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EXACT ELECTROMAGNETIC DUALITY

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Introduction

Electromagnetic duality is a very old idea, possibly predating Maxwell's equations. Although the route that has recently led to a precise and convincing formulation has been long, it has turned out to be of quite surprising interest. This is because it has synthesised many hitherto independent lines of thought, and so intriguingly interelated disparate ideas arising in the quest for a unified theory of particle physics valid in the natural space-time with three space and one time dimension. Despite the progress, final proof is lacking and likely to require further breakthroughs in fundamental mathematics.

Although nature does not seem to display exact electromagnetic duality, realistic theories could well be judiciously broken versions of the exact theory in which sufficient structure survives to explain such long-standing puzzles as quark confinement in the way advocated by Seiberg and Witten [1]. Spectacular support for their arguments comes from applications in pure mathematics where new insight has been gained into the classification of four-manifolds [2], transcending the celebrated work of Donaldson [3].

Here I shall review the developments leading up to the formulation of exact electromagnetic duality, taking the view that an understanding of this must precede that of the symmetry breaking.

The Original Idea

The apparent similarity between the electric and magnetic fields $\underline{E}$ and $\underline{B}$ was confirmed and made more precise by Maxwell's discovery of his equations. In vacuo, they can be written concisely as just two equations [4]:

$$\nabla.(\underline{E}+i\underline{B})=0, \tag{1a}$$

$$\nabla\wedge(\underline{E}+i\underline{B})=i\frac{\partial}{\partial t}(\underline{E}+i\underline{B}) \tag{1b}$$

at the expense of introducing a complex vector field $\underline{E}+i\underline{B}$. These equations display several symmetries whose physical importance became clear subsequently. They display Poincaré (rather than Galilean) symmetry, and, beyond that, conformal symmetry (with respect to space-time transformations preserving angles and not just lengths).

Unlike the Poincaré symmetry, the conformal symmetry is specific to four space-time dimensions. Even more sensitive to the precise space-time metric is the electromagnetic duality rotation symmetry of Maxwell's equations

$$\underline{E} + i\underline{B} \to e^{i\phi}(\underline{E} + i\underline{B}) \tag{2}$$

since only in $3+1$ dimensions do the electric and magnetic fields both constitute vectors so that the complex linear combination $\underline{E} + i\underline{B}$ appearing in (1) and (2) can be formed. It is the extension of the fascinating symmetry (2) of (1) that is the main theme of what follows.

Notice that we can form two real, quadratic expressions invariant with respect to (2) [4]:

$$\frac{1}{2}|\underline{E} + i\underline{B}|^2 = \frac{1}{2}(E^2 + B^2),$$

$$\frac{1}{2i}(\underline{E} + i\underline{B})^* \wedge (\underline{E} + i\underline{B}) = \underline{E} \wedge \underline{B},$$

respectively the energy and momentum densities of the electromagnetic field.

On the other hand, $\frac{1}{2}(\underline{E} + i\underline{B})^2$ is complex with real and imaginary parts given by

$$\frac{1}{2}(E^2 - B^2) + i\underline{E}.\underline{B}.$$

As the real part is the Lagrangian density, this shows that it forms a doublet under (2) when combined with $\underline{E}.\underline{B}$ which is a total derivative. Thus the Maxwell action forms a doublet with a "topological quantity" which is proportional to the instanton number in non-abelian theories.

We would like to generalise the electromagnetic duality rotation symmetry (2) to include matter. We could also consider generalisations to non-abelian gauge theories of the type which seem to unify the fundamental interactions. In either case we meet the same difficulty that the gauge potentials enter the equations of motion and that we do not know how to extend the transformation (2) to include them. Eventually we shall find a way of combining the two generalisations, thereby extending the symmetry.

There is another, familiar, difficulty with the equations of motion in non-abelian gauge theories; that they are conformally invariant (in $3+1$ dimensions). As a consequence, the gauge particles which are the quanta of the gauge potentials, should be massless. This is fine for the photon, but not for any other gauge particles. This problem seems to be rather general and deep: unified theories chosen according to geometric principles tend to exhibit unwelcome conformal symmetry. This occurs in string theory too, at least as a world sheet symmetry. As a consequence, there is a general problem of understanding the origin of mass through a geometrical mechanism for breaking conformal symmetry. We know of only two possible solutions, at first sight different, but related in what follows.

The first is the idea that mass arises from the vacuum spontaneously breaking some of the gauge symmetry via a "Higgs" scalar field [5,6,7]. The second is a principle due to Zamolodchikov [8] that we now discuss.

Zamolodchikov's Principle and Solitons

The second insight into the origin of mass comes from another area of physics. Yet, as we shall see, it seems connected to the first, in four dimensions at least. The conformal symmetry on the world sheet needed for the internal consistency of string theory hinders the emergence of a physically realistic mass spectrum in an otherwise unified theory. However, when string theory is abstracted to the study of "conformal field theory", and applied to the study of second order phase transitions of two dimensional materials, it is seen that the simple application of heat breaks the conformal symmetry controlling the behaviour at the critical temperature. In specific models it was learnt by Onsager, Baxter and their followers [9] that it is possible to supply heat while maintaining integrability (or solvability). Zamolodchikov [8] has elevated this observation to a principle which has rationalised the theory of solitons and more. In two dimensions, the local conservation laws characteristic of conformal symmetry (or augmented versions such as W-symmetry) are chiral. This means that the densities are either left moving or right moving (at the speed of light) and so can be added, multiplied or differentiated. If conformal symmetry is judiciously broken, a certain (infinite) subset of the chiral densities remain conserved, although no longer chirally so. Their conserved charges, that is their space integrals, generate an infinite dimensional extension of the Poincaré algebra in which the charges carry integer spins. The charges with spin plus or minus 1 are the conventional light cone components of momentum. The sinh-Gordon equation illustrates this nicely. It can be written

$$\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}+\frac{\mu^2}{2\beta}\left(e^{\beta\phi}-e^{-\beta\phi}\right)=0. \tag{3}$$

The last term, proportional to $e^{-\beta\phi}$, can be multiplied by a variable coefficient η so that $\eta=1$ yields (3), while $\eta=0$ yields the Liouville equation. Liouville exploited the conformal symmetry of his equation in order to solve it completely, long ago.

It is interesting to investigate the behaviour as η varies from 1 to 0. As long as $\eta>0$, a simple redefinition of the field ϕ by a displacement restores the sinh-Gordon form (3) but with μ replaced by $\mu\eta^{1/4}$. As $\mu\hbar$ is the mass of the particle which is the quantum excitation of ϕ, we see that it is singular as η approaches zero with critical exponent 1/4. So we see how mass arises from the breaking of conformal symmetry. This short discussion was classical but it extends to the quantum regime as envisaged by Zamolodchikov [8].

The sine-Gordon equation is obtained from the sinh-Gordon equation (3) by replacing β by $i\beta$. It then exhibits the symmetry

$$\phi\to\phi+\frac{2\pi}{\beta} \tag{4}$$

and, consequently, possesses an infinite number of vacuum solutions $\phi_n=\frac{2\pi n}{\beta}$, $n\in\mathbb{Z}$, all with the same minimum energy, zero. The particle of mass $\mu\hbar$ describes fluctuations about any of these vacua. But there also exist classical solutions which interpolate two successive vacua and which are stable with respect to fluctuations. These solutions can be motionless, describing a new particle, the soliton, at rest, or can be boosted to any velocity less than that of light. The jump in n, equal to ± 1, can be regarded as a topological quantum number, indicating either a soliton or an antisoliton. What is particularly remarkable is that one can consider a solution with an arbitrary number of solitons and/or antisolitons, initially well separated, but approaching each other, then colliding and finally emerging with velocities unchanged and energy profiles generally unscathed except for time advances relative to uninterrupted trajectories [10]. Thus the solitons persist in their structure despite their collisions and can legitimately be regarded as providing classical models of a particle with a finite mass and a structure of finite extent. This phenomenon is a very special feature of sine-Gordon theory that can be ascribed to the infinite number of conservation laws mentioned previously, themselves relics of conformal symmetry.

This sort of integrable field theory has two "sorts" of particle, the quanta of the fluctuation of the field ϕ (obtained by second quantisation) and the solitons which are classical solutions. Skyrme [11] was the first to ask whether these two "sorts" of particle are intrinsically different and found that they were not. His explanation was that, in the full quantum theory, it is possible to construct a new quantum field whose fluctuations are the solitons. The new field operator is obtained by an exponential expression in the original field ϕ

$$\psi_{\pm}(x) = e^{i\beta(\phi \pm \int_{-\infty}^{x} dx' \frac{\partial\phi}{\partial t})} \tag{5}$$

with two spin components (and a normal ordering understood). Coleman and Mandelstam [12,13] later confirmed that ψ satisfied the equations of motion of the massive Thirring model.

The construction (5) is an example of the vertex operator construction later to be so important in string theory and in the representation theory of infinite dimensional algebras (resembling quantum field theories).

There are multicomponent generalisations of the sine-Gordon equations called the affine Toda theories, likewise illustrating Zamolodchikov's principle in a nice way, and revealing the role of algebraic structures such as affine Kac Moody algebras [14]. Again there are particles created by each field component, now possessing interesting mass and coupling patterns (related to group theory). Remarkably, there are an equal number of soliton species and these display very similar properties [15].

Magnetic Charge and its Quantisation

Let us now return to the question of extending the electromagnetic duality rotation symmetry (2) to matter carrying electric and magnetic charges. Suppose first that matter can be regarded as being composed of classical point particles carrying typical electric and magnetic charges q and g, say. Then it is easy to include the source charges on the right hand side of Maxwell's equations (1) and to supplement (1) by the equations of motion for the individual particles subject to a generalised Lorentz force. This system maintains the symmetry (2) if, in addition,

$$q + ig \to e^{i\phi}(q + ig). \tag{6}$$

The price to be paid for achieving this is the inclusion of unobserved magnetic charge. We must therefore suppose that the failure to observe magnetic charge is either due to an associated very large mass or some other reason.

Turning from the classical to the quantum theory, we immediately find a difficulty, namely that the electromagnetic couplings of the matter wave functions require the introduction of gauge potentials, a procedure which is not straightforward in the presence of magnetic charge.

Nevertheless, in 1931, Dirac overcame this difficulty and showed that the introduction of magnetic charge could be consistent with the quantum theory, provided its allowed values were constrained [16]. His result was that a magnetic charge g_1, carrying no electric charge, could occur in the presence of an electric charge q_2, like the electron carrying no magnetic charge, provided

$$q_2 g_1 = 2\pi n\hbar \qquad\qquad n = 0, \pm1, \pm2, \ldots \tag{7}$$

As he pointed out, this condition had a stunning consequence: provided g_1 exists somewhere in the universe, even though unobserved, then any electric charge *must* occur in integer multiples of the unit $\frac{2\pi\hbar}{g_1}$, by (7). This quantisation of electric charge is indeed a feature of nature and this explanation is actually the best yet found. Although apparent alternative explanations, evading the necessity for magnetic charge, have appeared, they turn out to be unexpectedly equivalent to the above argument, as we shall see.

There is a problem with the Dirac condition (7), namely that it does not respect the symmetry (6). In fact (7) is not quite right because, although Dirac's argument is impeccable, there is an implicit assumption hidden within the situation considered. It took a surprisingly long time to rectify this and hence restore the symmetry (2) and (6), as we see later.

Given that this difficulty is overcome, we can seek a consistent quantum field theory with both electric and magnetic charges. Then, presumably, the particles carrying magnetic charge would have a structure determined by the theory, and hence a mass dependent on the charges carried. Just as the Maxwell energy density respected the symmetry (2), we would expect this mass formula to respect (6) so that

$$M(q, g) = M(|q + ig|) = M(\sqrt{q^2 + g^2}). \tag{8}$$

We now proceed to find such a theory.

Magnetic Monopoles (and dyons) as Solitons

We can now draw together several clues in the ideas already discussed. One concerns the quantisation of electric charge: since the electric charge operator, Q, is the generator of the $U(1)$ gauge group of Maxwell theory, its quantisation could be explained by supposing that it is actually a generator of a larger, simple gauge group (that could unify it with other interactions). If the larger group were $SU(2)$, for example, Q would then be a generator of an $SU(2)$ Lie algebra, that is, an internal angular momentum algebra. Consequently its eigenvalues would be quantised, thereby providing an alternative explanation of electric charge quantisation which apparently evades the need for magnetic charge.

However, we still have to furnish a mechanism selecting the direction of Q amongst the three $SU(2)$ directions. This can be achieved by a scalar field with three components $(\phi_1(x), \phi_2(x), \phi_3(x))$, like the $SU(2)$ gauge fields. This scalar field has to have the unusual feature of not vanishing in the vacuum, so that it can select the Q direction there. It is therefore a "Higgs" field providing the mechanism whereby the vacuum spontaneously breaks the $SU(2)$ gauge symmetry down to the $U(1)$ subgroup [5,6,7]. As well as this, it also breaks conformal symmetry, introducing mass for two of the gauge particles, leaving the photon massless.

There is a simple formula for the resultant masses of the gauge particles

$$M(q, 0) = a|q|, \tag{9}$$

where q is the eigenvalue of Q specifying the electric charge of a specific mass eigenstate. a constitutes a new fundamental parameter specifying the magnitude of the vacuum expectation value of the scalar Higgs field. Actually the mass formula (9) is much more general. Instead of the gauge group being $SU(2)$, it could have been any simple Lie group, G, say, and (9) holds as long as the Higgs field lies in the adjoint representation of G, like the gauge fields.

In the vacuum, the gauge group G is spontaneously broken to a subgroup

$$U(1)_Q \times K/Z, \tag{10}$$

where Q still generates the invariant $U(1)$ subgroup commuting with K. The denominator Z indicates a finite cyclic group in which the $U(1)_Q$ intersects K, and will not be important for what we have to say.

But this setup, a spontaneously broken gauge theory with a Higgs in the adjoint representation, is very much the analogue in four dimensions of the sine-Gordon theory in two dimensions. Instead of the symmetry relating the degenerate vacua being discrete, (4), it is now continuous, being the gauge symmetry, G, and it is again possible to trap nontrivial topologically stable field configurations of finite energy. Indeed in 1974, 't Hooft and Polyakov [17,18] found a classical soliton solution emitting a $U(1)$ magnetic flux with strength $4\pi\hbar/q$ in the $SU(2)$ theory with heavy gauge particles carrying electric charges $\pm q$. Thus there is a soliton which is a magnetic monopole whose charge indeed satisfies the Dirac condition (7). Thus the desired novelty of this explanation of electric charge quantisation is illusory as it reduces to Dirac's original argument [16]. For a more detailed review of the material in this section see [19].

However what we have done is inadvertently achieve our other aim, that of constructing a theory in which the magnetic monopoles have structure and a definite mass which can be calculated by feeding the field configuration into the energy density and integrating over space. The result is the following inequality, known as the "Bogomolny" bound [20],

$$M(0,g) \geq a|g|. \tag{11}$$

The similarity to the Higgs formula (9) prompts the question as to whether the inequality in (11) can be saturated to give equality. This is possible in the "Prasad-Sommerfield" limit in which the self interactions of the Higgs field vanish [21]. Then the lower bound in (11) is achieved if the fields satisfy certain first order differential equations, known as the "Bogomolny equations" [19]

$$\mathcal{E}_i = 0, \qquad \mathcal{D}_0\phi = 0, \qquad \mathcal{B}_i = \pm\mathcal{D}_i\phi, \tag{12}$$

where $\mathcal{E}_i$ and $\mathcal{B}_i$ denote the nonabelian electric and magnetic fields. Solutions to (12) have zero space momentum and therefore describe a magnetic monopole at rest, with mass $a|g|$ (if $|g|$ has its minimum least positive value).

The sine-Gordon solitons satisfy similar first order differential equations that imply that the mass can also be expressed as a surface term, but there is an important difference. This is that the Bogomolny equations (12) (unlike the first order sine-Gordon equations) can also be solved for higher values of the topological charge, here magnetic charge. When the magnetic charge is m times its least positive value, the space of solutions to (12), called the moduli space, form a manifold of $4m$ dimensions. $3m$ of these dimensions can be interpreted as referring to the space coordinates of m individual magnetic monopoles of like charge.

This means that m like magnetic monopoles can exist in arbitrary configurations of static equilibrium (unlike m sine-Gordon solitons, which must move). So, as they have no inclination to move relatively, like magnetic monopoles at rest must fail to exert forces on each other [22]. (This is reminiscent of the multi-instanton solutions to self-dual gauge theories: indeed the Bogomolny equations (12) can be interpreted as self-dual equations in four Euclidean dimensions).

The remaining m coordinates, one for each monopole, have a more subtle, but nevertheless, important interpretation: they correspond to degrees of freedom conjugate to the electric charge of each monopole. Because of this, it is possible for each magnetic monopole soliton to carry an electric charge, q, say [23]. In this case, they are called "dyons", following the terminology introduced by Schwinger [24]. Then the mass of an individual dyon is given by, [25],

$$M(q,g) = a|q+ig| = a\sqrt{q^2+g^2}. \tag{13}$$

The first remarkable fact about this formula is that it is universal. It applies equally to the dyon solitons of the theory and to the gauge particles, as it includes the Higgs formula (9). In fact it applies to all the particles of the theory created by the fundamental quantum fields, as it also includes the photon and Higgs particles which are both chargeless and massless. Thus, whatever G, (13) unifies the Higgs and Bogomolny formulae and is therefore democratic in the sense that it does not discriminate as to whether the particle considered arises as a classical soliton or as a quantised field fluctuation [26].

Secondly the mass formula (13) does indeed respect the electromagnetic duality rotation symmetry (6) as it has the structure (8).

Electromagnetic Duality Conjectures

We have seen that spontaneously broken gauge theories with adjoint Higgs (in the Prasad-Sommerfield limit) have remarkable properties, at least according to the naive arguments just outlined. Magnetically neutral particles occur as quantum excitations of the fields present in the action, whereas magnetically charged particles occur as solitons, that is, solutions to the classical equations of motion. Yet, despite this difference in description, all particles enjoy a universal mass formula (13).

Skyrme showed that, in two dimensions, the soliton of sine-Gordon theory could be considered as being created by a new quantum field obeying the equations of motion of the massive Thirring model [11,12,13]. Thus the same quantum field theory can be described by two distinct actions, related by the vertex operator transformation (5). It is natural to ask if something similar can happen in four dimensions, with the theory under consideration. There, the solitons carry magnetic charge with an associated Coulomb magnetic field. This suggests that the hypothetical quantum field operator, creating the monopole solitons, should couple to a "magnetic" gauge group with strength inversely related to the original "electric" gauge coupling because of Dirac's quantisation condition,

$$q_0 \to g_0 = \pm\frac{4\pi\hbar}{q_0}, \tag{14}$$

or possibly half this.

Thinking along these lines, two more specific conjectures were proposed in 1977. First, considering a more general theory, with a simple exact gauge symmetry group H, (i.e. not of the form (10)), Goddard, Nuyts and Olive established a non-abelian version of the Dirac quantisation condition (7) and used it to propose the conjecture that the magnetic, or dual group H^v could be constructed in two steps as follows [27].

(i) The Lie algebra of H^v is specified by saying that its roots α^v are the coroots of H :-

$$\alpha \to \alpha^v = \frac{2\alpha}{\alpha^2}. \tag{15a}$$

(ii) The global structure of the group H^v is specified by constructing its centre $Z(H^v)$ from that of H, $Z(H)$:-

$$Z(H) \to Z(H^v) = \frac{Z(\tilde{H})}{Z(H)}, \tag{15b}$$

where $\tilde{H}$ is the universal covering group of H, that is, the unique simply connected Lie group with the same Lie algebra as H.

This conjecture remains open. Notice the similarity between (15) and (14). In order to make progress, Montonen and Olive sought a more specific proposal in a simpler context, and considered spontaneously broken gauge theories of the type discussed above, but with the gauge group henceforth definitely chosen to be $SU(2)$ [26]. This is broken to $U(1)$ by a triplet Higgs field so that the mass formula (13) holds good.

The possible quantum states of the theory carry values of q and g which form an integer lattice when plotted in the complex $q + ig$ plane (with Cartesian coordinates (q,g)). Ignoring possible dyons, the single particle states correspond to five points of this lattice. The photon and the Higgs particle correspond to the origin (0,0), the heavy gauge particles $W^{\pm}$ to the points $(\pm q_0, 0)$. Thus the particles created by the fundamental fields in the original, "electric" action lie on the real, electric axis. The magnetic monopole solitons $M^{\pm}$ lie on the imaginary, magnetic axis at $(0, \pm g_0)$, while the dyons could lie on the horizontal lines through these two points. Since, at this stage, it is unclear what values of their electric charges are allowed, they will temporarily be omitted, to be restored later.

Now, if we follow the transformation (14) by a rotation through a right angle in the $q + ig$ plane, the five points just described are rearranged. This suggests that the "dual" or magnetic formulation of the theory with $M^{\pm}$ created by fields present in the action will also be a similar spontaneously broken gauge theory, but with the coupling constant altered by (14). In this new formulation it is the $W^{\pm}$ particles that would occur as solitons.

This is the Montonen-Olive electromagnetic duality conjecture in its original form [26]. In principle, it could be proven by finding the analogue of Skyrme's vertex operator construction (5) [11], but, even with present knowledge, this seems impossibly difficult. Notice that the sine-Gordon quantum field theory was described by two quite dissimilar actions whereas in the four dimensional theories the two hypothetical actions have a similar structure but refer to electric and magnetic formulations.

The magnitudes of physical quantities should agree whichever of the two actions is chosen as a starting point for their calculation. The conjecture will immediately pass at least two simple tests of this kind, showing that it is not obviously inconsistent. The first test concerns the mass formula (13) and is passed precisely because of the universal property that has already been emphasised.

A second test concerns the fact that, according to the existence of static solutions to the Bogomolny equations with magnetic charge $2g_0$ discussed earlier, an M^+M^+ pair exert no static forces on each other. This result is according to the electric formulation of the theory and ought to be confirmed in the magnetic formulation. This is equivalent to checking that there is no W^+W^+ force in the electric formulation. In the Born approximation, two Feynman diagrams contribute, photon and Higgs exchange. Using Feynman rules, one finds that photon exchange yields the expected Coulomb repulsion but that the second contribution precisely cancels the first. This can happen because the Higgs is massless in the Prasad-Sommerfield limit [26].

Thus, at the level considered, the conjecture is consistent, but there are more searching questions to be asked. Their answers will lead to a reformulation of the conjecture that passes even more stringent tests.

Catechism concerning the Duality Conjecture

The Montonen Olive electromagnetic duality conjecture immediately provokes the following questions:-
(1) How can the magnetic monopole solitons possess the unit spin necessary for heavy gauge particles?
(2) Will not the quantum corrections to the universal mass formula (13) vitiate it?
(3) Surely the dyon states, properly included, will spoil the picture just described?

Clearly the answers to the first two questions will depend on the choice of quantisation procedure, and presumably the most favourable one should be selected. The idea of what the appropriate choice was, and how it answered the first two questions came almost immediately, though understanding has continued to improve until the present. The answer to the third question remained a mystery until it was decisively answered by Sen in 1994, as we shall describe [28].

The immediate response of D'Adda, Di Vecchia and Horsley was the proposal that the quantisation procedure be supersymmetric [29]. The point is that the theory we have described is begging to be made supersymmetric since this can be achieved without spoiling any of the features we have described. For example, since the scalar and gauge fields lie in the same, adjoint representation of the gauge group they can lie in the same supermultiplet. The vanishing of the Higgs self-interaction implied by the Prasad-Sommerfield limit is then a consequence of supersymmetry. Because the helicity change between scalar and vector is one unit, the supersymmetry is presumably of the "extended" kind, with either $N = 2$ or $N = 4$ possible. Osborn was the first to advocate the second possibility [30].

The reason supersymmetry helps answer the first question is that, given that it holds in the full quantum theory, it must be represented on any set of single particle states carrying the same specific values of the charges and of energy and momentum. This is so, regardless of the nature of the particles, whether they are created by quantum fields or arise as soliton states, that is, whether or not they carry magnetic charge. When the extended supersymmetry algebra with N supercharges acts on massive states, the algebra is isomorphic to a Clifford algebra in a Euclidean space with $4N$ dimensions [31]. This algebra has a unique irreducible representation of 2^{2N} dimensions. This representation includes states whose helicity h varies over a range $\Delta h = N$ with intervals of 1/2. The limits of this range may differ but should not exceed 1 in magnitude if the states can be created by fields satisfying renormalisable equations of motion, according to the standard wisdom.

So, of necessity, the monopoles carry spin, quite likely unit spin. Secondly, quantum corrections tend to cancel in supersymmetric theories, essentially because the supersymmetric harmonic oscillator has no zero point energy. This is relevant to the second question because the small fluctuations about the soliton profile decompose into such oscillators, with the mass correction equal to the sum of zero point energies.

The structure of the representation theory raises some questions. According to the renormalisability criterion, the maximum range of helicity is $\Delta h = 1-(-1) = 2$, which is just consistent with $N = 2$ supersymmetry, but apparently forbids $N = 4$ supersymmetry. Another difficulty concerns the understanding of how the Higgs mechanism providing the mass of the gauge particles, works in the presence of supersymmetry. The point is that the expression on the right hand side of the supercharge anticommutator, $\gamma.P$, is a singular matrix when $P^2 = 0$, that is, for massless states. As a consequence, the supersymmetry algebra acting on massless states is isomorphic to an Euclidean algebra in $2N$ dimensions and so now possesses a unique irreducible representation of 2^N dimensions (the square root of the number in the massive case), with helicity range $\Delta h = N/2$. This now accommodates both $N = 2$ and $N = 4$ superalgebras but no more. Indeed it is the reason we cannot envisage a more extended supersymmetry, such as $N = 8$, which requires gravitons of spin 2, whose interactions are not renormalisable.

The question arises of how to reconcile the jump in the dimensions of the representations with the acquisition of mass for a given field content of scalar and gauge fields. The answer, due to Witten and Olive [32], is that something special happens precisely when the Higgs field lies in the adjoint representation, as we have assumed, and so can lie in the same supermultiplet as the gauge field. Then the electric charge, q, occurs as a central charge, providing an additional term on the right hand side of the supercharge anticommutator, thereby altering the structure of the algebra. The condition for a "short" representation, that is, of dimension 2^N, is now $P^2 = a^2q^2$, rather than $P^2 = 0$. Thus, providing the Higgs formula (9) holds, mass can be acquired without altering the dimension of the irreducible representation. Furthermore, magnetic charge can occur as yet another additional central charge with the condition for a "short" representation being simply the universal Bogomolny-Higgs formula (13). In particular, this means that this formula now has an exact quantum status as it follows from the supersymmetry algebra which is presumably an exact, quantum statement (though there may be subtle renormalisation effects) [32].

More on Supersymmetry and $N = 2$ versus $N = 4$

The possibility that, unlike the unextended supersymmetry algebra, the extended ones could be modified by the inclusion of central charges was originally noted by Haag, Łopuszanski and Sohnius [33], while the physical identification of these charges was due to Witten and Olive [32]. The confirmation of the result involved a new matter of principle. Hitherto supersymmetry algebras had been checked via the algebra of transformations of the fields entering the action. But since these will never carry magnetic charge in the electric formulation, this method will not detect the presence of magnetic charge in the algebra. Instead, it is necessary to manipulate all the charges explicitly, treating them as space integrals of local polynomials in the fields and their derivatives.

The supersymmetry algebras possess an automorphism (possibly outer) involving chiral transformations of the supercharges

$$Q^{\alpha}_{L,R} \to e^{\pm i\phi/2} Q^{\alpha}_{L,R} \qquad \alpha = 1, 2 \ldots N \tag{16}$$

where the suffices refer to the handedness. When the central charges are present, this automorphism requires them to simultaneously transform by (6) (at least in the

$N = 2$ case). Thus the electromagnetic duality rotation is now seen to relate to a chiral rotation of the supercharges, sometimes known as R-symmetry.

So far, the theory could possess either $N = 2$ or $N = 4$ supersymmetry and it is necessary to determine which, if possible. At first sight, $N = 2$ is simpler as there are precisely two central charges, q and g, as we have described, whereas in $N = 4$ there are more. However the $N = 4$ theory has one very attractive feature, namely that there is precisely one irreducible representation of the supersymmetry algebra fitting the renormalisability criterion $|h| \leq 1$, which, moreover, has to be "short", and therefore satisfy the universal mass formula (13) [30]. It follows that any dyon state must, willynilly, lie in a multiplet isomorphic to the one containing the gauge particles. Correspondingly there is only one supermultiplet of fields and, as a result, the supersymmetric action is unique apart from the values of the coupling constants.

However, there is an even more compelling reason for $N = 4$ supersymmetry which emerged some years later. In a series of papers it became apparent that the Callan-Symanzik β-function vanished identically in the unique $N = 4$ supersymmetric theory [34,35]. This was therefore the first example of a quantum field theory in four dimensions with this property. The vanishing has at least three remarkable consequences favourable to the ideas considered:

(1) As β controls the running of the coupling constant, its vanishing means that the gauge coupling constant does not renormalise. Presumably this applies in both the electric and magnetic formulations and it means that there is no question whether the Dirac quantisation applies to the bare or renormalised coupling constants, as these are the same (Rossi [36]).

(2) The trace of the energy momentum tensor is usually proportional to β times a local quantity and so it should vanish in this theory, indicating that the theory is exactly conformally invariant (if a vanishes). Thus the $N = 4$ supersymmetric gauge theory is the first known example of a quantum conformal field theory in four dimensions, to be compared with the rich spectrum of examples in two dimensions. Furthermore, the Higgs mechanism producing a nonzero value for the vacuum expectation value parameter a presumably provides an integrable deformation realising Zamolodchikov's principle in four dimensions [8]. Notice that the naive idea that conformal field theories should be more numerous in four rather than two dimensions seems to be false despite the fact that the conformal algebra has only fifteen rather than an infinite number of dimensions. Besides the $N = 4$ supersymmetric gauge theory, there are now a few other known conformal field theories in four dimensions, all supersymmetric gauge theories.

(3) Finally, just as the trace anomaly vanishes, so does the axial anomaly. In fact the two properties are related by a supersymmetry transformation. This means that the chiral symmetry (16) can be extended to the fields of the theory and is an exact symmetry for $N = 4$. Thus we have answered an earlier question and seen that, indeed, electromagnetic duality rotations can be extended to include matter, albeit in a very special case.

The second point above, concerning the realisation of Zamolodchikov's principle in four dimensions via a special sort of Higgs mechanism [8], raises questions about the nature of "integrability" in four dimensions. As far as is known, the $N = 4$ supersymmetry algebra is the largest extension of Poincaré symmetry there, but only provides a finite number of conservation laws, unlike the infinite number available in two dimensions. On the other hand, there are, apparently, monopole/dyon solutions with particle-like attributes (certainly if the duality conjecture is to be believed). But a complete and direct proof is lacking, even though the results for like monopoles are encouraging.

For each value of magnetic charge, the moduli space of solutions to the Bogomolny equations (12) forms a manifold whose points correspond to static configurations of distinct monopoles with total energy $a|g|$. The problem of describing their relative motion was answered by Manton [37], at least if it was slow. His idea follows from the analogy with a Newtonian point particle confined to move freely on a Riemannian manifold. It can remain at rest at any point of the manifold, but, if it moves, it follows a geodesic on the manifold determined by the Riemannian metric. He realised that the moduli spaces of the Bogomolny equations must possess such a metric and saw how to derive it from the action. Actually it has a hyperkähler structure which makes it very interesting mathematically. Moreover, Atiyah and Hitchin calculated the metric explicitly for the moduli space with double magnetic charge [38]. This is sufficient to determine the classical scattering of two monopoles at low relative velocity and yielded surprisingly involved behaviour, including a type of incipient breathing motion perpendicular to the scattering plane, visible on a video prepared by IBM.

Despite these beautiful results, there is no idea of how to describe relative motion of monopole solitons with unlike charge. The duality conjecture predicts the possibility of pair annihilation, unlike the sine-Gordon situation. This is why we say the soliton behaviour is incompletely understood. It is certainly more complicated than in two dimensions.

The Schwinger Quantisation Condition and the Charge Lattice

The remaining difficulty, one that has been repeatedly deferred, concerns the dyon spectrum. We know that there exist dyon solutions carrying magnetic charge, but we do not know what values of the electric charge are allowed. The problem is that the Dirac quantisation condition (7) does not determine this, nor does it respect the electromagnetic duality rotation (6) which is apparently so fundamental.

It was Schwinger and Zwanziger who independently resolved the problem [39,24]. They saw that Dirac's assumption that the monopole carried no electric charge was unjustified, and responsible for the difficulties. Instead, they applied Dirac's argument to two dyons, carrying respective charges (q_1, g_1) and (q_2, g_2), and found

$$q_1 g_2 - q_2 g_1 = 2\pi n\hbar, \qquad n = 0, \pm 1, \pm 2, \ldots. \tag{17}$$

This is known (somewhat unfairly) as the Schwinger quantisation condition and it does now respect the duality rotation symmetry (6) applied simultaneously to the two dyons. Notice that it is significant that the group $SO(2)$ has two invariant tensors, the

Kronecker delta entering the mass formula (13) and the antisymmetric tensor entering (17).

As mentioned earlier, the values of $q+ig$ realised by localised states composed of particles should lie at the points of a lattice in the complex plane. The origin of this lattice structure are the conservation laws for charge and the TCP theorem. The set of allowed values must be closed under both addition and reversal of sign as these operations can be realised physically by combining states and by TCP conjugation.

Without loss of generality, it can be assumed that there exist a subset of states carrying purely electric charge. As long as magnetic charge exists (17) implies that there is a minimum positive value, q_0, say. Then the allowed values of pure electric charge are nq_0, $n \in \mathbb{Z}$, that is, a discrete one dimensional lattice. Now let us examine the most general values of $q+ig$ allowed by the Schwinger quantisation condition, (17). By it, the smallest allowed positive magnetic charge, g_0 satisfies

$$g_0 = \frac{2\pi n_0 \hbar}{q_0}, \tag{18}$$

where n_0 is a positive integer dependent on the detailed theory considered. Now consider two dyons with magnetic charge g_0 and electric charges q_1 and q_2 respectively. By (17) and (18)

$$q_1 - q_2 = \frac{2\pi n\hbar}{g_0} = \frac{nq_0}{n_0}.$$

However, as there must consequently be a state with pure electric charge $q_1 - q_2$, n must be a multiple of n_0. Hence for any dyon with magnetic charge g_0, its electric charge

$$q = q_0\left(n + \frac{\theta}{2\pi}\right),$$

where θ is a new parameter of the theory which is, in a sense, angular since increasing it by 2π is equivalent to increasing n by one unit. So

$$q + ig = q_0(n+\tau),$$

where

$$\tau = \frac{\theta}{2\pi} + \frac{2\pi i n_0 \hbar}{q_0^2}. \tag{19}$$

Repeating the argument for more general states with magnetic charge mg_0

$$q + ig = q_0(m\tau + n), \qquad m, n \in \mathbb{Z}. \tag{20}$$

This is the charge lattice and it finally breaks the continuous symmetry (2) and (6) in a spontaneous manner [40]. This lattice has periods q_0 and $q_0\tau$ with ratio τ, (19). Notice that τ is a complex variable formed of dimensionless parameters dependent on the detailed theory. Its imaginary part is positive, being essentially the inverse of the fine structure constant.

So far, this part of the argument has been very general, but, given a specific theory, an important question for electromagnetic duality concerns the identification of the subset of the charge lattice that can be realised by single particle states, rather than multiparticle states.

It is easy to show that, if single particle states obey the universal mass formula (13), and are stable with respect to any two-body decay into lighter particles permitted by the conservation of electric and magnetic charge, then they must correspond to points of the charge lattice which are "primitive vectors".

A point P of the charge lattice is a primitive vector if the line OP contains no other points of the lattice strictly between O, the origin, and P. Thus the only primitive vectors on the real axis are $(\pm q_0, 0)$. Equivalently, a primitive vector is a point given by (20) in which the integers m and n are coprime (in saying this we must agree that 0 is divisible by any integer).

The proof of the assertion is simple: it relies on the fact that the mass of a particle at P is proportional to its Euclidean distance OP from the origin, by (13). So, by the triangle inequality, any particle is stable unless its two decay products correspond to points collinear with itself and the origin. This is impossible, providing the original particle corresponds to a primitive vector.

There are an infinite number of primitive vectors on the charge lattice, for example, all the points with $m = \pm 1$ or $n = \pm 1$. The corresponding masses can be indefinitely large. If $m = 2$, every second point is a primitive vector. If $m = 3$, every third point fails to be a primitive vector, and so on.

This result tells us what to expect for the spectrum of dyons, namely that they correspond to the primitive vectors off the real axis. Since the mass formula used in this argument is characteristic of supersymmetric gauge theories as discussed above, it ought to be possible to recover this result from consideration of the Bogomolny moduli spaces governing the static soliton solutions. This is what Sen achieved in 1994, [28], and a simplified explanation follows.

For $m = \pm 1$, the dyons relate to points in the $m = \pm 1$ moduli space since the single monopoles are solutions to the Bogomolny equations. However, as discussed earlier, the points of the $m = \pm 2$ moduli space correspond to configurations of a pair of like monopoles in static equilibrium. Thus the $m = \pm 2$ single particle states cannot be Bogomolny solutions. Instead they must be regarded as quantum mechanical bound states, with zero binding energy (in order to satisfy the mass formula). Remembering Manton's treatment of moving monopoles following geodesics on the moduli space determined by the hyperkähler metric thereon, it is clear that it is crucial to examine the spectrum of the Laplacian determined by this metric, as this is proportional to the quantum mechanical Hamiltonian [41]. In particular, zero modes in the discrete spectrum are sought. There is some subtlety, treated by Sen, concerning the fact that the quantum mechanics possesses $N = 4$ supersymmetry because the metric is hyperkähler, but using the Atiyah-Hitchin metric, Sen was able to solve for the zero modes, and show that only every other permitted value of the electric charge could occur. Thus the dyons with magnetic charge $2g_0$ do indeed correspond precisely to the primitive vectors on the charge lattice. For higher values of $|m|$, the explicit metric is not known, but Hodge's theorem relates the counting of the zero modes of the Laplacian on the moduli space to its cohomology, which can be determined without knowledge of the metric. (This argument is said to be due to Segal, unpublished).

These are the results that finally clear up the dyon problem and leave the electromagnetic duality conjecture in good shape, though a reassessment will be in order. Before discussing this, we ask whether the angle θ, occurring in (19), appears explicitly as a parameter in the action of the spontaneously broken gauge theory. Witten found the answer in 1979 [40]. Because the gauge group is non abelian, $SU(2)$, a term proportional to the instanton number, k can be added to the action, so that the Feynman weighting factor becomes:

$$exp\left(\frac{i\text{Action}}{\hbar}\right) \to exp\left(\frac{i\text{Action}}{\hbar} + \frac{i\tilde{\theta}}{2\pi}k\right). \qquad (21)$$

As k is proportional to an integral of $F\tilde{F}$ over space time, it is a surface term which cannot affect the classical equations of motion, but it does affect the quantum theory. Note that, like θ, $\tilde{\theta}$ is an angular variable as the theory is unaffected if it is increased by 2π. In fact the two angles are indeed equal as Witten showed by an elementary calculation of the electric and magnetic charges using Noether's theorem. Thus θ is what is known as the instanton or vacuum angle.

The above result has another consequence, yet again singling out the $N = 4$ supersymmetric theory as the only viable one for exact electromagnetic duality. This is because an application of the chiral rotation (16) to the fermion fields alters the Lagrangian density by an anomalous term proportional to the axial anomaly $\beta F\tilde{F}$. This means that the instanton angle can be altered by a redefinition of the fermion field, and so has no physical meaning, unless β, and hence the axial anomaly, vanishes. This forces us back to the $N = 4$ theory, with the conclusion that only in this theory does the charge lattice really make sense. Finally note that in this theory the integer n_0 occurring in (19) equals 2. This is because the $N = 4$ theory has only one supermultiplet which includes the gauge particle and hence must be an $SU(2)$ triplet. No doublets are allowed in $N = 4$, unlike $N = 2$.

Exact Electromagnetic Duality and the Modular Group

Armed with the new insight that the spectrum of single particle states correspond to the primitive vectors of the charge lattice, augmented by the origin, rather than the five points previously considered, we can see that the original Montonen-Olive conjecture was too modest. Instead of possessing two equivalent choices of action, the $N = 4$ supersymmetric gauge theory apparently possesses an infinite number of them, all with an isomorphic structure, but with different values of the parameters [28].

Roughly speaking, the reason is that it is the charge lattice that describes the physical reality. Choices of action correspond to choices of basis in the lattice, that is a pair of non collinear primitive vectors (or, a pair of periods). As the charge lattice is two dimensional, these choices are related by the action of the modular group, an infinite discrete group containing the previous transformation (14).

Let us choose a primitive vector in the charge lattice, represented by a complex number, q_0', say. Then we may ascribe short $N = 4$ supermultiplets of quantum fields to each of the three points $\pm q_0'$ and 0. The particles corresponding to the origin are massless and neutral whereas the particles corresponding to $\pm q_0'$ possess complex charge $\pm q_0'$ and mass $a|q_0'|$. We may form an $N = 4$ supersymmetric action with these fields. It is unique, given the coupling $|q_0'|$, apart from the vacuum angle whose specification requires a second primitive vector, $q_0'\tau'$, say, non-collinear with q_0'. The remaining single particle states are expected to arise as monopole solitons or as quantum bound states of them as discussed above.

Since the two non-collinear primitive vectors q_0' and $q_0'\tau'$ form an alternative basis for the charge lattice, they can be expressed as integer linear combinations of the original basis, q_0 and $q_0\tau$:

$$q_0'\tau' = aq_o\tau + bq_0, \tag{22a}$$

$$q_0' = cq_0\tau + dq_0, \tag{22b}$$

where

$$a, b, c, d \in \mathbb{Z}. \tag{22c}$$

Equally, $q_0\tau$ and q_0 can be expressed as integer linear combinations of $q_0'\tau'$ and q_0'. This requires that the matrix of coefficients in (22a) and (22b) has determinant equal to ± 1,

$$ad - bc = \pm 1. \tag{23}$$

By changing a sign we can take this to be plus one. Then the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

form a group, $SL(2, \mathbb{Z})$, whose quotient by its centre is called the modular group. Equation (22a) divided by (22b) yields

$$\tau' = \frac{a\tau + b}{c\tau + d}.$$

These transformations form the modular group and preserve the sign of the imaginary part of τ. This gives the relation between the values of the dimensionless parameters in the two choices of action corresponding to the two choices of basis. It is customary to think of the modular group as being generated by elements T and S where

$$T : \tau \to \tau + 1 \qquad S : \tau \to -\frac{1}{\tau}$$

According to (19), T increases the vacuum angle by 2π. This is obviously a symmetry of (21). If the vacuum angle vanishes, S precisely yields the transformation (14) previously considered.

Proof of the quantum equivalence of all the actions associated with each choice of basis in the charge lattice would presumably require a generalised vertex operator transformation relating the corresponding quantum fields. Since these transformations would represent the modular group the prospect is challenging.

Meanwhile it has been possible to evaluate the partition function of the theory on certain space-time manifolds, provided that the theory is simplified by a "twisting procedure" that renders it "topological". Vafa and Witten verified that the results indeed possessed modular symmetry [42].

Conclusion

According to the new results reviewed above, it now appears increasingly plausible that electromagnetic duality is realised exactly in the $N = 4$ supersymmetric $SU(2)$ gauge theory in which the Higgs field acquires a non-zero vacuum expectation value. This theory is a deformation of one of the very few exact conformal field theories in Minkowski space time. The supporting analysis involves an array of almost all the previously advanced ideas particular to quantum field theories in four dimensions, but awaits definitive proof.

Despite its remarkable quantum symmetry this theory is apparently not physical unless further deformed. Seiberg and Witten have proposed deformations such that enough structure remains as to offer an explanation of quark confinement, perhaps the outstanding riddle in quantum field theory [1,43].

More generally, the potential validity of exact electromagnetic duality in at least one theory means that quantum field theory in four dimensions is much richer than the sum of its parts, quantum mechanics and classical field theory. This is because the new symmetry is essentially quantum in nature with no classical counterpart. Moreover it relates strong to weak coupling regimes of the theory. Consequently, the new insight opens a veritable Pandora's box whose contents are now subject to urgent study.

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