BONN-TH-95-15
ITPSB 95-29
hep-th/9508110

Generalizations of the Andrews-Bressoud Identities for the $N = 1$ Superconformal Model $SM(2, 4\nu)$

Alexander Berkovich [1]

Physikalisches Institut der
Rheinischen Friedrich-Wilhelms Universität Bonn
Nussallee 12
D-53115 Bonn, Germany

and

Barry M. McCoy [2]

Institute for Theoretical Physics
State University of New York
Stony Brook, NY 11794-3840

8/95

[1] berkovic@axpib.physik.uni-bonn.de

[2] mccoy@max.physics.sunysb.edu

Abstract

We present generalized Rogers-Ramanujan identities which relate the fermi and bose forms of all the characters of the superconformal model $SM(2,4\nu)$. In particular we show that to each bosonic form of the character there is an infinite family of distinct fermionic $q-$ series representations.

1. Introduction

Generalized Rogers-Ramanujan identities are identities between $q-$series which are part of a very rich area of mathematics that includes:

1. $q-$series;
2. Partition identities;
3. Path space counting problems;
4. Modular functions;
5. Polynomial identities;
6. Conformal field theory;
7. Basic Hypergeometric Series;
8. Continued Fractions;
9. Dilogarithm identities;
10. Characters and branching functions of Kac-Moody algebras;
11. Solvable models of statistical mechanics;
12. Thermodynamic Bethe's Ansatz;
13. Holonomic Differential equations.

It is almost a "folk theorem" that there is a $1-1$ isomorphism between these concepts. This multiplicity of connections is both a blessing in the number of insights provided to mathematics and physics in the last 30 years and a curse in that the explanation of any new result needs to be made in many different forms if it is to be intelligible to all those interested in the subject. This general dilemma of presentation is well illustrated by the historical background of the subject of this talk.

The history begins with the generalized Rogers-Ramanujan identities that relate sums to products which are items 34 and 36 in the justly famous 1952 list of Lucy Slater [1]

$$\sum_{k=0}^{\infty}\frac{(-q^{\frac{1}{2}})_k q^{\frac{k(k+2)}{2}}}{(q)_k}=\frac{(-q^{\frac{1}{2}})_\infty}{(q)_\infty}\prod_{k=1}^{\infty}(1-q^{4k-\frac{1}{2}})(1-q^{4k-\frac{7}{2}})(1-q^{4k}) \tag{1.1}$$

and

$$\sum_{k=0}^{\infty}\frac{(-q^{\frac{1}{2}})_k q^{\frac{k^2}{2}}}{(q)_k}=\frac{(-q^{\frac{1}{2}})_\infty}{(q)_\infty}\prod_{k=1}^{\infty}(1-q^{4k-\frac{3}{2}})(1-q^{4k-\frac{5}{2}})(1-q^{4k}) \tag{1.2}$$

where we use the notation

$$(A)_k = \begin{cases} \prod_{j=0}^{k-1}(1-Aq^j) & \text{if } k=1,2,\cdots \\ 1 & k=0. \end{cases} \tag{1.3}$$

These identities were given a partition theoretic interpretation by Göllnitz [2] and Gordon [3] and this interpretation was soon thereafter generalized by Andrews [4] as the following theorem:

Let ν and a be integers with $0 < a \le \nu$. Let $C_{\nu,a}(n)$ be the number of partitions of n into parts which are neither $\equiv 2 \pmod 4$ nor $\equiv 0, \pm(2a-1) (\mathrm{mod}\ 4\nu)$. Let $D_{\nu,a}(n)$ denote the number of partitions of N of the form $n = \sum_{i=1}^{\infty} if_i$ with $f_1 + f_2 \le a-1$ and for all $1 \le i$

$$f_{2i-1} \le 1 \ \text{ and } f_{2i} + f_{2i+1} + f_{2i+2} \le \nu - 1. \tag{1.4}$$

Then $C_{\nu,a} = D_{\nu,a}(n)$.

The corresponding generalization of the analytic result of Slater (1.2) was only given by Andrews in 1975 [5]- [6] and the full generalization of Slater's results was finally given by Bressoud [7]–[8] who proved that for $s = 1, 3, 5, \cdots, 2\nu - 1$

$$\begin{aligned} &\sum_{n_2,\cdots,n_\nu=0}^{\infty} \frac{(-q^{\frac{1}{2}})_{N_2} q^{\frac{1}{2}N_2^2 + N_3^2 + \cdots + N_\nu^2 + \ N_{(s+3)/2} + N_{(s+5)/2} + \cdots N_\nu}}{(q)_{n_2}(q)_{n_3}\cdots\ (q)_{n_\nu}} \\ &= \frac{(-q^{\frac{1}{2}})_\infty}{(q)_\infty} \prod_{k=1}^{\infty} \ (1-q^{2\nu k})(1-q^{2\nu k - \frac{s}{2}})(1-q^{2\nu k - \frac{4\nu - s}{2}}) \end{aligned} \tag{1.5}$$

where

$$N_k = \sum_{j=k}^{\nu} n_j \tag{1.6}$$

The passage from (1.1)–(1.2) to (1.5) goes through partition identities, path identities, continued fractions, basic hypergeometric series and modular functions.

However many more extensions of these results are possible and the key to these extensions is the identification of the identities (1.5) with the $N = 1$ superconformal field theory $SM(2, 4\nu)$. This connection is made by using Jacobi's triple product identity

$$\sum_{j=0}^{\infty} z^j q^{j^2} = \prod_{j=0}^{\infty}(1-q^{2j+2})(1+zq^{2j+1})(1+z^{-1}q^{2j+1}), \tag{1.7}$$

to rewrite the product side of (1.5) as

$$\frac{(-q^{\frac{1}{2}})_\infty}{(q)_\infty} \sum_{j=-\infty}^{\infty} \ (q^{j(8\nu j + 4\nu - 2s)/2} - q^{(2j+1)(4\nu j + s)/2}) \tag{1.8}$$

and using the identity

$$(z)_N = \sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix}_q z^j q^{j(j-1)/2}, \tag{1.9}$$

where (for latter use) we use the slightly unconventional definition of the q-binomial coefficient

$$\begin{bmatrix} l \\ m \end{bmatrix} = \begin{cases} \frac{(q)_l}{(q)_m (q)_{l-m}} & \text{if } 0 \le m \le l \\ 1 & \text{if } m = 0, l \le -1 \\ 0 & \text{otherwise,} \end{cases} \tag{1.10}$$

to rewrite the sum side of (1.5) as

$$\sum_{m_1, n_2, \cdots, n_\nu \ge 0}^{\infty} \frac{q^{\frac{m_1^2}{2} - m_1 N_2 + \sum_{j=2}^{\nu} N_j^2 + \sum_{j=\frac{s+3}{2}}^{\nu} N_j}}{(q)_{n_2} \cdots (q)_{n_\nu}} \begin{bmatrix} N_2 \\ m_1 \end{bmatrix}_q. \tag{1.11}$$

These expressions are to be compared with the bosonic form of the characters of the general $N = 1$ superconformal model $SM(p, p')$ [10]

$$\hat{\chi}_{r,s}^{(p,p')}(q) = \hat{\chi}_{p-r,p'-s}^{(p,p')}(q) = \frac{(-q^{\epsilon_{r-s}})_\infty}{(q)_\infty} \sum_{j=-\infty}^{\infty} \left(q^{\frac{j(jpp'+rp'-sp)}{2}} - q^{\frac{(jp+r)(jp'+s)}{2}}\right) \tag{1.12}$$

where

$$\epsilon_a = \begin{cases} \frac{1}{2} & \text{if } a \text{ is even (Neveu-Schwarz (NS) sector)} \\ 1 & \text{if } a \text{ is odd (Ramond (R) sector)} \end{cases} \tag{1.13}$$

Here $r = 1, 2, \cdots, p-1$ and $s = 1, 2, \cdots, p'-1$ and p and $\frac{(p'-p)}{2}$ are coprime. Thus we see that (1.5) are the characters of the Neveu-Schwarz sector of the $N = 1$ superconformal field theory $SM(2, 4\nu)$. The sum expression (1.11) is seen to be in the canonical form of a fermionic representation of conformal field theory characters [11].

The question now naturally arises of what corresponds to the Andrews-Bressoud identities (1.5) in Ramond sector of the field theory and a partial answer to this was given by Melzer [12] who found generalizations of (1.11) for Ramond characters with $s = 2$ and 2ν.

Recently the authors, in collaboration with W. Orrick, have used the methods which were developed to prove polynomial and character identities for the minimal models $M(p, p+1)$ [13] to extend these results to all values of s in the Ramond sector [14]. Indeed, it was found that there was not one but two generalizations of (1.11). But even more can be done and subsequent investigation has revealed that in both the Neveu-Schwarz and Ramond sectors there are an infinite number of fermionic forms for the characters. The goal of this note is to present these results and to indicate their proof. In fact, we will go further and extend the character identities to the families of polynomial identities. Some of these polynomial identities were presented for the first time in [14] and some of them are new.

The plan of this note is as follows. In sec. 2 we will present both the recent results of [14] and our subsequent generalizations. The character identities are given in (2.33)-(2.35) and (2.37), (2.40). The most general polynomial identities are given in (2.42), (2.43). In sec. 3 we will indicate how the new results can be obtained from the formalism of [14]. In sec. 4 we conclude with a few comments about the future direction of this research.

2. Polynomial and Character Identities

In this section we will separately define the polynomials which generalize the fermionic and bosonic forms of the characters introduced in sec. 1. We will then present the relations between these two forms which generalize the Andrews-Bressoud identities (1.5).

2.1. Fermionic Polynomials

In [14] the set of variables $(\vec{m},\vec{n})$ were introduced which satisfy the relations

$$\begin{aligned} n_1+m_1 &= \frac{1}{2}(L+m_1-m_2)-a_1 \\ n_2+m_2 &= \frac{1}{2}(L+m_1+m_3)-a_2 \\ n_i+m_i &= \frac{1}{2}(m_{i-1}+m_{i+1})-a_i, \qquad \text{for } 3\le i\le \nu-1 \\ n_\nu+m_\nu &= \frac{1}{2}(m_{\nu-1}+m_\nu)-a_\nu \end{aligned} \tag{2.1}$$

where n_i and m_i are integers and the components a_i of the vector $\vec{a}$ are either integers or half integers. This system is closely related to the TBA equations for the XXZ-model ((3.9) of [15]) with anisotropy

$$\gamma=\pi\frac{(2\nu-1)}{4\nu}. \tag{2.2}$$

In terms of these variables we then introduce the following polynomials to generalize the Fermi form of the characters (1.11)

$$F^{(\nu,n)}_{r',s'}(L,q)=\sum_{\mathcal{D}_{r',s'}} q^{Qf+Lf_{n,s'}}\prod_{i=1}^{\nu}\begin{bmatrix} n_i+m_i\\ n_i\end{bmatrix}_q \tag{2.3}$$

where the quadratic form Qf and linear form $Lf_{n,s'}$ are

$$Qf = \frac{m_1^2}{2} - m_1 N_2 + \sum_{j=2}^{\nu} N_j^2 \quad \text{and} \quad Lf_{n,s'} = n\frac{m_1}{2} + \sum_{l=\nu-s'+1}^{\nu} N_l, \tag{2.4}$$

the ranges of r' and s' are

$$r' = 0,1,2,\cdots,\nu-2 \ \text{ and } \ s' = 0,1,2,\cdots,\nu-1, \tag{2.5}$$

the relation between s' and s is

$$s = \begin{cases} 2\nu - 2s' & \text{if } n \text{ is odd} \\ 2\nu - 2s' - 1 & \text{if } n \text{ is even,} \end{cases} \tag{2.6}$$

and the vector $\vec{a}$ is defined by

$$\begin{aligned} \vec{a} &= \vec{a}^{(r')} + \vec{a}^{(s')} \\ a_i^{(k)} &= \begin{cases} \frac{1}{2}(\delta_{i,\nu} - \delta_{i,\nu-k}) & \text{for } 0 \le k \le \nu-2 \\ \frac{1}{2}(\delta_{i,\nu} + \delta_{i,1}) & \text{for } k = \nu-1. \end{cases} \end{aligned} \tag{2.7}$$

The domain of summation, $\mathcal{D}_{r',s'}$ is best described in terms of $\vec{n}$ and m_ν which are subject to the constraint derived from (2.1)

$$L = (n_1 + a_1) + m_\nu + \sum_{i=2}^{\nu}(2i-3)(n_i + a_i). \tag{2.8}$$

All other variables are given by

$$m_1 = N_2 - n_1 \tag{2.9}$$

$$m_i = m_\nu + 2\sum_{j=i+1}^{\nu}(j-i)(n_j + a_j), \qquad i = 2,3,\cdots,\nu-1. \tag{2.10}$$

Keeping in mind that (from the definition (1.10))

$$\begin{bmatrix} \text{neg. int.} \\ 0 \end{bmatrix}_q = 1, \tag{2.11}$$

we define $\mathcal{D}_{r',s'}$ for $s' \ge r'$ as the union of the sets of solutions to (2.8) satisfying

$$\begin{aligned} 0:&\quad n_i,\ m_\nu \ge 0, \\ 1:&\quad n_\nu = 0, m_\nu = -2,\ n_1,\cdots,n_{\nu-1} \ge 0, \\ 2:&\quad n_\nu = n_{\nu-1} = 0,\ m_\nu = -4,\ n_1,\cdots n_{\nu-2} \ge 0, \\ &\qquad \cdots \\ r':&\quad n_\nu = n_{\nu-1} = \cdots = n_{\nu-r'+1} = 0, \quad m_\nu = -2r', \quad n_1,\cdots,n_{\nu-r'} \ge 0; \end{aligned} \tag{2.12}$$

and for $s' < r'$ the definition is the same as above with $r' \to s'$.

Using the asymptotic formula

$$\lim_{A\to\infty} \begin{bmatrix} A \\ B \end{bmatrix}_q = \frac{1}{(q)_B} \tag{2.13}$$

and the simple consequence of (2.1)

$$n_i + m_i = L + m_1 + n_i - 2\sum_{j=2}^{i}(j-1)(n_j + a_j) - 2\sum_{j=i+1}^{\nu}(i-1)(n_j + a_j); \quad i \geq 2 \tag{2.14}$$

along with (2.9), we find

$$\lim_{L\to\infty} F_{r',s'}^{(\nu,n)}(L,q) = F_{s'}^{(\nu,n)}(q) \tag{2.15}$$

which holds for all r' where

$$F_{s'}^{(\nu,n)}(q) = \sum_{m_1,n_2,\cdots,n_\nu \geq 0} \frac{q^{Qf+Lf_{n,s'}}}{(q)_{n_2}(q)_{n_3}\cdots(q)_{n_\nu}} \begin{bmatrix} N_2 \\ m_1 \end{bmatrix}_q. \tag{2.16}$$

2.2. Bosonic Polynomials

To generalize the bosonic character formula to a polynomial expression we introduce the $q-$trinomial coefficients $T_n(L,A;q^{\frac{1}{2}})$ of Andrews and Baxter [16] defined in terms of

$$\binom{L, A-n; q}{A}_2 = \sum_{j\geq 0} t_n(L,A;j), \qquad n \in Z \tag{2.17}$$

with

$$t_n(L,A;j) = \frac{q^{j(j+A-n)}(q)_L}{(q)_j(q)_{j+A}(q)_{L-2j-A}} \tag{2.18}$$

as

$$T_n(L,A;q^{\frac{1}{2}}) = q^{\frac{L(L-n)-A(A-n)}{2}} \binom{L, A-n; q^{-1}}{A}_2. \tag{2.19}$$

These trinomial coefficients satisfy the recursion relations proven in [22] and [14]

$$\begin{aligned} T_n(L,A;q^{\frac{1}{2}}) &= T_n(L-1,A+1;q^{\frac{1}{2}}) + T_n(L-1,A-1;q^{\frac{1}{2}}) \\ &\quad + q^{L-\frac{n+1}{2}} T_n(L-1,A;q^{\frac{1}{2}}) + (q^{L-1}-1) T_n(L-2,A;q^{\frac{1}{2}}). \end{aligned} \tag{2.20}$$

We note the elementary property

$$T_n(L,A;q^{\frac{1}{2}}) = T_n(L,-A;q^{\frac{1}{2}}) \tag{2.21}$$

and remark that

$$T_n(L,A;-q^{\frac{1}{2}}) = \begin{cases} (-1)^{L+A} T_n(L,A;q^{\frac{1}{2}}) & \text{for } n \text{ even} \\ T_n(L,A;q^{\frac{1}{2}}) & \text{for } n \text{ odd.} \end{cases} \tag{2.22}$$

Consequently, $T_n(L,A;q^{\frac{1}{2}})$ is actually a polynomial in q for n odd or for n even and $L+A$ even, while for n even and $L+A$ odd $T_n(L,A,q^{\frac{1}{2}})$ contains only odd powers of $q^{\frac{1}{2}}$.

We then have the following definition of bosonic polynomials:

1) For the Neveu-Schwarz sector with $n=0$

$$\begin{aligned} B^{(\nu,0)}_{r',s'}(L,q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{\nu j^2+(s'+\frac{1}{2})j} \Big(& T_0(L,2\nu j+s'-r';q^{\frac{1}{2}}) \\ & + T_0(L,2\nu j+s'+1+r';q^{\frac{1}{2}}) \Big); \end{aligned} \tag{2.23}$$

2) For the Ramond sector with $n=-1$

$$B^{(\nu,-1)}_{r',s'}(L,q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{\nu j^2+s'j} \sum_{i=-r'}^{r'} (-1)^{r'+i} T_1(L,2\nu j+s'+i;q^{\frac{1}{2}}). \tag{2.24}$$

For other $n \geq 1$ we define $B^{(\nu,n)}_{r',s'}$ from the recursion relation

$$B^{(\nu,n+2)}_{r',s'}(L,q) = B^{(\nu,n)}_{r',s'}(L+1,q) + q^{-(n+1)/2} B^{(\nu,n)}_{r',s'}(L,q). \tag{2.25}$$

Then using the relation ,which is derivable from (2.20),

$$\begin{aligned} & T_{-n}(L+1,A;q^{\frac{1}{2}}) + q^{-(n+1)/2} T_{-n}(L,A;q^{\frac{1}{2}}) \\ & = T_{-(n+2)}(L,A+1;q^{\frac{1}{2}}) + T_{-(n+2)}(L,A-1;q^{\frac{1}{2}}) \\ & \quad + (q^{(n+1)/2} + q^{-(n+1)/2}) T_{-(n+2)}(L,A;q^{\frac{1}{2}}) \end{aligned} \tag{2.26}$$

we find, for example, for $n=1$ in the Ramond sector

$$\begin{aligned} & B^{(\nu,1)}_{r',s'}(L,q) = \\ & \sum_{j=-\infty}^{\infty} (-1)^j q^{\nu j^2+s'j} \Big(T_{-1}(L,2\nu j+s'-r';q^{\frac{1}{2}}) + T_{-1}(L,2\nu j+s'+1+r';q^{\frac{1}{2}}) \\ & \quad + T_{-1}(L,2\nu j+s'-r'-1;q^{\frac{1}{2}}) + T_{-1}(L,2\nu j+s'+r';q^{\frac{1}{2}}) \Big); \end{aligned} \tag{2.27}$$

and for $n = 2$ in the Neveu-Schwarz sector

$$
\begin{aligned}
&B_{r',s'}^{(\nu,2)}(L,q) = \\
&\sum_{j=-\infty}^{\infty} (-1)^j q^{\nu j^2 + (s'+\frac{1}{2})j} \Big(T_{-2}(L, 2\nu j + s' - r' - 1; q^{\frac{1}{2}}) + T_{-2}(L, 2\nu j + s' - r' + 1; q^{\frac{1}{2}}) \\
&\quad T_{-2}(L, 2\nu j + s' + r' + 2; q^{\frac{1}{2}}) + T_{-2}(l, 2\nu j + s' + r'; q^{\frac{1}{2}}) \\
&\quad (q^{\frac{1}{2}} + q^{-\frac{1}{2}})[T_{-2}(L, 2\nu j + s' - r'; q^{\frac{1}{2}}) + T_{-2}(L, 2\nu j + s' + r' + 1; q^{\frac{1}{2}})] \Big).
\end{aligned}
\tag{2.28}
$$

In general, $B_{r',s'}^{(\nu,n)}(L,q)$ polynomials are given in terms of $T_{-n}-$trinomials.

Using the limiting formula

$$
\lim_{L\to\infty} T_n(L, A; q^{\frac{1}{2}}) = \begin{cases} \frac{(-q^{\frac{(1-n)}{2}})_\infty + (q^{\frac{(1-n)}{2}})_\infty}{2(q)_\infty} & \text{if } L - A \text{ is even} \\ \frac{(-q^{\frac{(1-n)}{2}})_\infty - (q^{\frac{(1-n)}{2}})_\infty}{2(q)_\infty} & \text{if } L - A \text{ is odd.} \end{cases}
\tag{2.29}
$$

and noting the special case

$$
\lim_{L\to\infty} T_1(L, A; q^{\frac{1}{2}}) = \frac{(-q)_\infty}{(q)_\infty}
\tag{2.30}
$$

we find the relation between the limit

$$
B_{s'}^{(\nu,n)}(q) = \lim_{L\to\infty} B_{r',s'}^{(\nu,n)}(L,q)
\tag{2.31}
$$

and the characters (1.12)

$$
\begin{aligned}
B_{s'}^{(\nu,0)}(q) &= \hat{\chi}_{1,2\nu-2s'-1}^{(2,4\nu)}(q) \\
B_{s'}^{(\nu,1)}(q) &= B_{s'}^{(\nu,-1)}(q) = \hat{\chi}_{1,2\nu-2s'}^{(2,4\nu)}(q)
\end{aligned}
\tag{2.32}
$$

which holds for all r'.

2.3. Bose/Fermi Identities

In [14] we found the following relation between the fermi and bose forms of the characters for $n = 0, \pm 1$:

1) For the Neveu-Schwarz sector with $n = 0$ we have (the original Andrews-Bressoud identities (1.5))

$$
B_{s'}^{(\nu,0)}(q) = F_{s'}^{(\nu,0)}(q);
\tag{2.33}
$$

2) For the Ramond sector with $n = 1$

$$B_{s'}^{(\nu,1)}(q) = \begin{cases} F_{s'}^{(\nu,1)}(q) + F_{s'-1}^{(\nu,1)}(q) & \text{for } s' \neq 0 \\ 2F_0^{(\nu,1)}(q) & \text{for } s' = 0; \end{cases} \tag{2.34}$$

3) For the Ramond sector with $n = -1$

$$F_{s'}^{(\nu,-1)}(q) = \begin{cases} B_{\nu-1}^{(\nu,-1)}(q) & \text{for } s' = \nu - 1 \\ B_{s'}^{(\nu,-1)}(q) + B_{s'+1}^{(\nu,-1)}(q) & \text{for } s' \neq \nu - 1 \end{cases} \tag{2.35}$$

or, equivalently

$$B_{s'}^{(\nu,-1)}(q) = \sum_{l=s'}^{\nu-1} (-1)^{l+s'} F_l^{(\nu,-1)}(q). \tag{2.36}$$

These results can be extended to all integers n as follows:

Neveu-Schwarz:

$$\begin{aligned} \vec{F}^{(\nu,-2n)}(q) &= \frac{1}{(-q^{\frac{1}{2}})_n} \prod_{j=1}^{n} K(\nu,-2j)\vec{B}^{(\nu,0)}(q) \quad \text{for} \quad n \geq 1 \\ \vec{F}^{(\nu,2n)}(q) &= q^{-n^2/2}(-q^{\frac{1}{2}})_n \prod_{j=0}^{n-1} K^{-1}(\nu,2j)\vec{B}^{(\nu,0)}(q) \quad \text{for} \quad n \geq 1 \end{aligned} \tag{2.37}$$

where we use the vector notation

$$\begin{aligned} \vec{F}_{r'}^{(\nu,n)}(L,q)|_{s'} = F_{r',s'}^{(\nu,n)}(L,q) \qquad \vec{B}_{r'}^{(\nu,n)}(L,q)|_{s'} = B_{r',s'}^{(\nu,n)}(L,s) \quad \text{and} \\ \vec{F}^{(\nu,n)}(q)|_{s'} = F_{s'}^{(\nu,n)}(q) \qquad \vec{B}^{(\nu,n)}(q)|_{s'} = B_{s'}^{(\nu,n)}(q) \end{aligned} \tag{2.38}$$

and the matrix $K(\nu,n)$ is defined for $0 \leq i,j \leq \nu - 1$ by

$$[K(\nu,n)]_{i,j} = \delta_{i,j}(\delta_{i,0} + \delta_{i,\nu-1} + q^{(n+1)/2} + q^{-(n+1)/2}) + \delta_{i,j+1} + \delta_{i,j-1} \tag{2.39}$$

Ramond:

$$\begin{aligned} \vec{F}^{(\nu,-1-2n)}(q) &= \frac{1}{(-q)_n} \prod_{j=1}^{n} K(\nu,-1-2j)C\vec{B}^{(\nu,-1)}(q) \quad \text{for} \quad n \geq 1 \\ \vec{F}^{(\nu,2n+1)}(q) &= (-1)_{n+1} q^{-n(n+1)/2} \prod_{j=0}^{n} K^{-1}(\nu,2j-1)C\vec{B}^{(\nu,-1)}(q) \quad \text{for} \quad n \geq 0 \end{aligned} \tag{2.40}$$

where for $0 \leq i, j \leq \nu - 1$

$$[C]_{i,j} = \delta_{i,j} + \delta_{i+1,j}. \tag{2.41}$$

For $n \geq -1$ these character identities can be extended to polynomial identities. Defining $B^{(\nu,n)}_{r',s'}(L,q)$ through (2.25) we have

$$\begin{aligned} \vec{F}^{(\nu,2n)}_{r'}(L,q) &= \prod_{j=0}^{n-1} K^{-1}(\nu,2j)\vec{B}^{(\nu,2n)}_{r'}(L,q), \qquad n \geq 1 \\ \vec{F}^{(\nu,2n+1)}_{r'}(L,q) &= \prod_{i=0}^{n} K^{-1}(\nu,2j-1)C\vec{B}^{(\nu,2n+1)}_{r'}(L,q), \qquad n \geq 0. \end{aligned} \tag{2.42}$$

along with the two identities proven in [14]

$$\begin{aligned} \vec{F}^{(\nu,0)}_{r'}(L,q) &= \vec{B}^{(\nu,0)}_{r'}(L,q) \\ \vec{F}^{(\nu,-1)}_{r'}(L,q) &= C\vec{B}^{(\nu,-1)}_{r'}(L,q). \end{aligned} \tag{2.43}$$

3. Sketch of Proof

The proof of the results of the previous section starts with the recursion relations established in [14] for the fermionic polynomials (2.3) which holds for all allowed values of s'

$$\begin{aligned} h^{(n)}_0(L) &= h^{(n)}_1(L-1) + (q^{L+\frac{n-1}{2}}+1)h^{(n)}_0(L-1) + (q^{L-1}-1)h^{(n)}_0(L-2), \\ h^{(n)}_{r'}(L) &= h^{(n)}_{r'-1}(L-1) + h^{(n)}_{r'+1}(L-1) + q^{L+\frac{n-1}{2}}h^{(n)}_{r'}(L-1) + (q^{L-1}-1)h^{(n)}_{r'}(L-2) \\ &\qquad \text{for } 1 \leq r' \leq \nu-3, \\ h^{(n)}_{\nu-2}(L) &= h^{(n)}_{\nu-3}(L-1) + q^{L+\frac{n-1}{2}}h^{(n)}_{\nu-2}(L-1) + q^{L-1}h^{(n)}_{\nu-2}(L-2); \end{aligned} \tag{3.1}$$

where we note that the first and the last equations follow from the middle equation if one introduces $h^{(n)}_{-1}(L)$ and $h^{(n)}_{\nu-1}(L)$ satisfying

$$h^{(n)}_{-1}(L) = h^{(n)}_0(L) \tag{3.2}$$

and

$$h^{(n)}_{\nu-1}(L) = h^{(n)}_{\nu-2}(L-1). \tag{3.3}$$

For $\nu = 2$ there is only the single equation

$$h^{(n)}_0(L) = (1 + q^{L+\frac{n-1}{2}})h^{(n)}_0(L-1) + q^{L-1}h^{(n)}_0(L-2). \tag{3.4}$$

These equations specify the fermionic polynomials (2.3) uniquely if in addition we impose the boundary conditions

$$
\begin{aligned}
&F_{r',s'}^{(\nu,n)}(0,q)=\delta_{r',s'}\\
&F_{r',s'}^{(\nu,n)}(1,q)=\delta_{s',r'+1}+\delta_{s',r'-1}+\delta_{s',0}\delta_{r',0}+q^{\frac{n+1}{2}}\delta_{r',s'}
\end{aligned}
\tag{3.5}
$$

It is important to observe that any set of functions $h_{r'}^{(n)}(L)$ defined by (3.1) are not independent but satisfy the relation

$$
h_{r}^{(n+2)}(L)=h_{r'}^{(n)}(L+1)+q^{-\frac{n+1}{2}}h_{r'}^{(n)}(L). \tag{3.6}
$$

To prove this for the generic equation of (3.1) we add (3.1) with $L\rightarrow L+1$ to (3.1) multiplied by $q^{-\frac{n+1}{2}}$ and define $X_{r'}^{(n)}(L)=h_{r'}^{(n)}(L+1)+q^{-\frac{n+1}{2}}h_{r'}^{(n)}(L)$ to find

$$
\begin{aligned}
X_{r'}^{(n)}(L)=X_{r'-1}^{(n)}(L-1)+X_{r'+1}^{(n)}(L-1)+q^{L+\frac{n-1}{2}}X_{r'}^{(n)}(L-1)+(q^{L-1}-1)X_{r'}^{(n)}(L-2)\\
\text{for}\quad 1\leq r'\leq \nu-3
\end{aligned}
\tag{3.7}
$$

which is the generic equation of (3.1) with $n\rightarrow n+2$. The remaining equations of (3.1) are treated similarly. Thus the relation (3.6) follows. In [14] we have demonstrated that the bosonic polynomials $B_{r',s'}^{(\nu,n)}(L,q)$ with $n=0,\pm1$ satisfy (3.1). The relation (3.6) shows that it is true for all $n\geq -1$.

We now proceed to establish the recursion relation

$$
\vec{F}_{r'}^{(\nu,n)}(L+1,q)+q^{\frac{-(n+1)}{2}}\vec{F}_{r'}^{(\nu,n)}(L,q)=K(\nu,n)\vec{F}_{r'}^{(\nu,n+2)}(L,q) \tag{3.8}
$$

with $K(\nu,n)$ given by (2.39). This is proven by first using (3.6) as applied to the fermionic polynomials (2.3) to show that the left hand side of (3.8) is expressed as a linear combination of $F_{r',s'}^{(\nu,n+2)}(L,q)$ for $0\leq s'\leq \nu-1$. The matrix K which specifies this linear combination is then determined by demanding that (3.8) hold for $L=0$ and 1 using the boundary conditions (3.5) along with (3.1).

We will now prove the results of sec. 2. First consider the character identities (2.37) and (2.40). Let $L\rightarrow\infty$ in (3.8) to obtain the recursion relation on the fermionic sums (2.16)

$$
(1+q^{-(n+1)/2})\vec{F}^{(\nu,n)}(q)=K(\nu,n)\vec{F}^{(\nu,n+2)}(q). \tag{3.9}
$$

If n is even this may be solved in terms of $\vec{F}^{(\nu,0)}$ as

$$\begin{aligned}\vec{F}^{(\nu,-2m)}(q) &= \frac{1}{(-q^{\frac{1}{2}})_m}\prod_{j=1}^{m}K(\nu,-2j)\vec{F}^{(\nu,0)}(q) \quad \text{for} \quad m\geq 1\\ \vec{F}^{(\nu,2m)}(q) &= q^{-m^2/2}(-q^{\frac{1}{2}})_m\prod_{j=0}^{m-1}K^{-1}(\nu,2j)\vec{F}^{(\nu,0)}(q) \quad \text{for} \quad m\geq 1\end{aligned} \tag{3.10}$$

from which (2.37) follows by using the bose/fermi identity (2.33).

Similarly if n is odd we solve (3.9) in terms of $\vec{F}^{(\nu,-1)}(q)$ and use the identity (2.35) which was proven in [14] to obtain (2.40).

The corresponding results for polynomials given by (2.42) is a simple consequence of the fermionic identity (3.8), the recursive definition (2.25) of the bosonic polynomials $B^{(\nu,n)}_{r',s'}(L,q)$ and identities (2.43).

4. Discussion

There are several features of the general theory of Rogers-Ramanujan type identities which are very clearly seen in the results summarized in sec. 2 and because of their importance deserve to be discussed in further detail.

The first of these features is the occurrence in the range of summation $\mathcal{D}_{r',s'}$ (2.12) of the fermionic sum (2.3) of solutions of the constraint (2.1) where some of the variable are negative. This is an entirely new aspect of fermionic sums which has first been seen in this problem. It is, however, a very general feature, which, once recognized, is seen to occur in many of the fermionic character formulae of nonunitary models $M(p,p')$. This was explicitly seen in [14] for the minimal model $M(2\nu-1,4\nu)$ whose character polynomials are obtained from the polynomials of $SM(2,4\nu)$ by the duality transformation $q\rightarrow\frac{1}{q}$. These extra terms are needed to extend the fermionic polynomials of the minimal model $M(p,p')$ from the special subset of characters considered in our previous study [19] to the general case.

The second feature of our results which deserves prominent discussion is the fact that we have developed a rather complete study of the phenomenon of linear combinations. Indeed, with exceptions such as the $n=0$ Neveu-Schwarz and the $n=1,\ s=2\nu$ and the $n=-1,\ s=2$ Ramond characters all of our formulae for the bosonic characters $\hat{\chi}^{(2,4\nu)}_{r,s}(q)$ involve linear combinations of two or more fermionic series of the canonical form (2.3). Such linear combinations of the canonical fermionic series have been seen before [11], [17], [9] and [18] in several special cases, but in this present study of the $SM(2,4\nu)$ model it is clear that these linear combinations are a mandatory integral part of the theory and that, indeed, it was the failure to consider linear combinations which prevented previous authors from finding a complete set of character formulae.

In our study [19] of the models $M(p,p')$ we explicitly restricted ourselves to cases where these linear combinations did not occur and the generalization of these results will certainly involve linear combinations. As an example of the sort of new phenomena which takes place we consider the dual transformation $q \rightarrow \frac{1}{q}$ which sends $SM(2,4\nu)$ into $M(2\nu-1,4\nu)$ [14]. For this transformation we define

$$\begin{aligned} \vec{f}_{r'}^{(\nu,n)}(q) &= \lim_{L\rightarrow\infty} q^{\frac{L(L+n)}{2}} \vec{F}_{r'}^{(\nu,n)}(L,\frac{1}{q}) \\ \vec{b}_{r'}^{(\nu,n)}(q) &= \lim_{L\rightarrow\infty} q^{\frac{L(L+n)}{2}} \vec{B}_{r'}^{(\nu,n)}(L,\frac{1}{q}) \end{aligned} \tag{4.1}$$

and find from [14]

$$\begin{aligned} \vec{b}_{r'}^{(\nu,0)}(q)|_{s'} &= q^{\frac{(s'-r')^2}{2}} \chi^{(2\nu-1,4\nu)}_{\nu-r'-1,2\nu-2s'-1}(q) + q^{\frac{(s'+r'+1)^2}{2}} \chi^{(2\nu-1,4\nu)}_{\nu+r',2\nu-2s'-1}(q) \\ \vec{b}_{r'}^{(\nu,-1)}(q)|_{s'} &= q^{\frac{(s'-r')(s'-r'-1)}{2}} \chi^{(2\nu-1,4\nu)}_{\nu-r'-1,2\nu-2s'}(q) + q^{\frac{(s'+r')(s'+r'+1)}{2}} \chi^{(2\nu-1,4\nu)}_{\nu+r',2\nu-2s'}(q) \end{aligned} \tag{4.2}$$

where

$$\chi^{(p,p')}_{r,s}(q) = \chi^{(p,p')}_{p-r,p'-s}(q) = \frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} (q^{j(jpp'+rp'-sp)} - q^{(jp+r)(jp'+s)}). \tag{4.3}$$

We also find the fermionic forms

$$\begin{aligned} \lim_{L\rightarrow\infty} q^{\frac{L(L+n)}{2}} F^{(\nu,n)}_{r',s'}(L,q^{-1}) = & \sum_{\tilde{\mathbf{m}}-\text{restrictions}} \frac{q^{\Phi_n(\tilde{\mathbf{m}},r',s')}}{(q)_{\tilde{m}_1}(q)_{\tilde{m}_2}} \\ & \times \prod_{i=3}^{\nu} \begin{bmatrix} ((1-\mathbf{B})\tilde{\mathbf{m}})_i - a_i^{(r')} - a_i^{(s')} \\ \tilde{m}_i \end{bmatrix}_q \end{aligned} \tag{4.4}$$

where

$$\tilde{\mathbf{m}}^t = (n_1, m_2, m_3, \cdots, m_\nu) \tag{4.5}$$

$$\Phi_n(\tilde{\mathbf{m}},r',s') = \frac{1}{2}\tilde{\mathbf{m}}\mathbf{B}\tilde{\mathbf{m}} + L_n(\tilde{\mathbf{m}},s') + C_n(r',s') \tag{4.6}$$

$$2L_n(\tilde{\mathbf{m}},s') = \tilde{m}_\nu - \tilde{m}_{\nu-s'} + \tilde{m}_1\delta_{s',\nu-1} + (2\tilde{m}_1+\tilde{m}_2)(n+\delta_{s',\nu-1}) \tag{4.7}$$

$$4C_n(r',s') = s' - r' + (1+2n)\delta_{s',\nu-1} \tag{4.8}$$

the matrix $\mathbf{B}$ defined by its matrix elements

$$(\mathbf{B})_{j,k} = \begin{cases} 2 & \text{for } j=k=1 \\ \delta_{k,2} & \text{for } j=1,\ 2\leq k\leq \nu \\ \delta_{j,2} & \text{for } k=1,\ 2\leq j\leq \nu \\ \frac{1}{2}\delta_{j,2}\delta_{k,2}+\delta_{j,k}-\frac{1}{2}\delta_{j,k+1}-\frac{1}{2}\delta_{j,k-1}-\frac{1}{2}\delta_{j,\nu}\delta_{k,\nu} & \text{otherwise,} \end{cases} \tag{4.9}$$

the inhomogeneous vectors $a_i^{(s')}$ and $a_i^{(r')}$ are given by (2.7) and the restrictions on the summation variables $\tilde{\mathbf{m}}$ are

$$\tilde{m}_i - \tilde{m}_\nu = \left(\vec{v}^{(s')} + \vec{v}^{(r')}\right)_{i-1} (\mathrm{mod}\ 2); \qquad i = 2, 3, \cdots, \nu \tag{4.10}$$

with

$$(\vec{v}^{(k)})_i = k\theta(1 \leq i \leq \nu - k - 1) + (\nu - 1 - i)\theta(k > 0)\theta(\nu - k - 1 < i \leq \nu - 1) \tag{4.11}$$

where $k = 0, 1, \cdots, \nu - 1$.

Then we find from (3.8) the dual analogue of (3.9)

$$\vec{f}_{r'}^{(\nu,n)}(q) = q^{\frac{n+1}{2}} K(\nu, n) \vec{f}_{r'}^{(\nu,n+2)}(q) \tag{4.12}$$

and thus find for $M(2\nu - 1, 4\nu)$ the results

$$\begin{aligned} \vec{f}_{r'}^{(\nu,-2n)}(q) &= q^{-n^2/2} \prod_{j=1}^{n} K(\nu, -2j) \vec{b}_{r'}^{(\nu,0)}, \quad \text{for } \ n \geq 1 \\ \vec{f}_{r'}^{(\nu,2n)}(q) &= q^{-n^2/2} \prod_{j=0}^{n-1} K^{-1}(\nu, 2j) \vec{b}_{r'}^{(\nu,0)} \quad \text{for } \ n \geq 1 \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} \vec{f}_{r'}^{(\nu.2n+1)}(q) &= q^{-\frac{n(n+1)}{2}} \prod_{j=0}^{n} K^{-1}(\nu, 2j-1) C \vec{b}_{r'}^{(\nu,-1)}(q) \quad \text{for } \ n \geq 0 \\ \vec{f}_{r'}^{(\nu,-1-2n)}(q) &= q^{-\frac{n(n+1)}{2}} \prod_{j=1}^{n} K(\nu, -1-2j) C \vec{b}_{r'}^{(\nu,-1)} \quad \text{for } \ n \geq 1 \end{aligned} \tag{4.14}$$

where we have used (6.23)-(6.25) of [14] rewritten in the form

$$\vec{f}_{r'}^{(\nu,0)}(q) = \vec{b}_{r'}^{(\nu,0)}(q) \ \text{ and } \ \vec{f}_{r'}^{(\nu,-1)}(q) = C\vec{b}_{r'}^{(\nu,-1)}(q). \tag{4.15}$$

We thus find that for the minimal model $M(2\nu-1, 4\nu)$ there are an infinite number of ways in which the characters $\chi_{r,s}^{(2\nu-1,4\nu)}(q)$ are given in term of linear combinations of fermionic

$q-$series. It is to be expected that this is a general feature of the theory of all of the models $M(p,p')$ and thus there is a great deal of generalization of the results of [19] which can be carried out.

The expression for the characters (4.2) of the $M(2\nu-1,4\nu)$ in terms of the bosonic polynomials $B^{(\nu,0)}_{r',s'}(L,\frac{1}{q})$ and $B^{(\nu,-1)}_{r',s'}(L,\frac{1}{q})$ (2.23)-(2.24) reveals yet another important feature of Rogers-Ramanujan type identities; namely the characters of $M(2\nu-1,4\nu)$ which have a well known polynomial representations in terms of $q-$binomials [20] also have a (different) representation in terms of $q-$trinomials. Further examples of trinomial representations of characters of minimal models are given by Andrews and Baxter [16] for the $M(2,7)$ model and by Warnaar [21] for the unitary model $M(p,p+1)$. Just as $q-$ binomials are a natural basis of functions for a spin $\frac{1}{2}$ XXZ system so are the $q-$trinomials the natural basis for a spin 1 XXZ system. Moreover there are extensions of the $q-$trinomials to $q-$multinomials [22] which are related to higher spins. It is expected, but not yet presented in the literature, that each order of multinomial will lead to a separate polynomial representation of the characters (4.3) of $M(p,p')$. Moreover, it is likely that spin-1 $SM(2,4\nu)$ polynomials presented here also have higher spin analogs.

As our final comment of this discussion we return from the technical extensions of Rogers-Ramanujan identities which have been suggested by the details of the computations on the $SM(2,4\nu)$ model to the "folk theorem" stated in the introduction.

The assertion of the "folk theorem" is that the 13 items on the list are not independent subjects but all originate in a more basic mathematical concept such that if properly formulated the intrinsic relations of the items on the list would be manifest in a general fashion which would obviate the need for detailed proofs of individual connections. It is our belief that this principle is counting as seen mathematically in combinatorics and physically in the notion of the statistics of excitations.

This fundamental importance of counting is embodied in the choice of variables (2.1) and has been made very explicit both the fermionic counting computations presented in [13] and [14]. This variable choice originates in the treatment of the thermodynamic Bethe's Ansatz equations of the spin $\frac{1}{2}$ XXZ chain of ref. [15]. It is this counting problem for a system with a finite number of fermionic excitations parameterized by L that unifies the first 12 items on the list of sec. 1.

But the constraint equations (2.1) are obtained as a very special case of the equations of [15] where the temperature T is set precisely to zero and L is kept finite. The $q-$counting problems of [13] and [14] are obtained from [15] in the limit $T\rightarrow 0$, $L\rightarrow\infty$ with TL fixed and in this limit all energy momentum relations of the excitations are linearized. But in the complete treatment of the thermodynamic Bethe's Ansatz the limit $L \to \infty$ at a fixed temperature is taken and the energy momentum relations are not linearized. The result is a set of nonlinear integral equations which explicitly involve the temperature T. It would thus appear as if item 12 on the list is much more general that the previous 11 items. The conclusion we draw from this is that the full significance of the combinatorial basis of the thermodynamic Bethe's Ansatz remains to be explored and that new Rogers-Ramanujan type identities which would incorporate the physical concept of the nonlinear dispersion relations are to be discovered.

Exactly what types of relations amongst TBA solutions should be implied by the counting problem cannot of course be speculated on in any detail. But surely it is of great importance that for the TBA equations of a particular $N = 2$ supersymmetric model it has been shown that TBA equations can be solved in terms of the Painlevé III differential equation [23]-[24]. It is our belief that there is a counting basis not only to the Painlevé equations but to all holonomic systems of equations which are obtained by a deformation procedure. It is this speculation which is implied by the inclusion of the last item on the list of sec. 1.

Acknowledgments

We would like to thank the entire organizing committee for the opportunity of participation in the conference "Physique et Combinatoire" and for the hospitality of the CIRM in the Faculté de Sciences de Luminy. This work was partially supported by NSF grant DMR 9404747 and by the Deutsche Forshungs Gemeinschaft.

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