

Static spherically symmetric monopole solutions in the presence of a dilaton field

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Abstract

A numerical study of static, spherically symmetric monopole solutions of a spontaneously broken $SU(2)$ gauge theory coupled to a dilaton field is presented. Regular solutions seem to exist only up to a maximal value of the dilaton coupling. In addition to the generalization of the 't Hooft-Polyakov monopole a discrete family of regular solutions is found, corresponding to radial excitations absent in the theory without dilaton.

The aim of this paper is to present a detailed numerical study of classical solutions of an SU(2) Yang-Mills-Higgs (YMH) theory coupled to a (massless) dilaton (DYM), when the Higgs field is in the adjoint representation. Dilaton fields appear naturally in low energy effective field theories derived from superstring models [1, 2, 3]. As previous studies have already shown [4, 5] the inclusion of a dilaton in a pure Yang-Mills (YM) theory has drastic consequences already at the classical level. In particular the dilaton Yang-Mills (DYM) theory possesses finite energy ‘particle-like’ solutions which are absent in the pure YM case. The same phenomenon happens in the Einstein-Yang-Mills system where non-singular finite energy solutions have been discovered some time ago [6].

The above YMH model without a dilaton field is known to possess nonsingular, finite energy solutions, describing magnetic monopoles [7]. In the present work we study static, spherically symmetric solutions of the DYM theory. Our results show a striking similarity to those obtained in the corresponding YMH theory coupled to gravity (EYM) [8]. In the DYM theory there is a finite energy abelian solution for all values of the dilaton coupling α . Based on numerical investigations there is a strong indication for the existence of a finite energy *nonabelian* monopole up to only a maximal value of α , α_m . The nonabelian monopole merges with the abelian one at a critical value of α , α_c . In limit when the dilaton decouples the nonabelian solution joins smoothly the ‘t Hooft-Polyakov monopole. If the Higgs self coupling, β , is in the interval $[0, 0.6]$ $\alpha_m(\beta)$ and $\alpha_c(\beta)$ are different and with close analogy to the EYM theory there are two different solutions if $\alpha_c < \alpha < \alpha_m$.

In addition to the ‘fundamental’ monopole there is good numerical evidence that a countable family of globally regular solutions exists for $0 < \alpha < \sqrt{3}/2$ independently of β . As their energy is higher they can be interpreted as excitations of the fundamental monopole. In the limit when the dilaton field decouples their energy diverges. There is another limit when the DYM theory reduces to the DYM model and then these excitations tend to the solutions discovered in refs. [4, 5].

The action of our model is given by

$$S = \int d^4x \left\{ \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - e^{2\kappa\varphi} \frac{1}{4g^2} F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2} (D_\mu \Phi)^a (D^\mu \Phi)^a - e^{-2\kappa\varphi} V(\Phi) \right\}, \quad (1)$$

where $a = 1, 2, 3$, and the Higgs potential is

$$V(\Phi) = \frac{\lambda}{8} (\Phi^a \Phi^a - v^2)^2. \quad (2)$$

The couplings of the scalar field φ has been chosen so that the action (1) possesses the following dilatational symmetry

$$x^\mu \rightarrow e^{\kappa c} x^\mu, \quad A_\mu \rightarrow e^{-\kappa c} A_\mu, \quad \varphi \rightarrow \varphi + c, \quad S \rightarrow e^{2\kappa c} S, \quad (3)$$

which can be used to fix the value of φ arbitrarily at any point. With the

$$\begin{aligned} \Phi \rightarrow v\Phi, \quad x \rightarrow \frac{1}{gv}x, \quad \varphi \rightarrow v\varphi, \quad S \rightarrow \frac{v}{g}S, \\ \lambda \rightarrow g^2\beta^2, \quad \kappa \rightarrow \frac{1}{v}\alpha \end{aligned} \quad (4)$$

rescaling of variables the only remaining dimensionless parameters are

$$\alpha = \frac{\kappa M_W}{g}, \quad \beta = \frac{M_H}{M_W}, \quad (5)$$

where $M_W = vg$ and $M_H = v\sqrt{\lambda}$ are the characteristic masses of the theory. Without further restrictions one can assume $\alpha \geq 0$. Let us mention here that the action of the DYM theory

(1) can be obtained from the Einstein-Yang-Mills-Higgs action by restricting the metric to the special conformstatic form $ds^2 = e^{2\phi} dt^2 - e^{-2\phi} dx^i dx^i$.

We use the minimal static spherically symmetric ansatz with zero ‘electric’ field

$$A_0^a \equiv 0, \quad A_i^a = \epsilon_{aij} \frac{x^j}{r^2} (W(r) - 1), \quad (6a)$$

$$\Phi^a = H(r) \frac{x^a}{r}, \quad (6b)$$

$$\varphi = \varphi(r). \quad (6c)$$

After the rescaling (4) the action (1) reduces to

$$E = \int dr \left\{ \frac{1}{2} r^2 \varphi'^2 + e^{2\alpha\varphi} \left[W'^2 + \frac{(W^2 - 1)^2}{2r^2} \right] + W^2 H^2 + \frac{r^2 H'^2}{2} + \frac{\beta^2}{8} r^2 e^{-2\alpha\varphi} (H^2 - 1)^2 \right\} \quad (7)$$

using the ansatz (6). Varying (7) we obtain the equations of motion:

$$(r^2 \varphi')' = 2\alpha e^{2\alpha\varphi} \left(W'^2 + \frac{(W^2 - 1)^2}{2r^2} \right) - \frac{\alpha\beta^2}{4} r^2 e^{-2\alpha\varphi} (H^2 - 1)^2, \quad (8a)$$

$$r^2 W'' = W(W^2 - 1 + r^2 e^{-2\alpha\varphi} H^2) - 2\alpha r^2 \varphi' W', \quad (8b)$$

$$(r^2 H')' = H \left(2W^2 + \frac{\beta^2}{2} r^2 e^{-2\alpha\varphi} (H^2 - 1) \right). \quad (8c)$$

Solutions regular at the origin must satisfy the following boundary conditions (b.c.):

$$\begin{aligned} H &= ar + O(r^2), \quad W = 1 - br^2 + O(r^3), \\ \varphi &= \varphi_0 + \alpha \left(2b^2 e^{2\alpha\varphi_0} - \frac{\beta^2}{24} e^{-2\alpha\varphi_0} \right) r^2 + O(r^3), \end{aligned} \quad (9)$$

i.e. there is a three parameter (a, b, φ_0) family of regular solutions at $r = 0$. The local existence can be proved following the procedure discussed in [9]

For $r \rightarrow \infty$ the corresponding ‘regular’ b.c. are

$$\varphi = \varphi_\infty - \frac{d}{r} + O\left(\frac{1}{r^2}\right), \quad (10a)$$

$$W = B e^{-\mu\rho} \left(1 + O\left(\frac{1}{r}\right) \right), \quad (10b)$$

$$H = 1 - \frac{C}{r} e^{-\mu\beta\rho} \left(1 + O\left(\frac{1}{r}\right) \right) \quad \text{for } \beta < 2 \quad (10c)$$

$$H = 1 - \frac{2B^2}{\mu^2(\beta^2 - 4)r^2} e^{-2\mu\rho} \left(1 + O\left(\frac{1}{r}\right) \right) \quad \text{for } \beta > 2, \quad (10d)$$

parametrized by $(\varphi_\infty, d, B, C)$, where $\mu = e^{-\alpha\varphi_\infty}$, $\rho = r + \alpha d \ln r$. For $\beta > 2$ we cannot fully parametrize the stable manifold at $r = \infty$. Exploiting the virial theorem and using Eqs. (8), (9) and (10) one obtains for the energy (7)

$$E = \frac{1}{\alpha} \int_0^\infty dr (r^2 \varphi')' = \frac{1}{\alpha} (r^2 \varphi') \Big|_0^\infty. \quad (11)$$

We discuss first an exact two parameter family of solutions of Eqs. (8) which is going to play an important role in our further analysis, since it describes the asymptotic behaviour of nonabelian solutions. Consider the singular abelian monopole, $W \equiv 0$, $H \equiv 1$ then the general solution of (8) takes the form

$$\varphi_a = -\frac{1}{\alpha} \ln \left| \frac{1}{A} \sinh \left(A \left(\frac{1}{M} + \frac{\alpha}{r} \right) \right) \right| \quad (12)$$

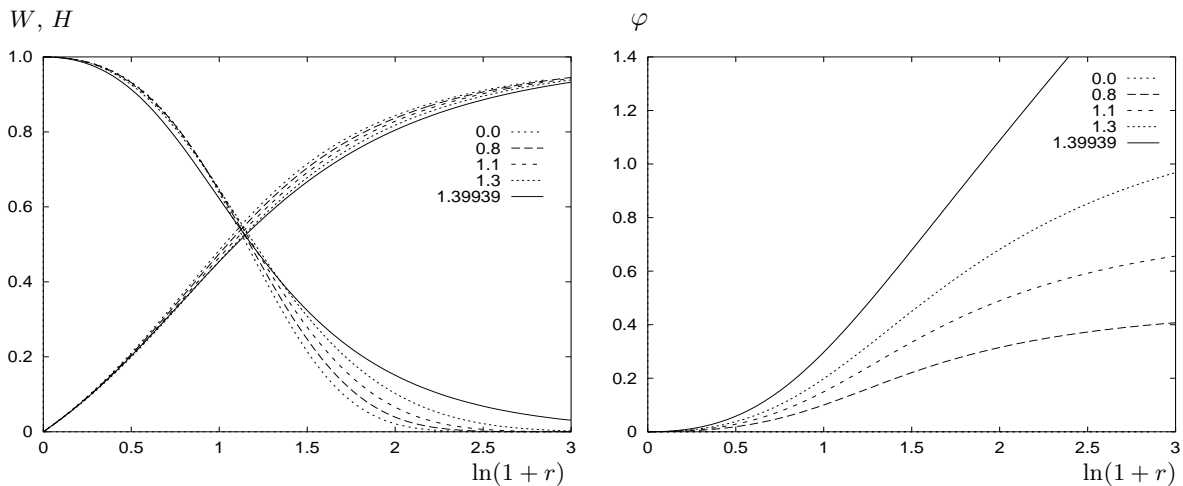


Figure 1: The α dependence of the fundamental monopole solution for $\beta = 0$

with $A^2, M \in \mathbf{R}$. For $M > 0$ and A real it is regular for all $r > 0$. Its energy, E Eq. (7), is infinite unless $A = 0$. For $A = 0$ the total energy of the solution (12) is finite, $E = M/\alpha$. When $M < 0$ and A is real the solution is singular at $r = -\alpha M$ and the energy diverges. If A is imaginary for any value of M the solution becomes singular at some $r > 0$.

Using the dilatational symmetry (3) from now on we set $\varphi_\infty = 0$. Note that then the energy scales as $E \rightarrow \exp(-\alpha\varphi_\infty)E$. The finite energy abelian solution for $\varphi_\infty = 0$ reads

$$\varphi = -\frac{1}{\alpha} \ln\left(1 + \frac{\alpha}{r}\right),$$

and its energy is simply $E = 1/\alpha$. It satisfies the following first order differential Eq.

$$r^2\varphi' = \pm e^{\alpha\varphi}, \quad (13)$$

which is derivable from a Bogomolny type bound [5] in the background $W \equiv 0, H \equiv 1$.

The asymptotic behaviour of finite energy nonabelian solutions is described by (12). With the help of Eq. (11) we arrive at the following useful formula for the total energy of globally regular nonabelian solutions:

$$E = \frac{1}{\alpha} \cosh(A/M), \quad (14)$$

where A, M are parameters of φ_a describing their asymptotic behaviour. For globally regular solutions of (8) satisfying the b.c. (9), (10) the energy (7) is a monotonically increasing function of the Higgs coupling β and monotonically decreases with the dilaton coupling α . (We remark here that the condition $\varphi_\infty = 0$ is essential to prove these facts.)

We have numerically integrated a suitably desingularized version of Eqs. (8) from $r = 0$ using the b.c. (9). We set φ_0 to zero in order to have to vary only two parameters. By adjusting the parameters a, b we suppressed the divergent modes in the asymptotic region according to (17). The resulting solutions can then be transformed to satisfy $\varphi_\infty = 0$ by the dilatational symmetry (3).

We found that there exist a fundamental nonabelian monopole solution for all β below a maximal value of α , α_m which smoothly joins the 't Hooft-Polyakov monopole in the $\alpha \rightarrow 0$ limit. Some solution curves for $\beta = 0$ are shown in Fig. 1. For the parameters see Table 1.

There is a critical α value, α_c , where the dilaton field becomes logarithmically divergent while W and H tend to some nontrivial functions as can be seen on Fig. 1. After shifting φ_∞ to 0 the fundamental monopole tends to the finite energy abelian one as $\alpha \rightarrow \alpha_c$. For $\beta^2 \leq 0.06 \alpha_c$ is seen to differ from α_m (see Table 2). It means that for a given α in the interval $[\alpha_c, \alpha_m]$ there

Table 1:

Parameters for the fundamental monopole solutions in Fig. 1				
α	a	b	ϕ_∞	$\alpha \cdot E$
0.0	0.33333333	0.16666666	0	0
0.1	0.33324422	0.16674419	0.050094	0.099853
0.2	0.33297626	0.16697893	0.100758	0.198827
0.5	0.33107418	0.16872021	0.262673	0.481732
0.7	0.32884174	0.17095758	0.387902	0.650049
0.8	0.32741363	0.17252229	0.460048	0.725582
1.0	0.32388447	0.17699856	0.638684	0.855180
1.1	0.32176986	0.18029088	0.758343	0.907479
1.2	0.31944319	0.18485325	0.919110	0.950109
1.3	0.31705890	0.19195967	1.172960	0.981707
1.4	0.31657705	0.20987214	1.950448	0.999523
1.4088	0.31898696	0.21924215	2.582925	1.000092
1.4	0.32380520	0.22987901	5.289654	1.000000
1.39939	0.32402298	0.23025275	8.461582	1.000000

Table 2:

β^2	α_c	α_m
0	1.39938	1.4088
0.02	1.37874	1.3803
0.03	1.36950	1.3702
0.04	1.36088	1.3612
0.043	1.35839	1.3586
0.05	1.35279	1.3529
0.06	α_m	1.3452
0.07	α_m	1.338
0.08	α_m	1.33
0.1	α_m	1.32
0.15	α_m	1.29
0.2	α_m	1.27
1	α_m	1.11
4	α_m	0.98
∞	α_m	0.89

are two different nonabelian solutions with different energies. The function $\alpha_m(\beta)$ decreases with increasing β from $\alpha_m(0) \approx 1.4088$ to $\alpha_m(\infty) \approx 0.89$.

There also seems to exist a countable family of globally regular monopole solutions indexed by the number zeros of $W(r)$ for all β and $0 < \alpha < \sqrt{3}/2$. They can be interpreted as radial excitations of the fundamental monopole. In the $\alpha \rightarrow 0$ limit after a suitable rescaling they can be identified with the previously discovered DYM solutions [4],[5]. We illustrate some of these excited solutions with one and two zeros for α various values on Figs. 2 and 3, while the corresponding parameters are listed in Table 3 and Table 4.

Table 3:

Parameters for the first excited monopole solution in Fig. 2				
α	a	$\alpha^2 b$	$\alpha \varphi_\infty$	$\alpha \cdot E$
DYM	—	0.26083015	1.711412	0.8038078
0.05	0.66718265	0.26286356	1.731264	0.8507008
0.1	0.77749002	0.27004539	1.803886	0.8904364
0.2	0.99147112	0.30099818	2.170909	0.9448188
0.3	1.00351436	0.33435729	2.730953	0.9720677
0.5	0.77902186	0.37296818	3.994232	0.9932192
0.7	0.59081576	0.39281185	5.919563	0.9992733
0.8	0.51698205	0.39916099	8.207917	0.9999547
0.866	0.47164657	0.40215328	26.31413	1.0000000

When $\alpha \rightarrow \sqrt{3}/2$ for the excited solutions the dilaton diverges logarithmically again while the zeros of W go rapidly to infinity, except for the innermost one. For $\sqrt{3}/2 < \alpha < 1$ there is a surviving solution with a single zero of W and with divergent φ existing up to $\alpha = 1$, where W develops an extra divergent mode, so for $\alpha > 1$ this solution is not expected to exist. There also exists another type of limiting solution when the number of zeros of W goes to infinity for all $0 < \alpha < \sqrt{3}/2$ and the dilaton field diverges logarithmically.

If one shifts, however φ_∞ to 0 then all of the excited monopoles merge with the finite energy abelian solution for $\alpha \rightarrow \sqrt{3}/2$. The fundamental and the excited monopoles up to six zeros are plotted on Fig. 4. In order to better display the higher zeros of W we plotted $\sqrt{r}W$. Notice that the newer zeros appear nearer and nearer to the origin.

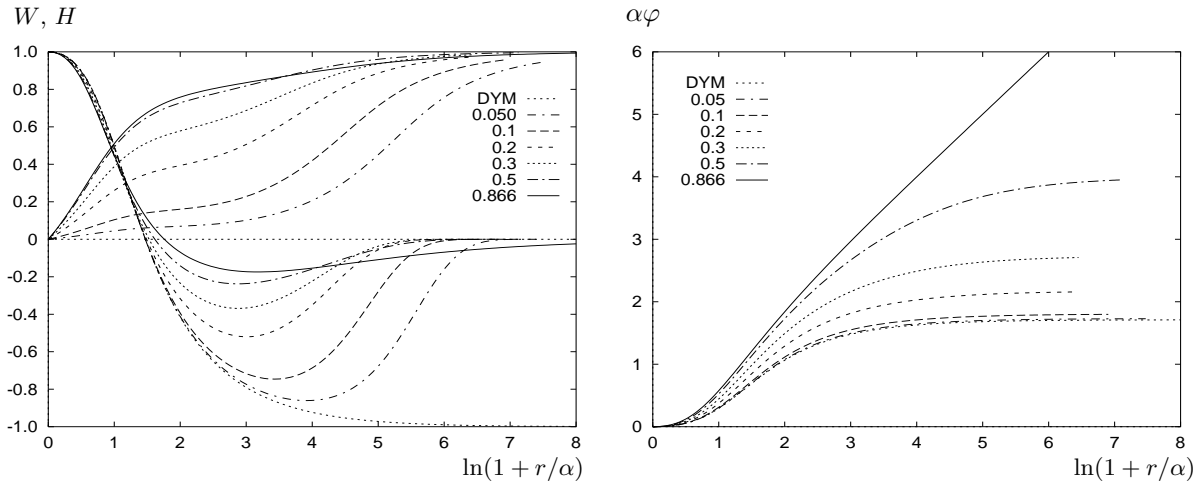


Figure 2: The α dependence of the first excited monopole solution for $\beta = 0$

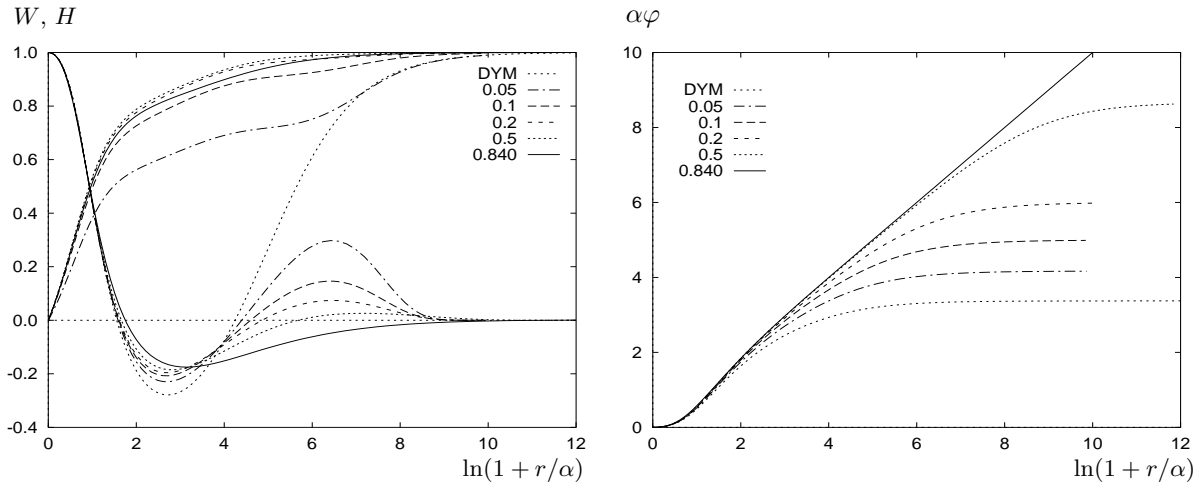


Figure 3: The α dependence of the second excited monopole solution for $\beta = 0$

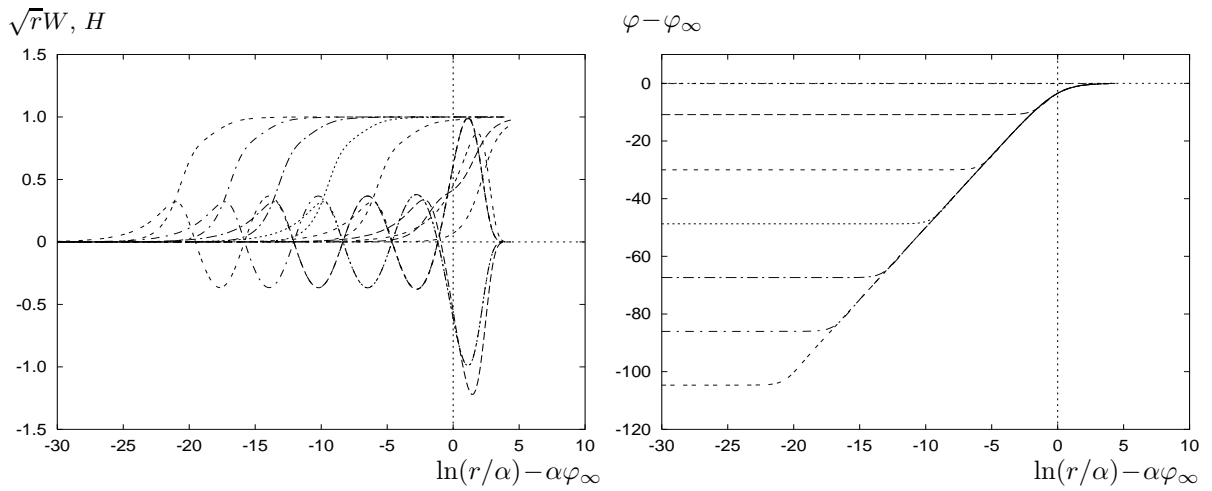


Figure 4: The fundamental and the first six excited monopole solutions for $\alpha = 0.2$ and $\beta = 0$

Table 4:

Parameters for the second excited monopole solution in Fig. 3				
α	a	$\alpha^2 b$	$\alpha \varphi_\infty$	$\alpha \cdot E$
DYM	—	0.35351801	3.373903	0.96559852
0.05	6.0251373	0.36759603	4.169777	0.99251583
0.1	3.9102367	0.37450939	4.996143	0.99673418
0.2	2.0946637	0.37885368	5.999793	0.99882187
0.3	1.4148625	0.38161925	6.815222	0.99949621
0.5	0.85071037	0.38762800	8.666195	0.99993064
0.7	0.60036405	0.39522735	12.25580	0.99999862
0.8	0.51818563	0.39944391	17.77525	1.00000000
0.84	0.48948129	0.40111050	25.77133	1.00000000

The structure of the solutions can be understood from the linearization of the field equations (8) around the abelian monopole (12). Using the

$$W = w(r), \quad H = 1 + \frac{h(r)}{r}, \quad \varphi = \varphi_a(r) + \psi, \quad (15)$$

variables, from the linearized field equations one finds for $1/M \ll \alpha/r \ll 1/|A|$ (where $\varphi_a \simeq \ln(r/\alpha)/\alpha$) that the solutions are well approximated by: $\psi \sim e^{\lambda_\psi \tau}$, $h \sim e^{\lambda_h \tau}$, $w \sim e^{\lambda_w \tau}$ where $\tau = \ln r$ and the ‘frequencies’ are

$$\lambda_\psi = (1; -2), \quad \lambda_h = \frac{1}{2}(1 \pm \sqrt{1 + 4\alpha^2 \beta^2}), \quad \lambda_w = -\frac{1}{2} \pm \sqrt{\alpha^2 - 3/4}. \quad (16)$$

So we see that in this ‘middle’ region W oscillates with an amplitude decaying like $1/\sqrt{r}$. If $M \rightarrow \infty$ (i.e. $\alpha E \rightarrow 1$) this region stretches out to infinity, while W has more and more zeros when $\alpha < \sqrt{3}/2$. This behaviour is a similar to the one found in the DYM case in the interval where φ grows logarithmically. If $\alpha > 1$ we do not expect the corresponding solution to exist. In the asymptotic region defined by $r \gg |\alpha A \coth(A/M)|$, $r \gg \mu^{-1}$ (where $\mu = |\sinh(A/M)/A|$ and $\varphi_a \simeq -\frac{1}{\alpha} \ln \mu$) the linear corrections $\beta \neq 0$ are characterized by $\psi \sim e^{\lambda_\psi \tau}$, $w \sim e^{\lambda_w \tau}$, $h \sim e^{\lambda_h \tau}$, with

$$\lambda_\psi = (0; -1), \quad \lambda_h = \pm \mu |\beta|, \quad \lambda_w = \pm \mu, \quad (17)$$

while $h \sim e^{\lambda_h \tau}$ and $\lambda_h = (0; +1)$ for $\beta = 0$.

The energy of the solutions goes rapidly to 1 if the number of oscillations of W increases (see Table 5). We have determined the energy of the solution by fitting the parameters A ,

Table 5:

Parameters of the solutions for $\alpha = 0.2$ and $\beta = 0$ in Fig. 4				
n	a	b	φ_∞	$\alpha \cdot E$
0	0.33297626	0.16697893	0.1007578	0.1988268
1	0.99147112	7.52495447	10.854544	0.9448188
2	2.09466373	9.47134195	29.998964	0.9988219
3	2.14396020	9.51995711	48.727339	0.9999720
4	2.14545164	9.52111922	67.372640	0.9999993
5	2.14549487	9.52114737	86.014682	1.0000000
6	2.14549609	9.52114805	104.6566	1.0000000

M in Eq. (12) in the asymptotic region using formula (14). The energy determined this way

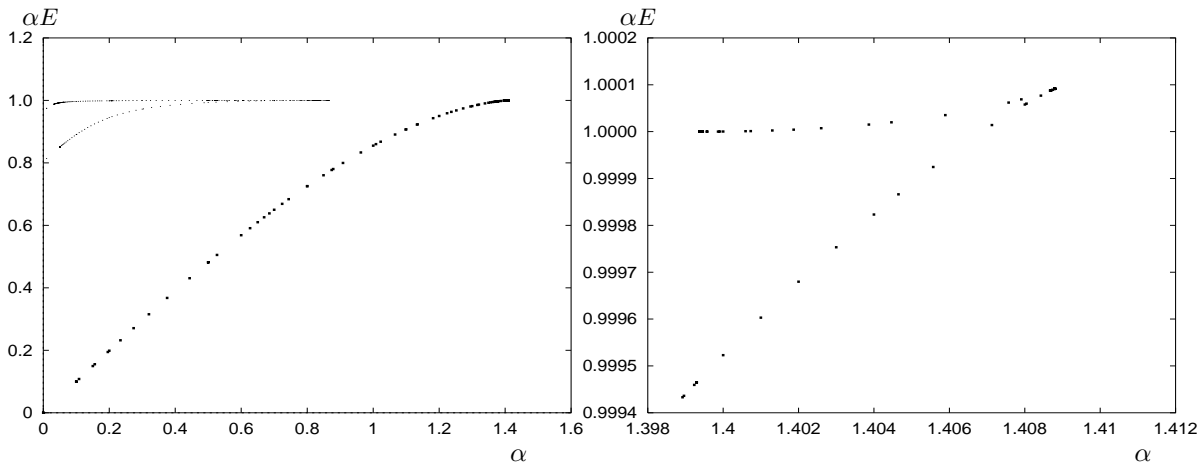


Figure 5: α dependence of the energy of the fundamental, the first, and the second excited monopole solutions, and its detailed view for the fundamental monopole near α_m for $\beta = 0$

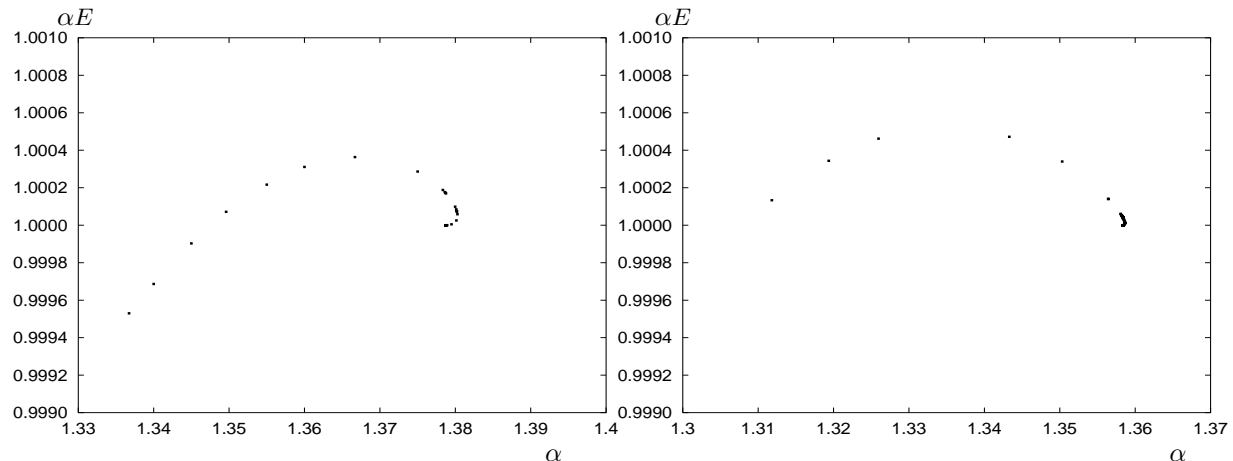


Figure 6: Detailed view of the α dependence of the energy of the fundamental monopole solution near α_m for $\beta^2 = 0.02$ and for $\beta^2 = 0.043$

contains only exponentially small corrections. We have also plotted the α dependence of the energy, $E(\alpha)$, (rescaled to $\varphi_\infty = 0$) on Fig. 5 for $\beta = 0$. The similarity of Fig. 5 to Fig 3 in Ref. [8] where the masses of *gravitating monopoles* are plotted as a function of the ‘gravitational coupling strength’, M_W/M_{Planck} , is indeed striking. On Fig. 6 $E(\alpha)$ for $\beta^2 = 0.02$ and 0.043 is shown.

For not too large β values ($\beta \leq 3$) αE of the fundamental monopole becomes larger than 1 unlike for the excited ones. For $\beta = \infty$ this maximum is 1.

We make finally some remarks on the stability of the solutions. The ’t Hooft-Polyakov monopoles are stable since they are solutions with minimal energy [10]. It is natural to expect the fundamental monopoles to remain stable in the DYM case for $\alpha > 0$. For sufficiently small β , however, where the mass of the fundamental monopole is larger than that of the abelian one, the nonabelian solution is expected to become unstable against large perturbations. If $\alpha_c \neq \alpha_m$ there is a bifurcation point where the linear stability of the solutions can change. In the EYM case this change of stability has been shown in [11]. The excited monopoles are expected to be unstable for all α since their energy is significantly larger than that of the fundamental ones. This heuristic argument is strengthened by the fact that in the $\alpha \rightarrow 0$ limit their counterparts are known to be unstable [5, 4].

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