Nuts have no hair

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Abstract

We show that the Riemannian Kerr solutions are the only Riemannian, Ricci-flat and asymptotically flat C^2-metrics $g_{\mu\nu}$ on a 4-dimensional complete manifold $\mathcal{M}$ of topology $R^2 \times S^2$ which have (at least) a 1-parameter group of periodic isometries with only isolated fixed points ("nuts") and with orbits of bounded length at infinity.

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The relevance of instantons, (here understood as being regular, real, Riemannian, Ricci-flat manifolds), in quantum gravity [1]-[4] has stimulated interest in theorems on the (non-) existence in particular of periodic such solutions and of solutions with isometries. It has been shown under rather general conditions that there are no non-trivial instantons on R^4 and on $\mathrm{R}^3 \times \mathrm{S}$ [5]. Known examples include the $\mathrm{R}^2 \times \mathrm{S}^2$ Kerr-NUT instanton [2, 4, 6] which (like many others) has been found by "Euclideanizing" the corresponding Lorentzian solution [7], and the adaption of Lorentzian uniqueness ("no-hair") theorems has been discussed as well [3]. The different character of the Riemannian case (the absence of horizons and ergospheres and the existence of singularity-free solutions) require and suggest, however, alternative approaches to the uniqueness problem. Based on a characterization of the Lorentzian Kerr metric in terms of complex quantities [8] which become real in the Riemannian case and also satisfy generalizations of "Israel"-type identities [9] we have obtained the following result. (We abbreviate "Riemannian" by "Riem." and "Lorentzian" by "Lor." henceforth. Greek indices go from 0 to 3).

Theorem. The Riem. Kerr solutions are the only Riem., Ricci-flat and asymptotically flat C^2-metrics $g_{\mu\nu}$ on a 4-dimensional complete manifold $\mathcal{M}$ of topology $\mathrm{R}^2 \times \mathrm{S}^2$ which have (at least) a 1-parameter group of periodic isometries with only isolated fixed points and with orbits of bounded length at infinity.

Introductory material and 2 Lemmas will precede the proof. Details of parts of our analysis and extensions thereof will be given elsewhere.

The condition of Ricci flatness ($R_{\mu\nu} = 0$) implies that $g_{\mu\nu}$ and the Killing field ξ^μ corresponding to the isometry μ_τ (τ is the group parameter) are analytic in harmonic coordinates [10]. The set $\mathcal{L}$ of fixed points of μ_τ has the following structure [2]. At every $q \in \mathcal{L}$ the differential $\mu_{\tau*}$ leaves invariant two 2-dimensional orthogonal subspaces T_q^+ and T_q^- of the tangent space T_q. If $\mu_{\tau*}$ acts as the identity on one of T_q^+ or T_q^- there is a 2-surface of fixed points called "bolt" which we exclude by assumption. If q is isolated it is called a "nut" after the Taub-NUT metric [11]. In this case $\mu_{\tau*}$ acts as rotations in each of T_q^+ and T_q^- with periods $\tau^\pm = 2\pi/\kappa^\pm$ (the smallest values of τ such that $\mu_{\tau*}X^\pm = X^\pm$ for $X^\pm \in T_q^\pm$). κ^+ and κ^- are also the skew eigenvalues of $\nabla_\mu \xi_\nu$ in an orthonormal frame and called "gravities" of the nut. As μ_τ is assumed to be periodic, there is a (smallest) τ^0 such that $\mu_{\tau^0 *}X = X$ for all $X \in T_q$ which implies that $\tau^+ p^+ = \tau^0 = \tau^- p^-$ for relative prime integers p^+ and p^-. Since μ_τ commutes with the exponential map, i.e. $\exp(\mu_{\tau*}X) = \mu_\tau(\exp X)$, the period of $\mu_{\tau*}X$ at q equals the periods of the orbits through all points of a geodesic emanating from q with tangent vector X (at least) as long as the exponential map is non-singular.

Let $\lambda = \xi^\mu \xi_\mu$ denote the norm and $\omega_\mu = \epsilon_{\mu\nu\sigma\tau}\xi^\nu \nabla^\sigma \xi^\tau$ the twist of ξ^μ. ($\epsilon_{\mu\nu\sigma\tau}$ is antisymmetric and $\epsilon_{0123} = (det\ g)^{1/2}$). By Ricci flatness, $\nabla_\mu(\lambda^{-2}\omega^\mu) = 0$. Hence the "nut charge" [2]

$$m_i^* = \frac{1}{8\pi}\int_{\mathcal{S}_i} \lambda^{-2}\omega_\mu dS^\mu = \frac{\pi}{2\kappa_i^+\kappa_i^-} \tag{1}$$

is independent of the compact 3-surface $\mathcal{S}_i$ which encloses the nut n_i and does not intersect others. The surface element dS^μ points outwards. The second part of (1) follows by Taylor-expanding ξ^μ at n_i and by shrinking $\mathcal{S}_i$ to n_i.

In Lemma 1 and in the Theorem we adopt a standard definition of asymptotic flatness (AF) [4, 12] and require $\mathcal{M}$ minus a compact set to be diffeomorphic to $\mathrm{R}^+ \times \mathrm{S} \times \mathrm{S}^2$ and the metric and its first and second derivatives to go to the flat metric and its derivatives, with the usual $1/r-$ falloff in coordinates adapted to the isometry, i.e. $\partial_\sigma g_{\mu\nu} = 0$. The definition implies that at infinity all orbits μ_τ have the same length which we call l_∞. (We remark, however, that the limit of the length function may be discontinuous when the limiting orbit is approached via orbits which wind repeatedly around the large $\mathrm{S}^2 \times \mathrm{S}$-surfaces of constant distance from a nut). In Lemma 2 we will require "local asymptotic flatness" (ALF) with the cyclic group Z [4, 12]. In this setting we can define the "dual mass" m^* [13] by considering the integral in (1) over the asymptotic region. We remark that $(\mathcal{M}, g_{\mu\nu})$ is AF iff it is ALF and $m^* = 0$. ξ^μ is normalized such that $\lambda \to 1$ at infinity.

Lemma 1. Under the requirements of the Theorem $\mathcal{M}$ has precisely 2 nuts n_1 and n_2 whose "gravities" $\kappa_1^\pm$ and $\kappa_2^\pm$ satisfy $\kappa_1^+ = \kappa_2^+$ and $\kappa_1^- = -\kappa_2^-$. (The choice of the labels + and − is a convention). Moreover, we have $l_\infty \geq \min(2\pi/\kappa^+, 2\pi/\kappa^-)$ where $\kappa^\pm = |\kappa_1^\pm| = |\kappa_2^\pm|$.

Proof. As $\mathcal{M}$ has topology $\mathrm{R}^2 \times \mathrm{S}^2$, it has Euler number $\chi = 2$ and signature $\tau = 0$. In the absence of bolts and using AF, the index theorem implies that χ is equal to the number of nuts and $3\tau = \kappa_1^+/\kappa_1^- + \kappa_1^-/\kappa_1^+ + \kappa_2^+/\kappa_2^- + \kappa_2^-/\kappa_2^+$. (See [14] for the compact case and [12, 15] regarding boundary terms). Together with (1) and $m_1^* + m_2^* = m^* = 0$ we obtain the first part of the lemma.

As $\mathcal{M}$ is not compact there is (at least one) $X_1 \in T_{n_1}$ and (at least one) $X_2 \in T_{n_2}$ such that $\gamma_1 = \exp(tX_1)$ and $\gamma_2 = \exp(tX_2)$, $t \in (0, \infty)$ approach infinity as minimizing (radial) geodesics [16]. As families $\mu_\tau(\gamma_1)$ and $\mu_\tau(\gamma_2)$ of such geodesics diverge in the asymptotic region, the exponential map remains non-singular in the limit. Hence l_∞ equals the periods of $\mu_{\tau *}X_1$ and $\mu_{\tau *}X_2$ which can be $\tau^\pm$ or $\tau^0 \geq \tau_\pm$. Thus the lemma holds.

From $R_{\mu\nu} = 0$, ω_μ is curl-free, i.e. $\nabla_{[\mu}\omega_{\nu]} = 0$. As $\mathcal{M}$ is simply connected, $\nabla_\mu \omega = \omega_\mu$ defines a scalar field ω globally and up to a constant which we choose such that ω vanishes at infinity. We also define $\mathcal{E}_\pm = \lambda \pm \omega$, $\mu = \frac{1}{2}(\nabla_\mu \xi_\nu)(\nabla^\mu \xi^\nu)$ and $\nu = \frac{1}{4}\epsilon_{\mu\nu\sigma\tau}(\nabla^\mu \xi^\nu)(\nabla^\sigma \xi^\tau)$ which satisfy, again from $R_{\mu\nu} = 0$,

$$\Box \mathcal{E}_\pm = 4(\mu \pm \nu) = \lambda^{-1} \nabla_\mu \mathcal{E}_\pm \nabla^\mu \mathcal{E}_\pm \geq 0. \tag{2}$$

The maximum principle and the asymptotic conditions imply $\mathcal{E}_\pm < 1$ and hence $\mathcal{E}_\pm = -\mathcal{E}_\mp + 2\lambda \geq -\mathcal{E}_\mp > -1$.

To simplify what follows we now foliate $\mathcal{M} \setminus \mathcal{L}$ by the orbits of μ_τ [2, 17]. We obtain a manifold $(\mathcal{N}, \gamma_{ij})$ where γ_{ij} is the pullback of $\gamma_{\mu\nu} = \lambda g_{\mu\nu} - \xi_\mu \xi_\nu$. (Tensors on $\mathcal{N}$ carry latin indices). We denote by D_i and R_{ij} the covariant derivative and the Ricci tensor with respect to γ_{ij} and introduce $w_\pm = (1 + \mathcal{E}_\pm)^{-1}(1 - \mathcal{E}_\pm)$, $\Theta = 1 - w_+ w_-$, $A_i = \frac{1}{2}(w_+ D_i w_- - w_- D_i w_+)$ and $\mathcal{D}^i_\pm = \Theta^{-1} D^i \mp 2\Theta^{-2} A^i$. Since $|\mathcal{E}_\pm| < 1$ we have $0 < w_\pm < \infty$ and $\Theta > 0$. On $\mathcal{N}$ the condition $R_{\mu\nu} = 0$ reads

$$D_i \mathcal{D}^i_\pm w_\pm = 0 \tag{3}$$

$$R_{ij} = 2\Theta^{-2} D_{(i} w_- D_{j)} w_+. \tag{4}$$

When $(\mathcal{M}, g_{\mu\nu})$ is ALF, $(\mathcal{N}, w_\pm, \gamma_{ij})$ is asymptotically flat in a standard sense (compare [21]).

In coordinates $r = \Re - m$ where $\Re$ is the radial "Boyer-Lindquist"- coordinate (equ. (2.13) of [18]) the Riem. Kerr-NUT metric reads

$$w_\pm = m_\pm (r \pm \alpha\ cos\theta)^{-1}, \tag{5}$$

$$\gamma_{ij} dx^i dx^j = (r^2 - m_+ m_- - \alpha^2)^{-1}(r^2 - m_+ m_- - \alpha^2 cos^2\theta) dr^2 + \\ +(r^2 - m_+ m_- - \alpha^2 cos^2\theta) d\theta^2 + (r^2 - m_+ m_- - \alpha^2) sin^2\theta d\phi^2. \tag{6}$$

Here $m = \frac{1}{2}(m_+ + m_-)$ and $m^* = \frac{1}{2}(m_+ - m_-)$ are the mass and the dual mass and α is another real constant. For $m^* = 0$ this is the Riem. Kerr metric for which $\xi^\mu = \partial/\partial\tau$ has 2 nuts at $r = \sqrt{m^2 + \alpha^2}$, $\theta = 0$ and $\theta = \pi$. In the Riem. Schwarzschild case ($m^* = \alpha = 0$) this vector has a bolt at $r = m$. For the Riem. Kerr metric Kruskal-like coordinates can be obtained by "Euclideanizing" (3.8) of [18].

Our characterization involves the pairs of quantities

$$k^4_\pm = D^i w_\pm D_i w_\pm, \tag{7}$$

$$B^\pm_{ij} = 4\Theta^{-2} \mathcal{C}[D_i D_j w_\pm - (3w_\pm^{-1} + \Theta^{-1} w_\mp) D_i w_\pm D_j w_\pm], \tag{8}$$

where $\mathcal{C}$ denotes the trace-free part and

$$C^{\pm}_{ijk} = 4\Theta^{-2}(D_i D_{[j} w_\pm D_{k]} w_\pm - \gamma_{i[j} u^{\pm}_{k]}), \tag{9}$$

where

$$u^{\pm}_k = \gamma^{ij} D_i D_{[j} w_\pm D_{k]} w_\pm. \tag{10}$$

On sets where $k^4_\pm \neq 0$ (3) and (4) imply, for each $\alpha \in \mathrm{R}$,

$$\begin{aligned} D_i \mathcal{D}^i_\pm \frac{k^{\alpha+1}_\pm}{w^\alpha_\pm} &= \alpha(\alpha+1)\frac{k^{\alpha-1}_\pm}{\Theta w^\alpha_\pm}(D_i k_\pm - \frac{k_\pm}{w_\pm} D_i w_\pm)(D^i k_\pm - \frac{k_\pm}{w_\pm} D^i w_\pm) + \\ &\quad + \frac{\alpha+1}{16}\frac{k^{\alpha-7}_\pm}{w^\alpha_\pm}\Theta^3 C^{\pm}_{ijk} C^{ijk}_\pm. \end{aligned} \tag{11}$$

For $\alpha \geq 0$, the r.h. sides of (11) are non-negative and for $\alpha = 3$ they can be written as $\frac{1}{8}\Theta^3 w_\pm^{-3} B^{\pm}_{ij} B^{ij}_\pm$.

When ξ^μ is hypersurface-orthogonal ($\omega = 0$) the objects $B^{\pm}_{ij}$ coincide and are, by virtue of (3) and (4), equal to certain functions $f_\pm(\lambda)$ times the Ricci tensors $R^{\pm}_{ij}$ with respect to the metrics $\gamma^{\pm}_{ij} = \frac{1}{16}\lambda^{-1}(1 \pm \lambda^{1/2})^4 \gamma_{ij}$. Likewise, for $\omega = 0$ each of $C^{\pm}_{ijk}$ reduces to the Cotton tensor which characterizes conformal flatness. The corresponding characterizations of the Lor. Schwarzschild metric and the restriction of (11) for certain values of α were employed in uniqueness proofs [9, 19]. In the general case $B^{\pm}_{ij}$, $k_\pm$ and $C^{\pm}_{ijk}$ have complex Lor. counterparts B_{ij}, k and C_{ijk} which have analogous properties. The latter two quantities have been employed in local characterizations of the Kerr metric among the AF ones [8] and of a larger class of metrics if the asymptotic assumption is dropped [20]. The methods of these papers can be straightforwardly applied in the Riem. case and yield the following result.

Lemma 2. An ALF C^2-solution $(w_\pm, \gamma_{ij})$ of (3) and (4) is isometric to a Riem. Kerr-NUT metric iff it satisfies one of (12), (13) or (14) (a pair of equations in each case) in a neighbourhood $\mathcal{U}$ of a point of $\mathcal{N}$:

$$\text{Either}\ \ k_+ = \sigma_+ w_+ \ \text{ or }\ w_+ = 0, \text{ and either }\ k_- = \sigma_- w_- \ \text{ or }\ w_- = 0, \tag{12}$$

where $\sigma_\pm > 0$ are constants.

$$B^{\pm}_{ij} = 0. \tag{13}$$

$$C^{\pm}_{ijk} = 0. \tag{14}$$

Proof. Degenerate cases in which either w_+ and w_- are functionally related or one of $w_\pm$ vanishes on $\mathcal{U}$ are easily disposed of. In the generic case, from (3) and (11), (12) implies (13) and (14). Conversely, (12) follows either by inserting (8) into $B^{\pm}_{ij} w^j_\pm = 0$, using also (3), or from $C^{\pm}_{ijk} = 0$ and the ALF conditions as in the Lor. case [8, 20].

The Riem. Kerr-NUT metric in the form (5), (6) is easily seen to satisfy (12). To show the converse we essentially follow [20] and define the vector field

$$l_i = -\frac{1}{2}\sigma_-^{-4}\sigma_+^{-4}w_-^{-3}w_+^{-3}\ \Theta\ \epsilon_{ijk}(D^j w_-)(D^k w_+) \tag{15}$$

which, from (3) and (13), is hypersurface-orthogonal ($\epsilon_{ijk}l^i D^j l^k = 0$) and Killing ($D_{(i}l_{j)} = 0$). Hence there exists a function r_0 on $\mathcal{U}$ such that $l^i = \partial/\partial r_0$ and the metric coefficients in the coordinates r_0 and $r_\pm = \frac{1}{2}(\sigma_+^{-2}w_+^{-1} \pm \sigma_-^{-2}w_-^{-1})$ are independent of r_0. Moreover, from (12) and (15) the metric is diagonal in these coordinates and

$$\gamma^{r_+r_+} + \gamma^{r_-r_-} = \gamma^{ij}D_i r_+ D_j r_+ + \gamma^{ij}D_i r_- D_j r_- = 1, \tag{16}$$

$$\gamma_{r_0r_0} = \gamma_{ij}l^i l^j = \gamma^{r_+r_+}\gamma^{r_-r_-}(r_+^2 - r_-^2 - \sigma_+^{-2}\sigma_-^{-2})^2. \tag{17}$$

Finally, inserting (8) into $(B_{ij}^+ w_-^j + B_{ij}^- w_+^j)D^i r_\pm = 0$ and using (16) and again (3) and (12) yields linear first order differential equations for $\gamma^{r_+r_+}$ or $\gamma^{r_-r_-}$. Integrating, we find (5) and (6) with $r_+ = r$, $r_- = \alpha\ cos\theta$, $r_0 = \alpha^{-1}\phi$ and $\sigma_\pm = |m_\pm|^{-1/2}$ where α is a constant of integration. This proves the lemma.

Proof of the Theorem. We prove the Theorem by integrating (11) for $\alpha = 1$. Rewriting the l.h. sides in terms of quantities defined above we find

$$\nabla_\mu[\frac{(1+\mathcal{E}_+)(1+\mathcal{E}_-)}{\lambda}(\nabla^\mu \frac{\sqrt{\mu \pm \nu}}{1-\mathcal{E}_\pm^2}) + \frac{\sqrt{\mu \pm \nu}}{2\lambda^2}(\nabla^\mu \mathcal{E}_\mp - \frac{1-\mathcal{E}_\mp^2}{1-\mathcal{E}_\pm^2}\nabla^\mu \mathcal{E}_\pm)] \geq 0, \tag{18}$$

and by Lemma 2 equality implies Kerr in the AF case. The vector pair in brackets, called $Y_\pm^\mu$, is singular at the nuts and on the sets $\mathcal{X}_\pm$ where $\mu \pm \nu = 0$. The latter are submanifolds of dimension ≤ 2 and invariant under μ_τ as can be shown from (2) like in the static Lor. case [19]. We note that at a nut $\sqrt{\mu \pm \nu} = |\kappa^+ \pm \kappa^-|$, and we assume first that none of the nuts is (anti-) self dual, viz. $|\kappa^+ \pm \kappa^-| \neq 0$. Applying the divergence theorem to (18) we get the bounds

$$0 \leq \int_\infty Y_\mu^\pm dS^\mu + \sum_{i=1,2}\int_{\mathcal{S}_i} Y_\mu^\pm dS^\mu + \int_{\mathcal{T}_\pm} Y_\mu^\pm dS^\mu \tag{19}$$

on surface integrals (with dS^μ directed outwards) over infinity, over small spheres $\mathcal{S}_i$ around the nuts and over small tubes $\mathcal{T}_\pm$ around $\mathcal{X}_\pm$. Again both bounds are simultaneously saturated for Kerr only. Performing the limits $\mathcal{S}_i \to n_i$ and $\mathcal{T}_\pm \to \mathcal{X}_\pm$ as carefully as done in [19] we find that the last pair of integrals in (19) is non-positive whereas the first two pairs can be evaluated using (1) and Lemma 1. We obtain

$$\begin{aligned} 0 &\leq -4\pi l_\infty \pm 8\pi m_1^*|\kappa_1^+ \pm \kappa_1^-| \pm 8\pi m_2^*|\kappa_2^+ \pm \kappa_2^-| \leq \\ &\leq -4\pi \min(2\pi/\kappa^+, 2\pi/\kappa^-) + (4\pi^2/\kappa^+\kappa^-)(|\kappa^+ + \kappa^-| - |\kappa^+ - \kappa^-|) = \\ &= 0 \end{aligned} \tag{20}$$

which also follows easily for (and excludes) (anti-) self-dual nuts. This finishes the proof.

Our result can possibly be generalized in various directions. Firstly, it might be possible to show directly (i.e. without using results of this paper) that geodesics emanating from nuts n_i with tangent vectors in the prefered subspaces $T^{\pm}_{n_i}$ either join the nuts or reach infinity. This would yield a stronger version of Lemma 1 (namely that l_∞ equals the period $\tau^\pm$ of the corresponding subspaces) without or under weaker assumptions on the topology of $\mathcal{M}$.

We also would like to allow "bolts". In fact, we can show as follows that $(\mathcal{M}, g_{\mu\nu})$ must be the Riem. Schwarzschild metric if $\mathcal{L}$ is connected. Since the twist scalar satisfies $\nabla_\mu(\lambda^{-2}\nabla^\mu\omega) = 0$ which is regular elliptic except at $\mathcal{L}$, ω must have its maximum and its minimum at $\mathcal{L}$ or at infinity. But extrema at the infinity of $\mathcal{M}$ or $\mathcal{N}$ can be ruled out by compactifying the end of $\mathcal{N}$ (as in the Lor. case [22]). Since ω is constant on $\mathcal{L}$ it must vanish identically, i.e. ξ^μ is hypersurface-orthogonal. The proof can now be completed via any of the Lor. methods [9, 19, 23], in particular again by integrating (18).

Of course Lemma 2 suggests that our uniqueness result might be extendable to the Kerr-NUT case. For this purpose we should assume ALF instead of AF, generalize Lemma 1 to include the boundary terms in the signature [15, 12] and note that the dual mass m^* no longer vanishes.

Furthermore, there presumably result still more general families of "half-Kerr-NUT" solutions (and of Lor. counterparts) by imposing only one of the "+" or "−" parts of (12), (13) or (14) (or corresponding Lor. equations). Under suitable asymptotic conditions a uniqueness result for the Riem. solutions might be obtained by integrating the corresponding part of (18).

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