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THE NATURE OF COSMIC TIME

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ABSTRACT

Hamilton-Jacobi theory provides a natural starting point for a covariant description of the gravitational field. Using a spatial gradient expansion, one may solve for the phase of the wavefunction by using a line-integral in superspace. Each contour of integration corresponds to a particular choice of time-hypersurface, and each yields the same answer. In this way, one can describe all time choices simultaneously. In an interesting application to cosmology, I compute large-angle microwave background anisotropies and the galaxy-galaxy correlation function associated with the scalar and tensor fluctuations of power-law inflation.

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1. Introduction

Hamilton-Jacobi (HJ) theory is a cornerstone of modern theoretical physics. It may be profitably applied to numerous problems in cosmology. Since a full quantum theory is lacking, a semi-classical analysis provides our best understanding of the gravitational field.

HJ theory provides an elegant formalism for computing density perturbations as well as microwave background fluctuations arising from the inflationary scenario.[1,2] It has also been successfully employed in deriving the Zel'dovich approximation[3] (which describes the formation of sheet-like structures in the Universe) from general relativity.[4] Numerous researchers have employed HJ methods in an attempt to recover the inflaton potential from cosmological observations.[5] Lastly, HJ techniques can be used to construct inflationary models that yield non-Gaussian primordial fluctuations;[6] such models could possibly resolve the problems of large scale structure in the Universe.[7]

I will focus on one particularly attractive feature of HJ theory: it provides a covariant formulation of the gravitational field.[8] In the semi-classical theory, the answer to the question of time is clear: time is arbitrary. HJ theory enables one to consider all such time choices simultaneously. I will now consider a simple analogy from potential theory which illuminates the general technique.

2. Potential Theory

The fundamental problem in potential theory is: given a force field $g^i(u_k)$ which is a function of n variables u_k, what is the potential $\Phi \equiv \Phi(u_k)$ (if it exists) whose gradient returns the force field,

$$\frac{\partial \Phi}{\partial u_i} = g^i(u_k) \quad ? \tag{1}$$

Not all force fields are derivable from a potential. Provided that the force field satisfies the integrability relation,

$$0 = \frac{\partial g^i}{\partial u_j} - \frac{\partial g^j}{\partial u_i} = \left[\frac{\partial}{\partial u_j}, \frac{\partial}{\partial u_i} \right] \Phi \, , \tag{2}$$

(i.e., it is curl-free), one may find a solution which is conveniently expressed using a line-integral

$$\Phi(u_k) = \int_C \sum_j dv_j \ g^j(v_l) \ . \tag{3}$$

If the two endpoints are fixed, all contours return the same answer. In practice, one employs the simplest contour that one can imagine: a line connecting the origin to the observation point u_k. Using s, $0 \le s \le 1$, to parameterize the contour, the line-integral may be rewritten as

$$\Phi(u_k) = \sum_{j=1}^{n} \int_0^1 ds \ u_j \ g^j(su_k) \ . \tag{4}$$

Similarly, in solving for the phase of the wavefunctional, one utilizes a line-integral in *superspace*.

3. Solving the Hamilton-Jacobi Equation for General Relativity

The Hamilton-Jacobi equation for general relativity is derived using a Hamiltonian formulation of gravity. One first writes the line element using the ADM 3+1 split,

$$ds^2 = \left(-N^2 + \gamma^{ij} N_i N_j\right) dt^2 + 2N_i dt \, dx^i + \gamma_{ij} \ dx^i dx^j \ , \tag{5}$$

where N and N_i are the lapse and shift functions, respectively, and γ_{ij} is the 3-metric. Hilbert's action for gravity interacting with a scalar field becomes

$$\mathcal{I} = \int d^4x \left(\pi^\phi \dot{\phi} + \pi^{ij} \dot{\gamma}_{ij} - N\mathcal{H} - N^i \mathcal{H}_i\right) . \tag{6}$$

The lapse and shift functions are Lagrange multipliers that imply the energy constraint $\mathcal{H}(x) = 0$ and the momentum constraint $\mathcal{H}_i(x) = 0$.

The object of chief importance is the generating functional $\mathcal{S} \equiv \mathcal{S}[\gamma_{ij}(x), \phi(x)]$. For each universe with field configuration $[\gamma_{ij}(x), \phi(x)]$ it assigns a number which can be complex. The generating functional is the 'phase' of the wavefunctional in the semi-classical approximation: $\Psi \sim e^{i\mathcal{S}}$. The probability functional, $\mathcal{P} \equiv |\Psi|^2$, is given by the square of the wavefunctional.

Replacing the conjugate momenta by functional derivatives of $\mathcal{S}$ with respect to the fields,

$$\pi^{ij}(x) = \frac{\delta \mathcal{S}}{\delta \gamma_{ij}(x)} \ , \qquad \pi^\phi(x) = \frac{\delta \mathcal{S}}{\delta \phi(x)} \ , \tag{7}$$

and substituting into the energy constraint, one obtains the Hamilton-Jacobi equation,

$$\mathcal{H}(x) = \quad \gamma^{-1/2} \frac{\delta \mathcal{S}}{\delta \gamma_{ij}(x)} \frac{\delta \mathcal{S}}{\delta \gamma_{kl}(x)} \left[2\gamma_{il}(x)\gamma_{jk}(x) - \gamma_{ij}(x)\gamma_{kl}(x)\right]$$

$$+\frac{1}{2}\gamma^{-1/2}\left(\frac{\delta\mathcal{S}}{\delta\phi(x)}\right)^2+\gamma^{1/2}V(\phi(x))$$
$$-\frac{1}{2}\gamma^{1/2}R+\frac{1}{2}\gamma^{1/2}\gamma^{ij}\phi_{,i}\phi_{,j}=0\ , \tag{8}$$

which describes how $\mathcal{S}$ evolves in superspace. R is the Ricci scalar associated with the 3-metric, and $V(\phi)$ is the scalar field potential. In addition, one must also satisfy the momentum constraint

$$\mathcal{H}_i(x)=-2\left(\gamma_{ik}\frac{\delta\mathcal{S}}{\delta\gamma_{kj}(x)}\right)_{,j}+\frac{\delta\mathcal{S}}{\delta\gamma_{lk}(x)}\gamma_{lk,i}+\frac{\delta\mathcal{S}}{\delta\phi(x)}\phi_{,i}=0\ , \tag{9}$$

which legislates gauge invariance: $\mathcal{S}$ is invariant under reparametrizations of the spatial coordinates.[9] (Units are chosen so that $c=8\pi G=\hbar=1$). Since neither the lapse function nor the shift function appears in eqs.(8,9) the temporal and spatial coordinates are *arbitrary*: HJ theory is *covariant*.

As a first step in solving eqs.(8,9), I will expand the generating functional

$$\mathcal{S}=\mathcal{S}^{(0)}+\mathcal{S}^{(2)}+\mathcal{S}^{(4)}+\dots\ , \tag{10}$$

in a series of terms according to the number of spatial gradients that they contain. The invariance of the generating functional under spatial coordinate transformations suggests a solution of the form,

$$\mathcal{S}^{(0)}[\gamma_{ij}(x),\phi(x)]=-2\int d^3x\gamma^{1/2}H\left[\phi(x)\right]\ , \tag{11}$$

for the zeroth order term $\mathcal{S}^{(0)}$. The function $H\equiv H(\phi)$ satisfies the separated HJ equation of order zero,[6]

$$H^2=\frac{2}{3}\left(\frac{\partial H}{\partial\phi}\right)^2+\frac{1}{3}V(\phi)\ , \tag{12}$$

which is an ordinary differential equation. Note that $\mathcal{S}^{(0)}$ contains no spatial gradients.

In order to compute the higher order terms, one introduces a change of variables, $(\gamma_{ij},\phi)\rightarrow(f_{ij},u)$:

$$u=\int\frac{d\phi}{-2\frac{\partial H}{\partial\phi}}\ ,\quad f_{ij}=\Omega^{-2}(u)\,\gamma_{ij}\ , \tag{13}$$

where the conformal factor $\Omega\equiv\Omega(u)$ is defined through

$$\frac{d\ln\Omega}{du}\equiv-2\frac{\partial H}{\partial\phi}\frac{\partial\ln\Omega}{\partial\phi}=H\ . \tag{14}$$

in which case the equation for $\mathcal{S}^{(2m)}$ becomes

$$\left.\frac{\delta\mathcal{S}^{(2m)}}{\delta u(x)}\right|_{f_{ij}}+\mathcal{R}^{(2m)}[u(x),f_{ij}(x)]=0\ . \tag{15}$$

The remainder term $\mathcal{R}^{(2m)}$ depends on some quadratic combination of the previous order terms (*i.e.*, it may be written explicitly[8]). For example, for $m = 1$, it is

$$\mathcal{R}^{(2)} = \frac{1}{2}\gamma^{1/2}\gamma^{ij}\phi_{,i}\phi_{,j} - \frac{1}{2}\gamma^{1/2}R\,. \tag{16}$$

Eq.(15) has the form of an infinite dimensional gradient. It may integrated using a line integral analogous to eq.(4):

$$\mathcal{S}^{(2m)} = -\int d^3x \int_0^1 ds\ u(x)\ \mathcal{R}^{(2m)}[su(x), f_{ij}(x)]\,. \tag{17}$$

Typically, $\mathcal{S}^{(2m)}$ is an integral of terms which contain the Ricci tensor and derivatives of the scalar field.[8]

The integrability condition for the HJ equation[10] follows from the Poisson bracket of the energy constraints evaluated at spatial points x and x',

$$\{\mathcal{H}(x^k), \mathcal{H}(x^{k'})\} = [\gamma^{ij}(x^k)\mathcal{H}_j(x^k) + \gamma^{ij}(x^{k'})\mathcal{H}_j(x^{k'})]\,\frac{\partial}{\partial x^i}\delta^3(x^k - x^{k'})\,. \tag{18}$$

In fact, alternative contours replacing the line-integral eq.(17) will correspond to different time-hypersurface choices. Provided that the generating functional is invariant under reparametrizations of the spatial coordinates, (e.g., $\mathcal{H}_i$ vanishes in the right-hand-side of eq.(18)), different time-hypersurface choices will lead to the same generating functional. Hypersurface invariance is closely related to gauge invariance.

4. Computing Large-Angle Microwave Background Fluctuations and Galaxy Correlations

In order to describe the fluctuations arising during the inflationary epoch, it is necessary to sum an infinite subset[1] of the terms $\mathcal{S}^{(2m)}$. In this case, one considers all terms which are quadratic in the Ricci tensor $\tilde{R}_{ij}$ of the conformal 3-metric $f_{ij}(x)$ defined in eq.(13). Once again, no explicit choice of time hypersurface is made.

However, when one compares theory with observations, there are indeed preferred gauges. The phase transition of photon-decoupling occurs essentially on a uniform temperature slice, $T \sim 4000K$, when protons combine with electrons to form neutral hydrogen. For adiabatic perturbations at large wavelengths, this slice is the same as a comoving, synchronous time hypersurface which Sachs and Wolfe[11] used in the computation of large-angle microwave background anisotropies.

The power-law inflationary model[12] provides an excellent example of HJ techniques. For this model, the scalar factor of the Universe evolves as $a \sim t^p$ which describes an inflationary epoch provided $p > 1$. The scalar field potential has an exponential form

$$V(\phi) = V_0 \exp\left(-\sqrt{\frac{2}{p}}\phi\right) \ . \tag{19}$$

Power-law inflation is of high interest for observational cosmology because it may produce copious amounts of primordial gravitational radiation,[13,14] which is in essence a quantum gravitational effect.

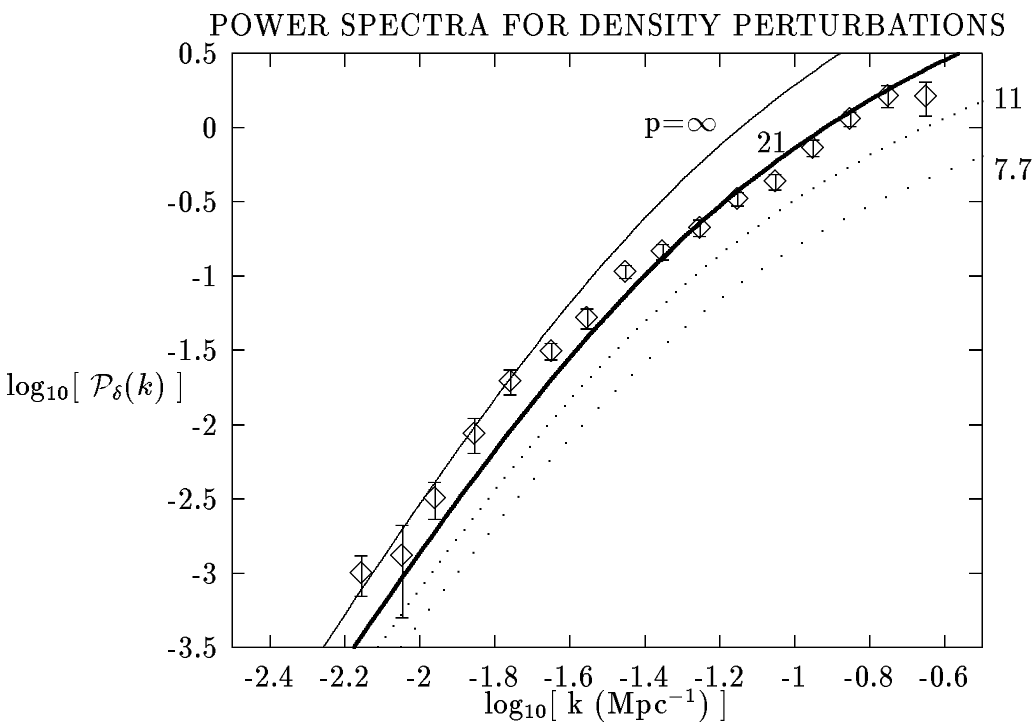

Fig. 1. For the present epoch, the power spectra for the linear density perturbation $\delta\rho/\rho$ is shown. The data points are the observed power spectrum derived from galaxy surveys. The curves are theoretical predictions of the power-law inflationary model for several values of p: $p = \infty$ is the standard cold-dark-matter model; $p = 21$ provides the best fit.

Fig.(1) illustrates the observed power spectrum,

$$\mathcal{P}_\delta(k) \equiv \frac{k^3}{2\pi^2}\int d^3x \ e^{-i\vec{k}\cdot\vec{x}} \ < \frac{\delta\rho(x)}{\rho}\frac{\delta\rho(0)}{\rho} > , \tag{20}$$

for the linear density perturbation at the present epoch; here k is the comoving wavenumber. The data points were compiled using eight galaxy surveys.[15] Also shown are the power spectra arising from power-law inflation for various values of p. After the inflationary epoch, I have assumed that the evolution of the fluctuations is described by the cold-dark-matter transfer function[16] where the present Hubble parameter is taken to be $H_0 = 50$ km s^{-1}Mpc^{-1}. With a correction for gravitational waves, the theoretical power spectra for density perturbations have been normalized using the 2-year DMR data set[17] of the Cosmic Background Explorer (COBE) satellite: $\sigma_{sky}(10^0) = 30.5 \pm 2.7\mu K \quad$ (68% confidence level).

The discrepancy between the galaxy surveys and the standard cold-dark-matter model ($p = \infty$) is quite severe at short length scales, $k > 10^{-1.4}$ Mpc^{-1}. The bold line, $p = 21$, provides the best fit to the observed data. The agreement is excellent at short scales. At longer scales, the theoretical model under-predicts the observed power but the deficit is not very severe. For $p = 21$, gravitational waves contribute 35% to σ^2_{sky}, the square of COBE's microwave anisotropy.

5. Summary

The question of time choice in general relativity is a difficult one, particularly for the quantum theory.[18] For semi-classical problems of interest to observational cosmology, one may construct a covariant formalism which treats all time choices on an equal footing. Power-law inflation with $p = 21$ yields a better fit to cosmological data than the standard cold-dark-matter model.

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References

1. D.S. Salopek and J.M. Stewart, *Phys. Rev. D* **51**, 517 (1995).
2. J.J. Halliwell and S.W. Hawking, *Phys. Rev. D* **31**, 1777 (1985).
3. Y.B. Zel'dovich, *Astron. & Astrophys.*, **5**, 84 (1970).
4. K.M. Croudace, J. Parry, D.S. Salopek and J.M. Stewart, *Astrophys. J.* **423**, 22 (1994); D.S. Salopek, J.M. Stewart and K.M. Croudace, *Mon. Not. Roy. Astr. Soc.*, **271**, 1005 (1994).
5. E.J. Copeland *et al* , *Phys. Rev. D* **48**, 2529 (1993).
6. D.S. Salopek, *Phys. Rev. D* **43**, 3214 (1991); *ibid* **45**, 1139 (1992).
7. L. Moscardini *et al*, *Astrophys. J.* **413**, 55 (1993).
8. J. Parry, D.S. Salopek and J.M. Stewart, *Phys. Rev. D*, **49**, 2872 (1994).
9. A. Peres, *Nuovo Cim.* **26**, 53 (1962).
10. V. Moncrief and C. Teitelboim, *Phys. Rev. D* **6**, 966 (1972).

11. R.K. Sachs and A.M. Wolfe, *Astrophys. J.* **147**, 73 (1967).
12. F. Lucchin and S. Matarrese, *Phys. Rev. D* **32**, 1316 (1985).
13. D.S. Salopek, *Phys. Rev. Lett.*, **69**, 3602 (1992); R.L. Davis *et al*, *Phys. Rev. Lett.*, **69**, 1856 (1992).
14. D.S. Salopek, Proc. of the Intern. School of Astrophys., "D. Chalonge", Third Course, Erice, Italy, September 4-16, 1994, ed. N. Sanchez (Kluwer, 1995).
15. J.A. Peacock and S.J. Dodds, *Mon. Not. Roy. Astron. Soc.* **267**, 1020 (1994).
16. P.J.E. Peebles, *Astrophys. J.* **263**, L1 (1982); J.R. Bond and G. Efstathiou, *Astrophys. J. Lett.* **285**, L45 (1984).
17. C.L. Bennett *et al*, *Astrophys. J.* **436**, 423 (1994).
18. K.V. Kuchař, in Proc. of 4th Canadian Conference on GR and Relativistic Astrophys., May 16-18 1991, ed. G. Kunstatter *et al*, p.211 (World Scientific, 1992).