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A renormalization group improved non-local gravitational effective Lagrangian

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Abstract

Renormalization group techniques are used in order to obtain the improved non-local gravitational effective action corresponding to any asymptotically free GUT, up to invariants which are quadratic on the curvature. The corresponding non-local gravitational equations are written down, both for the case of asymptotically free GUTs and also for quantum R^2-gravity. The implications of the results when obtaining the flux of vacuum radiation through the future null infinity are briefly discussed.

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The effective action has turned out to be a quite important subject in the study of different aspects of quantum field theory. Among the phenomena to which it has been applied successfully, we can mention symmetry breaking/restoration effects, phase transitions in general, models of quantum corrected field equations, etc.

Most of the studies of the effective action have been limited to a quasi-local approach (for a general introduction see [1, 2]), that is, they deal with almost constant background fields, as is the case of quantum gravity on a De Sitter background [3] —which is important in inflationary universes. Recently, some interest has arisen ([4]-[7], see [6] for an extense account) for the case of weak but very quickly varying background fields, which typically lead to non-local effective actions. In the present note, by using simple renormalization group (RG) methods —implemented by means of a Wilsonian procedure [8]— we are going to show how one can obtain in fact an improved non-local effective gravitational action for a big class of theories.

The starting point for our considerations will be a massless, multiplicatively renormalizable theory including scalar, spinor and vector fields on a classical gravitational background. The corresponding Euclidean Lagrangian has the following form

$$\begin{aligned} L &= L_m + L_{ext}, \\ L_m &= L_{YM} + \frac{1}{2}(\nabla_\mu \varphi)^2 + \frac{1}{2}\xi R \varphi^2 + \frac{1}{4!} f \varphi^4 + i\overline{\psi}(\gamma^\mu \nabla_\mu - h\varphi)\psi, \\ L_{ext} &= a_1 R^2 + a_2 C^2_{\mu\nu\alpha\beta} + a_3 G + a_4 \Box R. \end{aligned} \tag{1}$$

By choosing a specific gauge group, we can assume that some multiplets of the scalar, φ, and spinor, ψ, fields are given in some concrete representation of the gauge group.

We will assume that our theory (1) is a typical asymptotically free GUT in curved spacetime (for a general introduction, see [2]). In principle one could equally well consider other types of GUTs, what would not change qualitatively the conclusions of our study below.

The running coupling constants corresponding to the asymptotically free couplings of the theory (1) have the form [9, 10]

$$\begin{aligned} g^2(t) &= g^2 \left[1 + \frac{B^2 g^2 t}{(4\pi)^2} \right]^{-1}, \quad g^2(0) = g^2, \\ h^2(t) &= \kappa_1 g^2(t), \qquad f(t) = \kappa_2 g^2(t), \end{aligned} \tag{2}$$

where t is the RG parameter while κ_1 and κ_2 are numerical couplings defined by the specific features of the theory under consideration. We know of many examples of such theories, with gauge groups SU(N), O(N), E_6, etc. [9, 10]. Asymptotic freedom ($g^2(t) \to 0$, $t \to \infty$) is realized for all running couplings: gauge, Yukawa and scalar ones, as is easy to see from (2).

The study of asymptotically free GUTs in curved spacetime was started in Ref. [11] (for a review and detailed list of references see [2]). In the theories with one scalar multiplet, for the running scalar-graviton coupling constant one gets

$$\xi(t) = \frac{1}{6} + \left(\xi - \frac{1}{6}\right) \left[1 + \frac{B^2 g^2 t}{(4\pi)^2}\right]^b , \tag{3}$$

where $\xi(0) = \xi$ and where for the different GUTs the constant b can be either positive, negative or zero (see Ref. [2]).

The gravitational running couplings are defined by the following differential equations (we shall consider the gravitational equations in the Euclidean region)

$$\begin{aligned} \frac{da_1(t)}{dt} &= \frac{1}{(4\pi)^2} \left[\xi(t) - \frac{1}{6}\right]^2 \frac{N_s}{2}, \\ \frac{da_2(t)}{dt} &= \frac{1}{120(4\pi)^2} \left(N_s + 6N_f + 12N_A\right), \\ \frac{da_3(t)}{dt} &= -\frac{1}{360(4\pi)^2} \left(N_s + 11N_f + 62N_A\right), \end{aligned} \tag{4}$$

where N_s, N_f and N_A are the number of real scalars, Dirac spinors and vectors, respectively (notice that the running of $a_4(t)$ will not be meaningful for us, as we shall see below).

Owing to the fact that the theory under discussion is multiplicatively renormalizable, the effective Lagrangian satisfies the RG equation

$$\left(\mu \frac{\partial}{\partial \mu} + \beta_i \frac{\partial}{\partial \lambda_i} - \gamma_i \phi_i \frac{\partial}{\partial \phi_i}\right) L_{eff}(\mu, g_{\mu\nu}, \lambda_i, \phi_i) = 0, \tag{5}$$

where μ is the mass parameter, $\lambda_i = (g^2, h^2, f, \xi, a_1, a_2, a_3, a_4)$ is the set of all coupling constants, the β_i are the corresponding β-functions and $\phi_i = (A_\mu, \phi, \psi)$. The solution of Eqs. (5) by the method of the characteristics gives (for all quantum fields we consider a zero background field, $\phi_i = 0$):

$$L_{eff}(\mu, g_{\mu\nu}, \lambda_i) = L_{eff}(\mu\, e^t, g_{\mu\nu}, \lambda_i(t)), \tag{6}$$

where

$$\frac{d\lambda_i(t)}{dt} = \beta_i\left(\lambda_i(t)\right), \qquad \lambda_i(0) = \lambda_i. \tag{7}$$

Observe that for some of the coupling constants, the corresponding Eq. (7) has been written above explicitly (Eq. (4)), while for a subset of them Eqs. (7) have been actually solved (see Eqs. (2) and (3)).

Actually, the idea itself of a RG improvement procedure was suggested many years ago [12]. What we do here is to make use once more of this interesting concept. Physically, the meaning of expression (6) is the following: L_{eff} (called sometimes the Wilsonian effective action [8]) is obtained through the above equations provided its functional form for some value of t is known (usually it is the classical Lagrangian that serves as boundary condition at $t = 0$). Another difficulty is related with the choice of t, which cannot be given a unique definition due to the presence, in general, of several different efective masses (see the discussions in Refs. [13, 14] concerning that point, for curved and for flat spaces, respectively).

There are different approaches to the gravitational effective action (for a general introduction, see [1, 2]). In the literature, mainly the case of a local effective action has been discussed (i.e., the situation where the gravitational field is slowly varying). One-loop non-local effective actions have been considered in Refs. [4]-[7] (see also the references therein), in different contexts, but almost exclusively the case of a free scalar field theory has been taken into account.

We will be interested in the situation where the gravitational field is weak, but rapidly varying, e.g.

$$\nabla\nabla R >> R^2. \tag{8}$$

The non-local one-loop effective action for a free scalar field theory in this case has been calculated in Ref. [5] (see also [6, 7, 15]), up to the second order on curvature invariants. Such a calculation is quite tedious, moreover, its extension to other fields (especially, to interacting fields) is anything but trivial (see [6] for a discussion and list of references).

We will make use of this RG improvement technique in our calculation, what is going to yield a correspondingly more precise result than the one that has been obtained till now by means of previous approaches to the problem. First, all those calculations have been carried out in the one-loop approximation, while ours here will yield the RG improved effective Lagrangian to leading-log order (through summation of all possible logarithms) of perturbation theory, i.e., clearly beyond one-loop. Secondly, the theory under discussion had been usually restricted to scalar fields, while the considerations here will be applicable to any renormalizable theory on a curved background, including the ordinary renormalizable models of quantum gravity, as R^2-gravity (see [2] for a review). In particular we will present results for an arbitrary asymptotically free GUT in curved spacetime (see [2, 11]).

To begin, using the general expression (6) we can write explicitly the RG improved effective Lagrangian for the theory (1), employing the classical Lagrangian as boundary condition:

$$L_{eff} = a_1(t)R^2 + a_2(t)C^2_{\mu\nu\alpha\beta} + a_3(t)G + a_4(t)\Box R, \tag{9}$$

where the choice of RG parameter t will be described below. From the explicit one-loop calculation [10, 11], the RG parameter is found to be

$$t \sim \frac{1}{2} \ln \frac{-\Box + c_1 R}{\mu^2}, \tag{10}$$

where the constant c_1 is different in the different sectors (scalar, spinor and vector). By looking at (8) one can see that in order to get the dominant contribution we may just keep the first term in (10), i.e. $t \simeq (1/2)\ln(-\Box/\mu^2)$. From the explicit study of the non-local effective action [5]-[7] it follows that the thing one has to understand is the way non-local form factors act, as formal operators obeying the variational rules of finite matrices (in the Lorentzian region). Note also that the terms $a_4(t)\Box R$ and $a_3(t)G$ are still total derivatives after the RG improvement (compare with the other regime in [13] where these terms become important). Notice that a different way of understanding the appearance of the $-\Box$ under the logarithm is to resort to RG considerations in curved space [2], where we know that a scale transformations of the metric, $g_{\mu\nu} \to e^{-2t} g_{\mu\nu}$, ought to be performed. Since, under this transformation, $R^2 \to e^{4t} R^2$ and $\Box \to e^{2t}\Box$, the logarithm corresponding to those terms becomes relevant in the high-energy limit $t \to \infty$.

Finally, the RG improved non-local gravitational effective Lagrangian takes the form

$$\begin{aligned} L_{eff} &= R \left\{ a_1 - \frac{(\xi - 1/6)^2 N_s}{2B^2 g^2 (2b+1)} \left[\left(1 + \frac{B^2 g^2 \ln(-\Box/\mu^2)}{2(4\pi)^2} \right)^{2b+1} - 1 \right] \right\} R \\ &\quad + C_{\mu\nu\alpha\beta} \left[a_2 + \frac{\ln(-\Box/\mu^2)}{240(4\pi)^2} (N_s + 6N_f + 12 N_A) \right] C^{\mu\nu\alpha\beta}, \end{aligned} \tag{11}$$

where a_1 and a_2 are intital values for the corresponding effective couplings. Notice that with the above form factors the solutions fulfill the requeriment of asymptotic flateness [15]. As it has been discussed in Refs. [15, 16], the coefficients of the terms linear in $\ln(-\Box)$ give a measure of the energy radiation through the future null infinity.

Here we have obtained an effective Lagrangian, L_{eff}, which sums *all* the logarithms of perturbation theory, up to second order terms on curvature invariants on the background, of weak but quickly varying curvature. The theory under consideration is an asymptotically free GUT but, in principle, we can consider in the same way any other kind of renormalizable quantum field theory.

Notice, however, that the price one has to pay for the universality of the approach (i.e., for the possibility to write (11) for a variety of theories beyond the one-loop approximation) is the fact that we cannot proceed to higher orders in the curvature. The reason is that the terms as R^3, $R^4, \ldots$ are ultraviolet finite. At the same time, the ordinary technique to one-loop order [6, 7] gives the possibility, in principle, to calculate the non-local effective action up to any desired order in the curvature —although it is quite complicated, already in the case of the scalar theory. It turns out, therefore, that the two approaches complement each other quite well.

Using L_{eff} one can obtain the effective gravitational equations. Adding the quantum matter-induced effective Lagrangian (11) to the classical Einstein Lagrangian (without the cosmological constant, for simplicity), one gets the effective gravitational equations in close analogy with Refs. [6, 7, 15]. Before doing this, it is convenient to rewrite

$$C^2_{\mu\nu\alpha\beta} = G + 2R^2_{\mu\nu} - \frac{2}{3}R^2, \tag{12}$$

and to substitute it into Eq. (11). Then, one finds the following Euclidean effective gravitational equations

$$\begin{aligned} &-\frac{1}{8\pi G}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \left\{a_1 + \frac{(\xi - 1/6)^2 N_s}{2B^2g^2(2b+1)}\left[\left(1 + \frac{B^2g^2\ln(-\Box/\mu^2)}{2(4\pi)^2}\right)^{2b+1} - 1\right]\right. \\ &\left. -\frac{2}{3}a_2 - \frac{\ln(-\Box/\mu^2)}{360(4\pi)^2}(N_s + 6N_f + 12N_A)\right\}\left[4\nabla_\mu\nabla_\nu R - 4g_{\mu\nu}\Box R + \mathcal{O}(R^2)\right] \\ &+2\left[a_2 + \frac{\ln(-\Box/\mu^2)}{240(4\pi)^2}(N_s + 6N_f + 12N_A)\right]\left[2\nabla_\mu\nabla_\nu R - g_{\mu\nu}\Box R - 2\Box R_{\mu\nu} + \mathcal{O}(R^2)\right] = 0. \end{aligned} \tag{13}$$

Observe that in order to obtain the effective gravitational equations it is not necessary to take into account the $g_{\mu\nu}$-dependence of the form factors. As was discussed in Ref. [15], the effective gravitational equations can be used in order to study the problem of collapse.

To be remarked is the fact that the above approach works well for renormalizable models of quantum gravity too. In order to exemplify this, let us consider R^2-gravity under the form

$$L = \frac{1}{\lambda}\left(R_{\mu\nu} - \frac{1}{3}R^2\right) - \frac{\omega}{3\lambda}R^2. \tag{14}$$

Such a theory is multiplicatively renormalizable, being non-unitary in the perturbative approach (for a general review and a list of references, see [2]). The RG improved non-local effective Lagrangian corresponding to this theory, with the same gravitational background (8), can be easily constructed. The effective gravitational equations are (for simplicity, only leading-log terms have been kept)

$$\begin{aligned}
&\left[\frac{1}{\lambda}+\frac{133}{20(4\pi)^2}\ln(-\Box/\mu^2)\right]\left[\nabla_\mu\nabla_\nu R - g_{\mu\nu}\Box R - 2\Box R_{\mu\nu} + \mathcal{O}(R^2)\right] \\
&+2\left[-\frac{1}{3\lambda}-\frac{\omega}{3\lambda}+\left(\frac{10}{9}\omega^2+\frac{5}{3}\omega+\frac{5}{36}\right)\frac{\ln(-\Box/\mu^2)}{2(4\pi)^2}\right] \\
&\quad\times\left[4\nabla_\mu\nabla_\nu R - 4g_{\mu\nu}\Box R + \mathcal{O}(R^2)\right] = 0,
\end{aligned} \tag{15}$$

where λ and ω are the initial values for the corresponding effective couplings.

Following now Refs. [15, 16] (it is explained there in which form non-local effective action can be relevant for black-hole physics), we can discuss the implications that the above non-local gravitational action has concerning the flux of the vacuum radiation in an asymptotically free GUT. Working with the asymptotically flat (Lorentzian) solution of Eqs. (13) one may consider the congruence $u(x) = \text{const.}$ of the light rays that can reach the future null infinity $\mathcal{F}^+$. We shall denote, as in [15], by r the luminosity distance along rays and by $M(u)$ the Bondi mass at $\mathcal{F}^+$ (see [17]). Then, the final expression for the radiation corresponding to the vacuum energy in a spherically symmetric state has been found to be the following [15]:

$$\frac{dM(u)}{du} = -\frac{1}{4\pi}(w_1+2w_2)\frac{d^2}{d^2u}\int_{\mathcal{F}^-}^{\mathcal{F}^+} dr\, r\, R + \mathcal{O}(R^2), \tag{16}$$

where w_1 and w_2 are the coefficients of terms linear in $\ln(-\Box)$ of (11), that is

$$L_{eff} = \left\{R_{\mu\nu}\left[a_1 - \frac{2}{3}a_2 - \frac{w_1}{2(4\pi)^2}\ln\left(-\frac{\Box}{\mu^2}\right)\right]R^{\mu\nu} + R\left[2a_2 - \frac{w_2}{2(4\pi)^2}\ln\left(-\frac{\Box}{\mu^2}\right)\right]R\right\}, \tag{17}$$

where a_1 and a_2 can be taken to be zero and where $a_1(t)$ has been expanded up to terms linear on $\ln(-\Box)$. Taking into account the overall change of sign of L_{eff} in the Lorentzian region, from (4) we obtain

$$w_1 = \frac{1}{60}(N_s + 6N_f + 12N_A), \qquad w_2 = -\frac{1}{180}(N_s + 6N_f + 12N_A) + \frac{N_s}{2}\left(\xi - \frac{1}{6}\right). \tag{18}$$

In this way we can calculate the rate of the vacuum energy radiation through the future null infinity, taking into account corrections to the GUT under consideration. To be remarked is the fact that the choice of ξ can influence this rate of radiation significantly (18). Radiation disappears when the null surface $u = \text{const.}$ comes very close to the horizon [15, 16]. Then, in order to find the Hawking radiation [18] one has to calculate the next-to-leading correction in (17), namely the $\mathcal{O}(R^2)$-terms.

To summarize, using rather simple RG considerations, we have constructed a RG improved non-local gravitational Lagrangian corresponding to a general asymptotically free GUT and also to R^2-quantum gravity. The corresponding effective gravitational equations have been written down as well. It would be now of interest to study the applications of these equations to black hole physics in more detail, since they certainly modify a number of results obtained previously.

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