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We derive an exact expression for the single particle Green function in the Calogero-Sutherland model for all the rational values of the coupling β. The calculation is based on Jack polynomial techniques and the results are given in the thermodynamical limit. Two type of intermediate states contribute. The first one consists of a particle propagating out of the Fermi sea and the second one consists of a particle propagating in one direction, q particles in the opposite direction and p holes.

I. INTRODUCTION

The Calogero-Sutherland model describes interacting particles on a circle [2,3]. Recent interest in the model arose from its relation with the Haldane-Shastry chain ($\beta = 2$) and also with the random matrix theory. Moreover, it has proven to be a good model to study fractional statistics.

There has been several works devoted to derive the correlation functions in the Calogero-Sutherland model [11,12,8,6,9,7,10]. In particular, the density-density correlation functions were obtained. The Green function where the annihilation operator acts before the creation operator was also computed. Recently, Zirnbauer and Haldane obtained the Green function with the creation operator acting first for the special values $\beta = 1/2, 1, 2$ [14]. In this paper we compute this Green function in a simple way for all the rational values of β using Jack polynomial techniques. The method we use is closly related to our preceeding paper [8], but the computation is more involved. There are also new physical phenomenas: Contrary to what one might naively expect, the intermediate states do not consist only of one particle which propagates out of the Fermi sea. There is an additional contribution where one particle propagates in one direction, q particles in the opposite direction and p holes.

The model is defined on a circle of length L and the Hamiltonian given by :

$$\hat{H} = -\sum_{j=1}^{N} \frac{1}{2}\frac{d^2}{dx_j^2} + \beta(\beta-1)\frac{\pi^2}{L^2}\sum_{i<j}\frac{1}{\sin^2(\pi(x_i-x_j)/L)} \tag{1}$$

where we use units in which $\hbar = m = 1$. Its eigenfunctions are known and written :

$$\Psi(x_i) = \Delta^\beta(x)\Phi(x_i) \tag{2}$$

with :

$$\Delta(x) = \prod_{i<j}\sin\left(\frac{\pi(x_i-x_j)}{L}\right) \tag{3}$$

and $\Phi(x_i)$ is a symmetric function of the variables x_i. The functions $\Phi(x_i)$ are symmetric polynomials in the variables $z_j = e^{i2\pi x_j/L}$ and they are labeled by a partition λ. They are known as the Jack polynomials $J_\lambda(z_i;\beta)$. The properties of these polynomials were recently studied in the mathematical literature [16,18,15].

The emergence of fractional statistics is seen by considering the spectrum of $\hat{H}$. This spectrum is like a free spectrum and is described by a set of quasi-momenta k_i. The momentum and energy are given by additive laws :

$$Q_\alpha = \frac{2\pi}{L}\sum_{i=1}^{N} k_i, \quad E_\alpha = \frac{2\pi^2}{L^2}\sum_{i=1}^{N} k_i^2, \tag{4}$$

and interaction between the particles is encoded in the fact that the quasi-momenta k_i obey a generalized exclusion principle :

$$k_{i+1} - k_i = \beta + I_i. \tag{5}$$

where I_i are positive integers or zero. The ground state is described by the configuration of k_i's the most densely packed around the origin, $I_i = 0$. For N odd, the ground state momenta are :

$$k_i^{(0)} = \beta \left(i - \frac{N+1}{2} \right) .$$

By analogy with the fermion case ($\beta = 1$) we call this configuration the Fermi sea. To describe the elementary excitations it is convenient to multiply the quasi-momenta k_i by q so that they differ by integer numbers. Then, the Fermi sea can be described by a set of occupied quasi-momenta separated by $p-1$ unoccupied ones. In this occupation number representation, a particle corresponds to one 1 followed by $p-1$ zeroes and a hole corresponds to a sequence of q zeroes. Note that the effect of removing q particles from the Fermi sea is to create p holes (see figure 1).

a) ... 0 0 0 0 0 (1 0)(1 0)(1 0)(1 0)(1 0)(1 0)(1 0) 0 0 0 0 0 0 0 0 0 0 0 ...

b) ... (1 0)0 0 0 (1 0)(0 0 0)(1 0)(0 0 0)(1 0)(1 0) 0 0 0 (1 0)(1 0) 0 0 ...

-k F $\qquad\qquad$ k F

Fig.1:a) The ground state at $\beta = 2/3$ b) an excitation with 3 particles and 2 holes .

This description does not tell which type of excitations are produced when one acts with a specific operator. This information can only be obtained by considering the properties of the wave functions. In the next part, we use these properties to obtain the Green function form factor.

II. GREEN FUNCTION.

The quantity of interest here is the Green function with the creation operator acting before the annihilation operator. The results in this section follow the paper [8] where the notations are defined. It is written :

$$\begin{aligned} {}_N< 0|\Psi(x,t)\Psi^\dagger(0,0)|0>_N &= \sum_{|\alpha>} \frac{{}_N<0|\Psi(x,t)|\alpha>_{N+1}\,{}_{N+1}<\alpha|\Psi^\dagger(0,0)|0>_N}{{}_{N+1}<\alpha|\alpha>_{N+1}\,{}_N<0|0>_N} \\ &= \sum_{|\alpha>} \frac{|{}_N<0|\Psi(0,0)|\alpha>_{N+1}|^2}{{}_{N+1}<\alpha|\alpha>_{N+1}\,{}_N<0|0>_N} e^{-iE_\alpha t - iQ_\alpha x} \end{aligned} \tag{6}$$

where $|\alpha>_{N+1}$ is a complete basis of states with $N+1$ particles and $|0>_N$ has N particles. This complete set of states can be expressed in terms of Jack polynomials in $N+1$ variables :

$$|\alpha>_{N+1} = |n,\lambda> = \prod_{i=1}^{N+1} z_i^n J_\lambda(z_i;\beta)|0>_{N+1} \tag{7}$$

with

$$|0>_{N+1} = \prod_{i=1}^{N+1} z_i^{-\beta(N+1)/2} \prod_{i<j}(z_i - z_j)^\beta . \tag{8}$$

and

$$|0>_N = \prod_{i=1}^{N} z_i^{-\beta(N-1)/2} \prod_{i<j}(z_i - z_j)^\beta . \tag{9}$$

n is an arbitrary integer and the partition λ is such that $l(\lambda) < N+1$ in order to avoid double counting of states. The quasi-momenta k_i entering the expression of the energy and momentum (4) of the state $|\alpha>$ are given in terms of λ by :

$$k_i = \lambda_{N-i+1} + n + \beta\left(i - \frac{N+3}{2}\right). \tag{10}$$

The vacuum state $|0>_{N+1}$ has been chosen in such a way that the quasimomentum of the $N+1$ particle is at the left of the Fermi sea of the N particle vacuum. Another possible choice would have been to put this particle to the right by replacing the exponent of z_i by $-\beta(N-1)/2$ in the expression (8) of $|0>_{N+1}$. If β is an integer, these two states belong to the same Hilbert space and the two definitions give the same form factor in the thermodynamical limit. If β is not integer, we shall see that the two possible choices give different form factors in the thermodynamical limit.

The action of $\Psi(0,0)$ is to remove a particle at the positions $x_i = 0$, therefore setting the variable z_i to 1. The matrix element we need to compute is :

$$M_\alpha = {}_N< 0|\prod_i z_i^{n-\beta}(1-z_i)^\beta J_\lambda(1, z_j;\beta)|0>_N, \tag{11}$$

for all λ. It is possible to expand the Jack polynomial in $N+1$ variables on polynomials in N variables :

$$J_\lambda(1, z_i;\beta) = \sum_\nu J_{\lambda/\nu}(1;\beta) J_\nu(z_i;\beta) \tag{12}$$

and here $J_{\lambda/\nu}(1;\beta)$ is different from zero only when λ covers ν and when the difference between λ and ν is at most a box per column. When these conditions are satisfied, the coefficient is given by [18] :

$$J_{\lambda/\nu}(1;\beta) = \prod_{s\in C_{\lambda/\nu}} \left(\frac{\beta l(s) + a(s) + 1}{\beta(l(s)+1) + a(s)}\right)_\lambda \left(\frac{\beta(l(s)+1) + a(s)}{\beta l(s) + a(s) + 1}\right)_\nu. \tag{13}$$

The notations here are as follow: $s = (i,j)$ is a box on a Young tableau identified by its coordinates $1 \le j \le \lambda_i$. Given a box, s, $a(s)$ is the number of boxes on its right and $l(s)$ the number of boxes under it. The notation $(...)_\lambda$ indicates that the quantities are evaluated with respect to the partition λ and $C_{\lambda/\nu}$ denotes the set of columns of λ which have the same length as the corresponding column of ν.

The product in (11) can also be expanded on Jack polynomials with the relation [15] :

$$\prod_i (1-z_i)^\beta = \sum_\mu H^\beta(\beta;\mu) J_\mu(z_i;\beta) \tag{14}$$

with coefficient $H^\beta(\beta;\mu)$:

$$H^\beta(\beta;\mu) = \prod_{s\in\mu} \frac{j - 1 - \beta i}{1 + a(s) + \beta l(s)} \tag{15}$$

with notations defined above. The results of [15] are proven for β integer but it is simple to extend the proof to arbitrary rational β using Kaneko's generalized Selberg integrals [17]. This coefficient is zero whenever the partition μ has more than p legs or more than $(q-1)$ arms for $\beta = p/q$. This is at the origin of the selection rule for the intermediate states. Using these expansions and the orthogonality relation for the Jack polynomials [16] :

$${}_N< 0|J_\nu(z_i;\beta) J_\rho(\bar{z}_i;\beta)|0>_N = \delta_{\nu,\rho}\mathcal{N}_\rho(\beta, N) \tag{16}$$

we find the matrix element M_α :

$$M_\alpha = (-1)^{\beta N} \sum_\nu H^\beta(\beta;\nu+n) J_{\lambda/\nu}(1;\beta) \frac{\mathcal{N}_\nu(\beta, N)}{\sqrt{\mathcal{N}_0(\beta;N)\mathcal{N}_\lambda(\beta;N+1)}} \tag{17}$$

where $\nu + n$ is the partition ν with n columns added (or subtracted when n negative). In this expression $\nu + n$ is a partition with at most p legs, therefore $n \le p$.

The correlation function is now written :

$${}_N< 0|\Psi(x,t)\Psi(0,0)^\dagger|0>_N = \sum_{|\alpha>} |M_\alpha|^2 e^{iQ_\alpha x - iE_\alpha t}. \tag{18}$$

The evaluation of M_α in the thermodynamical limit is our main result. All the quantities entering (17) are known from the mathematical literature and can be also found in [8].

There is a first contribution coming from $n = p$ and $n = -r$ which will be denoted $G^{(1)}$. It this part, the only partitions contributing correspond to Young Tableaux, λ, having one "arm" or one part, $\lambda = (\lambda_1)$, with λ_1 an arbitrary positive integer or Young tableaux having N parts of length r denoted $\lambda = (r^N)$. Each type of tableaux account for a particle propagating on a different side of the Fermi sea. In the thermodynamical limit, we obtain:

$$G^{(1)}(x,t) = \rho_0 \beta \int_1^\infty dw \left(\frac{w-1}{w+1}\right)^{\beta-1} 2\cos(Qx) e^{-iEt} \tag{19}$$

where we used the variable :

$$\frac{\lambda_1}{\beta N} = \frac{w-1}{2} \tag{20}$$

and

$$E = \frac{\rho_0^2 \beta^2 \pi^2}{2} w^2, \quad Q = \rho_0 \beta \pi w, \tag{21}$$

with $\rho_0 = N/L$. This is interpreted as the propagation of one particle over the Fermi sea. For $\beta = 1$, the fermions, we find the correct result.

The contributions for $n = 1, ..., (p-1)$ are present for finite N but in the thermodynamic limit they are suppressed by factors of N. The only remaining contribution is for $n \le 0$. It comes from partitions λ which are of the form :

$$(r+\lambda_1, ..., r+\lambda_q, (r+p)^{\lambda'_p - q}, (r+p-1)^{\lambda'_{p-1}-\lambda'_p}, ..., r^{N-\lambda'_1}) \tag{22}$$

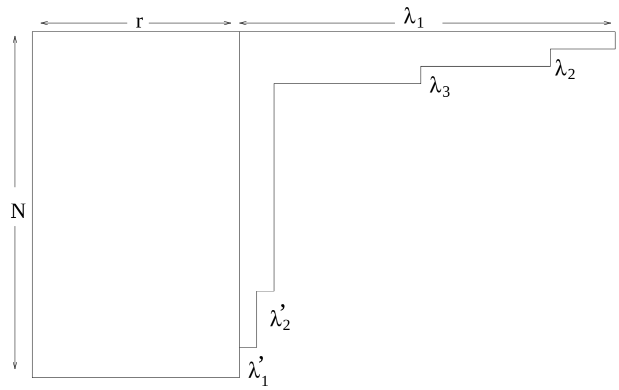

Fig.2: Young tableaux contributing to the correlation for $\beta = 2/3$.

Here the λ_i's $(i = 1, .., q)$ and $r = -n$ are non negative integers and correspond to q particles propagating in the forward direction and one particle in the backward direction outside the Fermi sea . The λ'_i $(i = 1, ..., p)$ are bounded by N and are related to the momenta of the holes inside the Fermi sea.

When we take the continuum limit with the following variables :

$$\frac{\lambda_i - (\beta-1)i}{\beta N} = \frac{w_i - 1}{2}, \quad \frac{\beta\lambda'_i - i}{\beta N} = \frac{1-v_i}{2}, \quad \frac{r}{\beta N} = -\frac{w_0+1}{2}, \quad \frac{\nu_i - (\beta-1)i}{\beta N} = \frac{\xi_i - 1}{2} \tag{23}$$

we obtain the second part of the correlation function . The expression is given by :

$$G^{(2)}(x,t) = C(\beta) \int_1^\infty \prod_{j=1}^q dw_j \int_{-\infty}^{-1} dw_0 \int_{-1}^1 \prod_{i=1}^p dv_i \; F(v_i, w_j) \; e^{-iE_\alpha t - Q_\alpha x}, \tag{24}$$

The energy and momentum are :

$$E_\alpha = \frac{\beta \pi^2 \rho_0^2}{2} \left(-\sum_{i=1}^p v_i^2 + \beta \sum_{i=0}^q w_i^2\right), \quad Q_\alpha = \pi \rho_0 \left(\sum_{i=1}^p v_i - \beta \sum_{i=0}^q w_i\right) \tag{25}$$

and the constant :

$$C(\beta) = \rho_0 \frac{\beta^{-pq+1}\Gamma(q/p)^p}{2p!q!\Gamma(p/q)^q} \prod_{i=1}^{p} \Gamma(qi/p)^{-2} \prod_{i=1}^{q-1} \Gamma(-pi/q)^{-2}. \tag{26}$$

The form factor $F(v_i, w_j)$ is equal to ;

$$F(v_i, w_j) = \frac{\prod_{1\le i<j\le p} |v_i - v_j|^{2/\beta} \prod_{0\le i<j\le q} |w_i - w_j|^{2\beta}}{\prod_{i=1}^{p}(1-v_i^2)^{1-1/\beta} \prod_{j=0}^{q}(w_j^2-1)^{1-\beta} \prod_{i,j}(v_i - w_j)^2} \mathcal{K}^2, \tag{27}$$

$$\mathcal{K} = \frac{\prod_{j=1}^{p}\prod_{i=0}^{q}(w_i - v_j)^{\beta}}{\prod_{k<j}(v_k - v_j)\prod_{0\le i<j\le q}(w_i - w_j)^{2\beta-1}} \int_{w_q}^{w_{q-1}} d\xi_{q-1} \cdots \int_{w_2}^{w_1} d\xi_1 \prod_{i<j}^{q-1}(\xi_i - \xi_j) \prod_{i=1}^{q-1}\prod_{j=0}^{q}(\xi_i - w_j)^{\beta-1}$$
$$\partial_{v_1} \ldots \partial_{v_p} \left[\prod_{k<j}(v_k - v_j) \prod_{j=1}^{p} \left(\prod_{i=0}^{q}(w_i - v_j)^{1-\beta} \prod_{i=1}^{q-1}(\xi_i - v_j)^{-1} \right) \right] \tag{28}$$

In the appendix we give the main lines of the derivation of $\mathcal{K}$ and we argue that this expression can be simplified to a single contour integral :

$$\mathcal{K} = (\beta - 1)\frac{\Gamma^q(\beta)}{2\pi i} \int_{\mathcal{C}_w} dz \frac{\prod_{i=1}^{p}(v_i - z)}{\prod_{j=0}^{q}(w_j - z)^{\beta}} \tag{29}$$

where $\mathcal{C}_w$ is a contour surrounding the points $w_1, \ldots, w_q$ as in the figure 3.

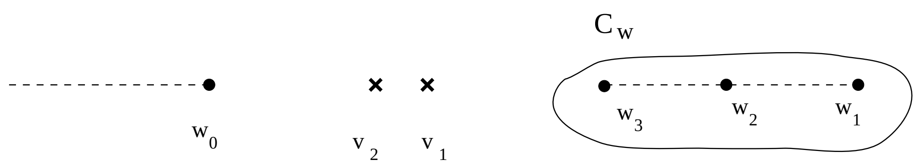

Fig.3: Contour of integration. The branch cuts of the integrand are represented by dashed lines.

When $\beta = 1/2$, this expression of $\mathcal{K}$ coincides with the result obtained by Zirnbauer and Haldane [14]. For β integer (or, equivalently, $q = 1$) this result is rigorous and proves the conjecture in [14]. The free fermion case $\beta = 1$ is special in the sense that the second contribution to the form factor vanishes.

We tested numerically the equality of the integrals (28) and (29) for $\beta = 1/2$ and $\beta = 3/2$.

This completes the computation of the form factor of the advanced Green function for all rational values of the coupling constant β.

III. CONCLUSIONS.

We computed the advanced Green function in the Calogero-Sutherland model using Jack polynomial techniques. The computation done here follows from the techniques already used in [8].

The final expression for the form factor we obtain is the same for β integer or not. There is however a big difference between the two cases. If β is an integer, the form factor is invarient under a parity transformation (change of sign of all the quasimomenta which define the intermediate state α). If β is not an integer, one particle propagates in the forward direction and q particle propagate in the backward direction but there is no contribution coming from a particle propagating in the backward direction and q particles in the forward direction. The form factor is thus not invariant under a parity transformation. The origin of this phenomena can be traced back to two possible inequivalent $N+1$ particles vacua. Had we chosen the other possibility, we would have obtained the form factor conjugated under parity. The physical meaning of this phenomena is not clear to us.

The results presented here simplify drastically in the continuum limit. Finding a way to do these computations directly in this limit would be of great interest. The complexity of the part described by $\mathcal{K}$ in our results show that simple bosonic vertex operators don't seem to reproduce the right answer. Note however that the integrals over the variables ξ in the first expression of $\mathcal{K}$ bear striking similarities with the screening operators which occur in the Coulomb Gaz representation of conformal field theories [22]

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APPENDIX A: SUMMATION.

Let us go back to the expression for the form factor :

$$M_\alpha = (-1)^{\beta N} \sum_\nu H^\beta(\beta;\nu+n) J_{\lambda/\nu}(1;\beta) \frac{\mathcal{N}_\nu(\beta,N)}{\sqrt{\mathcal{N}_0(\beta;N)\mathcal{N}_\lambda(\beta;N+1)}} \tag{A1}$$

and let us look at this expression for a fixed partition λ and a negative integer n. Then the coefficient $H^\beta(\beta;\nu+n)$ forces the partition $\nu+n$ to have p legs and $(q-1)$ arms. Since n is negative, the partition ν must have $r=-n$ columns and then p legs and $(q-1)$ arms.

The main difficulty of the computation is to make the summation in (A1) on the partitions ν such that $J_{\lambda/\nu}\neq 0$. As explained in [8] , only cases for which $\lambda'_i-\nu'_i=0,1$ are allowed. The case considered in figure 2 corresponds to $\beta=2/3$. Again, we characterize the partition λ by the numbers r,λ_i,λ'_j, which define the quasi-particle momenta in the continuum limit. In this case the partition λ is of the form :

$$(r+\lambda_1, r+\lambda_2, r+\lambda_3, (r+2)^{\lambda'_2-3}, (r+1)^{\lambda'_1-\lambda'_2}, r^{N-\lambda'_1}),$$

and ν can be read similarly from the figure :

$$(r+\nu_1, r+\nu_2, (r+2)^{\nu'_2-2}, (r+1)^{\nu'_1-\nu'_2}, r^{N-\nu'_1}).$$

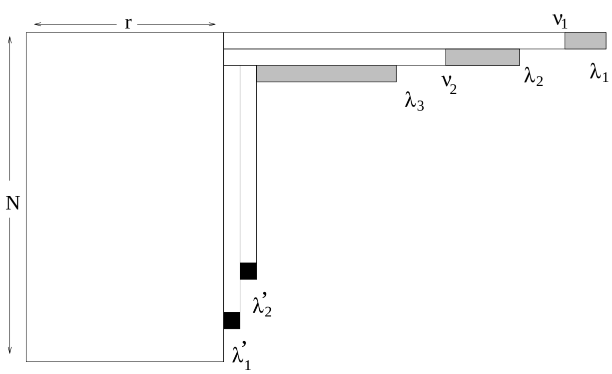

Fig.4: Young tableaux setting the notation.

When we keep ν_i fixed, we must sum over the possibilities $\nu'_i=\lambda'_i,\lambda'_i-1$ corresponding to the black boxes in the figure. The restriction that λ/ν is a one dimensional strip gives the following constraints on the ν_i's :

$$\lambda_2\leq\nu_1\leq\lambda_1,\quad \lambda_3\leq\nu_2\leq\lambda_2. \tag{A2}$$

If we use the variables appropriate for the continuum limit

$$\frac{\lambda_i-(\beta-1)i}{\beta N}=\frac{w_i-1}{2},\quad \frac{\beta\lambda'_i-i}{\beta N}=\frac{1-v_i}{2},\quad \frac{r}{\beta N}=-\frac{w_0+1}{2},\quad \frac{\nu_i-(\beta-1)i}{\beta N}=\frac{\xi_i-1}{2}, \tag{A3}$$

and we will use $a=\frac{2}{\beta N}$. The sum over the ν_i's in the intervals defined by (A2) become integrals over the continuum variables ξ_i's. This correspond to the shaded parts in the figure.

Let us concentrate on the sum over ν'_i, factorizing out the part which is independent of ν'_i. This sum results in the following expression :

$$Q_N = \frac{1}{a^p}\sum_I(-1)^{|I|}\prod_{i\in I}\left[\prod_{k=0}^{q}\frac{w_k - v_i-(\beta-1)a}{w_k-v_i}\prod_{k=1}^{q-1}\frac{\xi_k-v_i-a}{\xi_k-v_i}\prod_{j\notin I}\frac{v_j-v_i+a}{v_j-v_i}\right]. \tag{A4}$$

Here I is the ensemble for which $\lambda'_i = \nu'_i$ and we have to sum over all possibilities $I \subset \{1,...,p\}$. The symbol $|I|$ means the cardinal of the ensemble. Before taking the thermodynamical limit of this expression, we note the following property :

$$\lim_{a\to 0}\frac{1}{a^p}\sum_I(-1)^{|I|}\prod_{i\in I}\left[\frac{f(v_i-a)}{f(v_i)}\prod_{j\notin I}\frac{v_j-v_i+a}{v_j-v_i}\right] = \frac{\partial_{v_1}...\partial_{v_p}\prod_{i<j}(v_i-v_j)\prod_{i=1}^p f(v_i)}{\prod_{i<j}(v_i-v_j)\prod_{i=1}^p f(v_i)} \tag{A5}$$

This expression,which comes from the definition of the derivative, can be also written as :

$$\begin{aligned}\lim_{a_1,...,a_p\to 0}\frac{1}{a_1...a_p}\sum_I(-1)^{|I|}\prod_{i\in I}\frac{f(v_i-a_i)}{f(v_i)}\prod_{i\in I,j\in I}\left(\frac{v_j-v_i-a_j+a_i}{v_j-v_i}\right)\\ \prod_{i\in I,j\notin I}\left(\frac{v_j-v_i+a_i}{v_j-v_i}\right)\prod_{i\notin I,j\in I}\left(\frac{v_j-v_i-a_j}{v_j-v_i}\right)\end{aligned} \tag{A6}$$

and it is unchanged if a quantity of order one in a_i is added. Let $f(v)$ have the form $f(v) = f_1^{\alpha_1}(v)f_2^{\alpha_2}(v)$. To first order in a_i :

$$\frac{f(v_i-a_i)}{f(v_i)} \approx \frac{f_1(v_i-\alpha_1 a_i)f_2(v_i-\alpha_2 a_i)}{f_1(v_i)f_2(v_i)} = 1 - a_i\left(\alpha_1\frac{f_1'(v_i)}{f_1(v_i)}+\alpha_2\frac{f_2'(v_i)}{f_2(v_i)}\right)+\mathcal{O}(a_i^2).$$

Replacing this in (A5) and putting $a_1 = ... = a_p = a$ gives the large N limit of the sum in (A3), with $f_1 = \prod_{i=1}^q(w_i - v)$, $f_2 = \prod_{i=1}^{q-1}(\xi_i - v)$ and $\alpha_1 = 1-\beta$, $\alpha_2 = -1$.

$$Q = \frac{\partial_{v_1}...\partial_{v_p}\left[\prod_{k<j}(v_k-v_j)\prod_{j=1}^p\left(\prod_{i=0}^q(w_i-v_j)^{1-\beta}\prod_{i=1}^{q-1}(\xi_i-v_j)^{-1}\right)\right]}{\prod_{k<j}(v_k-v_j)\prod_{j=1}^p\left[\prod_{i=0}^q(w_i-v_j)^{1-\beta}\prod_{i=1}^{q-1}(\xi_i-v_j)^{-1}\right]} \tag{A7}$$

It remains to do the integrals over the variables ξ_i's. When we multiply (A7) by the part depending explicitly on ν_i and factor out the part which is independent of ν_i we obtain the following multiple integral :

$$\begin{aligned}I(\beta) = \frac{\prod_{j=1}^p\prod_{i=0}^q(w_i-v_j)^{\beta-1}}{\prod_{k<j}(v_k-v_j)}\int_{w_q}^{w_{q-1}}d\xi_{q-1}\cdots\int_{w_2}^{w_1}d\xi_1\prod_{i<j}^{q-1}(\xi_i-\xi_j)\prod_{i=1}^{q-1}\prod_{j=0}^q(\xi_i-w_j)^{\beta-1}\\ \partial_1...\partial_p\left[\prod_{k<j}(v_k-v_j)\prod_{j=1}^p\left(\prod_{i=0}^q(w_i-v_j)^{1-\beta}\prod_{i=1}^{q-1}(\xi_i-v_j)^{-1}\right)\right]\end{aligned} \tag{A8}$$

Doing explicitly these integrals is not an easy task. However, a careful analysis of this expression suggests it can take the simpler form of a single contour integral. In the following we present the arguments leading to this conjecture.

The integral I can be regarded as a complex function of the variables $w_0, ..., w_q$ and $v_1, ..., v_p$. We try to obtain the limit of I when the points $w_1, ..., w_q$ collapse onto the point w and $v_1, ..., v_p \to v$. Due to the antisymmetry of the integrand as a function of $\xi_1, ..., \xi_{q-1}$, changing counterclockwise w_{i-1} and w_i, $(i = 2, ..., q)$ just multiplies the integral by a phase $e^{i\pi(2\beta-1)}$. We expect then branch points at $w_i = w_j$. The singular part of the integral responsible of these branch points can be calculated:

$$\begin{aligned}I_0 &= \int_{w_q}^{w_{q-1}}d\xi_{q-1}\cdots\int_{w_2}^{w_1}d\xi_1\prod_{i<j}^{q-1}(\xi_i-\xi_j)\prod_{i=1}^{q-1}\prod_{j=1}^{q}(\xi_i-w_j)^{\beta-1}\\ &= \frac{\Gamma^q(\beta)}{\Gamma(q\beta)}\prod_{1\le i<j\le q}(w_i-w_j)^{2\beta-1}R(w_1,...,w_q)\end{aligned} \tag{A9}$$

The function $R(w_1, ..., w_q)$ is symmetric, rational, homogeneous of order zero and it has no singularities, so it is constant. Evaluating the integral in the 'nested' limit $w_1 \to (w_2 \to ... \to (w_q \to w)...)$ gives $R(w_1, ..., w_q) = 1$.

Further simplification arises when we specialize the variables $v_i = v$. This operation is delicate when done on an expression like (A7), so it is easier to restart from the discrete version (A4) and put $v_i = v + ai$. The sum reduces over the ensembles $I = \{1, ...s\},\ s \leq p$:

$$
\begin{aligned}
Q_N &= \frac{1}{a^p} \sum_{s=0}^{p} (-1)^s \binom{p}{s} \prod_{i=1}^{s} \left[\prod_{k=0}^{q} \frac{w_k - v - ai - (\beta - 1)a}{w_k - v - ai} \prod_{k=1}^{q-1} \frac{\xi_k - v - ai - a}{\xi_k - v - ai} \right] \\
&= \frac{1}{G(v)} \sum_{s=0}^{p} (-1)^s \binom{p}{s} G(v + as) \qquad \text{(A10)}
\end{aligned}
$$

where

$$
G(v) = \prod_{k=0}^{q} \left(\frac{w_k - v}{a} - \beta + 1 \right)_{\beta - 1} \prod_{k=1}^{q-1} \left(\frac{\xi_k - v}{a} - 1 \right)
$$

Here the notation $(x)_n = \Gamma(x+n)/\Gamma(x)$ is used. In the last term of the equality (A10) one recognizes the discrete p^{th} derivative of $G(v)$. Using Stirling's formula and the homogeneity of G we get in the large N limit :

$$
\frac{(-1)^p}{\prod_{i=0}^{q} (w_i - v)^{\beta - 1} \prod_{i=1}^{q-1} (\xi_i - v)} \frac{d^p}{dv^p} \left[\prod_{i=0}^{q} (w_i - v)^{\beta - 1} \prod_{i=1}^{q-1} (\xi_i - v) \right]. \qquad \text{(A11)}
$$

Note that the powers $\beta - 1$ and 1 have opposite sign in (A7) and (A11). Now, we can take the limit of I when the w_i's collapse to w and all the v_i's are set equal to v. Without the constant and the factor $\prod_{i<j} (w_i - w_j)^{2\beta - 1}$, the result is :

$$
(w - w_0)^{(\beta - 1)(q - 1)} (w - v)^{1 - qp} \frac{d^p}{dv^p} \left[(w - v)^{p-1} (w_0 - v)^{p/q - 1} \right] \qquad \text{(A12)}
$$

and this can be rewritten in the following form :

$$
(\beta - 1)(w_0 - w)^{2p - q} (w - v)^{-pq} (w_0 - v)^{-p} \frac{d^{p-1}}{dw^{p-1}} \left[\frac{(v - w)^p}{(w_0 - w)^\beta} \right]. \qquad \text{(A13)}
$$

The equality of the two formulas can be checked by expanding the products under the derivatives and using the properties of the binomial coefficients. Then we write the derivative as a contour integral :

$$
\frac{d^{p-1}}{dw^{p-1}} \left[\frac{(v - w)^p}{(w_0 - w)^\beta} \right] = \frac{(p-1)!}{2\pi i} \int_{\mathcal{C}_w} dz \frac{(v - z)^p}{(z - w)^p (w_0 - z)^\beta}. \qquad \text{(A14)}
$$

The conjecture is that we can undo the specialization of the variables w_j and v_i to obtain

$$
I = (\beta - 1) \frac{\Gamma^q(\beta)}{2\pi i} \prod_{0 \leq i < j}^{q} (w_i - w_j)^{2\beta - 1} \prod_{i=1}^{p} \prod_{j=0}^{q} (w_j - v_i)^{-1} \int_{\mathcal{C}_w} dz \frac{\prod_{i=1}^{p} (v_i - z)}{\prod_{i=1}^{q} (z - w_i)^\beta (w_0 - z)^\beta} \qquad \text{(A15)}
$$

where $\mathcal{C}_w$ is a contour in the complex plane encircling $w_1, ..., w_q$. This function has the good analyticity properties we can expect from (A8).

For β integer (or $q = 1$) this result is rigorous. In this case, there is no variable of integration ξ and Q has simple poles at $w_i = v_j$, the residu being a polynomial of order $2q - 1 = 1$ in each v_i. This justify the passage from (A13) to (A15). $\beta = 1/2$ is another case to test our conjecture. The contour of integration $\mathcal{C}_w$ can be deformed to obtain $\mathcal{C}_{w_0}$. As $\beta < 1$ we have to subtract the divergent contribution of the contour at infinity, we thus obtain :

$$
I = \frac{1}{4} \prod_{i=0}^{2} \frac{1}{w_i - v} \int_{-\infty}^{w_0} \frac{dy}{\sqrt{w_0 - y}} \left(\frac{v - y}{\sqrt{(w_1 - y)(w_2 - y)}} - 1 \right) \qquad \text{(A16)}
$$

which is the result Zirnbauer and Haldane obtained using a supersymmetric calculation [14].

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