a = -1$ . For an arbitrary shear tensor one may deduce from (16) that

$$\Gamma^a{}_{ob} = 0 \quad (a \neq b), \quad (17)$$

and hence the tetrad is Fermi-propagated along  $u_a$ . Condition (17) can also be shown to be valid for the case of degenerate shear. This is achieved by performing a rotation in the degenerate plane and using this tetrad freedom to obtain (17), see [6]. Furthermore for dust the vanishing of vorticity in a shear eigenframe is equivalent to the conditions

$$\Gamma^0{}_{ab} = 0. \quad (18)$$

Now by (17) and the diagonality of  $E_{ab}$ , it follows that also  $\dot{E}_{ab}$  is diagonal:

$$\dot{E}_{ab} = (E_a - E_b)\Gamma^a{}_{ob} = 0, \quad a \neq b.$$

Then from the time development equation (7) it follows that ‘‘curl H’’ is also diagonal i.e.,

$$J^{mt} = h_a^{(m}\eta^{t)rsd}u_r H^a{}_{s;d} = 0, \quad m \neq t. \quad (19)$$

Consistent with the rest of the notation, we denote  $J_{aa}$  by  $J_a$ . Then the fact it is trace free becomes  $J_1 + J_2 + J_3 = 0$ .

### 3.1 Propagation equations

Equations (1)-(9) governing the evolution of irrotational dust may now be written as a set of propagation equations tied to a set of constraint equations.

A direct conversion of equations (1), (2), (3) and (7) into a tetrad system which is an eigenframe for both the shear tensor and the electric Weyl tensor yields the following time-evolution equations:

$$\begin{aligned} \dot{\rho} &= -\rho\Theta, \\ \dot{\Theta} &= -\frac{1}{3}\Theta^2 - (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{1}{2}\rho, \\ \dot{\sigma}_\mu &= -(\sigma_\mu)^2 - \frac{2}{3}\Theta\sigma_\mu + \frac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - E_\mu, \\ \dot{E}_\mu &= -\Theta E_\mu - \frac{1}{2}\rho\sigma_\mu - J_\mu + 3\sigma_\mu E_\mu - (\sigma_1 E_1 + \sigma_2 E_2 + \sigma_3 E_3), \end{aligned} \quad (20)$$

where  $\mu = 1, 2, 3$  and no sum is carried out. The propagation equation for the magnetic Weyl tensor  $H_{ab}$  follows from the Bianchi identity (9) and has the form

$$\begin{aligned}
\dot{H}_{11} &= -\Theta H_{11} + 3\sigma_1 H_{11} + I_{11} - (\sigma_1 H_{11} + \sigma_2 H_{22} + \sigma_3 H_{33}), \\
\dot{H}_{22} &= -\Theta H_{22} + 3\sigma_2 H_{22} + I_{22} - (\sigma_1 H_{11} + \sigma_2 H_{22} + \sigma_3 H_{33}), \\
\dot{H}_{33} &= -\Theta H_{33} + 3\sigma_3 H_{33} + I_{33} - (\sigma_1 H_{11} + \sigma_2 H_{22} + \sigma_3 H_{33}), \\
\dot{H}_{23} &= -\Theta H_{23} - \frac{3}{2}\sigma_1 H_{23} + I_{23}, \\
\dot{H}_{12} &= -\Theta H_{12} - \frac{3}{2}\sigma_3 H_{12} + I_{12}, \\
\dot{H}_{31} &= -\Theta H_{31} - \frac{3}{2}\sigma_2 H_{31} + I_{31}.
\end{aligned} \tag{21}$$

It is important to note that in each of these equations, the left hand side is in terms of the covariant derivative  $\dot{T} = T_{;a}u^a$ , and so in general involves rotation coefficients; for example, in the obvious notation, the left-hand side of the 3rd equation (20) is

$$\dot{\sigma}_\nu = \sigma_{\nu,0} - 2\sigma_\nu \Gamma^\nu_{0\nu} \quad (\text{no sum}) \tag{23}$$

and that of the 4th equation is

$$\dot{E}_\nu = E_{\nu,0} - 2E_\nu \Gamma^\nu_{0\nu} \quad (\text{no sum}). \tag{24}$$

### 3.2 Ricci Constraints

The tetrad form of the Ricci constraints (4), (5) are as follows: the “(0,  $\nu$ )” field equations (4) take the form

$$\begin{aligned}
\frac{2}{3}\partial_1\Theta &= \partial_1\sigma_1 + (\sigma_1 - \sigma_2)\Gamma^2_{21} + (\sigma_1 - \sigma_3)\Gamma^3_{31}, \\
\frac{2}{3}\partial_2\Theta &= \partial_2\sigma_2 + (\sigma_2 - \sigma_1)\Gamma^1_{12} + (\sigma_2 - \sigma_3)\Gamma^3_{32}, \\
\frac{2}{3}\partial_3\Theta &= \partial_3\sigma_3 + (\sigma_3 - \sigma_1)\Gamma^1_{13} + (\sigma_3 - \sigma_2)\Gamma^2_{23}.
\end{aligned} \tag{25}$$

The “ $H_{ab}$ ” equations (5) take the form

$$\begin{aligned}
H_{11} &= \Gamma^1_{23}(\sigma_3 - \sigma_1) - \Gamma^1_{32}(\sigma_2 - \sigma_1), \\
H_{22} &= \Gamma^2_{31}(\sigma_1 - \sigma_2) - \Gamma^2_{13}(\sigma_3 - \sigma_2), \\
H_{33} &= \Gamma^3_{12}(\sigma_2 - \sigma_3) - \Gamma^3_{21}(\sigma_1 - \sigma_3);
\end{aligned} \tag{26}$$

$$\begin{aligned}
H_{23} &= \frac{1}{2} [\partial_1(\sigma_2 - \sigma_3) + \Gamma^3_{31}(\sigma_1 - \sigma_3) - \Gamma^2_{21}(\sigma_1 - \sigma_2)] , \\
H_{31} &= \frac{1}{2} [\partial_2(\sigma_3 - \sigma_1) + \Gamma^3_{32}(\sigma_3 - \sigma_2) - \Gamma^1_{12}(\sigma_1 - \sigma_2)] , \\
H_{12} &= \frac{1}{2} [\partial_3(\sigma_1 - \sigma_2) + \Gamma^1_{13}(\sigma_1 - \sigma_3) - \Gamma^2_{23}(\sigma_2 - \sigma_3)] . \quad (27)
\end{aligned}$$

### 3.3 Form of $H_{ab}$

Given our choice of a tetrad frame which diagonalizes  $\sigma_{ab}$  (and hence according to theorem 1,  $\text{div}H = 0$ ) we show below that consistency of the Ricci constraint requires that  $H_{ab} = 0$ .

To calculate time derivatives of the constraints we first start from the covariant form of the constraint equation, say  $(G)$  and then take the time derivative as defined by  $(\dot{G}) \equiv (G)_{;a}u^a$ . We commute derivatives and use the constraint equations; the result is then converted to tetrad form. We give an example of such a calculation in Appendix A (using the ' $H_{ab}$ ' constraint). This approach avoids any direct calculations of the time derivatives of Ricci coefficients  $\Gamma^a_{bc}$ .

The time derivative of the “ $(0, \nu)$ ” constraint (4) in covariant form is

$$0 = h^{eb} \left[ \frac{2}{3} (\Theta_{;b}) \cdot - h^d{}_c (\sigma_b{}^c{}_{;d}) \cdot \right] . \quad (28)$$

If we use

$$(\Theta_{;b}) \cdot = (\dot{\Theta})_{;b} - \Theta_{;p}u^p{}_{;b} , \quad (29)$$

$$(\sigma_b{}^c{}_{;d}) \cdot = (\dot{\sigma}_b{}^c)_{;d} - \sigma_b{}^c{}_{;p}u^p{}_{;d} + R^c{}_{qp} \sigma_b{}^q u^p - R^q{}_{bpd} \sigma_q{}^c u^p \quad (30)$$

and substituting into (28) we get

$$\begin{aligned}
0 &= h^{eb} \left[ (\dot{\Theta})_{;b} - \Theta_{;p}u^p{}_{;b} - h^d{}_c \left\{ (\dot{\sigma}_b{}^c)_{;d} - \sigma_b{}^c{}_{;p}u^p{}_{;d} \right\} \right] \\
&\quad - h^{eb} h^d{}_c \left\{ R^c{}_{qp} \sigma_b{}^q u^p - R^q{}_{bpd} \sigma_q{}^c u^p \right\} . \quad (31)
\end{aligned}$$

We may now convert (31) into tetrad form and use (20) and (25). The Riemann tensor is given in terms of the Weyl tensor in Appendix A. On further simplification (see that Appendix for details) we obtain

$$0 = (\sigma_2 - \sigma_3) H_{23} , \quad 0 = (\sigma_3 - \sigma_1) H_{31} , \quad 0 = (\sigma_1 - \sigma_2) H_{12} . \quad (32)$$



The time derivative of the “ $H_{ab}$ ” constraint (4) in covariant form may be written as

$$\dot{H}_{ad} = h^{(t}{}_a H^s)_{d\eta}{}^c{}_{fbs} u^f \left[ \left( \dot{\sigma}^b{}_t \right)_{;c} - \sigma^b{}_{t;p} u^p{}_{;c} + R^b{}_{apc} \sigma^a{}_t u^p - R^q{}_{tpc} \sigma^b{}_q u^p \right] . \quad (33)$$

The tetrad form of (33) for  $a \neq b$  yields

$$0 = \sigma_1 H_{23} , \quad 0 = \sigma_2 H_{31} , \quad 0 = \sigma_3 H_{12} \quad (34)$$

(see the Appendix B for details).

For nonzero shear conditions (32) and (34) yields

$$H_{12} = H_{23} = H_{31} = 0, \quad (35)$$

and thus the “magnetic” part of the Weyl tensor  $H_{ab}$  is also diagonal. Through the  $\dot{H}$  equations, this implies  $curl E$  too is diagonal. We henceforth write  $H_{aa}$  as  $H_a$ ; then the trace free property of  $H_{ab}$  is

$$H_1 + H_2 + H_3 = 0 . \quad (36)$$

The tetrad form of (33) for  $a = b$  yields (see Appendix B)

$$\sigma_1 H_1 = \sigma_2 H_2 = \sigma_3 H_3 , \quad (37)$$

We point out that diagonal property (35) of  $H_{ab}$  was not used in obtaining (37). We now introduce a constant  $\lambda$  that relates the shear eigenvalue  $\sigma_1 = \sigma$  to  $\sigma_2$  as follows

$$\sigma_1 = \sigma, \quad \sigma_2 = \lambda \sigma . \quad (38)$$

If the shear tensor is degenerate in the  $e_1, e_2$  plane then  $\lambda = 1$ . Now equation (37)  $\sigma_1 H_1 = \sigma_2 H_2$  prescribes the following relation on the eigenvalues  $H_1, H_2$  of the magnetic Weyl tensor:

$$H_1 = \lambda H, \quad H_2 = H . \quad (39)$$

The trace-free property yields

$$\sigma_3 = -(1 + \lambda)\sigma, \quad H_3 = -(1 + \lambda)H , \quad (40)$$

so now from (37) if we use  $\sigma_1 H_1 = \sigma_3 H_3$  we obtain

$$0 = (1 + \lambda + \lambda^2)\sigma H \quad (41)$$

The following cases satisfy (41)

**Case A:**  $H = 0$  with  $\sigma \neq 0$  and  $\lambda$  remaining arbitrary. This case include degenerate shear,  $\lambda = 1$ , and is studied in [2, 7, 8, 9] and [5]. Variables defined in terms of the above tetrad frame allow development of useful phase- plane portraits, which show *inter alia* that the generic collapse configuration is a cigar or spindle rather than a pancake.

**Case B:**  $\sigma = 0$ . For this case all the shear eigenvalues vanish and this implies  $H = 0$  and  $E = 0$  and the model is FRW.

**Case C:**  $(1 + \lambda + \lambda^2) = 0$ . The values of  $\lambda$  are complex. The magnitudes of the shear tensor  $\sigma_{mag}$  and the magnetic Weyl tensor  $H_{mag}$  have the tetrad form

$$\begin{aligned} \sigma_{mag} &= \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2). \\ H_{mag} &= H_1^2 + H_2^2 + H_3^2. \end{aligned} \quad (42)$$

On using (38) and (40) for this case we get

$$\sigma_{mag} = (1 + \lambda + \lambda^2)\sigma^2 = 0, \quad H_{mag} = (1 + \lambda + \lambda^2)H^2 = 0 \quad (43)$$

and hence both shear and the magnetic Weyl tensor are zero, leading to the same results as case B above.

Thus we can formulate the following

**Theorem:**

For irrotational dust the divergence of the magnetic Weyl tensor vanishes,  $div H = 0$  (or equivalently the shear tensor is consistently diagonalizable) if and only if the magnetic Weyl tensor vanishes,  $H_{ab} = 0$ .

## 4 Conclusion

For irrotational dust  $\rho \neq 0$  the existence of a tetrad frame which is a principal frame for the shear tensor  $\sigma_{ab}$  and its covariant time derivative  $\dot{\sigma}_{ab}$  requires that the divergence of the magnetic Weyl tensor vanish. We have shown

here that if we employ this frame then the magnetic Weyl tensor itself vanishes;  $H_{ab} = 0$ , as a consistency requirement of the Ricci constraints. This establishes a new property, that the only consistent solutions for irrotational dust with vanishing  $div H$  are those described as the ‘silent’ universe [2],[5]. Hence gravitational waves interacting with irrotational dust will have to have  $div H \neq 0$ , contrary to the usual result of linearised theory.

The key point here is that (assuming the fluid is irrotational), our result comes from the second term in the ‘div H’ equation (8); but when we linearise, that term is necessarily second order. Thus always  $Div H = 0$  (to first order) in the linear case. Hence our exact result comes from a term which plays no role in the linearised theory, if we discard all second order terms in all equations; in this case the surviving Riemann terms would be disregarded (as they always consist of a Weyl tensor component, which is first order, multiplying a first order quantity: see (32), (34), (37)).

However this argument needs to be treated with care. One needs to recall that second order terms can only be dropped from an equation if there is a non-zero first order term present, so that the second-order term is negligible relative to the first-order one. However this argument cannot be applied to the key equation (8): both terms are second order, and as there is no first order term, we have to ensure that the equation is true to second order accuracy. Thus even in linearised theory, we cannot ignore this second-order equation. The following intriguing situation results: if we have a linearised solution where  $H_{ab} \neq 0$  and  $div H$  is non-zero but second order, we can presumably get a consistent solution. If however  $H_{ab} \neq 0$  with  $div H$  *exactly* zero, the solution will not be consistent - because of the above proof.

Thus in the linear theory, it is possible to have models where  $Div H = 0$  to second order but  $H_{ab} \neq 0$ , and indeed that is usually assumed for gravitational waves. Our result (for irrotational dust only) shows that in this case in fact we must have  $div H \neq 0$ , although it is second order. We have therefore an example of linearization instability in that the usual process of linearization leads to a different answer than the exact result - which in fact constrains the linearised solution, even though this is usually not commented on.

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## References

- [1] Bruni, M., Dunsby, P. K. S., and Ellis, G. F. R., 1992, *Ap.J* **34**,395
- [2] Bruni, M., Matarrese, S., and Pantano, O., 1994, to appear in *Ap.J.*  
Bruni, M., Matarrese, S., and Pantano, O., 1994, to appear in *Phys. Rev. Lett.*
- [3] Barnes, A., and Rowlingson, R. R. 1989, *Class. Quantum Grav.*, **6**, 949.
- [4] Ellis, G. F. R. 1971, in *General Relativity and Cosmology*, ed. R. K. Sachs (New York: Academic Press).
- [5] Lesame, W. M., Dunsby, P. K. S., and Ellis, G. F. R., 1994, *Integrability condition for irrotational dust with a purely electric Weyl tensor: A tetrad analysis*, to appear, *Phys Rev D*.
- [6] Lesame, W. M., 1995, *Irrotational dust with a purely magnetic Weyl tensor*, UCT Preprint (1995).
- [7] Matarrese, S., Pantano, O., and Saez, D. 1993, *Phys. Rev. D* **47**, 1311.
- [8] Matarrese, S., Pantano, O., and Saez, D. 1994, *Phys. Rev. Lett.*, **72**, 320.
- [9] Matarrese, S., Pantano, O., and Saez, D. 1994, *MNRAS* **271**, 3, 513.

## A Time propagation of the “ $(0, \mu)$ ” constraint

The covariant form of the “ $(0, \mu)$ ” constraint (4) is

$$h^e_b \left( \frac{2}{3} \Theta^{;b} - h^d_c \sigma^{bc}_{;d} \right) = 0 , \quad (44)$$

with tetrad form

$$\begin{aligned}
\frac{2}{3}\partial_1\Theta &= \partial_1\sigma_1 + (\sigma_1 - \sigma_2)\Gamma^2_{21} + (\sigma_1 - \sigma_3)\Gamma^3_{31} , \\
\frac{2}{3}\partial_2\Theta &= \partial_2\sigma_2 + (\sigma_2 - \sigma_1)\Gamma^1_{12}(\sigma_2 - \sigma_3)\Gamma^3_{32} , \\
\frac{2}{3}\partial_3\Theta &= \partial_3\sigma_3 + (\sigma_3 - \sigma_1)\Gamma^1_{13} + (\sigma_3 - \sigma_2)\Gamma^2_{23} ,
\end{aligned} \tag{45}$$

The covariant time propagation of (44) is (31),

$$\begin{aligned}
0 &= h^{eb} \left[ \frac{2}{3}(\Theta_{;b})^\cdot - h^d{}_c(\sigma_b{}^c{}_{;d})^\cdot \right] \\
&= h^{eb} \left[ (\dot{\Theta})_{;b} - \Theta_{;p}u^p{}_{;b} - h^d{}_c \left\{ (\dot{\sigma}_b{}^c)_{;d} - \sigma_b{}^c{}_{;p}u^p{}_{;d} \right\} \right] \\
&\quad - h^{eb}h^d{}_c \left\{ R^c{}_{qp}d\sigma_b{}^q u^p - R^q{}_{bpd}\sigma_q{}^c u^p \right\} .
\end{aligned} \tag{46}$$

We now write the Riemann tensor  $R^s{}_{mpc}$  in the terms of the Weyl tensor as

$$\begin{aligned}
R^s{}_{mpc} &= C^s{}_{mpc} + \frac{1}{2}(g^s{}_p R_{cm} + g^s{}_c R_{pm} - g_{mp}R^s{}_c + g_{mc}R^s{}_p) \\
&\quad - \frac{R}{6}(g^s{}_p g_{cm} - g^s{}_c g_{pm})
\end{aligned} \tag{47}$$

where

$$\begin{aligned}
C^s{}_{mpc} &\equiv (\eta^s{}_{mij}\eta_{pckl} + g^s{}_{mij}g_{pckl})u^i u^k E^{bd} \\
&\quad + (\eta^s{}_{mij}g_{pckl} + \eta_{pckl}g^s{}_{mij})u^i u^k H^{bd} ;
\end{aligned} \tag{48}$$

and

$$g^s{}_{mij} \equiv g^s{}_i g_{mj} - g^s{}_j g_{mi} . \tag{49}$$

The two terms involving the Riemann tensor  $R^a{}_{bcd}$  in (46) may be written in covariant form as

$$\begin{aligned}
(RT)^e &= -h^{eb}h^d{}_c [R^c{}_{qp}d\sigma_b{}^q u^p - R^q{}_{bpd}\sigma_q{}^c u^p] \\
&= -h^{eb}h^d{}_c u^p \left[ \frac{1}{2}\sigma^q{}_b (g^c{}_p R_{qd} + g^c{}_d R_{qp} - g_{qp}R^c{}_d + g_{qd}R^c{}_p) \right. \\
&\quad - \frac{1}{2}\sigma^c{}_q (g^q{}_p R_{bd} + g^q{}_d R_{bp} - g_{bp}R^q{}_d + g_{bd}R^q{}_p) \\
&\quad \left. - \frac{R}{6}\sigma^q{}_b (g^c{}_p g_{qd} - g^c{}_d g_{qp}) + \frac{R}{6}\sigma^c{}_q (g^q{}_p g_{bd} - g^q{}_d g_{bp}) \right] \\
&- h^{eb}h^d{}_c u^p u^i u^k E^{jl} [\sigma^q{}_b \eta^c{}_{qij} \eta_{pdkl} + \sigma^q{}_b (g^c{}_i g_{qj} - g^c{}_j g_{qi}) (g_{pk}g_{dl} - g_{pl}g_{dk})]
\end{aligned}$$

$$\begin{aligned}
& -\sigma^c_q \eta^q_{bij} \eta_{pdkl} - \sigma^c_q (g^q_i g_{bj} - g^q_j g_{bi}) (g_{pk} g_{dl} - g_{pl} g_{dk}) \\
& - h^{eb} h^d_c u^p u^i u^k H^{jl} [\sigma^q_b \eta^c_{qij} (g_{pk} g_{dl} - g_{pl} g_{dk}) - \sigma^c_q \eta^q_{bij} (g_{pk} g_{dl} - g_{pl} g_{dk}) \\
& + \sigma^q_b \eta_{pdkl} (g^c_i g_{qj} - g^c_j g_{qi}) - \sigma^c_q \eta_{pdkl} (g^q_i g_{cj} - g^q_j g_{ci})] . \tag{50}
\end{aligned}$$

The only non-vanishing terms in (50) are

$$\begin{aligned}
(RT)^e & = -h^{eb} h^d_c u^p u^i u^k H^{jl} [\sigma^q_b \eta^c_{qij} g_{pk} g_{dl} - \sigma^c_q \eta^q_{bij} g_{pk} g_{dl}] \\
& = \eta^c_{qij} \sigma^{eq} H^j_c u^i - h^{eb} \eta^q_{bij} \sigma_{ql} H^{jl} u^i . \tag{51}
\end{aligned}$$

The tetrad form of (51) becomes

$$(RT)^e = \eta^c_{q0j} \sigma^{eq} H^j_c - \eta^q_{e0j} \sigma_{ql} H^{jl} . \tag{52}$$

and from (52) for  $e = 1$  we obtain

$$\begin{aligned}
(RT)^1 & = \eta^c_{q0j} \sigma^{1q} H^j_c - \eta^q_{10j} \sigma_{ql} H^{jl} \\
& = \eta^3_{102} \sigma^1 H^2_3 + \eta^2_{103} \sigma^1 H^3_2 - \eta^3_{102} \sigma_3 H^{23} - \eta^2_{103} \sigma_2 H^{32} \\
& = H_{23} (\sigma_3 - \sigma_2) . \tag{53}
\end{aligned}$$

Similarly from (52) for  $e = 2, 3$  we get

$$\begin{aligned}
(RT)^2 & = H_{31} (\sigma_1 - \sigma_3) , \\
(RT)^3 & = H_{12} (\sigma_2 - \sigma_1) . \tag{54}
\end{aligned}$$

We now focus on the tetrad form of the time propagation equation (46) for  $e \equiv \underline{1}$ ; in expanded form this becomes

$$\begin{aligned}
0 & = \frac{2}{3} [\partial_1 \dot{\theta} - \theta_1 \partial_1 \theta] - [\partial_1 \dot{\sigma}_1 - \theta_1 \partial_1 \sigma_1] \\
& \quad - \Gamma^2_{21} [(\dot{\sigma}_1 - \dot{\sigma}_2) - \theta_2 (\sigma_1 - \sigma_2)] - \Gamma^3_{31} [(\dot{\sigma}_1 - \dot{\sigma}_3) - \theta_3 (\sigma_1 - \sigma_3)] \\
& \quad + H_{23} (\sigma_3 - \sigma_2) , \tag{55}
\end{aligned}$$

where the last term on the right hand side is the contribution of the Riemann term  $(RT)^1$  calculated in (53). On using  $\dot{\theta}$  and  $\dot{\sigma}_\mu$  from equation (20) we get

$$\begin{aligned}
0 & = \frac{2}{3} \left[ -\frac{2}{3} \theta \partial_1 \theta - \partial_1 (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{1}{2} \partial_1 \rho - (\sigma_1 + \frac{1}{3} \theta) \partial_1 \theta \right] \\
& \quad - \left[ -2\sigma_1 \partial_1 \sigma_1 - \frac{2}{3} \theta \partial_1 \sigma_1 - \frac{2}{3} \sigma_1 \partial_1 \theta + \frac{1}{3} \partial_1 (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \sigma_1 \partial_1 \sigma_1 \right]
\end{aligned}$$

$$\begin{aligned}
& -\partial_1 E_1 - (\sigma_1 + \frac{1}{3}\theta)\partial_1 \sigma_1] \\
& -\Gamma^2_{21} \left[ -(\sigma_1^2 - \sigma_2^2) - \frac{2}{3}\theta(\sigma_1 - \sigma_2) - (E_1 - E_2) - (\sigma_2 + \frac{1}{3}\theta)(\sigma_1 - \sigma_2) \right] \\
& -\Gamma^3_{31} \left[ -(\sigma_1^2 - \sigma_3^2) - \frac{2}{3}\theta(\sigma_1 - \sigma_3) - (E_1 - E_3) - (\sigma_3 + \frac{1}{3}\theta)(\sigma_1 - \sigma_3) \right] \\
& + H_{23}(\sigma_3 - \sigma_2) , \tag{56}
\end{aligned}$$

where we also used  $u_{a;b} = \sigma_{ab} + \frac{1}{3}\theta h_{ab}$  and  $\theta_\mu = \sigma_\mu + \frac{1}{3}\theta$ . We now apply to (56) the “ $(0, \mu)$ ” constraint (45) and the “*div E*” constraint written here in tetrad form as

$$\begin{aligned}
\frac{1}{3}\partial_1 \rho &= \partial_1 E_1 + (E_1 - E_2)\Gamma^2_{21} + (E_1 - E_3)\Gamma^3_{31} + H_{23}(\sigma_3 - \sigma_2) , \\
\frac{1}{3}\partial_2 \rho &= \partial_2 E_2 + (E_2 - E_1)\Gamma^1_{12} + (E_2 - E_3)\Gamma^3_{32} + H_{31}(\sigma_1 - \sigma_3) , \\
\frac{1}{3}\partial_3 \rho &= \partial_3 E_3 + (E_3 - E_1)\Gamma^1_{13} + (E_3 - E_2)\Gamma^2_{23} + H_{12}(\sigma_2 - \sigma_1) . \tag{57}
\end{aligned}$$

to obtain the following form of (56)

$$\begin{aligned}
0 &= 3\sigma_1\partial_1\sigma_1 - \partial_1(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \\
&\quad - \Gamma^2_{21}(\sigma_1 - \sigma_2)(\sigma_3 - \sigma_2) - \Gamma^3_{31}(\sigma_1 - \sigma_3)(\sigma_2 - \sigma_3) . \tag{58}
\end{aligned}$$

The first term on the right of (58) simplifies as follows

$$\begin{aligned}
3\sigma_1\partial_1\sigma_1 - \partial_1(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) &= 3\sigma_1\partial_1\sigma_1 - 2\sigma_1\partial_1\sigma_1 - \partial_1(\sigma_2^2 + \sigma_3^2) \\
&= \sigma_1\partial_1\sigma_1 - \partial_1(\sigma_2^2 + \sigma_3^2) \\
&= \frac{1}{2}\partial_1(-\sigma_1^2 - 2\sigma_2^2 - 2\sigma_3^2) \\
&= \frac{1}{2}\partial_1[(\sigma_2 - \sigma_3)^2 - 2\sigma_2^2 - 2\sigma_3^2] \\
&= -\frac{1}{2}\partial_1(\sigma_2^2 - 2\sigma_2\sigma_3 + \sigma_3^2) \\
&= -\frac{1}{2}\partial_1(\sigma_2 - \sigma_3)^2 \\
&= -(\sigma_2 - \sigma_3)\partial_1(\sigma_2 - \sigma_3) , \tag{59}
\end{aligned}$$

so now (59) in (58) gives

$$\begin{aligned}
0 &= (\sigma_2 - \sigma_3) \left[ \partial_1(\sigma_3 - \sigma_2) + \Gamma^2_{21}(\sigma_1 - \sigma_2) - \Gamma^3_{31}(\sigma_1 - \sigma_3) \right] \\
&= (\sigma_2 - \sigma_3)H_{23} , \tag{60}
\end{aligned}$$

where in the last step we used the “ $H_{23}$ ” constraint (27)

$$H_{23} = \frac{1}{2} \left[ \partial_1(\sigma_2 - \sigma_3) - \Gamma^2_{21}(\sigma_1 - \sigma_2) + \Gamma^3_{31}(\sigma_1 - \sigma_3) \right] . \tag{61}$$

Similar relations to (60) follows for  $e = 2, 3$  in (46) these are

$$\begin{aligned} 0 &= (\sigma_3 - \sigma_1)H_{31} , \\ 0 &= (\sigma_1 - \sigma_2)H_{12} , \end{aligned} \quad (62)$$

Thus we have derived the required equations (32).

## B The time derivative of the $H_{ab}$ constraint

We first write the covariant form (5) of the “ $H_{ab}$ ” constraint as

$$\begin{aligned} H_{ad} &= -h^t_a h^s_d \sigma^b_{(t}{}^c \eta_{s)fb} u^f . \\ &= -\frac{1}{2} h^t_a h^s_d u^f \left[ \sigma^b_{t;c} \eta_{sfb}{}^c + \sigma^b_{s;c} \eta_{tfb}{}^c \right] . \end{aligned} \quad (63)$$

If we use

$$(\sigma^b_{s;c})^\cdot = (\dot{\sigma}^b_s)_{;c} - \sigma^b_{s;p} u^p{}_{;c} + R^b_{qpc} \sigma^q_s u^p - R^q_{spc} \sigma^b_q u^p , \quad (64)$$

the time propagation of (63) becomes

$$\begin{aligned} \dot{H}_{ad} &= -\frac{1}{2} h^t_a h^s_d u^f \left[ \eta_{sfb}{}^c \left\{ (\dot{\sigma}^b_t)_{;c} - \sigma^b_{t;p} u^p{}_{;c} \right\} + \eta_{tfb}{}^c \left\{ (\dot{\sigma}^b_s)_{;c} - \sigma^b_{s;p} u^p{}_{;c} \right\} \right] \\ &\quad -\frac{1}{2} h^t_a h^s_d u^f \left[ \eta_{sfb}{}^c \left\{ R^b_{qpc} \sigma^q_t u^p - R^q_{tpc} \sigma^b_q u^p \right\} + \eta_{tfb}{}^c \left\{ R^b_{qpc} \sigma^q_s u^p - R^q_{spc} \sigma^b_q u^p \right\} \right] . \end{aligned} \quad (65)$$

Using the Riemann tensor  $R^s_{mpc}$  expression as given in the previous Appendix, the two terms involving the Riemann tensor  $R^a_{bcd}$  in (65) may be written in covariant form as

$$\begin{aligned} (RT)_{ad} &= -\frac{1}{2} h^t_a h^s_d u^f \left[ \eta_{sfb}{}^c \left\{ R^b_{qpc} \sigma^q_t u^p - R^q_{tpc} \sigma^b_q u^p \right\} \right. \\ &\quad \left. + \eta_{tfb}{}^c \left\{ R^b_{qpc} \sigma^q_s u^p - R^q_{spc} \sigma^b_q u^p \right\} \right] \\ &= -\frac{1}{2} h^t_a h^s_d u^f \eta_{sfb}{}^c \left[ \frac{1}{2} \sigma^q_t \left( g^b_p R_{qc} + g^b_p R_{qp} - g_{qp} R^b_c + g_{qc} R^b_p \right) \right. \\ &\quad - \frac{1}{2} \sigma^b_q \left( g^q_p R_{tc} + g^q_c R_{tp} - g_{tp} R^q_c + g_{tc} R^q_p \right) \\ &\quad \left. - \frac{R}{6} \sigma^q_t \left( g^b_p g_{qc} - g^b_c g_{qp} \right) + \frac{R}{6} \sigma^b_q \left( g^q_p g_{tc} - g^q_c g_{tp} \right) \right] \\ &\quad + u^i u^k E^{jl} \left\{ \sigma^q_t \eta^b_{qij} \eta_{pckl} + \sigma^q_t \left( g^b_i g_{qj} - g^b_j g_{qi} \right) \left( g_{pk} g_{cl} - g_{pl} g_{ck} \right) \right\} \end{aligned}$$



$$\begin{aligned}
& -\sigma^b_q \eta^q_{tij} \eta_{pckl} - \sigma^b_q (g^q_i g_{tj} - g^q_j g_{ti}) (g_{pk} g_{cl} - g_{pl} g_{ck}) \} \\
& + u^i u^k H^{jl} \left\{ \sigma^q_t \eta^b_{qij} (g_{pk} g_{cl} - g_{pl} g_{ck}) - \sigma^b_q \eta^q_{tij} (g_{pk} g_{cl} - g_{pl} g_{ck}) \right. \\
& \quad \left. + \sigma^q_t \eta_{pckl} (g^b_i g_{qj} - g^b_j g_{qi}) - \sigma^b_q \eta_{pckl} (g^q_i g_{tj} - g^q_j g_{ti}) \right\} \\
& - \frac{1}{2} h^t_a h^s_d u^f \eta_{tfc} \left[ \frac{1}{2} \sigma^q_s (g^b_p R_{qc} + g^b_p R_{qp} - g_{qp} R^b_c + g_{qc} R^b_p) \right. \\
& \quad - \frac{1}{2} \sigma^b_q (g^q_p R_{sc} + g^q_c R_{sp} - g_{sp} R^q_c + g_{sc} R^q_p) \\
& \quad \left. - \frac{R}{6} \sigma^q_s (g^b_p g_{qc} - g^b_c g_{qp}) + \frac{R}{6} \sigma^b_q (g^q_p g_{sc} - g^q_c g_{sp}) \right] \\
& + u^i u^k E^{jl} \left\{ \sigma^q_s \eta^b_{qij} \eta_{pckl} + \sigma^q_s (g^b_i g_{qj} - g^b_j g_{qi}) (g_{pk} g_{cl} - g_{pl} g_{ck}) \right. \\
& \quad \left. - \sigma^b_q \eta^q_{sij} \eta_{pckl} - \sigma^b_q (g^q_i g_{sj} - g^q_j g_{si}) (g_{pk} g_{cl} - g_{pl} g_{ck}) \right\} \\
& + u^i u^k H^{jl} \left\{ \sigma^q_s \eta^b_{qij} (g_{pk} g_{cl} - g_{pl} g_{ck}) - \sigma^b_q \eta^q_{sij} (g_{pk} g_{cl} - g_{pl} g_{ck}) \right. \\
& \quad \left. + \sigma^q_s \eta_{pckl} (g^b_i g_{qj} - g^b_j g_{qi}) - \sigma^b_q \eta_{pckl} (g^q_i g_{sj} - g^q_j g_{si}) \right\} . \quad (66)
\end{aligned}$$

The only non-vanishing terms in (66) are those containing  $H^{jl}$  i.e.,

$$\begin{aligned}
(RT)_{ad} & = -\frac{1}{2} h^t_a h^s_d u^f \eta_{sfb}{}^c u^i u^k H^{jl} \left( \sigma^q_t \eta^b_{qij} g_{pk} g_{cl} - \sigma^b_q \eta^q_{tij} g_{pk} g_{cl} \right) \\
& \quad - \frac{1}{2} h^t_a h^s_d u^f \eta_{tfc} u^i u^k H^{jl} \left( \sigma^q_s \eta^b_{qij} g_{pk} g_{cl} - \sigma^b_q \eta^q_{sij} g_{pk} g_{cl} \right) \\
& = -\frac{1}{2} h^t_a h^s_d u^f u^p u^i u^k H^j{}_c \left[ \eta_{sfb}{}^c (g_{pk} \eta^b_{qij} \sigma^q_t - g_{pk} \eta^q_{tij} \sigma^b_q) \right. \\
& \quad \left. + \eta_{tfc}{}^c (g_{pk} \eta^b_{qij} \sigma^q_s - g_{pk} \eta^q_{sij} \sigma^b_q) \right] \\
& = \frac{1}{2} h^t_a h^s_d u^f u^i H^j{}_c \left[ \eta_{sfb}{}^c (\eta^b_{qij} \sigma^q_t - \eta^q_{tij} \sigma^b_q) \right. \\
& \quad \left. + \eta_{tfc}{}^c u^f (\eta^b_{qij} \sigma^q_s - \eta^q_{sij} \sigma^b_q) \right] . \quad (67)
\end{aligned}$$

Equation (67) represent the covariant form of the Riemann term. The tetrad form of (67) for **diagonal** elements  $a = d$  becomes

$$(RT)_{aa} = \eta_{a0b}{}^c \eta^b_{a0j} H^j{}_c (\sigma_a - \sigma_b) . \quad (68)$$

and hence

$$(RT)_{11} = H_3(\sigma_1 - \sigma_2) + H_2(\sigma_1 - \sigma_3) ,$$

$$\begin{aligned}
(RT)_{22} &= H_1(\sigma_2 - \sigma_3) + H_3(\sigma_2 - \sigma_1), \\
(RT)_{33} &= H_2(\sigma_3 - \sigma_1) + H_1(\sigma_3 - \sigma_2).
\end{aligned} \tag{69}$$

The tetrad form of the time propagation equation (65) for  $a = d$  now becomes

$$\begin{aligned}
\dot{H}_{aa} &= -\eta_{a0b}{}^c \left[ \partial_c \dot{\sigma}^b{}_a - \theta_c \partial \sigma^b{}_a + \Gamma^b{}_{ca} \{(\dot{\sigma}_a - \dot{\sigma}_b)\} - \theta_c(\sigma_a - \sigma_b) \right] \\
&+ (RT)_{aa} \quad (\text{no sum over } a).
\end{aligned} \tag{70}$$

For  $a = 1$  (70) becomes

$$\dot{H}_{11} = \Gamma^3{}_{21} [(\dot{\sigma}_1 - \dot{\sigma}_3) - \theta_2(\sigma_1 - \sigma_3)] - \Gamma^2{}_{31} [(\dot{\sigma}_1 - \dot{\sigma}_2) - \theta_3(\sigma_1 - \sigma_2)] + (RT)_{11}. \tag{71}$$

On using  $\dot{\sigma}_\mu$  in (20) and constraint (26) we get

$$\dot{H}_{11} = -\theta H_{11} + I_{11} + (RT)_{11}. \tag{72}$$

If we compare with the propagation equation for  $\dot{H}_{11}$  in (21) i.e.,

$$\dot{H}_{11} = -\theta H_{11} + I_{11} + 3\sigma_1 H_{11} - (\sigma_1 H_{11} + \sigma_2 H_{22} + \sigma_3 H_{33}), \tag{73}$$

we get

$$\begin{aligned}
&3\sigma_1 H_{11} - (\sigma_1 H_{11} + \sigma_2 H_{22} + \sigma_3 H_{33}) \\
&= (RT)_{11} = H_3(\sigma_1 - \sigma_2) + H_2(\sigma_1 - \sigma_3).
\end{aligned} \tag{74}$$

We note however that

$$\begin{aligned}
3\sigma_1 H_{11} - (\sigma_1 H_{11} + \sigma_2 H_{22} + \sigma_3 H_{33}) &= 2\sigma_1 H_{11} - \sigma_2 H_{22} - \sigma_3 H_{33} \\
&= [\sigma_1 H_{11} - \sigma_2 H_{22}] + [\sigma_1 H_{11} - \sigma_3 H_{33}] \\
&= [\sigma_1 H_{11} + (\sigma_1 + \sigma_3) H_{22}] + [\sigma_1 H_{11} + (\sigma_1 + \sigma_2) H_{33}] \\
&= [\sigma_1 (H_{11} + H_{22}) + \sigma_3 H_{22}] + [\sigma_1 (H_{11} + H_{33}) + \sigma_2 H_{33}] \\
&= -\sigma_1 H_{33} + \sigma_3 H_{22} - \sigma_1 H_{22} + \sigma_2 H_{33} \\
&= H_{33}(\sigma_2 - \sigma_1) + H_{22}(\sigma_3 - \sigma_1) \\
&= -(RT)_{11}.
\end{aligned} \tag{75}$$

Thus (74) together with (75) gives

$$0 = 3\sigma_1 H_{11} - (\sigma_1 H_{11} + \sigma_2 H_{22} + \sigma_3 H_{33}), \tag{76}$$

$$0 = (RT)_{11} = H_3(\sigma_1 - \sigma_2) + H_2(\sigma_1 - \sigma_3). \tag{77}$$

Similar results hold for  $a = d = 2$  and  $a = d = 3$  respectively

$$0 = 3\sigma_2 H_{22} - (\sigma_1 H_{11} + \sigma_2 H_{22} + \sigma_3 H_{33}) , \quad (78)$$

$$0 = (RT)_{22} = H_1(\sigma_2 - \sigma_3) + H_3(\sigma_2 - \sigma_1) . \quad (79)$$

$$0 = 3\sigma_3 H_{33} - (\sigma_1 H_{11} + \sigma_2 H_{22} + \sigma_3 H_{33}) , \quad (80)$$

$$0 = (RT)_{33} = H_2(\sigma_3 - \sigma_1) + H_1(\sigma_3 - \sigma_2) . \quad (81)$$

If we write (77),(79),(81) together we get

$$0 = (RT)_{11} = H_3(\sigma_1 - \sigma_2) + H_2(\sigma_1 - \sigma_3) , \quad (82)$$

$$0 = (RT)_{22} = H_1(\sigma_2 - \sigma_3) + H_3(\sigma_2 - \sigma_1) , \quad (83)$$

$$0 = (RT)_{33} = H_2(\sigma_3 - \sigma_1) + H_1(\sigma_3 - \sigma_2) , \quad (84)$$

which are consistent with each other.

Similarly if we write equations (76),(78),(80) which are equivalent to (77),(79),(81) we get

$$0 = 3\sigma_1 H_{11} - (\sigma_1 H_{11} + \sigma_2 H_{22} + \sigma_3 H_{33}) , \quad (85)$$

$$0 = 3\sigma_2 H_{22} - (\sigma_1 H_{11} + \sigma_2 H_{22} + \sigma_3 H_{33}) , \quad (86)$$

$$0 = 3\sigma_3 H_{33} - (\sigma_1 H_{11} + \sigma_2 H_{22} + \sigma_3 H_{33}) , \quad (87)$$

$$(88)$$

from which it follows that

$$\sigma_1 H_{11} = \sigma_2 H_{22} = \sigma_3 H_{33} . \quad (89)$$

Equation (89), which is the required equation (37), is a condition arising from consistency requirement on the “ $H_{ad}$ ” for  $a = d$ .

Similarly for **non-diagonal** elements  $a \neq b$  equation (67)

$$(RT)_{23} = \frac{3}{2}\sigma_1 H_{23} , (RT)_{31} = \frac{3}{2}\sigma_2 H_{31} , (RT)_{12} = \frac{3}{2}\sigma_3 H_{12} , \quad (90)$$

and the time propagation equation (65) yields

$$\begin{aligned} \sigma_1 H_{23} &= 0 , & (a = 2, d = 3) ; \\ \sigma_2 H_{13} &= 0 , & (a = 1, d = 3) ; \\ \sigma_3 H_{12} &= 0 , & (a = 1, d = 2) . \end{aligned} \quad (91)$$

which is the required equation (34).