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Collective Motion of Micro-organisms from Field Theoretical Viewpoint

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ABSTRACT

We analyze the collective motion of micro-organisms in the fluid and consider the problem of the red tide. The red tide is produced by the condensation of the micro-organisms, which might be a similar phenomenon to the condensation of the strings. We propose a model of the generation of the red tide. By considering the interaction between the micro-organisms mediated by the velocity fields in the fluid, we derive the Van der Waals type equation of state, where the generation of the red tide can be regarded as a phase transition from the gas of micro-organisms to the liquid.

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String theory is a candidate of the unified theory of elementary particles and their interactions. If we can derive from the string theory at the Planck scale the field contents and the mass ratios at TeV scale, it might be possible to confirm that nature is really described by the string theory. In order to get a stronger confidence, however, we need to know the dynamics of the string theory, *e.g.*, the mechanisms of the compactification, supersymmetry breaking *etc.*

In this respect it is important to study a system which has some of the properties similar to string theories. The techniques obtained from string theories might be applied to analyze such a system and some implications to string theory might be derived from the analysis of the system. One of such a systems is the micro-organisms swimming in the fluid. The problem of the swimming of the micro-organisms was considered from the gauge theoretical viewpoint by Shapere and Wilczek [1][2] and from the string and membrane theoretical viewpoint in Refs.[3] and [4].

In this paper, we consider the problem of the red tide. The red tide is generated by the condensation of the micro-organisms, which might be a similar phenomenon to the condensation of the strings. Having auch a view, we propose a model of the red tide, where we take into account the interaction between the micro-organisms mediated by the velocity fields in the fluid and derive the Van der Waals type equation. In this model, the generation of the red tide is regarded as a phase transition from the gas of micro-organisms to the liquid.

The Reynolds number is very small for the micro-organisms swimming in the fluid and the Navier-Stokes equation is linearized. Then the equations of motion for an incompressible fluid are given by

$$\nabla \cdot \mathbf{v} = 0 \ , \tag{1}$$

$$\Delta \mathbf{v} = \frac{1}{\eta} \nabla p \quad \text{or} \quad \Delta(\nabla \times \mathbf{v}) = 0 \ . \tag{2}$$

Here $\mathbf{v}$, p and η are the velocity field, the pressure and the coefficient of viscosity, respectively. In two dimensions, these equations (1) and (2) have the following forms:

$$\partial_z v_{\bar{z}} + \partial_{\bar{z}} v_z = 0 \ , \tag{3}$$

$$4\eta \partial_z \partial_{\bar{z}} v_{\bar{z}} = \partial_{\bar{z}} p \ , \quad \text{and} \quad 4\eta \partial_z \partial_{\bar{z}} v_z = \partial_z p \ . \tag{4}$$

It is an interesting point that the equations of motion (3) and (4) can be derived from the QED-like Lagrangean in the Landau gauge:

$$\mathcal{L} = 2\eta(\partial_z v_{\bar{z}} - \partial_{\bar{z}} v_z)^2 - p(\partial_z v_{\bar{z}} + \partial_{\bar{z}} v_z) \ . \tag{5}$$

Furthermore, the Lagrangean (5) can be regarded as the entropy density of the fluid, that will be shown next.

Following the standard textbook of the fluid dynamics,[4] we find the time-derivative of the fluid entropy S is given by

$$\dot{S} = \int dV \Big\{ \frac{\sigma'_{ik}}{2T} \Big(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \Big) - \frac{\mathbf{q} \cdot \mathrm{grad} T}{T^2} \Big\} \ . \tag{6}$$

Here σ'_{ik} is the stress tensor (in three dimensions):

$$\sigma'_{ik} = \eta \Big(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v_l}{\partial x_l} \Big) + \zeta \delta_{ik} \frac{\partial v_l}{\partial x_l} \ . \tag{7}$$

If we assume that the temperature T is globally defined; $\mathrm{grad} T = 0$ and the fluid is incompressible (1), the equation (6) can be rewritten as

$$\dot{S} = \int dV \frac{\eta}{2T} \Big\{ \Big(\frac{\partial v_i}{\partial x_k} - \frac{\partial v_k}{\partial x_i} \Big)^2 + p \frac{\partial v_l}{\partial x_l} + \frac{\partial}{\partial x_i} \Big(v_k \frac{\partial v_i}{\partial x_k} \Big) \Big\} \ . \tag{8}$$

Here we have imposed the condition of the incompressibility by the multiplier field p. The resulting expression is equivalent to the action (5) of QED in the Landau gauge if we neglect the surface term.

The micro-organism appears as a boundary condition.

$$\mathbf{v}(\mathbf{x} = \mathbf{X}(t;\xi)) = \dot{\mathbf{X}}(t;\xi) \ . \tag{9}$$

Here the shape of the surface of a micro-organism is described by a mapping function; $\mathbf{X}(t;\xi)$. Here ξ parametrizes the surface of the micro-organism. The equation (9) tells that there is no slipping between the surface of a micro-organism and the sticky fluid. In the following, we only consider the two dimensional case for simplicity. The extension to the three dimensional one is tedious but straightforward. Let a micro-organism exist at the origin

[4] We follow the notation of Landau-Lifsitz's one

$z = 0$. Then by solving the equations (3), (4) under the boundary condition (9), we obtain

$$v_{\bar{z}} = \phi_1(z) - z\overline{\phi_1'(z)} + \overline{\phi_2(z)} \ . \tag{10}$$

Here ϕ_i's are analytic functions:

$$\phi_1(z) = \sum_{k<0} a_k z^{k+1} \ , \qquad \phi_2(z) = \sum_{k<-1} b_k z^{k+1} \ . \tag{11}$$

The coefficients a_k and b_k can be determined by the boundary condition (9). By evaluating the singularity of the velocity fields at the origin, we obtain

$$\begin{aligned} & -4\partial_z\partial_{\bar{z}} v_{\bar{z}} + \frac{1}{\mu}\partial_{\bar{z}} p \\ & = 4\pi\Big\{ \sum_{k<-2} \frac{2(-1)^k a_k}{(-k-2)!}\delta^{(-k-1)}(\bar{z})\delta(z) \\ & + \sum_{k\leq -3} \frac{\bar{b}_k(-1)^k}{(-k-2)!}\delta^{(-k-1)}(\bar{z})\delta(z) - \sum_{k\leq -3} \frac{b_k(-1)^k}{(-k-2)!}\delta(\bar{z})\delta^{(-k-1)}(z)\Big\} \ . \end{aligned} \tag{12}$$

Note that the l.h.s in Eq.(12) is given from the Lagrangean (5) by the variation with respect of the velocity fields. Therefore the r.h.s. represents the source terms and we find that the micro-organisms can be expressed as multi-pole source.

When there are two micro-organisms, we can evaluate the interaction action $S^{\rm int}$ between them by using Eq.(8);

$$\begin{aligned} S^{\rm int} &= -\frac{16\eta}{kT}\int dt \sum_{k,l\leq -2} \pi(k+1)(l+1)\frac{(-1)^l(-l-k-2)!}{(-k-1)!(-l-1)!} \\ & \quad \times(a_k^{(1)}a_l^{(2)}z_0^{k+l+1}\bar{z}_0 + \bar{a}_k^{(1)}\bar{a}_l^{(2)}z_0\bar{z}_0^{k+l+1}) \ . \end{aligned} \tag{13}$$

Here i in $a_n^{(i)}$ and $b_n^{(i)}$ expresses i-th ($i = 1, 2$) micro-organism, z_0 is the relative coordinate between the two micro-organisms and k is the Boltzman constant.

We now consider a model of the generation of the red tide. For simplicity, we assume N micro-organisms with radius r_0 and mass m move by using a single mode a_l ($l \neq -1$) in Eq.(11). Then the micro-organism can rotate but cannot translate by themselves. Then the partition function of the system is given by

$$Z = \frac{1}{N!}\int_0^\infty {\rm e}^{-\frac{1}{kT}\sum_{i=1}^N \frac{p_i^2}{2m}}\Omega^{\rm fluid} dz_1 d\bar{z}_1 \cdots dp_{z_N} dp_{\bar{z}_N} \ . \tag{14}$$

Here $\Omega^{\rm fluid} = {\rm e}^{-S^{\rm int}}$ can be evaluated by using Eq.(13),

$$\begin{aligned} S^{\rm int} &= \sum_{i,j} \frac{\phi(\mathbf{r}_{ij})}{kT} \\ &= -\frac{16\eta}{kT} \int dt \sum_{i,j} \pi(l+1)^2 \frac{(-1)^l(-2l-2)!}{(-l-1)!^2} \\ &\qquad \times (a_l^{(i)} a_l^{(j)} z_{ij}^{2l+1} \bar{z}_{ij} + \bar{a}_k^{(i)} \bar{a}_l^{(j)} z_{ij} \bar{z}_{ij}^{2l+1}) \ . \end{aligned} \tag{15}$$

Here $\mathbf{r}_{ij} = (z_{ij}, \bar{z}_{ij})$ is the relative coordinate between the i-th micro-organism and j-th one. After the momentum integration, we obtain the following expression

$$Z = (2\pi mkT)^N \Omega(N,T,V) \ . \tag{16}$$

Here V is the volume of the system. If we assume the interaction is small, ${\rm e}^{-\phi(\mathbf{r}_{ij})} - 1 \ll 1$, $\Omega(N,T,V)$ can be approximated by the two body interaction;

$$\Omega(N,T,V) = \frac{V^N}{N!} \Big\{ 1 + \frac{N^2}{2V} \int_0^{2\pi} d\theta \int_0^\infty dr\, r \Big({\rm e}^{-\frac{\phi(\mathbf{r}_{ij})}{kT}} - 1 \Big) + \cdots \Big\} \ . \tag{17}$$

Here $z_{ij} = r{\rm e}^{i\theta}$. By integrating over the relative angle θ between two micro-organisms, the second term in Eq.(17) has the following form;

$$\int_0^{2\pi} d\theta \int_0^\infty dr\, r \Big({\rm e}^{-\frac{\phi(\mathbf{r}_{ij})}{kT}} - 1 \Big) = 2\pi \int_0^\infty dr\, r \Big\{ I_0\Big(\frac{C_l}{kT} r^{2l+2} \Big) - 1 \Big\} \ . \tag{18}$$

Here C_l is a positive quantity defined by

$$C_l \equiv 32\eta k\pi(l+1)^2 \frac{(-2l-2)!}{\{(-l-1)!\}^2} \int dt |a_l^{(i)}(t) a_l^{(j)}(t)| \tag{19}$$

and $I_0(x) \equiv \int_0^{2\pi} \frac{d\theta}{2\pi} {\rm e}^{x\cos\theta}$ is a deformed Bessel function. Since $I_0(x)$ is a monotonically increasing function, Eq.(18) tells that the interaction between two micro-organisms becomes effectively attractive one after being averaged with respect to θ.

Since the micro-organism has a finite radius r_0, the potential energy becomes effectively $+\infty$ inside the radius r_0; $r < r_0$. Therefore the equation (18) should be modified by

$$\begin{aligned} &\int_0^{2\pi} d\theta \int_0^\infty dr\, r \Big({\rm e}^{-\frac{\phi(\mathbf{r}_{ij})}{kT}} - 1 \Big) \\ &\qquad = 2\pi \int_0^{r_0} dr\, r(-1) + 2\pi \int_{r_0}^\infty dr\, r \Big\{ I_0\Big(\frac{C_l}{kT} r^{2l+2} \Big) - 1 \Big\} \ . \end{aligned} \tag{20}$$

Furthermore if we consider the case that $\frac{C_l}{kT} r_0^{2l+2} \ll 1$, we can approximate the deformed Bessel function $I_0\Big(\frac{C_l}{kT} r^{2l+2}\Big) \sim 1 + \frac{1}{4}\Big(\frac{C_l}{kT} r^{2l+2}\Big)^2$ and we obtain

$$\int_0^{2\pi} d\theta \int_0^{\infty} dr\, r\Big(\mathrm{e}^{-\frac{\phi(\mathbf{r}_{ij})}{kT}} - 1\Big) = \frac{2}{N}\Big\{b - \frac{a}{(RT)^2}\Big\} \ . \tag{21}$$

Here a and b are positive quantities defined by

$$a \equiv -\frac{\pi N^2}{2(2l+3)}\Big(\frac{C_l}{2}\Big)^2 r_0^{4l+6} \ , \tag{22}$$

$$b \equiv \frac{N}{2}\pi r_0^2 \ , \tag{23}$$

and $R = Nk$ The b is nothing but the volume occupied by all (N) the micro-organisms. Since the free energy F and the pressure (of micro-organism) p are defined by

$$F \equiv -Nk \ln Z \ , \tag{24}$$

$$p \equiv -\Big(\frac{\partial F}{\partial V}\Big)_T \ , \tag{25}$$

we obtain

$$p + \frac{a}{V^2 RT} = \frac{RT}{V}\Big(1 + \frac{b}{V}\Big) \tag{26}$$

We can replace the factor $\Big(1 + \frac{b}{V}\Big)$ in the r.h.s. of Eq.(26) by $\Big(1 - \frac{b}{V}\Big)^{-1}$ since the effect of b can be considered to be the decrease of the volume V of the system by the occupation of the micro-organisms. Then we obtain the following Van der Waals type equation of state for the micro-organism gas;

$$p = \frac{RT}{V - b} - \frac{a}{V^2 RT} \ . \tag{27}$$

In the usual Van der Waals equation, the second term in Eq.(27) is not $-\frac{a}{V^2 RT}$ but $-\frac{a}{V^2}$. The qualitative feature of the equation obtained here is, however, similar to the usual Van der Waals equation. If we consider p-V curve with fixed T, there appears a region where $\frac{\partial p}{\partial V} > 0$ when $T < T_c = \frac{2}{3R}\sqrt{\frac{a}{3b}}$. The appearance of such an unphysical region implies the existence of the phase transition between liquid and gas. The phase transition would be interpreted as the generation of the red tide. For fixed $T < T_c$, the phase transition from gas to liquid occurs when the volume decreases, which would correspond to the increase of the number density of the micro-organisms. Furthermore T_c becomes higher as a becomes bigger. Since a is proportional to $|a_l|^4$, the red tide might be generated if the micro-organisms become to move more actively when the temperature of the water rises higher in summer.

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