July, 1995

A comment on the theory of turbulence without pressure proposed by Polyakov

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Abstract

Owing to its lack of derivability, the dissipative anomaly operator appearing in the theory of turbulence without pressure recently proposed by Polyakov appears to be quite elusive. In particular, we give arguments that seem to lead to the conclusion that an anomaly in the first equation of the sequence of conservation laws cannot be *a priori* excluded.

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In a recent paper [1] (see also [2]) Polyakov has put forward a method to treat turbulence with exact field theoretical methods, in the case when the effect of pressure is negligible. The work has been inspired in a paper by Chekhlov and Yakhot [3], where new results concerning Burgers' turbulence have been given. The starting point for this one-dimensional case is Burgers equation

$$\begin{aligned} u_t + u u_x &= \nu u_{xx} + f(xt), \\ < f(x,t) f(x',t') > &= \kappa(x-x')\,\delta(t-t'), \end{aligned} \tag{1}$$

where κ is a function that defines the spatial correlation of the random forces. Equation (1) is the one dimensional version of the Navier-Stokes equation with a random force of white noise type and with zero pressure. Physical applications of equations of this type include the study of crystal growth [4] and galaxy formation [5].

For the generating functional

$$Z(\lambda_1 x_1| \ldots |\lambda_N x_N) = \left\langle \exp \sum \lambda_j u(x_j t) \right\rangle , \tag{2}$$

one obtains

$$\dot{Z} + \sum \lambda_j \frac{\partial}{\partial \lambda_j} \left(\frac{1}{\lambda_j} \frac{\partial Z}{\partial x_j} \right) = \sum \lambda_j \left\langle [f(x_j t) + \nu u''] \exp \sum \lambda_k u(x_k t) \right\rangle , \tag{3}$$

and further [1]

$$\dot{Z} + \sum \lambda_j \frac{\partial}{\partial \lambda_j} \left(\frac{1}{\lambda_j} \frac{\partial Z}{\partial x_j} \right) = \sum \kappa(x_i - x_j) \lambda_i \lambda_j Z + D, \tag{4}$$

where D is the dissipation term

$$D = \nu \sum \lambda_j \left\langle u''(x_j t) exp \sum \lambda_k u(x_k t) \right\rangle . \tag{5}$$

If the viscosity ν were zero one would have a closed differential equation for Z. To reach the inertial range one must, however, keep ν infinitesimal but non-zero. The anomaly mechanism mentioned above implies that infinitesimal viscosity produces a finite effect, whose computation is one of the main objectives in [1]. In a first stage, the inviscid equations (5) have been considered ($\nu = 0$). Then, modulo the stirring force and the viscosity, one has the sequence of conservation laws for Eq. (1)

$$\frac{\partial}{\partial t}(u^n) + \frac{n}{n+1} \frac{\partial}{\partial x}(u^{n+1}) \approx 0, \qquad n = 1, 2, 3, \ldots \tag{6}$$

the sign $\approx$ meaning precisely that the viscosity and the stirring force terms are dropped out [1].

As discussed by Polyakov in detail, Eq. (5) can be interpreted as a relation for the generating functionals $\langle u^{n_1}(x_1) \ldots u^{n_k}(x_k) \rangle$, involving both the stirring force and the viscosity. The latter presents a problem. The rule is that in any equation involving space points separated by a distance larger than a, the viscosity can be put equal to zero.

And here comes the specific situation we want to deal with. In principle, it seems legitimated to use the inviscid limit for the first equation, $n = 1$, of (6), because in this case one can make use of the steady state condition

$$\frac{d}{dt}\langle u(x_1)\dots u(x_N)\rangle = 0, \qquad | x_i - x_j | \gg a. \tag{7}$$

The problems start, in principle, with the case $n = 2$, because then one has to take a time derivative of the product of two (or more) u's at the same point.

This problem can be solved, in the case $n = 2$, by making the replacement

$$u^2(x) \Longrightarrow u(x+\frac{y}{2})u(x-\frac{y}{2}), \qquad | x_i - x_j | \gg y \gg a \tag{8}$$

and by leting $y \to 0$ only after the viscosity is taken to zero. Using then the inviscid equations for $n = 1$ one can write

$$-\frac{d}{dt}[u(x_1)u(x_2)] \approx \frac{1}{2}\left\{\frac{\partial}{\partial x_1}\left[u^2(x_1)u(x_2)\right] + \frac{\partial}{\partial x_2}\left[u^2(x_2)u(x_1)\right]\right\}, \quad x_{1,2} = x \pm \frac{y}{2}. \tag{9}$$

By employing simple algebraic identities, and the following expression

$$\frac{\partial}{\partial y}\left[u^3(x_1) - u^3(x_2)\right] = \frac{1}{2}\frac{\partial}{\partial x}\left[u^3(x_1) + u^3(x_2)\right] \longrightarrow \frac{\partial}{\partial x}u^3(x), \tag{10}$$

one gets

$$-\frac{d}{dt}[u(x_1)u(x_2)] \approx \frac{2}{3}\frac{\partial}{\partial x}u^3(x) + a_0(x), \tag{11}$$

where $a_0(x)$ is a dissipative anomaly operator, given by [1]

$$a_0(x) = \lim_{y\to 0}\frac{1}{3}\frac{\partial}{\partial y}[u(x_1) - u(x_2)]^3. \tag{12}$$

It is here crucial to observe that the anomaly would be zero if $u(x)$ were differentiable. *However*, as remarked in [1], the steady state condition clearly prevents this from being true. Indeed, one of the consequences of Eq. (5) is that in the steady-state situation one has

$$\frac{d}{dt}\left\langle u^2\right\rangle = \kappa(0) - \langle a_0\rangle = 0, \tag{13}$$

and the celebrated Kolmogorov relation holds

$$\left\langle [u(x_1) - u(x_2)]^3\right\rangle \propto \kappa(0)(x_1 - x_2). \tag{14}$$

The value of the anomaly defines the limiting contribution of the viscous term in the steady state

$$\lim_{\nu\to 0}\ \nu u(x)u''(x) = -a_0(x). \tag{15}$$

Notice, again, that the fact that the anomaly $a_0(x)$ is non vanishing (together with its important consequences, as the Kolmogorov relation) depends solely on the non-differentiability of the function $u(x)$. Simple considerations —the first of which could be pure symmetry— can lead us easily to the conclusion that an anomaly of the same type can be also present in the first of the equations. In fact, its absence has not been proven in [1], but just the compatibility of the general argument with the fact that it can be zero (the whole argument has been termed by Polyakov himself a *consistent conjecture* [1]).

Crude symmetry considerations yield

$$-\frac{d}{dt}\,u(x) \approx \frac{1}{2}\frac{\partial}{\partial x}u^2(x) + \lim_{y\to 0}\frac{1}{2}\frac{\partial}{\partial y}\left[u(x_1)-u(x_2)\right]^2, \qquad (16)$$

where, again the non-differentiability of the function $u(x)$ permits the anomaly term (the second one on the rhs) to be non-zero. Another consideration that leads to the same result can be put under a similar form as the derivation of the anomaly $a_0(x)$, by just point-splitting the $x-$derivative of $u^2(x)$ (what is *not* a trivial matter at all, given the non-differentiability of $u(x)$), in the way

$$-\frac{1}{2}\frac{\partial}{\partial x}\,u^2(x) \longrightarrow \frac{1}{2}\frac{\partial}{\partial x}\left[u(x_1)u(x_2)\right], \qquad (17)$$

and proceeding with the same kind of manipulations as in [1], one gets (16). In particular, the new anomaly term —which following the Polyakov's labeling we could call $a_{-1}(x)$— i.e.

$$a_{-1}(x) = \lim_{y\to 0}\frac{1}{2}\frac{\partial}{\partial y}[u(x_1)-u(x_2)]^2, \qquad (18)$$

is also clearly seen to be non-vanishing in general. This is realized by direct calculation of the derivative as a quotient of differences and by considering different possible ways of taking the two limits involved, namely the one of the derivative itself and the limit $y \to 0$. Notice that this non-vanishing is naively even more strong than in the cases $n = 2$ and further, because differentiability would here yield an infintesimum of first order only, while in the case $n = k$ it would be of the corresponding order k.

It is difficult to give an immediate meaning to this possible anomaly. Of course, it would modify Kolmogorov relation and all the subsequent anomalies (starting from $a_0(x)$), since it would contribute a term in the derivation of the relations for u^2 and all the subsequent u^k.

Acknowledgments

It is a pleasure to thank the members of the Department of Theoretical Physics of the University of Trento for warm hospitality, specially Sergio Zerbini, Luciano Vanzo, Guido Cognola, Ruggero Ferrari and Marco Toller. This investigation has been supported by DGICYT (Spain), by CIRIT (Generalitat de Catalunya) and by INFN (Italy).

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