

Reconstructing the Inflaton Potential — an Overview

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Abstract

We review the relation between the inflationary potential and the spectra of density (scalar) perturbations and gravitational waves (tensor perturbations) produced, with particular emphasis on the possibility of reconstructing the inflaton potential from observations. The spectra provide a potentially powerful test of the inflationary hypothesis; they are not independent but instead are linked by consistency relations reflecting their origin from a single inflationary potential. To lowest-order in a perturbation expansion there is a single, now familiar, relation between the tensor spectral index and the relative amplitude of the spectra. We demonstrate that there is an infinite hierarchy of such consistency equations, though observational difficulties suggest only the first is ever likely to be useful. We also note that since observations are expected to yield much better information on the scalars than on the tensors, it is likely to be the next-order version of this consistency equation which will be appropriate, not the lowest-order one. If inflation passes the consistency test, one can then confidently use the remaining observational information to constrain the inflationary potential, and we survey the general perturbative scheme for carrying out this procedure. Explicit expressions valid to next-lowest order in the expansion are presented. We then briefly assess the prospects for future observations reaching the quality required, and consider simulated data sets motivated by this outlook.

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I Introduction

Observational cosmology is entering a new era where it is becoming possible to make detailed quantitative tests of models of the early universe for the first time. Such observations are presently the most plausible route towards learning some of the details of physics at extremely high energies, and the possibility of testing some of the speculative ideas of recent years has generated much excitement.

One of the most important paradigms in early universe cosmology is that of cosmological inflation, which postulates a period of accelerated expansion in the universe's distant past (Starobinsky, 1980; Guth, 1981; Sato, 1981; Albrecht and Steinhardt, 1982; Hawking and Moss, 1982; Linde, 1982a, 1983). Although originally introduced as a possible solution to a host of cosmological conundrums such as the horizon, flatness and monopole problems, by far the most useful property of inflation is that it generates spectra of both density perturbations (Guth and Pi, 1982; Hawking, 1982; Linde, 1982b; Starobinsky, 1982; Bardeen, Steinhardt, and Turner, 1983) and gravitational waves (Starobinsky, 1979; Abbott and Wise, 1984a). These extend from extremely short scales to scales considerably in excess of the size of the observable universe. During inflation the scale factor grows quasi-exponentially, while the Hubble radius remains almost constant. Consequently the wavelength of a quantum fluctuation – either in the scalar field whose potential energy drives inflation or in the graviton field – soon exceeds the Hubble radius. The amplitude of the fluctuation therefore becomes ‘frozen’. Once inflation has ended, however, the Hubble radius increases faster than the scale factor, so the fluctuations eventually reenter the Hubble radius during the radiation- or matter-dominated eras. The fluctuations that exit around 60 e -foldings or so before reheating reenter with physical wavelengths in the range accessible to cosmological observations. These spectra provide a distinctive signature of inflation. They can be measured in a variety of different ways including the analysis of microwave background anisotropies, velocity flows in the universe, clustering of galaxies and the abundances of gravitationally bound objects of various types (for reviews, see Efstathiou (1990); Liddle and Lyth (1993a)).

Until the measurement of large angle microwave background anisotropies by the COsmic Background Explorer (COBE) satellite (Smoot et al., 1992; Wright et al., 1992; Bennett et al., 1994, 1996; see White, Scott, and Silk (1994) for a general discussion of the microwave background), such observations covered a fairly limited range of scales, and it was satisfactory to treat the prediction of a generic inflationary scenario as giving rise to a scale-invariant (Harrison–Zel’dovich) spectrum of density perturbations (Harrison, 1970; Zel’dovich, 1972) and a negligible amplitude of gravitational waves (though even then, it was recognized that the scale-invariance was only approximate (Bardeen et al., 1983)). Since the detection by COBE, however, the spectra are now constrained over a range of scales covering some four orders of magnitude from one megaparsec up to perhaps ten thousand megaparsecs. Moreover, shortly after the COBE detection, a number of authors reexamined the possibility that a significant fraction of the signal could be due to gravitational waves (Krauss and White, 1992; Davis et al., 1992; Salopek, 1992; Liddle and Lyth, 1992; Lidsey and Coles, 1992; Lucchin, Matarrese, and Mollerach, 1992; Souradeep and Sahni, 1992; Adams et al., 1993; Dolgov and Silk, 1993).

Thus, the inflationary prediction must now be considered with much greater care, even

in order to deal with *present* observations. At the next level of accuracy, one finds that different inflation models make different predictions for the spectra, which can be viewed as differing magnitudes of variation from the scale-invariant result. In the simplest approximation the spectra are taken to be power-laws. Hence, modern observations discriminate between different inflationary models, and are already sufficient to rule out some models completely (see e. g. Liddle and Lyth, 1992) and substantially constrain the parameter space of others (Liddle and Lyth, 1993a). Future observations will make even stronger demands on theoretical precision, and will certainly tightly constrain inflation.

These deviations from highly symmetric situations such as a scale-invariant spectrum provide an extremely distinctive way of probing inflation. This is considerably more powerful than employing historically emphasised predictions such as a spatially flat universe. Although a spatially flat universe is indeed a typical (but not inevitable, see e.g., Sasaki et al. (1993); Bucher, Goldhaber, and Turok (1995)) outcome of inflation, it appears unlikely that this feature will be unique to inflation. Moreover, the power that observations such as microwave background anisotropies provides may be sufficient to override the rather subjective arguments often made against inflation models because of their apparent ‘unnaturalness’. Regardless of whether a model appears natural or otherwise, it should be the observations which decide whether it is correct or not.

In a wide range of inflationary models, the underlying dynamics is simply that of a single scalar field — the *inflaton* — rolling in some underlying potential. This scenario is generically referred to as *chaotic inflation* (Linde, 1983, 1990b) in reference to its choice of initial conditions. This picture is widely favored because of its simplicity and has received by far the most attention to date. Furthermore, many superficially more complicated models can be rewritten in this framework. In view of this we shall concentrate on such a type of model here.

The generation of spectra of density perturbations and gravitational waves has been extensively investigated in these theories. The usual strategy is an expansion in the deviation from scale-invariance, formally expressed as the *slow-roll expansion* (Steinhardt and Turner, 1984; Salopek and Bond, 1990; Liddle, Parsons, and Barrow, 1994). At the simplest level of approximation, the spectra can be expressed as power-laws in wavenumber; further accuracy entails calculation of the deviations from this power-law approximation.

A crucial aspect of the two spectra is that they are not independent. In a general sense, this is clear since they correspond at the formal level to two continuous functions that both have an origin in the single continuous function expressing the scalar field potential. Such a link was noted in the simplest situation, where the spectra are approximated by power-laws, by Liddle and Lyth (1992); the general situation where the two are linked by a *consistency equation* was expounded in Copeland et al. (1993b, henceforth CKLL1), and an explicit higher-order version of the simplest equation was found by Copeland et al. (1994a, henceforth CKLL2). If one had complete expressions for the entire problem, the consistency relation would be represented as a differential equation relating the two spectra. However, we shall argue that it is preferable to express the spectra via an order-by-order expansion. In this case one obtains a finite set of *algebraic* expressions which represent the coefficients of an expansion of the full differential equation. The familiar situation is a single consistency equation that relates the gravitational wave spectral index to the relative amplitudes of the spectra. This is a result of the lowest-order expansion. The

general situation of multiple consistency equations does not seem to have been expounded before, though a second consistency equation did appear in Kosowsky and Turner (1995). In practice, the observational difficulties associated with measurements of the details of the gravitational wave spectrum make it extremely unlikely that any but the first consistency equation shall ever be needed.

Given a particular set of observations of some accuracy, one can attempt the bold task of reconstructing the inflaton potential from the observations. In fact, the situation one hopes for is stronger than a simple reconstruction, the language of which suggests the possibility of finding a suitable potential regardless of the observations. With sufficiently good observations, one can first test whether the consistency equation is satisfied; in situations where observations make this test non-trivial it provides a very convincing vindication of the inflationary scenario. Thus emboldened, one could then go on to use the remaining, non-degenerate, information to constrain features of the inflaton potential. Figure 1 illustrates this procedure schematically.

The main obstacle in reconstruction is the limited range of scales accessible. Although the observations may span up to four orders of magnitude, the expansion of the universe is usually so fast during inflation that this typically translates into only a brief range of scalar field values. One should therefore not overexaggerate the usefulness of this approach in determining the detailed structure of physics at high energy, but one should bear in mind that this may be the only observational information available of any kind at such energies.

A second obstacle is that one doesn't observe the primordial spectra directly, but rather after they have evolved considerably. Although this is a linear problem (except on the shortest scales) and hence computationally tractable, the evolution necessarily depends on the various cosmological parameters, such as the expansion rate and the nature of any dark matter. The form of the initial spectra must be untangled from their influence. We shall discuss this in some detail in Section VII.

Earlier papers discuss two possible ways of treating observational data. The bolder strategy is to use estimates of the spectra as functions of scale (Hodges and Blumenthal, 1990; Grishchuk and Solokhin, 1991; CKLL1). In practice, however, this approach founders through the lack of theoretically derived exact expressions for the spectra produced by an arbitrary potential. We shall therefore argue in this review in favor of the alternative approach, which is usually called perturbative reconstruction (Turner, 1993a; Copeland et al., 1993a; CKLL1; Turner, 1993b; CKLL2; Liddle and Turner, 1994). In this approach, the consistency equation and scalar potential are determined as an expansion about a given point (regarded either as a single scale in the spectra or as a single point on the potential), allowing reconstruction of a region of the potential about that point. This has the considerable advantage that one can terminate the series when either theoretical or observational knowledge runs out.

The outline of this review is as follows. We devote two Sections to a review of the inflation driven by a (slowly) rolling scalar field. We begin by considering the classical scalar field dynamics and then proceed to discuss the generation of the spectra of density perturbations and gravitational waves. Because an accurate derivation of the predicted spectra is crucial to this programme, we provide a detailed account of the most accurate calculation presently available, due to Stewart and Lyth (1993). In Section IV we consider the simplest possible scenario allowing reconstruction, and introduce the notion of the consistency

equation. Section V reviews the present state-of-the-art, where next-order corrections are incorporated into all expressions. One hopes that observational accuracy will justify this more detailed analysis, though this depends upon which (if any) inflation model proves correct. Section VI then expands on this by describing the full perturbative reconstruction framework, illustrating how much information can be obtained from which measurements and demonstrating that one can write a hierarchy of consistency equations. We then briefly illustrate worked examples on simulated data in Section VII. Before concluding, we devote a section to an examination of other proposals for constraining the inflaton potential, without using large-scale structure observations.

II Inflationary Cosmology and Scalar Fields

A The fundamentals of inflationary cosmology

Observations indicate that the density distribution in the universe is nearly smooth on large scales, but contains significant irregularities on small scales. These correspond to a hierarchy of structures including galaxies, clusters and superclusters of galaxies. One of the most important questions that modern cosmology must address is why the observable universe is almost, but not quite exactly, homogeneous and isotropic on sufficiently large scales.

The hot big bang model is able to explain the current expansion of the universe, the primordial abundances of the light elements and the origin of the cosmic microwave background radiation; for a review of all these successes see Kolb and Turner (1990). However, this model as it stands is unable to explain the origin of structure in the universe. This problem is related to the well known flatness problem (Peebles and Dicke, 1979) and is essentially a problem of initial data. It arises because the entropy in the universe is so large, $S \approx 10^{88}$ (Barrow and Matzner, 1977). One expects this quantity to be of order unity since it is a dimensionless constant.

This paradox can be made more quantitative in the following way. The dynamics of a Friedmann–Robertson–Walker (FRW) universe containing matter with density ρ and pressure p is determined by the Einstein acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3m_{\text{Pl}}^2}(\rho + 3p), \quad (2.1)$$

the Friedmann equation

$$H^2 = \frac{8\pi}{3m_{\text{Pl}}^2}\rho - \frac{k}{a^2}, \quad (2.2)$$

and the mass conservation equation

$$\dot{\rho} + 3H(\rho + p) = 0, \quad (2.3)$$

where $a(t)$ is the scale factor of the universe, $H \equiv \dot{a}/a$ is the Hubble expansion parameter, a dot denotes differentiation with respect to cosmic time t , m_{Pl} is the Planck mass and $k = 0, -1, +1$ for spatially flat, open, or closed cosmologies, respectively. Units are chosen such that $c = \hbar = 1$.

The Friedmann equation (2.2) may be expressed in terms of the Ω -parameter. This parameter is defined as the ratio of the energy density of the universe to the critical energy density ρ_c that is just sufficient to halt the current expansion:

$$\Omega \equiv \frac{\rho}{\rho_c}, \quad \rho_c \equiv \frac{3m_{\text{Pl}}^2 H^2}{8\pi}. \quad (2.4)$$

The current observational values for these parameters are $\rho_c = 1.88h^2 \times 10^{-29} \text{ g cm}^{-3}$ and $H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1}$ where conservatively we have $0.4 \leq h \leq 0.8$. Eq. (2.2) simplifies to

$$\Omega - 1 = \frac{k}{a^2 H^2}, \quad (2.5)$$

and this implies that

$$\frac{\Omega - 1}{\Omega} = \frac{3m_{\text{Pl}}^2}{8\pi} \frac{k}{\rho a^2}. \quad (2.6)$$

Now, for a radiation-dominated universe, the equation of state is given by $\rho = 3p = \pi^2 g_\rho T^4/30$ at some temperature T , where $g_\rho = \mathcal{O}(10^2)$ represents the total number of relativistic degrees of freedom in the matter sector at that time. Thus the scale factor grows as $a(t) \propto t^{1/2}$ when $k = 0$ and the expansion rate is given by

$$H = 1.66g_\rho^{1/2} \left(\frac{T^2}{m_{\text{Pl}}} \right) = \frac{1}{2t}. \quad (2.7)$$

Eq. (2.7) yields the useful expression

$$\left(\frac{t}{\text{sec}} \right) \approx \left(\frac{T}{\text{MeV}} \right)^{-2} \quad (2.8)$$

and substituting Eqs. (2.7) and (2.8) into Eq. (2.6) implies that

$$\left| \frac{\Omega - 1}{\Omega} \right| \approx \frac{10^{43}}{S^{2/3}} \left(\frac{t}{\text{sec}} \right) \approx \frac{10^{37}}{S^{2/3}} \left(\frac{\text{GeV}}{T} \right)^2, \quad (2.9)$$

where $S \approx 10^{88}$ is the entropy contained within the present horizon. The large amount of entropy in the universe therefore implies that Ω must have been very close to unity at early times. Indeed, we find that $\Omega = 1 \pm 10^{-16}$ just one second after the big bang, the time of nucleosynthesis.

The flatness problem is therefore a problem of understanding why the (classical) initial conditions corresponded to a universe that was so close to spatial flatness. In a sense, the problem is one of fine-tuning and although such a balance is possible in principle, one nevertheless feels that it is unlikely. On the other hand, the flatness problem arises because the entropy in a comoving volume is conserved. It is possible, therefore, that the problem could be resolved if the cosmic expansion was non-adiabatic for some finite time interval $t \in [t_i, t_f]$ during the early history of the universe.

This point was made explicitly by Guth in his seminal paper of 1981. He postulated that the entropy changed by an amount

$$S_f = Z^3 S_i \quad (2.10)$$

during this time interval, where Z is a numerical factor. In Guth's original model, this entropy production occurred at, or just below, the energy scale $T_{\text{GUT}} = \mathcal{O}(10^{17})$ GeV associated with the Grand Unified (GUT) phase transition. This corresponds to a timescale $t \approx 10^{-40}$ s. Eq. (2.9) then implies that the flatness problem is solved, in the sense that $|\Omega_i^{-1} - 1| = \mathcal{O}(1)$, if $Z \geq 10^{28}$. It can be shown that the other problems of the big bang model, such as the horizon and monopole problems are also solved if Z satisfies this lower bound (Guth, 1981).

Guth called this process of entropy production *inflation*, because the volume of the universe also grows by the factor Z^3 between $t = t_i$ and $t = t_f$. Indeed, the expansion of the universe during the inflationary epoch is very rapid. Further insight into the nature of this expansion may be gained by considering Eq. (2.6). This expression implies that the quantity $(\Omega^{-1} - 1)\rho a^2$ is conserved for an arbitrary equation of state. It follows, therefore, that

$$(\Omega_i^{-1} - 1)a_i^2 \rho_i = (\Omega_f^{-1} - 1)a_f^2 \rho_f \quad (2.11)$$

and, if we assume that the standard, big bang model is valid for $t > t_f$, we may deduce that (Lucchin and Matarrese, 1985b)

$$\rho_i a_i^2 |\Omega_i^{-1} - 1| \approx 10^{-56} \rho_f a_f^2 |\Omega_0^{-1} - 1|. \quad (2.12)$$

Since our current observations imply that $|\Omega_0^{-1} - 1| = \mathcal{O}(1)$, the flatness problem is solved if $\rho_f a_f^2 \gg \rho_i a_i^2$. However, Eq. (2.2) implies that the quantity $3\dot{a}^2 - (8\pi/m_{\text{Pl}}^2)\rho a^2$ is also conserved. Consequently, this inequality is satisfied if $\dot{a}_f > \dot{a}_i$. Thus, a necessary condition for inflation to proceed is that the scale factor of the universe *accelerates* with respect to cosmic time:

$$\ddot{a}(t) > 0. \quad (2.13)$$

This is in contrast to the decelerating expansion that arises in the big bang model.

The question now arises as to the nature of the energy source that drives this accelerated expansion. It follows from Eq. (2.1) that Eq. (2.13) is satisfied if $\rho + 3p < 0$ and this is equivalent to violating the strong energy condition (Hawking and Ellis, 1973). The simplest way to achieve such an antigravitational effect is by the presence of a homogeneous scalar field, ϕ , with some self-interaction potential $V(\phi) \geq 0$. In the FRW universe, such a field is equivalent to a perfect fluid with energy density and pressure given by

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad (2.14)$$

and

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi), \quad (2.15)$$

respectively. Other matter fields play a negligible role in the evolution during the inflation, so their presence will be ignored. In this case, Eqs. (2.2) and (2.3) are given by

$$H^2 = \frac{8\pi}{3m_{\text{Pl}}^2} \left(\frac{1}{2}\dot{\phi}^2 + V(\phi) \right) - \frac{k}{a^2} \quad (2.16)$$

and

$$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi), \quad (2.17)$$

where here and throughout a prime denotes differentiation with respect to ϕ . Hence, $-\rho \leq p \leq \rho$ and we have the inflationary requirement $\ddot{a} > 0$ as long as $\dot{\phi}^2 < V$. Inflation is thus achieved when the matter sector of the theory applicable at some stage in the early universe is dominated by vacuum energy.

Recently, an alternative inflationary scenario — the pre-big bang cosmology — has been developed whereby the accelerated expansion is driven by the kinetic energy of a scalar field rather than its potential energy (Gasperini and Veneziano, 1993a,b, 1994). If the field is non-minimally coupled to gravity in an appropriate fashion, this kinetic energy can produce a sufficiently negative pressure and a violation of the strong energy condition (Pollock and Sahdev, 1989; Levin, 1995a). Such couplings arise naturally within the context of the string effective action. However, models of this sort inherently suffer from a ‘graceful exit’ problem due to the existence of singularities in both the curvature and the scalar field motion (Brustein and Veneziano, 1994; Kaloper, Madden and Olive, 1995, 1996; Levin, 1995b; Easther, Maeda, and Wands, 1996). Moreover, a satisfactory mechanism for generating structure formation and microwave background anisotropies in these models has yet to be developed, although it is possible that such inhomogeneities may be generated by quantum fluctuations in the electromagnetic field (Gasperini, Giovannini and Veneziano, 1995).

In view of this, we shall restrict our discussion to potential-driven models. We will focus in this work on some of the general features of the chaotic inflation scenario (Linde, 1983, 1990b). Although Linde’s original paper considered a specific potential (a quartic one), the theme was much more general. We adopt the modern usage of *chaotic inflation* to refer to any model where inflation is driven by a single scalar field slow-rolling from a regime of extremely high potential energy. The phrase does not imply any particular choice of potential. Most, though not quite all, modern inflationary models fall under the umbrella of this definition. Since the precise identity of the scalar field driving the inflation is unknown, it is usually referred to as the *inflaton field*.

In the chaotic inflation scenario, it is assumed that the universe emerged from a quantum gravitational state with an energy density comparable to that of the Planck density. This implies that $V(\phi) \approx m_{\text{Pl}}^4$ and results in a large friction term in the Friedmann equation (2.16). Consequently, the inflaton will slowly roll down its potential, i.e., $|\ddot{\phi}| \ll H|\dot{\phi}|$ and $\dot{\phi}^2 \ll V$. The condition for inflation is therefore satisfied and the scale factor grows as

$$a(t) = a_i \exp \left(\int_{t_i}^t dt' H(t') \right). \quad (2.18)$$

The expansion is quasi-exponential in nature, since $H(\phi) \approx 8\pi V(\phi)/3m_{\text{Pl}}^2$ is almost constant, and the curvature term k/a^2 in Eq. (2.16) is therefore rapidly redshifted away. The kinetic energy of the inflaton gradually increases as it rolls down the potential towards the global minimum. Eventually, its kinetic energy dominates over the potential energy and inflation comes to an end when $\dot{\phi}^2 \approx V(\phi)$. The field then oscillates rapidly about the minimum and the couplings of ϕ to other matter fields then become important. It is these oscillations that result in particle production and a reheating of the universe.

The simplest chaotic inflation model is that of a free field with a quadratic potential, $V(\phi) = m^2\phi^2/2$, where m represents the mass of the inflaton. During inflation the scale factor grows as

$$a(t) = a_i e^{2\pi(\phi_i^2 - \phi^2(t))} \quad (2.19)$$

and inflation ends when $\phi = \mathcal{O}(1) m_{\text{Pl}}$. If inflation begins when $V(\phi_i) \approx m_{\text{Pl}}^4$, the scale factor grows by a factor $\exp(4\pi m_{\text{Pl}}^2/m^2)$ before the inflaton reaches the minimum of its potential (Linde, 1990b). One can further show that the mass of the field should be $m \approx 10^{-6} m_{\text{Pl}}$ if the microwave background constraints are to be satisfied. This implies that the volume of the universe will increase by a factor of $Z^3 \approx 10^{3 \times 10^{12}}$ and this is more than enough inflation to solve the problems of the hot big bang model.

It is important to emphasize that in this scenario the initial value of the scalar field is randomly distributed in different regions of the universe. On the other hand, one need only assume that a small, causally connected, region of the pre-inflationary universe becomes dominated by the potential energy of the inflaton field. Indeed, if the original domain is only one Planck length in extent, its final size will be of the order $10^{10^{12}}$ cm; for comparison, the size of the observable universe is approximately 10^{28} cm.

In conclusion, therefore, the chaotic inflationary scenario represents a powerful framework within which specific inflationary models can be discussed. The essential features of each model — such as the final reheat temperature and the amplitude of scalar and tensor fluctuations — are determined by the specific form of the potential function $V(\phi)$. This in turn is determined by the particle physics sector of the theory.

Unfortunately, however, there is currently much theoretical uncertainty in the correct form of the unified field theory above the electroweak scale. This has resulted in the development of a large number of different inflationary scenarios and the identity of the inflaton field is therefore somewhat uncertain. Possible candidates include the Higgs bosons of grand unified theories, the extra degrees of freedom associated with higher metric derivatives in extensions to general relativity, the dilaton field of string theory and, more generally, the time-varying gravitational coupling that arises in scalar-tensor theories of gravity.

It is not the purpose of this review to discuss the relative merits of different models, since this has been done elsewhere (Kolb and Turner, 1990; Linde, 1990b; Olive, 1990; Liddle and Lyth, 1993a). Traditionally, a specific potential with a given set of coupling constants is chosen. The theoretical predictions of the model are then compared with large-scale structure observations. The region of parameter space consistent with such observations may then be identified (Liddle and Lyth, 1993a). However, it is difficult to select a unique inflationary model by this procedure due to the large number of plausible models available.

In view of the above uncertainties and motivated by recent and forthcoming advances in observational cosmology, our aim will be to address the question of whether *direct* insight into the nature of the inflaton potential may be gained by studying the large-scale structure of the universe. We therefore assume nothing about the potential except that it leads to an epoch of inflationary expansion.

We will proceed in the remainder of this Section by reviewing a formalism that allows the classical dynamics of the scalar field during inflation to be studied in full generality. This formalism may then be employed to discuss the generation of quantum fluctuations in the inflaton and gravitational fields.

B Scalar field dynamics in inflationary cosmology

In view of the discussion in the previous Subsection, we will assume throughout this work that the universe was dominated during inflation by a single scalar field ϕ with a self-

interaction potential $V(\phi)$, the form of which it is our aim to determine. We shall further assume that gravity is adequately described by Einstein's theory of general relativity. We shall therefore employ the four-dimensional action

$$S = - \int d^4x \sqrt{-g} \left[\frac{m_{\text{Pl}}^2 R}{16\pi} - \frac{1}{2} (\nabla\phi)^2 + V(\phi) \right], \quad (2.20)$$

where R is the Ricci curvature scalar of the space-time with metric $g_{\mu\nu}$ and $g \equiv \det g_{\mu\nu}$.

Actually, these restrictions are not as strong as they seem. For example, even theories such as hybrid inflation, which feature multiple scalar fields, are usually dynamically dominated by only one degree of freedom (Linde, 1990a, 1991, 1994; Copeland et al., 1994b; Mollerach, Matarrese, and Lucchin, 1994). Many other models invoke extensions to general relativity, and much effort has been devoted to studying inflation in the Bergmann-Wagoner class of generalized scalar-tensor theories (Bergmann, 1968; Wagoner, 1970) and higher-order pure gravity theories in which the Einstein-Hilbert lagrangian is replaced with some analytic function $f(R)$ of the Ricci curvature. Such theories can normally be rewritten via a conformal transformation as general relativity plus one or more scalar fields, again with the possibility that only one such field is dynamically relevant (Higgs, 1959; Whitt, 1984; Barrow and Cotsakis, 1988; Maeda, 1989; Kalara, Kaloper, and Olive, 1990; Lidsey, 1992; Wands, 1994).

We are unable to discuss models where more than one field is dynamically important in the reconstruction context. While considerable progress has been made recently in understanding the perturbation spectra from these models (Starobinsky and Yokoyama, 1995; García-Bellido and Wands, 1995, 1996; Sasaki and Stewart, 1996; Nakamura and Stewart, 1996), the extra freedom of the second field thwarts any attempt at finding a unique reconstruction, though it is possible to find some general inequalities relating the spectra (Sasaki and Stewart, 1996). These problems arise both because there is no longer a unique trajectory, independent of initial conditions, into the minimum of the potential, and because with a second field one can generate isocurvature perturbations as well as adiabatic ones. Fortunately, it appears that it is hard, though not impossible, to keep models of this kind consistent with observation, as the density perturbations tend to be large whatever the energy scale of inflation (García-Bellido, Linde, and Wands, 1996). A completely different way of using two fields is to drive successive periods of inflation, as in the double inflation scenario (Polarski and Starobinsky, 1995 and refs therein). This can impose very sharp features in the spectra which, although rather distinctive, are not amenable to the perturbative approach that reconstruction requires.

As we saw above, the accelerated expansion during inflation causes the spatial hypersurfaces to rapidly tend towards flatness. Moreover, any initial anisotropies and inhomogeneities in the universe are washed away beyond currently observable scales by the rapid expansion. Since only the final stages of the accelerated expansion are important from an observational point of view, we can assume that the space-time metric may be described as a spatially flat FRW metric, given by

$$ds^2 = L^2(t) dt^2 - e^{2\alpha(t)} [dx^2 + dy^2 + dz^2], \quad (2.21)$$

where $L(t)$ represents the lapse function and $a(t) = e^{\alpha(t)}$ is the scale factor of the universe.

By taking this metric, we prevent ourselves from studying reconstruction in the recently discovered versions of inflation giving an open universe (Gott, 1982; Gott and Statler, 1984; Sasaki et al., 1993; Bucher et al., 1995; Linde, 1995; Linde and Mezhlumian, 1995). In fact these models have not yet been developed sufficiently to provide the information we need — in particular the gravitational wave spectrum has not been predicted — and the generalization of the reconstruction program to these models must await further developments.

Our analysis will however apply in full to low-density cosmological models where the spatial geometry is kept flat by the introduction of a cosmological constant (or similar mechanism). Our discussion is entirely focussed on the initial spectra, which are independent of the material composition of the universe at late times. Of course, in such a cosmology the details of going between these spectra and actual observables will be changed, and the impact of this on reconstruction has been studied by Turner and White (1995).

Substitution of the metric ansatz Eq. (2.21) into the theory given by Eq. (2.20) leads to an Arnowitt, Deser and Misner (ADM) (1962) action of the form

$$S = \int dt U L e^{3\alpha} \left[-\frac{3m_{\text{Pl}}^2}{8\pi} \frac{\dot{\alpha}^2}{L^2} + \frac{1}{2} \frac{\dot{\phi}^2}{L^2} - V(\phi) \right], \quad (2.22)$$

where $U \equiv \int d^3\mathbf{x}$ is the comoving volume of the universe and a dot denotes differentiation with respect to t . Without loss of generality we may normalize the comoving volume to unity.

In recent years, considerable progress in the treatment of scalar fields within the environment of the very early universe has been made. The approach we adopt in this work is to view the scalar field itself as the dynamical variable of the system (Grishchuk and Sidorav, 1988; Muslimov, 1990; Salopek and Bond, 1990, 1991; Lidsey, 1991b). This allows the Einstein-scalar field equations to be written as a set of first-order, non-linear differential equations.

The Hamiltonian constraint $\mathcal{H} = 0$ is derived by functionally differentiating the action Eq. (2.22) with respect to the non-dynamical lapse function. One arrives at the Hamilton-Jacobi equation

$$-\frac{4\pi}{3m_{\text{Pl}}^2} \left(\frac{\partial S}{\partial \alpha} \right)^2 + \left(\frac{\partial S}{\partial \phi} \right)^2 + 2e^{6\alpha} V(\phi) = 0, \quad (2.23)$$

where the momenta conjugate to α and ϕ are $p_\alpha = \partial S / \partial \alpha = -3m_{\text{Pl}}^2 e^{3\alpha} \dot{\alpha} / 4\pi L$ and $p_\phi = \partial S / \partial \phi = e^{3\alpha} \dot{\phi} / L$, respectively. This equation follows from the invariance of the theory under reparametrizations of time. The classical dynamics of this model is determined by the real, separable solution

$$S = -\frac{m_{\text{Pl}}^2}{4\pi} e^{3\alpha} H(\phi), \quad (2.24)$$

where $H(\phi)$ satisfies the differential equation (Grishchuk and Sidorav, 1988; Muslimov, 1990; Salopek and Bond, 1990, 1991)

$$\left(\frac{dH}{d\phi} \right)^2 - \frac{12\pi}{m_{\text{Pl}}^2} H^2(\phi) = -\frac{32\pi^2}{m_{\text{Pl}}^4} V(\phi). \quad (2.25)$$

In the gauge $L = 1$, substitution of ansatz Eq. (2.24) into the expressions for the conjugate momenta implies that

$$H(\phi) = \dot{\alpha} \quad ; \quad -\frac{m_{\text{Pl}}^2}{4\pi} \frac{dH}{d\phi} = \dot{\phi}. \quad (2.26)$$

Thus, $H(\phi)$ represents the Hubble expansion parameter expressed as a function of the scalar field ϕ . It follows immediately from the second of these expressions that $\dot{H} < 0$. Consequently, the physical Hubble radius H^{-1} increases with time as the inflaton field rolls down its potential. The Hubble radius can only remain constant if the inflaton field is trapped in a meta-stable false vacuum state; this is forbidden in the context of ‘old’ inflation as it can never successfully escape this state, but may be possible in the context of single-bubble open inflationary models which are outside the scope of this paper (see, for example, Sasaki et al., 1993; Bucher et al., 1994; Bucher, Goldhaber, and Turok, 1995; Linde, 1995).

The solution to Eq. (2.25) depends on an initial condition, the value of H at some initial ϕ (Salopek and Bond, 1990, 1991). If we are to obtain unique results, the late-time evolution (that is, the evolution during which the perturbations we see are generated) of H in terms of the scalar field must be independent of the initial condition chosen, and fortunately one can easily show that this is the case (Salopek and Bond, 1990; Liddle et al., 1994); the late-time behavior is governed by an inflationary ‘attractor’ solution, which is approached exponentially quickly during inflation.

The Hamilton–Jacobi formalism we have outlined is equivalent to the more familiar version of the equations of motion given by Eqs. (2.16) and (2.17) (for $k = 0$). Eq. (2.25) is equivalent to the time–time component of the Einstein field equations and therefore represents the Friedmann equation (2.16). In the form given by Eqs. (2.16) and (2.17), $\dot{\phi}$ is an initial condition at some value of t ; in the Hamilton–Jacobi formalism the equivalent freedom allows one to specify H at some initial value of ϕ .

The above analysis of the Hamilton–Jacobi formalism assumes implicitly that the value of the scalar field is a monotonically varying function of cosmic time. In particular, it breaks down if the field undergoes oscillations (though one can attempt to patch together separate solutions). As a result, this formalism is not directly suitable for investigating the dynamics of a field undergoing oscillations in a minimum of the potential, for example. However, the scalar and tensor fluctuations relevant to large-scale structure observations are generated when the field is still some distance away from the potential minimum. Moreover, the piece of the potential corresponding to these scales is relatively small, so it is reasonable to assume that the potential is a smoothly decreasing function in this regime. The scalar field will therefore roll down this part of the potential in an unambiguous fashion. In the following, we will assume, without loss of generality, that $\dot{\phi} > 0$, so that $H'(\phi) < 0$. This choice allows us to fix the sign of any prefactors that arise when square roots appear.

In principle, the Hamilton–Jacobi formalism enables us to treat the dynamical evolution of the scalar field exactly, at least at the classical level. In practice, however, the separated Hamilton–Jacobi equation, Eq. (2.25), is rather difficult to solve. On the other hand, the analysis can proceed straightforwardly once the functional form of the expansion parameter $H(\phi)$ has been determined. This suggests that one should view $H(\phi)$ as the fundamental quantity in the analysis (Lidsey, 1991a, 1993). This is in contrast to the more traditional approaches to inflationary cosmology, whereby the particle physics sector of the model —

as defined by the specific form of the inflaton potential $V(\phi)$ — is regarded as the input parameter. In the reconstruction procedure, however, the aim is to determine this quantity from observations, so one is free to choose other quantities instead. It proves convenient to express the scalar and tensor perturbation spectra in terms of $H(\phi)$ and its derivatives.

Unfortunately, exact expressions for these perturbations have not yet been derived in full generality. All calculations to date have employed some variation of the so-called ‘slow-roll’ approximation (Steinhardt and Turner, 1984; Salopek and Bond, 1990; Liddle and Lyth, 1992; Liddle et al., 1994). It is important to emphasize that there are two different versions of the slow-roll approximation, with their attendant slow-roll parameters ϵ , η , etc, depending on whether one is taking the potential or the Hubble parameter as the fundamental quantity — the differences are described in considerable detail in Liddle et al. (1994). Here we are defining them in terms of the Hubble parameter.

We represent the slow-roll approximation as an expansion in terms of quantities derived from appropriate derivatives of the Hubble expansion parameter. Since *at a given point* each derivative is independent, there are in general an infinite number of these terms. Typically, however, only the first few enter into any expressions of interest. We define the first three as¹:

$$\epsilon(\phi) \equiv \frac{3\dot{\phi}^2}{2} \left[V + \frac{1}{2}\dot{\phi}^2 \right]^{-1} = \frac{m_{\text{Pl}}^2}{4\pi} \left(\frac{H'(\phi)}{H(\phi)} \right)^2, \quad (2.27)$$

$$\eta(\phi) \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} = \frac{m_{\text{Pl}}^2}{4\pi} \frac{H''(\phi)}{H(\phi)} = \epsilon - \frac{m_{\text{Pl}} \epsilon'}{\sqrt{16\pi\epsilon}}, \quad (2.28)$$

$$\xi(\phi) \equiv \frac{m_{\text{Pl}}^2}{4\pi} \left(\frac{H'(\phi)H'''(\phi)}{H^2(\phi)} \right)^{1/2} = \left(\epsilon\eta - \left(\frac{m_{\text{Pl}}^2 \epsilon}{4\pi} \right)^{1/2} \eta' \right)^{1/2}. \quad (2.29)$$

One need not be concerned as to the sign of the square root in the definition of ξ ; it turns out that only ξ^2 , and not ξ itself, will appear in our formulae (Liddle et al., 1994). We emphasize that the choice $\dot{\phi} > 0$ implies that $\sqrt{\epsilon} = -\sqrt{m_{\text{Pl}}^2/4\pi} H'/H$.

Modulo a constant of proportionality, ϵ measures the relative contribution of the field’s kinetic energy to its total energy density. The quantity η , on the other hand, measures the ratio of the field’s acceleration relative to the friction acting on it due to the expansion of the universe. The slow-roll approximation applies when these parameters are small in comparison to unity, i. e. $\{\epsilon, |\eta|, |\xi|\} \ll 1$; this corresponds to being able to neglect the first term in Eq. (2.25) and its first few derivatives. Inflation proceeds when the scale factor accelerates, $\ddot{a} > 0$, and this is precisely equivalent to the condition $\epsilon < 1$. Inflation ends once ϵ exceeds unity. It is interesting that the conditions leading to a violation of the strong energy condition are uniquely determined by the magnitude of ϵ alone. In principle, inflation can still proceed if $|\eta|$ or $|\xi|$ are much larger than unity, though normally such values would drive a rapid variation of ϵ and bring about a swift end to inflation.

¹Note that the definition of the third parameter is different to that made in CKLL2, $\xi_{\text{CKLL2}} = (m_{\text{Pl}}^2/4\pi)H'''/H'$. The two are related by $\xi^2 = \epsilon\xi_{\text{CKLL2}}$. The former definition has proven awkward; because of the derivative on the denominator it need not be small in the scale-invariant limit (though the combination $\sqrt{\epsilon\xi_{\text{CKLL2}}}$ must be). We choose to use this better definition, as introduced by Liddle et al. (1994) who give further details and a collection of useful formulae.

For specific results, we shall not go beyond these three parameters. However, in general one can define a full hierarchy of slow-roll parameters (Liddle et al., 1994):

$$\begin{aligned}\beta_n &\equiv \left\{ \prod_{i=1}^n \left[-\frac{d \ln H^{(i)}}{d \ln a} \right] \right\}^{\frac{1}{n}}, \\ &= \frac{m_{\text{Pl}}^2}{4\pi} \left(\frac{(H')^{n-1} H^{(n+1)}}{H^n} \right)^{\frac{1}{n}},\end{aligned}\tag{2.30}$$

where $\beta_1 \equiv \eta$, $\beta_2 \equiv \xi$, etc, and a superscript (m) indicates the m -th derivative with respect to ϕ . The ϵ parameter has to be defined separately, though it may be referred to as β_0 .

These slow-roll parameters, along with analogues defined in terms of the potential, can be used as the basis for a slow-roll expansion to derive arbitrarily accurate solutions given a particular choice of potential. However, this formalism is not necessary when making general statements about inflation without demanding a specific potential.

The amount of inflationary expansion within a given timescale is most easily parametrized in terms of the number of e -foldings that occur as the scalar field rolls from a particular value ϕ to its value ϕ_e when inflation ends:

$$N(\phi, \phi_e) \equiv \int_t^{t_e} H(t) dt = -\frac{4\pi}{m_{\text{Pl}}^2} \int_{\phi}^{\phi_e} d\phi \frac{H(\phi)}{H'(\phi)}.\tag{2.31}$$

Thus, with the help of Eq. (2.31), we may relate the value of the scale factor $a(\phi) = e^{\alpha(\phi)}$ at any given epoch during inflation directly to the value of the scale factor at the end of inflation, a_e :

$$a(\phi) = a_e \exp[-N(\phi)].\tag{2.32}$$

An extremely useful formula is that which connects the two epochs at which a given scale equals the Hubble radius, the first during inflation when the scale crosses outside and the second much nearer the present when the scale crosses inside again. A comoving scale k crosses outside the Hubble radius at a time which is $N(k)$ e -foldings from the end of inflation, where

$$N(k) = 62 - \ln \frac{k}{a_0 H_0} - \ln \frac{10^{16} \text{ GeV}}{V_k^{1/4}} + \ln \frac{V_k^{1/4}}{V_{\text{end}}^{1/4}} - \frac{1}{3} \ln \frac{V_{\text{end}}^{1/4}}{\rho_{\text{reh}}^{1/4}}.\tag{2.33}$$

The subscript ‘0’ indicates present values; the subscript ‘ k ’ specifies the value when the wave number k crosses the Hubble radius during inflation ($k = aH$); the subscript ‘end’ specifies the value at the end of inflation; and ρ_{reh} is the energy density of the universe after reheating to the standard hot big bang evolution. This calculation assumes that instantaneous transitions occur between regimes, and that during reheating the universe behaves as if matter-dominated.

It is fairly standard to make a generic assumption about the number of e -foldings before the end of inflation at which the scale presently equal to the Hubble radius crossed outside during inflation; most commonly one sees this number taken as either 50 or 60. Within the context of making predictions from a given potential this can have a slight effect on results, but it is completely unimportant as regards reconstruction.

What we do need for reconstruction is a measure of how rapidly scales pass outside the Hubble-radius as compared to the evolution of the scalar field; this is essential for calculating such quantities as the spectral indices of scalar and tensor perturbations. The formal definition we take of a scale matching the Hubble radius is that $k = aH$. Then one can write

$$k(\phi) = a_e H(\phi) \exp[-N(\phi)], \quad (2.34)$$

where $N(\phi)$ is given by Eq. (2.31). Differentiating with respect to ϕ therefore yields

$$\frac{d \ln k}{d\phi} = \frac{4\pi}{m_{\text{Pl}}^2} \frac{H}{H'} (\epsilon - 1). \quad (2.35)$$

This concludes our discussion on the classical dynamics of the scalar field during inflation. In the following Section, we will proceed to discuss the consequences of quantum fluctuations that arise in both the inflaton and graviton fields.

III The Quantum Generation of Perturbations

During inflation, the inflaton and graviton fields undergo quantum-mechanical fluctuations. The most important observational consequences of the inflationary scenario derive from the significant effects these perturbations may have on the large-scale structure of the universe at the present epoch. In this Section we shall discuss how these fluctuations arise and present expressions for their expected amplitudes. Since the inflaton and gravitational perturbations are produced in a similar fashion, we shall begin with a qualitative description of the effects of the former. We shall then proceed with an extensive account of the calculation of both spectra by Stewart and Lyth (1993), which is the most accurate analytic treatment presently available.

A Qualitative discussion

Fluctuations in the inflaton field lead to a stochastic spectrum of density (scalar) perturbations (Guth and Pi, 1982; Hawking, 1982; Linde, 1982b; Starobinsky, 1982; Bardeen et al., 1983; Lyth, 1985; Mukhanov, 1985; Sasaki, 1986; Mukhanov, 1989; Salopek, Bond, and Bardeen, 1989). Physically, these arise because the inflaton field reaches the global minimum of its potential at different times in different places in the universe. This results in a time shift in how quickly the rollover occurs. Thus, constant $\delta\rho$ does not correspond to a constant-time hypersurface; in other words, there is a density distribution produced by the kinetic energy of the inflaton field for a given constant-time hypersurface. It is widely thought that these density perturbations result in the formation of large-scale structure in the universe via the process of gravitational instability. They may also be responsible for anisotropic structure in the temperature distribution of the cosmic microwave background radiation.

Typically, the inflationary scenario predicts that the spectrum of density perturbations should be gaussian and scale-dependent. This is certainly true for the class of models that we shall be considering here, in which it is assumed that the inflaton field is weakly coupled. However, one should bear in mind that the prediction of gaussianity is not generic to all

inflationary models; it is possible to contrive models with nongaussian perturbations by introducing features in just the right part of the inflationary potential (Allen, Grinstein, and Wise, 1987; Salopek and Bond, 1990).

The historical viewpoint on the scale-dependence of the fluctuations was that they were of scale-invariant (Harrison–Zel’dovich) form, though it had been recognized that the scale-invariance was only approximate (Bardeen et al., 1983; Lucchin and Matarrese, 1985a). This is because the scalar field must be undergoing some kind of evolution if inflation is to end eventually, and this injects a scale-dependence into the spectra. As we shall see, this effect should be easy to measure.

To take advantage of accurate observations, it is imperative that the spectra be calculated as accurately as possible. However, let us first make a qualitative discussion of the generation mechanism.

In a spatially flat, isotropic and homogeneous universe, the Hubble radius, $H^{-1}(t)$, represents the scale beyond which causal processes cannot operate. The relative size of a given scale to this quantity is of crucial importance for understanding how the primordial spectrum of fluctuations is generated. Quantities such as the power spectrum are defined via a Fourier expansion as functions of *comoving* wavenumber k , and the combination k/aH appears in many equations. Different physical behavior occurs depending on whether this quantity is much greater or smaller than unity.

Inflation is defined as an epoch during which the scale factor accelerates, and so the comoving Hubble radius, $(aH)^{-1}$, must necessarily decrease. This is an important feature of the inflationary scenario, because it means that physical scales will grow more rapidly than the Hubble radius. As a result, a given mode will start within the Hubble radius. In this regime the expansion is negligible and the microphysics in operation at that epoch will therefore be relevant. This is determined by the usual flat-space quantum field theory for which the vacuum state of the scalar field fluctuations is well understood. As the inflationary expansion proceeds, however, the mode grows much more rapidly than the Hubble radius (in physical coordinates) and soon passes outside it. One can utilize a Heisenberg picture of quantum theory to say that the operators obey the classical equations of motion, and so the evolution of the vacuum state can be followed until it crosses outside the Hubble radius. At this point the microphysics effectively becomes ‘frozen’. It turns out that the asymptotic state is not a zero-particle state — particles are created by the gravitational field. Corresponding perturbations in the gravitational field itself are also generated, so a spectrum of gravitational wave (tensor) fluctuations is independently produced by the same mechanism.

Once inflation is over, the comoving Hubble radius begins to grow. Eventually, therefore, the mode in question is able to come back inside the Hubble radius some time after inflation. The overall result is that perturbations arising from fluctuations in the inflaton field can be imprinted onto a given length scale during the inflationary epoch when that scale first leaves the Hubble radius. These will be preserved whilst the mode is beyond the Hubble radius and will therefore be present when the scale re-enters during the radiation-dominated or matter-dominated eras.

B Quantitative analysis

If one is to take full advantage of the observations to the extent one hopes, it is crucial to have extremely accurate predictions for the spectra induced by different inflationary models. For example, microwave background theorists have set themselves the stringent goal of calculating the radiation angular power spectrum (the C_l discussed later in this paper) to within one percent (Hu et al., 1995), in the hope that satellite observations may one day provide extremely accurate measurements of the anisotropies across a wide range of angular scales (Tegmark and Efstathiou, 1996). This involves a detailed treatment with a host of subtle physical effects. If inflationary models are to capitalize on this sort of accuracy, it is essential to have as accurate a determination as possible of the initial spectra which are to be input into such calculations. Given that the slow-roll parameters are typically at least a few percent, that implies that a determination of the spectra to at least one order beyond leading order in the slow-roll expansion is desired.

The calculations we make are based on linear perturbation theory. Since the observed anisotropies are small, this approximation is considerably more accurate than the slow-roll approximation, and we need not attempt to go beyond it, though it is possible to extend calculations beyond linear perturbation theory (Durrer and Sakellariadou, 1994).

Before proceeding, however, let us clarify a notational point. In earlier literature, especially CKLL2 and Liddle and Turner (1994), orders were referred to as first-order, second-order etc. However, we feel this can be misleading, because it might suggest that all terms containing say two slow-roll parameters in any given expression are supposed to be neglected. This is not the intention, because in many expressions the lowest-order term already contains one or more powers of the slow-roll parameters. Because differentiation respects the order-by-order expansion, while multiplying each term by a slow-roll parameter, it is always valid to take terms to the same number of orders, however many slow-roll parameters the actual terms possess. Therefore, in order to clarify the meaning, we choose to always employ the phrase lowest-order to indicate the term containing the least number of powers of the slow-roll parameters, however many that may be for a specific expression. The phrase next-to-lowest order, abbreviated to next-order, then indicates correction terms to this which contain one further power of the slow-roll parameters than the lowest-order terms.

The calculation of the spectra to next-order has been provided by Stewart and Lyth (1993). Because of its crucial importance, we shall devote quite some time to describing it. The basic principle is to start with the one known situation where the spectra can be calculated exactly, that of power-law inflation. This corresponds to each of the slow-roll parameters having the same constant value. To next-order, a general inflationary potential can be considered via an expansion in $(\epsilon - \eta)$ about a power-law inflation model with the same ϵ ; as we shall see, it is an adequate approximation to treat ϵ and η as different constant values.

In fact, the logic we develop is slightly different to that of Stewart and Lyth (1993). They computed an exact solution for the situation where ϵ and η are treated as exactly constant with different values. Formally, this situation does not exist as ϵ precisely constant implies $\epsilon = \eta$. They then treated power-law inflation as an exact special case of this situation, and a general inflation model to next-order as an expansion about their more general result.

Logically, it is more accurate to expand directly about the exact power-law inflation result, but nevertheless the final answer is guaranteed to be the same.

1 Scalar perturbations

Throughout the calculations to derive the spectra of scalar and tensor fluctuations, the space-time representing our universe is decoupled into two components, representing the background and perturbation contributions. The background part is taken to be the homogeneous and isotropic FRW metric. This is a reasonable assumption to make in view of the high degree of spatial uniformity in the temperature of the cosmic microwave background. In this paper we assume the background is also spatially flat with a line element given by Eq. (2.21). The perturbed sector of the metric then determines by how much the actual universe deviates from this idealization.

Four quantities are required to specify the general nature of a scalar perturbation. These may be denoted by A , B , Ψ and E and these are functions of the space and time coordinates. It has been shown by Bardeen (1980) and by Kodama and Sasaki (1984) that the most general form of the line element for the background and scalar metric perturbations is given by

$$ds^2 = a^2(\tau) \left[(1 + 2A)d\tau^2 - 2\partial_i B dx^i d\tau - [(1 - 2\Psi)\delta_{ij} + 2\partial_i \partial_j E] dx^i dx^j \right], \quad (3.1)$$

where $\tau \equiv \int dt/a(t)$ is conformal time.

The perturbations can be measured by the intrinsic curvature perturbation of the co-moving hypersurfaces, which has the form

$$\mathcal{R} = -\Psi - \frac{H}{\dot{\phi}} \delta\phi, \quad (3.2)$$

during inflation, where $\delta\phi$ represents the fluctuation of the inflaton field and $\dot{\phi}$ and H are calculated from the background field equations Eqs. (2.25)-(2.26). To proceed, we follow Mukhanov, Feldman and Brandenberger (1992) and introduce the gauge-invariant potential

$$u \equiv a \left[\delta\phi + \frac{\dot{\phi}}{H} \Psi \right]. \quad (3.3)$$

It also proves convenient to introduce the variable

$$z \equiv \frac{a\dot{\phi}}{H}, \quad (3.4)$$

and it follows immediately that

$$u = -z\mathcal{R}. \quad (3.5)$$

The evolution of the perturbations is determined by the Einstein action. The first-order perturbation equations of motion are given by a second-order action. Hence, the gravitational and matter sectors are separated and each expanded to second-order in the perturbations. The result for the gravitational component is simplified by employing the ADM form of the action (Arnowitt, Deser, and Misner, 1962; Misner et al., 1973). The

action for the matter perturbations, on the other hand, can be calculated by expanding the Lagrangian as a Taylor series about a fixed value of the scalar field, applying the background field equations and integrating by parts. Mukhanov et al. (1992) show that the full action for linear scalar perturbations is given by

$$S = \int d^4x \mathcal{L} = \frac{1}{2} \int d\tau d^3\mathbf{x} \left[(\partial_\tau u)^2 - \delta^{ij} \partial_i u \partial_j u + \frac{z_{\tau\tau}}{z} u^2 \right], \quad (3.6)$$

where a subscript τ denotes partial differentiation with respect to conformal time. For further details the reader is referred to Mukhanov (1989) and Makino and Sasaki (1991).

Formally, this is equivalent to the action for a scalar field in flat space-time with a time-dependent effective mass $m^2 = -z_{\tau\tau}/z$. This equivalence implies that one can consider the quantum theory in an analogous fashion to that of a scalar field propagating on Minkowski space-time in the presence of a time-varying external field (Grib, Mamaev, and Mostepanenko, 1980). The time-dependence has its origin in the variation of the background space-time (Birrell and Davies, 1982).

The momentum canonical to u is given by

$$\pi(\tau, \mathbf{x}) = \frac{\partial \mathcal{L}}{\partial(u_\tau)} = u_\tau(\tau, \mathbf{x}), \quad (3.7)$$

and the theory is then quantized by promoting u and its conjugate momentum to operators that satisfy the following commutation relations on the $\tau = \text{constant}$ hypersurfaces:

$$[\hat{u}(\tau, \mathbf{x}), \hat{u}(\tau, \mathbf{y})] = [\hat{\pi}(\tau, \mathbf{x}), \hat{\pi}(\tau, \mathbf{y})] = 0, \quad (3.8)$$

$$[\hat{u}(\tau, \mathbf{x}), \hat{\pi}(\tau, \mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}). \quad (3.9)$$

We expand the operator $\hat{u}(\tau, \mathbf{x})$ in terms of plane waves

$$\hat{u}(\tau, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left[u_k(\tau) \hat{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + u_k^*(\tau) \hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}} \right], \quad (3.10)$$

and the field equation for the coefficients u_k is derived by setting the variation of the action Eq. (3.6) with respect to u equal to zero. It is given by (Mukhanov, 1985, 1988; Stewart and Lyth, 1993)

$$\frac{d^2 u_k}{d\tau^2} + \left(k^2 - \frac{1}{z} \frac{d^2 z}{d\tau^2} \right) u_k = 0. \quad (3.11)$$

These modes are normalized so that they satisfy the Wronskian condition

$$u_k^* \frac{du_k}{d\tau} - u_k \frac{du_k^*}{d\tau} = -i, \quad (3.12)$$

and this condition ensures that the creation and annihilation operators $\hat{a}_{\mathbf{k}}^\dagger$ and $\hat{a}_{\mathbf{k}}$ satisfy the usual commutation relations for bosons:

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{l}}] = [\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{l}}^\dagger] = 0, \quad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{l}}^\dagger] = \delta^{(3)}(\mathbf{k} - \mathbf{l}). \quad (3.13)$$

The vacuum is therefore defined as the state that is annihilated by all the $\hat{a}_{\mathbf{k}}$, i. e., $\hat{a}_{\mathbf{k}}|0\rangle = 0$.

The modes $u_k(\tau)$ must have the correct form at very short distances so that ordinary flat space-time quantum field theory is reproduced. Thus, in the limit that $k/aH \rightarrow \infty$, the modes should approach plane waves of the form

$$u_k(\tau) \rightarrow \frac{1}{\sqrt{2k}} e^{-ik\tau}. \quad (3.14)$$

In the opposite (long wavelength) regime where k can be neglected in Eq. (3.11), we see immediately that the growing mode solution is

$$u_k \propto z, \quad (3.15)$$

with no dependence on the behavior of the scale factor (except insofar as implicitly through the definition of z).

Ultimately, the quantity in which we are interested is the curvature perturbation \mathcal{R} . We expand this in a Fourier series

$$\mathcal{R} = \int \frac{d^3 \mathbf{k}}{(2\pi)^{3/2}} \mathcal{R}_{\mathbf{k}}(\tau) e^{i\mathbf{k}\cdot\mathbf{x}}. \quad (3.16)$$

The power spectrum $\mathcal{P}_{\mathcal{R}}(k)$ can then be defined in terms of the vacuum expectation value

$$\langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{l}}^* \rangle = \frac{2\pi^2}{k^3} \mathcal{P}_{\mathcal{R}} \delta^{(3)}(\mathbf{k} - \mathbf{l}), \quad (3.17)$$

where the prefactor is in a sense arbitrary but is chosen to obey the usual Fourier conventions. The left-hand side of this expression may be evaluated by combining Eqs. (3.5), (3.13) and (3.16):

$$\langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{l}}^* \rangle = \frac{1}{z^2} |u_k|^2 \delta^{(3)}(\mathbf{k} - \mathbf{l}), \quad (3.18)$$

yielding

$$\mathcal{P}_{\mathcal{R}}^{1/2}(k) = \sqrt{\frac{k^3}{2\pi^2}} \left| \frac{u_k}{z} \right|. \quad (3.19)$$

For modes well outside the horizon, the growing mode of u_k will dominate and so the spectrum will approach a constant value. It is this value that we are aiming to calculate.

In order to provide a solution, we need an expression for $z_{\tau\tau}/z$. This can be straightforwardly obtained as

$$\frac{1}{z} \frac{d^2 z}{d\tau^2} = 2a^2 H^2 \left[1 + \epsilon - \frac{3}{2}\eta + \epsilon^2 - 2\epsilon\eta + \frac{1}{2}\eta^2 + \frac{1}{2}\xi^2 \right], \quad (3.20)$$

and despite its appearance as an expansion in slow-roll parameters, this expression is exact.

Exact solution for power-law inflation

So far, all the expressions we have written down have been exact. However, we have reached the limit of analytic progress for general circumstances. The desired situation then is to obtain an exact solution for some special case, about which a general expansion can be

applied in terms of the slow-roll parameters. Such an exact solution is the case of power-law inflation, which we now derive².

Power-law inflation, where the scale factor expands as $a(t) \propto t^p$, corresponds to the particularly simple case where the Hubble parameter is exponential in ϕ (Lucchin and Matarrese, 1985a, 1985b):

$$H(\phi) \propto \exp\left(\sqrt{\frac{4\pi}{p}} \frac{\phi}{m_{\text{Pl}}}\right). \quad (3.21)$$

It follows that the slow-roll parameters are not only constant but equal; we are primarily interested in

$$\epsilon = \eta = \xi = \frac{1}{p}. \quad (3.22)$$

With a constant ϵ , an integration by parts

$$\tau = \int \frac{da}{a^2 H} = -\frac{1}{aH} + \int \frac{\epsilon da}{a^2 H}, \quad (3.23)$$

supplies the conformal time as

$$\tau = -\frac{1}{aH} \frac{1}{1-\epsilon}. \quad (3.24)$$

Thus, τ is negative during inflation, with $\tau = 0$ corresponding to the infinite future.

Since the slow-roll parameters in Eq. (3.20) are constant, Eq. (3.11) simplifies to a Bessel equation of the form

$$\left[\frac{d^2}{d\tau^2} + k^2 - \frac{(\nu^2 - \frac{1}{4})}{\tau^2} \right] u_k = 0, \quad (3.25)$$

where

$$\nu \equiv \frac{3}{2} + \frac{1}{p-1}. \quad (3.26)$$

The appropriately normalized solution with the correct asymptotic behavior at small scales is therefore given by³

$$u_k(\tau) = \frac{\sqrt{\pi}}{2} e^{i(\nu+1/2)\pi/2} (-\tau)^{1/2} H_\nu^{(1)}(-k\tau), \quad (3.27)$$

where $H_\nu^{(1)}$ is the Hankel function of the first kind of order ν .

Ultimately, we are interested in the asymptotic form of the solution once the mode is well outside the horizon. Taking the limit $k/aH \rightarrow 0$ yields the asymptotic form

$$u_k \rightarrow e^{i(\nu-1/2)\pi/2} 2^{\nu-3/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{1}{\sqrt{2k}} (-k\tau)^{-\nu+1/2}, \quad (3.28)$$

and substituting this into Eq. (3.19) gives the asymptotic form of the power spectrum

$$\mathcal{P}_{\mathcal{R}}^{1/2}(k) = 2^{\nu-1/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} (\nu-1/2)^{1/2-\nu} \frac{1}{m_{\text{Pl}}^2} \frac{H^2}{|H'|} \Big|_{k=aH}, \quad (3.29)$$

²It is at this point that our construction of the expansion begins to differ in logical construction from Stewart and Lyth (1993), though the final result will agree.

³The choice of phase factor ensures that the behavior described by Eq. (3.14) is reproduced at short scales, and the factor of $\sqrt{\pi}/2$ implies that condition Eq. (3.12) is satisfied.

where we have employed Eq. (3.24) to substitute for $k\tau$. A subtle point is that, despite the appearance of this equation, the calculated value for the spectrum is *not* the value when the scale crosses outside the Hubble radius. Rather, it is the asymptotic value as $k/aH \rightarrow 0$, but rewritten in terms of the values which quantities had at Hubble radius crossing.

This exact expression for the asymptotic power spectrum was first derived in an earlier paper by Lyth and Stewart (1992). It is one of only two known exact solutions, and is the only one for a realistic inflationary scenario. The other known exact solution, found by Easther (1996), arises in an artificial model designed to permit exact solution, and while of theoretical interest is excluded by observations.

Slow-roll expansion for general potentials

Having obtained an exact solution, we can now make an expansion about it. The power-law inflation case corresponded to the slow-roll parameters being equal, and hence exactly constant; we now wish to allow them to be different which means they will pick up a time dependence.

At this stage, there is no need to require that the parameter ϵ be small, for the exact solution exists for all $\epsilon < 1$. However, the deviation of all higher slow-roll parameters from ϵ must indeed be small, since the differences vanish for the exact solution. Let us label the first of these as $\zeta = \epsilon - \eta$. There are in general an infinite number of such small parameters in the expansion but we shall only need this one.

The first step is to find a more general equation for τ . By integrating by parts in the manner of Eq. (3.23) an infinite number of times, one can obtain

$$\tau = -\frac{1}{aH} \frac{1}{1-\epsilon} - \frac{2\epsilon\zeta}{aH} + \text{expansion in slow-roll parameters } \zeta \text{ etc.}, \quad (3.30)$$

where ϵ can now have arbitrary time dependence. This is all very well, but even via an expansion in small ζ one cannot analytically solve Eq. (3.11) for a general time-dependent ϵ ; we must resort to a situation where $aH\tau$ can be taken as constant for each k -mode (though not necessarily the same constant for different k). The relevant equation to study is the exact relation

$$\dot{\epsilon}/H = 2\epsilon\zeta. \quad (3.31)$$

What we are aiming to do is to shift the time dependence of ϵ to next-order in the expansion, so that it can be neglected. This is achieved by assuming that ϵ is a small parameter as well as ζ (that is, that both ϵ and η are small), in which case one can expand to lowest-order to get

$$\tau = -\frac{1}{aH}(1 + \epsilon). \quad (3.32)$$

We will return to the question of the error in assuming constant ϵ shortly.

Having this expression for τ , we can now immediately use Eq. (3.20), which must also be truncated to first-order. This gives the same Bessel equation Eq. (3.25), but now with ν given by

$$\nu = \frac{3}{2} + 2\epsilon - \eta. \quad (3.33)$$

The assumption that treats ϵ as constant also allows η to be taken as constant, but crucially, ϵ and η need no longer be the same since we are consistent to first-order in their difference. The differences between further slow-roll parameters and ϵ lead to higher order effects, and so incorporating ϵ and η in this manner is applicable to an arbitrary inflaton potential to next-order. The same solution Eq. (3.29) can be used with the new form of ν , but for consistency it should be expanded to the same order. This gives the final answer, which is true for general inflation potentials to this order, of (Stewart and Lyth, 1993)

$$P_{\mathcal{R}}^{1/2}(k) = [1 - (2C + 1)\epsilon + C\eta] \frac{2}{m_{\text{Pl}}^2} \frac{H^2}{|H'|} \Big|_{k=aH}, \quad (3.34)$$

where $C = -2 + \ln 2 + \gamma \simeq -0.73$ is a numerical constant, γ being the Euler constant originating in the expansion of the Gamma function. Since the slow-roll parameters are to be treated as constant, they can also be evaluated at horizon crossing.

Let us now return to the question of the error in assuming ϵ is constant. The crucial aspect is that the variation of ϵ is only important around $k = aH$. In either of the two extreme regimes the evolution of u_k (in relation to z) is independent of it (Eqs. (3.14) and (3.15)). Assuming the variation of ϵ is only important for some unspecified but finite number of e -foldings, Eq. (3.31) measures that change (per e -folding). As long as we are assuming ϵ small as well as ζ , that change is next-order and can be neglected along with all the other next-order terms we did not attempt to include.

Finally, one can see from the complexity of this calculation the obstacles to obtaining general expressions which go to yet another higher order. This would involve finding some way of solving the Bessel-like equation in the situation where its coefficients could not be treated as constant.

This concludes our discussion on the generation of scalar perturbations during inflation. In the remainder of this Section we will present the analogous result for the tensor fluctuations.

2 Gravitational waves

The propagation of weak gravitational waves on the FRW background was investigated by Lifshitz (1946). Quantum fluctuations in the gravitational field are generated in a similar fashion to that of the scalar perturbations discussed above. A gravitational wave may be viewed as a ripple in the background space-time metric Eq. (2.21) and in general the linear tensor perturbations may be written as $g_{\mu\nu} = a^2(\tau)[\eta_{\mu\nu} + h_{\mu\nu}]$, where $|h_{\mu\nu}| \ll 1$ denotes the metric perturbation and $\eta_{\mu\nu}$ is the flat space-time metric (Bardeen, 1980; Kodama and Sasaki, 1984). In the transverse-traceless gauge, we have $h_{00} = h_{0i} = \partial^i h_{ij} = \delta^{ij} h_{ij} = 0$, and there are two independent polarization states (Misner et al., 1973). These are usually denoted as $\lambda = +, \times$.

The gravitons are the propagating modes associated with these two states. The classical dynamics of the gravitational waves is determined by expanding the Einstein-Hilbert action to quadratic order in $h_{\mu\nu}$ and it can be shown that this action takes the form (Grishchuk, 1974, 1977)

$$S_g = \frac{m_{\text{Pl}}^2}{64\pi} \int d\tau d^3\mathbf{x} a^2(\tau) \partial_\mu h^i_j \partial^\mu h_i^j. \quad (3.35)$$

It proves convenient to introduce the rescaled variable

$$P^i_j(x) \equiv (m_{\text{Pl}}^2/32\pi)^{1/2} a(\tau) h^i_j(x), \quad (3.36)$$

and substitution of this expression into the action Eq. (3.35) implies that

$$S_g = \frac{1}{2} \int d\tau d^3 \mathbf{x} \left[(\partial_\tau P_i^j) (\partial^\tau P^i_j) - \delta^{rs} (\partial_r P_i^j) (\partial_s P^i_j) + \frac{a_{\tau\tau}}{a} P_i^j P^i_j \right], \quad (3.37)$$

where we have ignored a total derivative. This expression resembles the equivalent action Eq. (3.6) for the scalar perturbations. Indeed, we may interpret Eq. (3.37) as the action for two scalar fields in Minkowski space-time each with an effective mass squared given by $a_{\tau\tau}/a$. This equivalence between the two actions implies that the procedure for quantizing the tensor fluctuations is essentially the same as in the scalar case.

We perform a Fourier decomposition of the gravitational waves by expanding P^i_j :

$$P^i_j = \sum_{\lambda=+,\times} \int \frac{d^3 \mathbf{k}}{(2\pi)^{3/2}} v_{\mathbf{k},\lambda}(\tau) \epsilon^i_j(\mathbf{k}; \lambda) e^{i\mathbf{k}\cdot\mathbf{x}}. \quad (3.38)$$

In this expression $\epsilon^i_j(\mathbf{k}; \lambda)$ is the polarization tensor and satisfies the conditions $\epsilon_{ij} = \epsilon_{ji}$, $\epsilon_{ii} = 0$, $k^i \epsilon_{ij} = 0$ and $\epsilon^i_j(\mathbf{k}, \lambda) \epsilon_i^{j*}(\mathbf{k}, \lambda') = \delta_{\lambda\lambda'}$. The analysis is further simplified if we choose $\epsilon_{ij}(-\mathbf{k}, \lambda) = \epsilon_{ij}^*(\mathbf{k}, \lambda)$, since this ensures that $v_{\mathbf{k},\lambda} = v_{-\mathbf{k},\lambda}^*$. We may consider each polarization state separately. The effective graviton action during inflation therefore takes the form

$$S_g = \frac{1}{2} \sum_{\lambda=+,\times} \int d\tau d^3 \mathbf{k} \left[(\partial_\tau |v_{\mathbf{k},\lambda}|)^2 - \left(k^2 - \frac{a_{\tau\tau}}{a} \right) |v_{\mathbf{k},\lambda}|^2 \right]. \quad (3.39)$$

We quantize by interpreting $v_{\mathbf{k},\lambda}(\tau)$ as the operator

$$\hat{v}_{\mathbf{k},\lambda}(\eta) = v_k(\eta) \hat{a}_{\mathbf{k},\lambda} + v_k^*(\eta) \hat{a}_{-\mathbf{k},\lambda}^\dagger, \quad (3.40)$$

where the modes v_k satisfy the normalization condition Eq. (3.12) and have the form given by Eq. (3.14) as $aH/k \rightarrow 0$. This ensures that the creation and annihilation operators satisfy

$$[\hat{a}_{\mathbf{k},\lambda} \hat{a}_{\mathbf{l},\sigma}^\dagger] = \delta_{\lambda\sigma} \delta^{(3)}(\mathbf{k} - \mathbf{l}), \quad \hat{a}_{\mathbf{k},\lambda} |0\rangle, \quad (3.41)$$

and the spectrum of gravitational waves $\mathcal{P}_g(k)$ is then defined by

$$\langle \hat{v}_{\mathbf{k},\lambda} \hat{v}_{\mathbf{l},\lambda}^* \rangle = \frac{m_{\text{Pl}}^2 a^2}{32\pi} \frac{2\pi^2}{k^3} \mathcal{P}_g \delta^{(3)}(\mathbf{k} - \mathbf{l}). \quad (3.42)$$

The field equation for u_k , derived by varying the action Eq. (3.39), is

$$\frac{d^2 v_k}{d\tau^2} + \left(k^2 - \frac{1}{a} \frac{d^2 a}{d\tau^2} \right) v_k = 0, \quad (3.43)$$

and the scale factor term can be written as

$$\frac{1}{a} \frac{d^2 a}{d\tau^2} = 2a^2 H^2 \left(1 - \frac{1}{2} \epsilon \right). \quad (3.44)$$

This puts us in a very similar situation to that for the density perturbations. The situation is simplified since a appears directly in the equation of motion rather than z , but the strategy is exactly the same.

For power-law inflation we can again solve exactly by writing

$$\frac{a_{\tau\tau}}{a} = \frac{1}{\tau^2} \left(\mu^2 - \frac{1}{4} \right), \quad (3.45)$$

where

$$\mu \equiv \frac{3}{2} + \frac{1}{p-1}. \quad (3.46)$$

For power-law inflation ν and μ coincide, though in general they do not. The appropriate solution for v_k is given by Eq. (3.27), as before, after replacing ν with μ . It follows, therefore, that

$$\mathcal{P}_g^{1/2}(k) = \frac{2}{\sqrt{\pi}} 2^{\mu-1/2} \frac{\Gamma(\mu)}{\Gamma(3/2)} (\mu - 1/2)^{1/2-\mu} \frac{H}{m_{\text{Pl}}} \Big|_{k=aH}, \quad (3.47)$$

where \mathcal{P}_g has been multiplied by a factor of 2 to account for the two polarization states. This exact solution was first obtained by Abbott and Wise (1984a) and we note that for power-law inflation

$$\frac{\mathcal{P}_g^{1/2}}{\mathcal{P}_{\mathcal{R}}^{1/2}} = \frac{4}{\sqrt{p}} = 4\sqrt{\epsilon}. \quad (3.48)$$

The final step is to carry out the expansion in the same way as in the scalar case to yield the slow-roll expression for the tensor spectrum. This gives

$$\mu = \frac{3}{2} + \epsilon, \quad (3.49)$$

and hence

$$\mathcal{P}_g^{1/2}(k) = [1 - (C+1)\epsilon] \frac{4}{\sqrt{\pi}} \frac{H}{m_{\text{Pl}}} \Big|_{k=aH}. \quad (3.50)$$

IV Lowest-Order Reconstruction

In the previous section we discussed the derivation of expressions for the two *initial* spectra $\mathcal{P}_{\mathcal{R}}^{1/2}$ and $\mathcal{P}_g^{1/2}$, which were accurate to next-order in the slow-roll parameters. Before proceeding, let's relate our notation to other notations that the reader may be familiar with, which concern the present-day spectra. In order to derive these, one needs the transfer functions $T(k)$ and $T_g(k)$ for both scalars (Efstathiou, 1990) and tensors (Turner, White, and Lidsey, 1993) respectively, which describe the suppression of growth on scale k relative to the infinite wavelength mode. The transfer functions in general depend on a whole range of cosmological parameters, as discussed later. The present-day spectrum of density perturbations, denoted $P(k)$, is given by

$$\frac{k^3}{2\pi^2} P(k) = \left(\frac{k}{aH} \right)^4 T^2(k) \mathcal{P}_{\mathcal{R}}(k), \quad (4.1)$$

while the energy density (per octave) in gravitational waves is

$$\Omega_g(k) = \frac{1}{24} T_g^2(k) \mathcal{P}_g(k). \quad (4.2)$$

These expressions apply to a critical density universe; for models with a cosmological constant they require generalization (see Turner and White, 1996). Note though that in the following Sections, we shall always be working with (rescaled versions of) the initial spectra, and not with the present-day spectra.

In this Section, we shall concentrate on the lowest-order situation, where all expressions are truncated at the lowest-order. This is not equivalent to assuming that ϵ and η are zero, for in some expressions, such as the spectral indices, the lowest-order terms contain ϵ and η , as we shall see. This approximation can be regarded as being extremely useful for the present state of observations. However, optimistically one hopes that future observations, particularly satellite-based high resolution microwave background anisotropy observations, will require a higher degree of accuracy as discussed in Section V.

A The consistency equation and generic predictions of inflation

In the forthcoming analysis it will prove convenient to work with rescaled expressions for the spectra $\mathcal{P}_{\mathcal{R}}^{1/2}$ and $\mathcal{P}_g^{1/2}$ which we will use throughout the rest of the paper. To lowest-order we obtain

$$A_S(k) \equiv 2\mathcal{P}_{\mathcal{R}}^{1/2}/5 = \frac{4}{5} \frac{H^2}{m_{\text{Pl}}^2 |H'|} \Big|_{k=aH}, \quad (4.3)$$

$$A_T(k) \equiv \mathcal{P}_g^{1/2}/10 = \frac{2}{5\sqrt{\pi}} \frac{H}{m_{\text{Pl}}} \Big|_{k=aH}. \quad (4.4)$$

The specific choice of normalizations is arbitrary⁴. The above choice ensures that A_S coincides precisely with the quantity δ_H as defined by Liddle and Lyth (1993a, Eq. (3.6)). This parameter may be viewed as the density contrast at Hubble-radius-crossing. The normalization for the tensor spectrum is then chosen so that to lowest-order $\epsilon = A_T^2/A_S^2$.

During inflation the scalar field slowly rolls down its self-interaction potential. This causes the Hubble parameter to vary as a function of cosmic time and therefore with respect to the scale at Hubble-radius crossing. The expressions for the perturbations therefore acquire a dependence on scale and it is conventional to express this variation in terms of *spectral indices*. In general, these indices are themselves functions of scale and there appear to be two ways in which they may be defined. In the first case, one may simply write the power spectra as

$$A_S^2(k) = A_S^2(k_0) \left(\frac{k}{k_0}\right)^{\tilde{n}(k)-1} \quad ; \quad A_T^2(k) = A_T^2(k_0) \left(\frac{k}{k_0}\right)^{\tilde{n}_T(k)}. \quad (4.5)$$

⁴We remark that these expressions have different prefactors to those contained in our original papers, CKLL1 and CKLL2; while one normalization is as valid as any other, the normalizations chosen in those papers were atypical of the literature. Those used here conform more readily with the conventions employed in the existing literature and in particular with the Stewart and Lyth (1993) calculation. In fact, the numerical difference is only 0.3%. The ratio of the tensor and scalar amplitudes is unaffected by this change.

Although these definitions are completely general, they do require a specific choice of k_0 to be made. This feature implies that the definitions are non-local, a considerable drawback. A more suitable alternative is to define the spectral indices differentially via

$$n(k) - 1 \equiv \frac{d \ln A_S^2}{d \ln k}, \quad (4.6)$$

$$n_T(k) \equiv \frac{d \ln A_T^2}{d \ln k}. \quad (4.7)$$

We shall adopt this second choice in this work. The two definitions coincide for power-law spectra, where the indices are constant. In general, however, they are inequivalent.

At the level of approximation we are considering in this Section, the spectral indices may be expressed directly in terms of the slow-roll parameters ϵ and η . One calculates the first derivatives of the amplitudes from Eqs. (4.3) and (4.4) with respect to ϕ and converts to derivatives with respect to wavenumber with the help of Eq. (2.35). It is straightforward to show that

$$n(k) - 1 = 2\eta - 4\epsilon, \quad (4.8)$$

$$n_T(k) = -2\epsilon. \quad (4.9)$$

The conventional statement attached to these expressions is that inflation predicts spectra which to the presently desired accuracy can be approximated as power-laws; that is, that the slow-roll parameters can be treated as constants. While this statement is formally correct, it requires some discussion. In particular it is important to realize that the power-law approximation has no direct connection to the slow-roll approximation, but rather is a statement that the relevant observations cover only a limited range of scale and do so with limited accuracy. As far as the derivations of the spectra are concerned, the approximation is that for each scale the parameters can be treated as constant while that scale crosses outside the Hubble radius. However, in this ‘adiabatic’ approximation, there is no need for those constant values to remain the same from scale to scale. Thus, the expressions for the spectra can be applied across the complete range of scales. Although they are an approximation at each scale, the approximation does not deteriorate when one attempts to study a wider range of scales. The feature that dictates whether the spectra can be treated as power-laws is that the range of scales over which observations can be made is quite small, in terms of the range of ϕ values, and taking additional derivatives of the spectra introduces into the lowest-order result an extra power in the slow-roll parameters. For example, although differentiating Eq. (4.8) gives the correct lowest-order expression for $dn/d \ln k$, this will be of order ϵ^2 and hence a small effect over the short range of scales large-scale structure samples. Were large-scale structure able to sample, for example, scales encompassing twenty orders of magnitude rather than four, the approximation by power-law would be liable to break down for typical inflation models. With high accuracy observations, the power-law approximation represented by these lowest-order expressions may prove inadequate even over the short range of accessible scales.

We emphasize that the spectral indices do not have to satisfy the exact power-law result $n - 1 = n_T$ at this level of approximation. Each spectrum is uniquely specified by its amplitude and spectral index. The overall amplitude is a free parameter determined by

the normalization of the expansion rate H during inflation (or equivalently the scalar field potential V). On the other hand, the relative amplitude of the two spectra is given by

$$\frac{A_T^2}{A_S^2} = \epsilon. \quad (4.10)$$

Thus, there exists a simple relationship between the relative amplitude and the tensor spectral index:

$$n_T = -2 \frac{A_T^2}{A_S^2}. \quad (4.11)$$

This is the lowest-order consistency equation and represents an extremely distinctive signature of inflationary models. It is difficult to conceive of such a relation occurring via any other mechanism for the generation of the spectra.

Since it is possible for the spectra to have different indices, the assumption that their ratio is fixed can be true only for a limited range of scales, but the correction enters at a higher order in the slow-roll parameters.

This expression is often written in a slightly different form in order to bring the right hand side closer to observations. Since the spectra can be defined with arbitrary prefactors, they themselves have no definite significance. The environment in which each spectrum may have an effect that allows direct comparison is in large angle microwave background anisotropies. In this case the scalar and tensor fluctuations each contribute independently to the expected value of the microwave multipoles, C_l (defined and discussed in more detail in Section VII), and in the approximation where only the Sachs-Wolfe term is included and perfect matter–domination at last scattering assumed, this enables one to write the lowest-order consistency equation as (Liddle and Lyth, 1992, 1993a)

$$\frac{C_l^T}{C_l^S} = -6.2n_T. \quad (4.12)$$

This equation applies for moderate values of l corresponding to scales that are sufficiently small for the curvature of the last scattering surface to be negligible and yet are large enough to be well above the Hubble radius at decoupling⁵.

Eqs. (4.8), (4.9) and (4.10) contain all the information one requires to determine the generic behavior of inflationary models at this order. Moreover, the current status of observational data is such that they are sufficient to allow a reasonable degree of precision to be attained in the study of large-scale structure and microwave background anisotropies. In the forthcoming years, however, data quality will inevitably improve and a higher degree of accuracy in the theoretical calculations will therefore be required. Indeed, high precision microwave anisotropy experiments are likely to be the first type of observation demanding just such an improvement in accuracy.

⁵The exact number in this relation is sometimes written in different ways. It was first evaluated exactly as $25(1 + 48\pi^2/385)/9$ in the scale-invariant limit by Starobinsky (1985). This is numerically equal to 6.2. There is no regime where this strictly holds, as corrections from the ‘Doppler’ peak and from the Universe being not perfectly matter dominated at last scattering intervene before the asymptote is reached. Other authors evaluate only part of the expression to approximate it as 2π , or even 6. Finally, many authors consider the ratio of contributions to the quadrupole $l = 2$. In this case there is a geometrical correction from the curvature of the last scattering surface which make the factor close to 7.

In the next Section we shall show how these improvements may be implemented. For the purposes of our present discussion, however, there are only two input parameters that need to be determined before one can proceed to investigate inflation-inspired models of structure formation (Liddle and Lyth, 1993b). The key points are that (a) the density perturbation spectrum has a power-law form and that (b) some fraction of the large angle microwave anisotropies might be due to gravitational waves. These conditions represent two completely independent parameters, but fortunately, they are the only two new parameters one requires in the lowest-order approximation. This is true even though one has complete freedom in choosing the functional form of the underlying inflationary potential. A large number of papers have now investigated the implications of these inflationary parameters for structure formation models such as Cold Dark Matter and Mixed Dark Matter models. Some only consider the possibility of tilt (Bond, 1992; Liddle, Lyth, and Sutherland, 1992; Cen et al., 1992, Pogosyan and Starobinsky, 1995) and some also allow for gravitational waves (Liddle and Lyth, 1993b; Schaefer and Shafi, 1994; Liddle et al., 1996).

One can classify the generic behavior of all inflationary models consistent with the lowest-order approximation into six separate categories, as summarized in Table 1. Each sector is characterized by the direction of the tilt away from scale invariant density perturbations and by the relative amplitude of the gravitational waves. In general, spectra with $n > 1$ increase the short-scale power of the density perturbation spectrum. Such spectra were named *blue* spectra by Mollerach et al. (1994). Conversely, those spectra with $n < 1$ subtract short scale power⁶. It is a general feature of inflation that $n < 1$ is easier to produce than $n > 1$. The reason for this follows from the definition Eq. (4.8) for the scalar spectral index. To lowest-order, a necessary and sufficient condition for the spectrum to be blue is simply that $\eta > 2\epsilon$. Since ϵ is positive by definition, this condition is not easy to satisfy and this is particularly so during the final stages of inflation where ϵ must necessarily begin to approach unity. However, specific inflation models have been constructed for each possibility, with the exception of a blue spectrum accompanied by a large gravitational wave amplitude. This last possibility, while still technically possible, is particularly hard to realize because it requires a large ϵ overpowered by a yet larger η .

B Reconstructing the potential

In CKLL1 we developed a framework initiated by Hodges and Blumenthal (1990) that one might call *functional reconstruction*. In this approach one views the observations as determining the spectra explicitly as functions of scale. Hodges and Blumenthal (1990) considered only scalar perturbations, and then Grishchuk and Solokhin (1991) made an investigation, considering only the tensors, with the aim of determining the time evolution of the Hubble parameter. In CKLL1, we provided a unified treatment of both scalars and tensors. The ultimate aim of such a procedure is to then process the functions through the differential equations describing the evolution of the universe during inflation. One thereby determines the potential driving inflation as a function of the scalar field. If such a procedure could be carried out exactly, the quantities in the consistency equation would also be functions of scale.

An important point worth emphasising here is that only by including the tensors can a

⁶We resist calling them *red* since the usual definition of red spectra is $n < 0$, not $n < 1$.

full reconstruction be achieved. The scalar perturbations only determine the potential up to an unknown constant. As the underlying equations are non-linear, different choices of the constant lead not just to a rescaling of the potential but to an entirely new functional form. Thus, there are many potentials which lead to the same scalar spectrum, and hence no unique reconstruction of the potential from the scalar spectrum. Any piece of knowledge concerning the tensors is enough to break this degeneracy.

From a practical point of view, one finds that the functional reconstruction procedure is not very useful, although it does allow some theoretical insight to be gained. The reason is that *exact* formulae for the amplitudes of the spectra do not exist for an *arbitrary* inflaton potential. Consequently, even though the classical dynamics of the scalar field can be accounted for exactly, one must input the information on the spectra using results that depend directly on the slow-roll expansion. At some level, it is inconsistent to treat the dynamics exactly and the perturbations approximately, so formally one should truncate both at the same order of approximation. Indeed, the next-order calculations we provide in the following Section show that this joint truncation is indeed preferable. In general, the next-order correction to the magnitude of the potential arising from the spectra has an opposite sign and is slightly larger than the correction to the dynamics. In effect, therefore, an exact treatment of the dynamics actually leads to a less accurate answer than that obtained by treating the entire problem to lowest-order in slow-roll!

We therefore advocate an alternative approach that may be referred to as *perturbative reconstruction*. The fundamental idea behind perturbative reconstruction follows directly from the fact that the scalar field must roll sufficiently slowly down its potential if inflation is to proceed at all. This is important for the following reason. Typically, the modes that ultimately lead to observational effects within our universe first crossed the Hubble radius somewhere between 50 and 60 e -foldings before inflation came to an end. (The precise number of e -foldings depends on the final reheating temperature, but this does not affect the general features of the argument). During these 10 e -foldings of inflationary expansion, the change in the value of the inflaton field is typically small. In effect, therefore, the position of the field in the potential would have remained essentially fixed at some specific value ϕ_0 . It follows, that cosmological and astrophysical observations can only yield information regarding this small segment of the potential. Hence, it is consistent to expand the underlying inflationary potential as a Taylor series about the point ϕ_0 . The use of such a procedure to lowest-order was suggested by Turner (1993a), Copeland et al. (1993a) and CKLL1. Turner (1993b) then included a next-order term in the potential. The formalism was then developed fully to next-order in CKLL2, including a next-order term in the derivatives as well as the potential and outlining the framework for the general expansion. This framework was recast into a more observationally-based language by Liddle and Turner (1994) who further discussed the meaning of the order-by-order expansion.

Perturbative reconstruction can be performed in a controlled way using the slow-roll expansion order-by-order. The dynamics can be treated to arbitrary order in this expansion by employing the formalism developed by Liddle et al. (1994). In contrast, however, the treatment of perturbations is presently available only to next-order. In this case there seems no obvious framework by which one can establish an order-by-order expansion, and even just obtaining terms to one higher order is a very difficult task.

Modulo questions of convergence, the perturbative reconstruction procedure successfully

encodes functional reconstruction in the sense that perturbative reconstruction performed to infinite order is formally equivalent to functional reconstruction. Perturbative reconstruction can also be rewritten as an expansion in the observed spectra. The advantage of considering an expansion of this type is that it indicates exactly how the features in the observed spectra yield information on the inflationary potential. Such an explicit account of the observational expansion has not been given before.

Before launching into specific calculation, however, it will be helpful to identify each observable quantity with some order in the slow-roll expansion. This may be achieved by considering which slow-roll parameters occur in the lowest-order term. Thus, one may employ the lowest-order expressions for the spectra. One sees by direct differentiation that the information associated with the accumulation of observables is as follows: H gives A_T^2 , ϵ gives A_S^2 and n_T , η gives n and $dn_T/d\ln k$, ξ gives $dn/d\ln k$ and $d^2n_T/d\ln k^2$, and so on. The key feature is that the tensor spectrum always remains one step above the scalar one. Furthermore, we shall see that an additional derivative of the Hubble parameter for each order is required to obtain a higher order expression for each observable.

We shall now proceed to derive expressions for the potential and its first two derivatives correct to lowest-order in the slow-roll expansion. We consider the Taylor series

$$V(\phi) = V(\phi_0) + V'(\phi_0)\Delta\phi + \frac{1}{2}V''(\phi_0)\Delta\phi^2 + \dots, \quad (4.13)$$

about the point ϕ_0 . At this order, the Hamilton-Jacobi equation (2.25) reduces to $V(\phi) = 3m_{\text{Pl}}^2 H^2(\phi)/8\pi$, so the derivatives in this expansion may be expressed directly in terms of the slow-roll parameters from Eqs. (2.27) and (2.28). It is only consistent to expand the potential to quadratic order, because the third derivative will contain terms that are of the same order as terms that were neglected in the original expressions for the amplitudes. In other words, the lowest-order expressions do not permit any higher derivatives to be obtained.

It follows by direct substitution, therefore, that Eq. (4.13) may be written as

$$V(\phi) = \frac{3m_{\text{Pl}}^2 H_0^2}{8\pi} \left[1 - (16\pi\epsilon_0)^{1/2} \frac{\Delta\phi}{m_{\text{Pl}}} + 4\pi(\epsilon_0 + \eta_0) \frac{(\Delta\phi)^2}{m_{\text{Pl}}^2} + \mathcal{O}\left(\frac{(\Delta\phi)^3}{m_{\text{Pl}}^3}\right) \right], \quad (4.14)$$

where a subscript 0 implies that quantities are to be evaluated at $\phi = \phi_0$. Hence, H_0 represents the expansion rate when the scale corresponding to this value of the scalar field first crossed the Hubble radius during inflation.

We write the coefficients that arise in this expansion in terms of the spectra by employing the expressions Eqs. (4.3) and (4.4) for the amplitudes, the definition Eq. (4.8) for the scalar spectral index and the definitions of the slow-roll parameters. We find that

$$V(\phi_0) = \frac{75m_{\text{Pl}}^4}{32} A_T^2(k_0), \quad (4.15)$$

$$V'(\phi_0) = -\frac{75\sqrt{\pi}}{8} m_{\text{Pl}}^3 \frac{A_T^3(k_0)}{A_S(k_0)}, \quad (4.16)$$

$$V''(\phi_0) = \frac{25\pi}{4} m_{\text{Pl}}^2 A_T^2(k_0) \left[9 \frac{A_T^2(k_0)}{A_S^2(k_0)} - \frac{3}{2}(1 - n_0) \right], \quad (4.17)$$

where k_0 is the scale at which the amplitude and spectral indices are determined and n_0 is the scalar spectral index at k_0 . As already implied by Eq. (4.8), if n exceeds one the potential must be convex ($V'' > 0$) at the point being probed. However, n being less than one says nothing definite about convexity or concavity.

Perturbative reconstruction can be possible even if it ultimately transpires that the observations necessary to test the consistency equation non-trivially cannot acquire sufficient accuracy. Similar work on reconstruction to this level of approximation has been done by Adams and Freese (1995), Mielke and Schunck (1995) and Mangano, Miele, and Stornaiolo (1995).

However, it is clear that a determination of the gravitational wave amplitude on at least one scale is essential for the reconstruction program to work. Presently, such a quantity has not been directly determined, but we may nevertheless draw some interesting conclusions from the above calculation. In particular, there are a number of limiting cases to Eq. (4.14) that are of interest. Firstly, when $\epsilon = \eta$, Eq. (4.14) is the expansion for the exponential potential $V \propto \exp(-\sqrt{16\pi\epsilon}\phi/m_{\text{Pl}})$. (Without loss of generality we may perform a linear translation on the value of the scalar field such that $\phi_0 = 0$). Secondly, the potential has the form

$$V(\phi) = \Lambda \left[1 + 2\pi(n-1)\phi^2/m_{\text{Pl}}^2 \right], \quad (4.18)$$

in the limiting case where $\epsilon \ll 1$. This class of potentials produces a negligible amount of gravitational waves, but a tilted scalar perturbation spectrum. The tilt arises because the curvature of the potential is significant. The direction of the tilt, as determined by the sign of $(n-1)$, depends on whether the effective mass of the inflaton field is real or imaginary.

The dynamics of inflation driven by a potential of the form Eq. (4.18) for $n > 1$ has an interesting property. The kinetic energy of the inflaton field is determined from $H'(\phi)$ via the second expression in Eq. (2.26). As the field rolls down the potential towards $\phi = 0$, H' gradually decreases whilst H tends towards a positive constant. Hence, the field slows down as it approaches the minimum, but it loses kinetic energy in such a way that it can never reach the minimum in a finite time. Hence, the de Sitter universe is a stable attractor for this model and consequently the inflationary expansion can never end.

There are two ways of circumventing this difficulty. Firstly, one can argue that the potential only resembles Eq. (4.18) over the small region corresponding to cosmological scales. This is rather unsatisfactory, however, since it requires ad-hoc fine-tuning of the potential and therefore goes against the overall spirit of inflation. A much more plausible suggestion is that the first term of Eq. (4.18) arises because a *second* scalar field is being held captive in a false vacuum state. This is the case, for example, in Linde's Hybrid Inflation scenario (Linde, 1991, 1994; Copeland et al., 1994b), and an associated instability can end inflation.

We end this section by quoting formulae appropriate to the situation where one is given the potential and must calculate the predicted spectra; in general, one cannot analytically find the $H(\phi)$ corresponding to a given $V(\phi)$. In order to obtain the spectra, one uses the Friedmann equation Eq. (2.25) and its derivatives in combination with the slow-roll approximation. To lowest-order, the spectral indices were first given by Liddle and Lyth (1992), and are

$$n-1 = -6\epsilon_V + 2\eta_V, \quad (4.19)$$

$$n_T = -2\epsilon_V, \quad (4.20)$$

where

$$\epsilon_V = \frac{m_{\text{Pl}}^2}{16\pi} \left(\frac{V'}{V} \right)^2 \quad ; \quad \eta_V = \frac{m_{\text{Pl}}^2}{8\pi} \frac{V''}{V}, \quad (4.21)$$

are slow-roll parameters defined from the potential and differ slightly from the definitions made in terms of the Hubble parameter used in the rest of this paper (see Liddle et al. (1994) for more details). It is also possible to write down next-order expressions for the spectral indices in terms of the potential (Stewart and Lyth, 1993; Kolb and Vadas, 1994). Expressions such as these written in terms of the potential only make sense because of the existence of the inflationary attractor.

V Next-Order Reconstruction

The level of accuracy discussed in the previous Section, while perfectly adequate at present, is unlikely to be sufficient once high resolution microwave background anisotropy experiments are carried out. The theoretical benchmark for calculating the radiation power spectrum from a matter power spectrum has been set at one percent in order to cope with such observations (Hu et al., 1995). If inflation is to take advantage of this level of accuracy, it is vital that the initial power spectrum can be considered to at least a similar level of accuracy. At the very least, this will require the next-order expressions for the spectra, which represent the highest level of accuracy presently achieved.

For many potentials, the next-order corrections may be small, perhaps smaller than the likely observational errors on the lowest-order terms. We shall see this in the simulated example later in this paper. In such a case the next-order calculation is still useful, because it serves as an estimate of the theoretical error bar on the calculation, which can be contrasted with the observational error.

We devote this Section to describing the next-order results in detail.

A The consistency equations

Let's first consider the next-order version of the lowest-order consistency equation Eq. (4.11). The best available calculations of the perturbation spectra are those by Stewart and Lyth (1993) containing the next-order, which we reviewed extensively in Section III. To this order, the amplitudes for the scalar and tensor fluctuations are given by

$$A_S(k) = \frac{4}{5m_{\text{Pl}}^2} [1 - (2C + 1)\epsilon + C\eta] \frac{H^2}{|H'|} \Big|_{k=aH}, \quad (5.1)$$

$$A_T(k) = \frac{2}{5\sqrt{\pi}} [1 - (C + 1)\epsilon] \frac{H}{m_{\text{Pl}}} \Big|_{k=aH}, \quad (5.2)$$

respectively, where we choose the same normalizations for A_S and A_T as in Section IV. We recall that $C \simeq -0.73$ is a constant. Once again, the right-hand sides of these expressions are to be evaluated when the scale in question crosses the Hubble radius during inflation.

Throughout the remainder of this Section we shall be quoting results that feature a leading term and a correction term, the next-order term, which is one order higher in the

slow-roll parameters. We shall utilize the symbol “ \simeq ” to indicate this level of accuracy. The correction terms shall be placed in square brackets, so the lowest-order equations can always be obtained by setting the square brackets equal to unity, except in Eqs. (5.3) and (5.20) where it needs to be set to zero.

To next-order, the scalar and tensor spectral indices may be expressed in terms of the first three slow-roll parameters by differentiating Eqs. (5.1) and (5.2) with respect to wavenumber k and employing Eq. (2.35). Some straightforward algebra yields (Stewart and Lyth, 1993)

$$1 - n \simeq 4\epsilon - 2\eta + \left[8(C + 1)\epsilon^2 - (6 + 10C)\epsilon\eta + 2C\xi^2 \right], \quad (5.3)$$

$$n_T \simeq -2\epsilon [1 + (3 + 2C)\epsilon - 2(1 + C)\eta]. \quad (5.4)$$

A very useful relationship may be derived by considering the ratio of the tensor and scalar amplitudes and replacing the derivative of the Hubble expansion rate with ϵ . We find that

$$\epsilon \simeq \frac{A_T^2}{A_S^2} [1 - 2C(\epsilon - \eta)]. \quad (5.5)$$

This relationship is the next-order generalization of Eq. (4.10). It plays a central role in deriving the next-order expressions for the potential and its first two derivatives in terms of observables. Moreover, substitution of this expression into Eq. (5.4) implies that

$$n_T \simeq -2\frac{A_T^2}{A_S^2} [1 + 3\epsilon - 2\eta]. \quad (5.6)$$

Now, since all the quantities in the square brackets of this expression are accompanied by a lowest-order prefactor, they may be converted into observables by applying the lowest-order expressions Eqs. (4.8) and (4.10). We conclude, therefore, that

$$n_T \simeq -2\frac{A_T^2}{A_S^2} \left[1 - \frac{A_T^2}{A_S^2} + (1 - n) \right]. \quad (5.7)$$

This is the next-order version of the lowest-order consistency equation $n_T = -2A_T^2/A_S^2$, given first in CKLL2 and translated into more observational language by Liddle and Turner (1994). It is interesting to remark that the corrections entering at next-order depend only on the relative amplitudes of the spectra and on n . They do *not* depend on n_T or on any of the derivatives of the indices, because they can be consistently removed using the lowest-order version of the same equation. This has an important consequence that has only been implicit in the literature thus far. We anticipate that n will be considerably easier to measure than n_T . It is reasonable to suppose, therefore, that if one has enough observational information to test the lowest-order consistency equation, one will also have sufficient data to test the next-order version as well. In other words, the situation where only the quantities in the lowest-order consistency equation are known is unlikely to arise. Consequently, one should employ the next-order consistency equation when testing the inflationary scenario, rather than the more familiar version given by Eqs. (4.11) or (4.12).

Another new feature of extending the observables to allow reconstruction at this order is that one has an entirely new consistency equation, being the lowest-order version of the

derivative of the original consistency equation. One calculates $dn_T/d\ln k$ by differentiating Eq. (5.4) with respect to scale k and employing Eqs. (2.27) and (2.35). One finds that

$$\frac{dn_T}{d\ln k} \simeq -4\epsilon(\epsilon - \eta). \quad (5.8)$$

Conversion of this expression into observables follows immediately by substituting in the lowest-order results Eqs. (4.8) and (4.10), giving

$$\frac{dn_T}{d\ln k} \simeq 2 \frac{A_T^2}{A_S^2} \left(2 \frac{A_T^2}{A_S^2} + (n-1) \right). \quad (5.9)$$

This equation was derived by Kosowsky and Turner (1995), though they did not explicitly recognize it as a new consistency equation. Unfortunately, the observables appearing in the above expression are far from promising as regards using it.

B Reconstruction of the potential to next-order

Now that we have discussed the formalism necessary for calculating the dynamics and perturbation spectra up to next-order in the slow-roll expansion, we shall proceed to consider the reconstruction of the inflationary potential at this improved level of approximation.

We begin by deriving expressions for the potential and its derivatives directly from the field equation Eq. (2.25) and the definitions Eqs. (2.27) – (2.29) for the slow-roll parameters. Successive differentiation of Eq. (2.25) with respect to the scalar field yields the exact relations

$$V = \frac{m_{\text{Pl}}^2 H^2}{8\pi} (3 - \epsilon), \quad (5.10)$$

$$V' = -\frac{m_{\text{Pl}} H^2}{\sqrt{4\pi}} \epsilon^{1/2} (3 - \eta), \quad (5.11)$$

$$V'' = H^2 (3\epsilon + 3\eta - (\eta^2 + \xi^2)), \quad (5.12)$$

Our immediate aim is to consider these expressions at a single point ϕ_0 and rewrite them in terms of observable quantities. The amplitude of the potential is derived by substituting Eqs. (5.2) and (5.5) into Eq. (5.10):

$$V(\phi_0) \simeq \frac{75 m_{\text{Pl}}^4}{32} A_T^2(k_0) \left[1 + \left(\frac{5}{3} + 2C \right) \frac{A_T^2(k_0)}{A_S^2(k_0)} \right], \quad (5.13)$$

$$\simeq \frac{75 m_{\text{Pl}}^4}{32} A_T^2(k_0) \left[1 + 0.21 \frac{A_T^2(k_0)}{A_S^2(k_0)} \right]. \quad (5.14)$$

At this stage, it is interesting to consider how this result would be altered if one treated the scalar field dynamics in full generality rather than truncating at next-order. It follows from the general expression Eq. (5.10) for the potential that the numerical factor on the next-order term in the last expression of Eq. (5.13) would become $-1/3$. What this means is that the next-order correction to the potential that is due to the spectra dominates the dynamical corrections. This is true for all inflationary models. Since the sign of the spectral

correction is opposite to that of the dynamical ones, the overall sign of the correction is reversed.

Since the potential's first derivative contains η , we need information regarding the value of the scalar spectral index at k_0 if we are to obtain $V'(\phi)$. We replace the H^2 term in Eq. (5.11) by substituting the tensor amplitude Eq. (5.2) and collecting together the terms containing $\{\epsilon, \eta\}$ to linear order. These may then be written in terms of the spectra via the lowest-order expressions Eqs. (4.8) and (4.11). The result is

$$\begin{aligned} V'(\phi_0) &\simeq -\frac{75\sqrt{\pi}}{8}m_{\text{Pl}}^3\frac{A_T^3(k_0)}{A_S(k_0)}[1+(C+2)\epsilon+(C-1/3)\eta], \\ &\simeq -\frac{75\sqrt{\pi}}{8}m_{\text{Pl}}^3\frac{A_T^3(k_0)}{A_S(k_0)}[1+1.27\epsilon-1.06\eta], \\ &\simeq -\frac{75\sqrt{\pi}}{8}m_{\text{Pl}}^3\frac{A_T^3(k_0)}{A_S(k_0)}\left[1-0.85\frac{A_T^2(k_0)}{A_S^2(k_0)}+0.53(1-n_0)\right]. \end{aligned} \quad (5.15)$$

The calculation for $V''(\phi_0)$ is much more involved. A new observable is needed to determine ξ ; the easiest example being the rate of change of the scalar spectral index. This will be substantially harder to measure, though, and it is fortunate that it only enters at next-order. (However, it would enter at leading order in $V'''(\phi_0)$, as mentioned in CKLL2 and derived fully in Liddle and Turner (1994)). We can obtain the next-order correction to $V''(\phi_0)$ directly in terms of the slow-roll parameters by employing Eqs. (5.2) and (5.12). We find that

$$V''(\phi_0) \simeq \frac{75\pi}{4}m_{\text{Pl}}^2A_T^2(k_0)(\epsilon+\eta)\left[1+(2C+2)\epsilon-\frac{1}{3}\left(\frac{\eta^2+\xi^2}{\eta+\epsilon}\right)\right]. \quad (5.16)$$

To proceed, we must convert the prefactor $(\epsilon+\eta)$ into observables, accurate to next-order. To accomplish this we must employ the next-order result Eq. (5.3) for the scalar spectral index. A straightforward rearrangement of this latter equation yields

$$\begin{aligned} \epsilon+\eta &\simeq 3\epsilon\left[1+\frac{4}{3}(C+1)\epsilon-\left(\frac{3+5C}{3}\right)\eta+\frac{C}{3}\frac{\xi^2}{\epsilon}\right]-\frac{1-n_0}{2}, \\ &\simeq 3\frac{A_T^2}{A_S^2}\left[1+\frac{1}{3}(4-2C)\epsilon+\frac{1}{3}(C-3)\eta+\frac{C}{3}\frac{\xi^2}{\epsilon}\right]-\frac{1-n_0}{2}, \end{aligned} \quad (5.17)$$

where the second expression follows after substitution of Eq. (5.5). Substituting this into Eq. (5.16) yields

$$\begin{aligned} V''(\phi_0) &\simeq \frac{225\pi}{4}m_{\text{Pl}}^2\frac{A_T^4(k_0)}{A_S^2(k_0)}\left[1+\frac{4C+10}{3}\epsilon+\frac{C-3}{3}\eta+\frac{C}{3}\frac{\xi^2}{\epsilon}\right], \\ &\quad -\frac{75\pi}{8}m_{\text{Pl}}^2A_T^2(k_0)(1-n_0)[1+(2C+2)\epsilon]-\frac{25\pi}{4}m_{\text{Pl}}^2A_T^2(k_0)(\eta^2+\xi^2), \end{aligned} \quad (5.18)$$

where the last term is entirely next-order. Note that there are two lowest-order terms. An interesting case is $\eta = -\epsilon$, corresponding to $H \propto \phi^{1/2}$, for which the lowest-order term vanishes identically and the final term of Eq. (5.16) is the only one to contribute. The

second derivative of the potential is the lowest derivative at which it is possible for the expected lowest-order term to vanish.

The final step is to convert the next-order terms into the observables. As they are already next-order, one only needs the lowest-order term in their expansion to complete the conversion. From the lowest-order expression for the scalar spectral index, one finds to lowest-order that

$$\frac{\xi^2}{\epsilon} \simeq -\frac{1}{2\epsilon} \left. \frac{dn}{d \ln k} \right|_{k_0} + 5\eta - 4\epsilon. \quad (5.19)$$

Note that the derivative of the spectral index is of order ϵ^2 . Finally, substitution of Eqs. (4.8), (4.11) and (5.19) into Eq. (5.18) yields

$$V''(\phi_0) \simeq \frac{25\pi}{4} m_{\text{Pl}}^2 A_T^2(k_0) \left\{ 9 \frac{A_T^2(k_0)}{A_S^2(k_0)} - \frac{3}{2}(1 - n_0) + \left[(36C + 2) \frac{A_T^4(k_0)}{A_S^4(k_0)} - \frac{1}{4}(1 - n_0)^2 - (12C - 6) \frac{A_T^2(k_0)}{A_S^2(k_0)}(1 - n_0) - \frac{1}{2}(3C - 1) \left. \frac{dn}{d \ln k} \right|_{k_0} \right] \right\}, \quad (5.20)$$

where the first two terms in the curly brackets represent the lowest-order contribution.

Before we conclude this section, it is worth remarking on a point that has perhaps been implicit in the existing literature but has not been stated explicitly before. A determination of each successive derivative of the potential requires an extra piece of observational information. In particular, for the case of lowest-order perturbative reconstruction, we conclude that the first term in the Taylor expansion requires only A_T , but the second requires both A_S and A_T . The third term, on the other hand, needs both of these together with n_0 . The ability to make the observations therefore dictates how many derivatives we can determine. On the other hand, a comparison of the lowest-order and next-order expressions for the derivatives implies the following: the new piece of information necessary for the derivation of the lowest-order term in V' is also sufficient to yield the next-order term in V . Likewise, the next observation will give the lowest-order term in V'' and this is enough to give the next-order term in V' . Furthermore, it is also sufficient, *in principle*, to give the third-order term in V . We stress *in principle* because the theoretical machinery has not been developed to allow the calculation of a third-order term in the potential or its derivatives to be performed. Hence, while observational limitations constrain how high a derivative we can reach, it may be theoretical rather than observational limitations which prevent higher accuracy in the lower derivatives. This will be the case even though the necessary observational information may become available.

Table 2 lists the inflation parameters required for reconstruction of a given derivative of the potential. Reconstruction requires the inflation parameters in terms of observables. Relations between inflationary parameters and observables are given in Tables 3 and 4. A combination of information from Table 2 and Table 4 results in Table 5, the observables needed to reconstruct a given derivative of the potential to a certain order. Although we know the information required for the next-to-next order given in Table 5, we don't know the coefficients of the expansion.

VI The Perturbative Reconstruction Framework

Although the next-order results of the previous Section represents the theoretical state-of-the-art, it is possible to see how the general pattern goes. We discuss this in this Section and also introduce an expansion of the observations corresponding to perturbative reconstruction.

A A variety of expansions

During reconstruction, there are three types of expansion being carried out. There is an expansion in terms of observables, an expansion in terms of slow-roll parameters and an expansion of the potential itself.

Since the underlying theme behind the reconstruction program is that one is driven by observations, let us first consider what information might be available. The reconstruction program assumes some measurements of $A_S(k)$ and $A_T(k)$ are available over some range of scales. In practice, the likely range of observations for the scalars will probably be no greater than $-5 < \ln(k/k_0) < 5$, with a much shorter range for the tensors. In accordance with the perturbative reconstruction strategy, the spectra should be expanded about some scale k_0 which corresponds to the scale at horizon crossing when $\phi = \phi_0$. The appropriate expansion is in terms of $\ln(k/k_0)$, and of course it makes best sense to carry out the expansion about a wavenumber close to the middle of the available data.

In general, the expansions can be written as

$$\ln A_S^2(k) = \ln A_S^2(k_0) + (n(k_0) - 1) \ln \frac{k}{k_0} + \frac{1}{2} \left. \frac{dn}{d \ln k} \right|_{k_0} \ln^2 \frac{k}{k_0} + \dots, \quad (6.1)$$

$$\ln A_T^2(k) = \ln A_T^2(k_0) + n_T(k_0) \ln \frac{k}{k_0} + \frac{1}{2} \left. \frac{dn_T}{d \ln k} \right|_{k_0} \ln^2 \frac{k}{k_0} + \dots, \quad (6.2)$$

where the coefficients continue as far as the accuracy of observations permit. There is no obligation for the two series to be the same length. Indeed, we anticipate that information associated with the scalars will be considerably easier to obtain in practice.

The range of $\ln k$ over which data are available leads to the range of ϕ over which the reconstruction converges well. Notice that since we believe $\ln(k/k_0)$ can be somewhat greater than unity, convergence of this type of series will only occur if the successive coefficients become smaller. Fortunately, we have already seen in Section IV that the lowest-order inflationary predictions attach an extra slow-roll parameter to each higher derivative of the spectra taken, so convergence can still occur as long as the slow-roll parameters are smaller than $1/\max|\ln(k/k_0)|$. This forms a good guide as to how wide a range of scales can be addressed via perturbative reconstruction. The observation that the spectral index (at least of the scalars) is not too far from unity suggests that the slow-roll parameters are small. Hence, the observational expansion might continue to converge well outside the range of $\ln k$ actually observed. The equivalent statement regarding the potential would be to say that if it is reconstructed very smoothly for the range $\Delta\phi$ corresponding to observations, one should feel fairly confident in continuing the extrapolation of the potential beyond the region where direct observations were available (though in a practical sense this does not correspond to any extra information).

The observational expansion discussed above is closely related to the slow-roll expansion. In particular, we may consider the expansion of the spectra at a given k in terms of slow-roll parameters, as discussed in Section III. A qualitative comparison of the two expansions then yields a general pattern. Each term from the scalars allows the determination of one extra slow-roll parameter. With regard to the tensors, a single piece of information (presumably the amplitude) is necessary before one can proceed at all, as we have discussed previously. Beyond that, however, extra terms for the tensors do not provide new slow-roll parameters. Instead, they lead to degenerate information and hence consistency relationships. If one has the first two terms for the tensors and the first scalar term, one can test the single familiar consistency equation $n_T = -2A_T^2/A_S^2$. Further tensor terms result in a whole hierarchy of consistency equations, as we shall discuss further in the next Subsection.

By including terms consisting of products of more and more slow-roll parameters, one builds up a more accurate answer. However, there are two separate factors that prevent arbitrary accuracy from being obtained. The first is observational limitations. For a practical observational data set with error bars, the observational expansion discussed above can only be carried out to some term, beyond which the coefficients are determined as being consistent with zero within the errors. (If the error bars are still small when this happens, it may still correspond to useful information). This reflects directly on the number of slow-roll parameters ϵ , η , ξ , etc, that one can measure. In general, however, there are an infinite number of slow-roll parameters, and formally they are all of the same order (meaning that for a ‘generic’ potential, one expects them all to be of similar size). This appears to be rather problematic, since a finite number of terms in the observational expansion cannot constrain an infinite number of slow-roll parameters. Fortunately, however, only a finite (and usually small) number of such terms ever appear when a specific expression is considered.

The second restriction is that current technical knowledge concerning the generation of the spectra, as reviewed in Section III, only allows the calculation of a lowest-order term plus a correction involving single slow-roll parameters. In general, one anticipates further corrections including products of two or more slow-roll parameters, but that has not been achieved. It follows, therefore, that the number of derivatives in the potential that may be calculated is determined by observational restrictions, whilst the accuracy of each derivative is also constrained by theoretical considerations.

It should be emphasized that once an expression written as an expansion in slow-roll parameters has been found, it can be differentiated an arbitrary number of times. It is interesting that the derivatives are accurate to the same number of orders in the slow-roll parameters. This follows because differentiation respects the order-by-order expansion. However, differentiation introduces higher and higher slow-roll parameters from the infinite hierarchy. An important point here is that the ‘lowest-order’ can be a product of any number of slow-roll parameters; the phrase is not synonymous with setting the slow-roll parameters all to zero.

Having started with the observations, we now come round to the crux of the reconstruction process: the inflaton potential. In perturbative reconstruction, one aims to calculate the potential and as many of its derivatives as possible *at a single point* to some level of accuracy in slow-roll parameters. The ultimate goal is to use this information to reconstruct some portion of the potential about this point, by carrying out some expansion of $V(\phi)$

about the point ϕ_0 . The simplest strategy is to use a Taylor series

$$V(\phi) = V(\phi_0) + V'(\phi_0)\Delta\phi + \frac{1}{2}V''(\phi_0)\Delta\phi^2 + \dots, \quad (6.3)$$

and we shall only consider that case here. The literature does include more ambitious strategies such as Padé approximants and these may become useful when specific data are available (Liddle and Turner, 1994). The success of this expansion is governed by how far away from ϕ_0 one hopes to go, which ultimately arises from the range of observations one has available, as well as on how accurately the individual derivatives are determined.

This expression shows us that perturbative reconstruction of the potential actually involves *two* expansions. We have already seen that the potential is obtained up to some accuracy in the slow-roll expansion. However, for reconstruction to be successful, it is also imperative to consider how accurate the expansion in $\Delta\phi$ might be. Determining the coefficients of only the first one or two terms may be completely useless if $\Delta\phi$ turns out to be large.

The key to investigating this is to rewrite $\Delta\phi$ in terms of $\Delta \ln k$, the range of scales over which observations can realistically be expected to cover⁷. Broadly speaking this corresponds to the interval from 1 Mpc to about 10^4 Mpc, so assuming a center point in the middle of this region implies a range for $\Delta \ln k$ between ± 5 . This may be biased through tensor data only being available on large scales, though it will also be of considerably lower quality than the scalar data. The relationship that allows one to achieve the comparison between $\Delta\phi$ and $\Delta \ln k$ is the exact formula Eq. (2.35) presented earlier

$$\frac{d\phi}{d \ln k} = \frac{m_{\text{Pl}}^2 H'}{4\pi H} \frac{1}{\epsilon - 1} = \frac{m_{\text{Pl}}}{\sqrt{4\pi}} \frac{\sqrt{\epsilon}}{\epsilon - 1}, \quad (6.4)$$

together with its derivatives. One can then expand $\Delta\phi$ in terms of $\Delta \ln k$, expanding each coefficient up to some order in the slow-roll expansion. Such an expansion begins

$$\begin{aligned} \Delta\phi &= -\frac{m_{\text{Pl}}}{\sqrt{4\pi}} \sqrt{\epsilon} [1 + \epsilon + \dots] \Delta \ln k \\ &\quad + \frac{m_{\text{Pl}}}{\sqrt{16\pi}} \sqrt{\epsilon} [\epsilon - \eta + \dots] (\Delta \ln k)^2 + \dots, \end{aligned} \quad (6.5)$$

where, for illustrative purposes, the first coefficient has been given to next-order in slow-roll and the second one to lowest-order. The signs are chosen in accordance with our convention that $V' < 0$.

For clarity we shall employ β to represent a generic slow-roll parameter. One can then schematically represent the double expansion (one in $\Delta\phi$ and one in the slow-roll parameters), as

$$\begin{aligned} \frac{V(\phi)}{A_T^2(k_0)} &\sim [1 + \beta + \dots], \\ &\quad + \beta \Delta \ln k [1 + \beta + \dots] \{1 + \beta + \beta \Delta \ln k + \dots\}, \\ &\quad + \beta^2 (\Delta \ln k)^2 [1 + \beta + \dots] \{1 + \beta + \beta \Delta \ln k + \dots\}, \end{aligned} \quad (6.6)$$

⁷Turner (1993b) and Liddle and Turner (1994) carried out a similar analysis using ΔN , the number of e -foldings. This is perfectly valid but somewhat harder to interpret in terms of observable scales since it is only formally equivalent in a lowest-order approximation. In this work, however, we desire a simple interpretation of the next-order results.

where numerical constants have not been displayed. The square brackets represent the expansion of the potential and its derivatives at ϕ_0 , while the curly brackets represent the $\Delta\phi$, which itself is written as an expansion in $\Delta \ln k$ with coefficients expanded in slow-roll.

For the slow-roll expansion to make sense, we need $\beta \ll 1$. One can see from the schematic layout of Eq. (6.6) that convergence of the expansion will fail unless $\beta\Delta \ln k \ll 1$, as successively higher-order terms will otherwise become more and more important. However, we have agreed that $\Delta \ln k$ itself need not be small. In regions where it is, it is clear that the best results are obtained by calculating the low derivatives of the potential as accurately as possible. In regions where $\Delta \ln k$ is not small, however, it is more fruitful to calculate higher derivatives.

B The consistency equation hierarchy

In the previous Subsection, we stated that there exists an infinite hierarchy of consistency equations. It is not difficult to see why such a hierarchy should exist. Even though exact expressions for the spectra as a function of scale are not presently available, one can imagine having such expressions, at least in principle. In this case, one could then write down a consistency equation in the full functional reconstruction framework that applied over all available scales. This equation could then be represented in the perturbative reconstruction framework by performing a Taylor (or similar) expansion on both sides of it. The perturbative consistency equations could then be derived by equating the coefficients of the expansions. The key idea here is that the full functional consistency equation and all its derivatives must be satisfied at the point about which perturbative reconstruction is being attempted. The equality of each derivative at this point, however, represents a separate piece of information.

In Section IV we presented the consistency equation Eq. (4.11) for lowest-order perturbative reconstruction. The connection between the tensor–scalar ratio and the tensor spectral index was first presented by Liddle and Lyth (1992) and has been much discussed in the literature. This consistency equation is simply the (unknown) full functional consistency equation applied at a single point, and moreover, it is the version of that equation truncated to lowest-order in slow-roll. Indeed, it does not require a determination of n and it corresponds to the lowest, non-trivial truncation of the expansion of the observed spectra.

The next order in slow-roll introduces n and $dn_T/d \ln k$. This not only supplies enough information to impose a next-order version of the original consistency equation, but is also enough to impose a lowest-order version of the *derivative* of the consistency equation. The next-order versions of the original consistency equation were supplied by CKLL2 and Liddle and Turner (1994) and we discussed these in Section V. We also discussed the lowest-order version of the derivative of the consistency equation in that Section. This equation was first given by Kosowsky and Turner (1995).

This pattern continues at all orders in the expansion. One can ask why this has not been emphasised before. One reason is that until now a clear understanding has not been established regarding the type of *observational* information that appears at each order in the expansion. At the same stage that one introduces n in the slow-roll expansion, one should also introduce the rate of change of the tensor spectral index. The latter does not provide any new information regarding the reconstruction, in the same way that n_T did

not provide new information at lowest-order in slow-roll. However, it is subject to the new consistency equation. Researchers have not paid attention to the new consistency equation because it requires $dn_T/d\ln k$ and it seems very unlikely that this could ever be measured.

This concludes our discussion of the theoretical framework for perturbative reconstruction. In the following Section, therefore, we shall discuss whether the observations are likely to reach an adequate level of sophistication in the foreseeable future and then consider a worked example that illustrates how the reconstruction programme might be applied in practice.

VII Worked Examples of Reconstruction

A Prospects for reconstruction

In this Subsection, we shall consider the long-term prospects for reconstructing the inflaton potential. It is clear that one must determine the amplitudes of the *primordial* power spectra of scalar and tensor fluctuations on at least one scale, together with the slope of the scalar spectrum at that scale. Such information would provide enough information to reconstruct the potential and its first two derivatives to lowest-order. However, a measurement of n_T is also required if one is to test the inflationary hypothesis via the consistency equation. If such information becomes available at all, it will probably be *after* A_S , A_T and n have themselves been determined, so reconstructing to lowest-order should prove easier to accomplish than testing the scenario via the consistency equation.

It is convenient to separate the full cosmological parameter space into two sectors. The first contains the inflationary parameters essential for reconstructing the potential and testing the consistency equations. They are

$$(A_S, r, n, n_T, \dots), \quad (7.1)$$

where all are evaluated at k_0 , and the list extends to as many derivatives of the spectra as one wishes to consider. The tensor-scalar ratio $r \equiv 12.4A_T^2/A_S^2$ is defined so that $r = 1$ corresponds to an equal contribution to large angle microwave anisotropies from the scalar and tensor fluctuations, as follows from Eqs. (4.11) and (4.12)).

The second set consists of the other cosmological parameters:

$$(\Omega_0, \Omega_\Lambda, \Omega_{\text{CDM}}, \Omega_{\text{HDM}}, \Omega_B h^2, h, z_R, \dots), \quad (7.2)$$

where the Ω represent the densities in matter of various sorts, respectively the total matter density, cosmological constant, cold dark matter, hot dark matter and baryonic matter. Here z_R represents the redshift of recombination; it may be that this single parameter is adequate or the full ionization history may have to be taken into account. In the standard cold dark matter (CDM) model these parameters take the values $(A_S(k_0), 0, 1, 0)$ and $(1, 0, 0.95, 0, 0.0125, 0.5)$ respectively (further parameters concerning derivatives of the spectral indices in the first set being zero); that is, the scalar amplitude is the only free parameter available to fit to observations. The standard ionization history of the universe is also assumed.

Experiments measuring microwave background anisotropies offer the most promising route towards acquiring such information to within the desired level of accuracy. Although

redshift surveys provide valuable insight into the nature of the scalar spectrum at the present epoch, uncertainties in the mass-to-light ratio of galaxy distributions imply that it is very difficult to determine the primordial spectrum from these observations alone. There are further complications associated with uncertainties in the type of non-baryonic dark matter in the universe. These can lead to significant modifications in the form of the transfer function. One crucial advantage that microwave background experiments have, however, is that the level of anisotropy above 10 arcmin is almost independent of whether the dark matter is hot or cold (Seljak and Bertschinger, 1994; Stompor, 1994; Ma and Bertschinger, 1995; Dodelson, Gates, and Stebbins, 1996). Moreover, as we shall see in Section VIII, a direct detection of the stochastic background of gravitational waves by laser interferometers seems highly improbable. Thus, the microwave background anisotropies appear to be the only practical route at present towards determining the gravitational wave amplitude.

It is conventional to expand the temperature distribution on the sky in terms of spherical harmonics

$$\frac{\Delta T}{T_0} = \sum_{l=2}^{\infty} \sum_{m=-l}^l a_{lm}(r) Y_{lm}(\mathbf{x}), \quad (7.3)$$

where the monopole and dipole terms have been subtracted out and $T_0 = 2.726\text{K}$ is the present mean background temperature. The l -th multipole corresponds loosely to an angular scale of π/l , and a comoving length scale of $100h^{-1}$ Mpc at the last scattering surface subtends an angle of about one degree (for $\Omega_0 = 1$).

Inflation predicts that the a_{lm} are gaussian random variables, with a rotational invariant expectation value for their variance $C_l \equiv \langle |a_{lm}|^2 \rangle$. The radiation power spectrum is defined to be $l(l+1)C_l$; this is exactly constant in the case of a scale-invariant density perturbation spectrum ($n = 1, r = 0$) when the Sachs-Wolfe effect is the sole source of anisotropy (Sachs and Wolfe, 1967; Bond and Efstathiou, 1987). In general, both tensor and scalar perturbations contribute to the observed radiation power spectrum, and for inflation these contributions are independent, so $C_l = C_l^S + C_l^T$.

Accurate calculations of the C_l from both scalar and tensor modes require numerical solutions using a Boltzmann code (Bond and Efstathiou, 1987), and this can now be done to an extremely high accuracy, of around one percent or so (Hu et al., 1995). A recent innovation is a new algorithm based on an integral solution of the Boltzmann equation (Seljak and Zaldarriaga, 1996a), which obtains this level of accuracy at much less computational expense. In principle high quality observations can approach this accuracy though the question of foreground remains a delicate one (Hu et al., 1995; Tegmark and Efstathiou, 1996) and so the true observational accuracy will be less. These types of numerical study seem essential for high accuracy work, although they are complemented by analytical approaches, which can be made both for scalars (Hu and Sugiyama, 1995) and for tensors. The latter case is the easier for two reasons; firstly, only gravitational effects need to be considered and secondly, gravitational waves redshift away once they are inside the Hubble radius, so their main influence is only on the lower multipoles, up to $l \simeq 100$. Analytic studies, of increasing sophistication, have been made by Abbott and Wise (1984a, 1984b), Starobinsky (1985), Turner, White, and Lidsey (1993), Atrio-Barandela and Silk (1994), Allen and Koranda (1994), Koranda and Allen (1994) and Wang (1996). These results show good agreement with the numerical calculations of Crittenden et al. (1993a) and Dodelson, Knox, and Kolb

(1994), who evolve the photon distribution function by applying first-order perturbation theory to the general relativistic Boltzmann equation for radiative transfer.

With this calculational power in place, there are two main obstacles to determining the primordial spectra. These are known as ‘cosmic variance’ and ‘cosmic confusion’, respectively.

Cosmic Variance: A given inflationary model predicts the quantities $C_l = \langle |a_{lm}|^2 \rangle$, but the observed multipoles measured from a single point in space are $a_l^2 = \sum_{m=-l}^{+l} |a_{lm}|^2 / 4\pi$. These only represent a single realization of the C_l . It is well known that a finite sampling of events generated from a random process leads to an intrinsic uncertainty in the variance even if the experiment is perfectly accurate; this is sometimes called sample variance. In the limit of full sky coverage this uncertainty is known as cosmic variance.

More precisely, the a_l^2 are a sum of $2l + 1$ Gaussian random variables and therefore have a probability distribution that is a χ^2 distribution with $2l + 1$ degrees of freedom. Thus, for each multipole there are $2l + 1$ samples, so the uncertainty in the C_l is given by

$$\frac{\Delta C_l}{C_l} = \sqrt{\frac{2}{2l + 1}}. \quad (7.4)$$

This implies that cosmic variance is proportional to $l^{-1/2}$ and is therefore less significant on smaller angular scales. However, for any given experiment, the beam width limits how high an l can be obtained before experimental noise intervenes, and anyway in standard cosmological models the predicted signal cuts off rapidly beyond $l \sim 1000$ due to the finite thickness of the last scattering surface. Thus, the information on the tensor components is limited because there is very little signal in near-scale invariant models for $l \geq 200$ where the effects of cosmic variance are less significant.

Cosmic Confusion: The anisotropy below $l \leq 60$ is essentially determined by the inflationary parameters in Eq. (7.1), and by Ω_0 and Ω_Λ , since it is dominated by the purely gravitational terms rather than the details of the matter content of the universe. On the other hand, the anisotropies are highly model dependent for $l > 60$ due to the complexity of the operating physical processes. In particular, the precise level of anisotropy in this range depends sensitively on the values of the cosmological parameters listed in Eq. (7.2). Bond et al. (1994) have suggested that different sets of values for these parameters sometimes lead to power spectra which are extremely similar (for a review see Steinhardt, 1994). This leads to degeneracies in determined parameters, which Bond et al. refer to as ‘cosmic confusion’. Cosmic confusion is problematic for the reconstruction program and the degeneracy must be lifted before it can proceed. Fortunately, things have moved on since the Bond et al. discussion, and it is now acknowledged that observations can be carried out at such a high accuracy that the degeneracy is lifted (Hu et al., 1995, Jungman et al., 1996). Tegmark and Efstathiou (1996) have found that the microwave background anisotropies can be determined to very high precision even in the presence of multi-component foreground noise by the COBRAS/SAMBA satellite.

It should also be noted that other methods are available for determining cosmological parameters. For example, the primordial light element abundances imply that $0.009 \leq \Omega_B h^2 \leq 0.022$ and these limits may become stronger as observations of deuterium in quasar

absorption lines improve (Olive et al., 1990; Copi, Schramm, and Turner, 1995). Furthermore, an accurate measurement of h , certainly to within 10% , seems achievable with the Hubble Space Telescope (Freedman et al., 1994), whilst polarization of the microwave background may provide insight into the ionization history of the universe (Crittenden, Davis, and Steinhardt, 1993b; Frewin, Polnarev, and Coles, 1994; Crittenden, Coulson, and Turok, 1994; Kosowsky, 1996). There has also recently been improved understanding of the possibility of using polarization to probe gravitational waves (Kamionkowski, Kosowsky, and Stebbins, 1996; Seljak and Zaldarriaga, 1996b; Zaldarriaga and Seljak, 1996). Because gravitational waves typically contribute more (relative to density perturbations) to the polarization than to the total anisotropy, and indeed because one can identify a combination of the polarization parameters which cannot be induced by density perturbations at all, it may ultimately be possible to use polarization to do better than the cosmic-variance limited studies of the temperature alone which we discuss below.

In view of this, it is important to consider to what degree the next generation of satellites will be able to determine the inflationary parameters in Eq. (7.1). Knox and Turner (1994) have considered what might be deduced from two experiments A and B whose window functions are centered around $l_A \approx 55$ and $l_B \approx 200$, respectively. Experiment B only measures anisotropy due to the scalar fluctuations, whereas A will be sensitive to both scalar and tensor fluctuations. They considered ‘standard’ cosmological parameters $h = 0.5$, $\Omega_B \approx 0.05$, $\Omega_\Lambda = 0$ and a scale-invariant spectrum. They concluded that if the tensor-scalar ratio $r \geq 0.14$, one should be able to rule out $r = 0$ with 95% confidence 95% of the time. Thus, the gravitational wave amplitude should be quantitatively measurable for $r \geq 0.14$. If n is reduced, the limit is improved slightly to $r \geq 0.1$. Knox and Turner (1994) further conclude that full-sky measurements on angular scales 0.5° and 3° should acquire the sensitivity required for making such a detection.

For reconstruction to proceed at lowest-order, however, one also requires C_l^S for some l and also the spectral index n . Knox (1995) has simulated a set of microwave background experiments within the context of chaotic inflation driven by a ϕ^4 potential. This model predicts $n = 0.94$, $n_T = -0.04$ and $r = 0.28$. He considers a third measurement made on a smaller angular scale than those of A and B . It is this measurement that determines C_l^S and this may be combined with the measurement at the intermediate scale l_B to determine the slope n . Finally, r is inferred by identifying the ‘excess power’ arising in measurement A with the gravitational waves. He concludes that the quantity $C_2^S 130^{1-n}$ could be measured to an accuracy of $\pm 0.3\%$ and the error in the slope of the scalar spectrum could be as small as ± 0.02 . If $n \approx 1$, the error on r is ± 0.1 and improves slightly for smaller n . A full-sky experiment designed with current technology and with a $20'$ beam should be able to achieve such precision.

However, these results are derived on the assumption that the cosmological parameters have been accurately determined by other means. Indeed, to achieve the above precision on r and n , one requires the errors in $\Omega_B h^2$ to be no more than 10 % and 6%, respectively (Knox, 1995). Furthermore, the Hubble parameter will have to be determined to within 6% or 14% respectively if $\Omega_\Lambda = 0.8$ and the uncertainty in Ω_Λ must be below 7%.

More recently, Jungman et al. (1996) have carried out an analysis where all inflationary and cosmological parameters are allowed to vary. They confirm the expectation that

the estimates provided by Knox (1995) are very optimistic. If all the other cosmological parameters are left completely free, it is impossible to get any useful information on the gravitational waves at all — the required value of r is somewhat larger than mentioned above, and n_T would have to be extremely large. However, that represents a somewhat pessimistic assessment, because certainly many of the cosmological parameters will be constrained by other types of observations, and more importantly one may also feel content to live within a subset of cosmological parameter space (for example, critical density universes with only cold dark matter).

The accuracy to which the above parameters can be observationally determined will decide whether the information is good enough to push any of the expressions beyond lowest-order. Another possibility is that a more sophisticated observable may become available; Kosowsky and Turner (1995) have considered the possibility that $dn/d\ln k$ might be observable in the microwave background. For most models this seems unlikely as the effect will be small, but there do exist inflationary models leading to an effect that is large enough to be observable. Whether this parameter generates any degeneracies with other inflationary or cosmological parameters in the shape of the C_l remains to be addressed.

B Toy model reconstructions with simulated data

We devote this subsection to carrying out a worked example of reconstruction on a faked data set, to indicate the kind of accuracy that might be possible. We have tried to make the outcome of analyzing the simulated data at least reasonably indicative of the sort that high resolution microwave background experiments might achieve, based on the analysis by Knox (1995) [see also Jungman et al., 1996]. However, our approach is strictly a toy model; it is not intended to bear any resemblance to what one might actually do with high accuracy observations. It seems very unlikely that observations such as CMB anisotropies might be used to directly estimate the k -space spectra (though such an approach is common with galaxy redshift surveys); the expectation is that if suitable quality data are obtained then the appropriate procedure will be to push the theory forward from the spectra rather than try to calculate the primordial spectra directly from the observations. That is, some analysis such as a likelihood analysis would be used to find best fitting parameters such as the amplitude and spectral indices of the scalars and tensors directly. Knox (1995) has taken some first steps in this direction.

Perturbative reconstruction requires an expansion of the observations about a single scale, which will end up corresponding to the location ϕ_0 on the potential about which it is to be reconstructed. As discussed earlier, an expansion of the logarithm of the spectra in terms of the logarithm of the wavenumber is the best way to proceed. It will always make the most sense to choose the scale k_0 about which the expansion is done to be near the ‘central’ point of the logarithmic k -interval⁸. Thus we write

$$\ln A_S^2(k) = \ln A_S^2(k_0) + (n(k_0) - 1) \ln \frac{k}{k_0} + \frac{1}{2} \left. \frac{dn}{d \ln k} \right|_{k_0} \ln^2 \frac{k}{k_0} + \dots, \quad (7.5)$$

⁸The word ‘central’ is in quotes to indicate that the effective center point of the data may be biased through tensors only being available on large scales, plus scale-dependent error bars on both scalars and tensors. The word is intended to refer to the point best determined by the data assuming the type of fit attempted.

$$\ln A_T^2(k) = \ln A_T^2(k_0) + n_T(k_0) \ln \frac{k}{k_0} + \frac{1}{2} \left. \frac{dn_T}{d \ln k} \right|_{k_0} \ln^2 \frac{k}{k_0} + \dots, \quad (7.6)$$

where we have written in explicitly the observational quantities to which the coefficients of the expansion correspond.

A given observational program produces some finite set of data with error bars, such as a list of galaxy redshifts and sky positions, or a pixel map of the microwave sky. As we said above, it is unlikely to be a useful strategy to try and obtain the power spectra from these, and then use these to reconstruct. Rather, one should push the theory towards the data by parametrizing the spectra and fitting for those parameters, as has been done so successfully with COBE. Other parameters which affect the data interpretation, such as the cosmological parameters, can be fixed or simultaneously fitted as required. The general reconstruction framework we have described indicates an efficient parametrization of the spectra that could be used.

Despite the above, for our illustrative examples we have chosen to simulate data for the spectra themselves, as it is the simplest thing to do. Enough is known (Knox, 1995; Jungman et al., 1996) about the capabilities of CMB satellites in particular to enable a fairly realistic example (in terms of the observational uncertainties) to be constructed. To do anything else would obscure the principal issues. Our aim therefore is to simulate a set of data, with errors, for the spectra, which when fitted give similar errors on parameters to those expected had we carried out the full task of simulating say a microwave sky and fitting directly for the spectral parameters. It is well outside the scope of this paper to attempt a realistic simulation of what future data might actually look like.

As a simple test, we have simulated fake data sets for two different models, as follows:

1. A power-law inflation model with power-law index $p = 21$, chosen to yield $n-1 = n_T = -0.1$. Since power-law inflation can be solved exactly we know the precise amplitude of the spectra corresponding to a given normalization of the spectra, Eqs. (3.29) and (3.47). This particular model has been advocated by White et al. (1995) as providing a good fit to the current observational data.
2. An intermediate inflation model (Barrow and Liddle, 1993), which gives a scale-invariant spectrum of density perturbations but still possesses significant gravitational waves. We choose a version where scalars and tensors contribute equally to COBE (to be precise, their contributions to the tenth multipole are chosen to be the same). In this case, a precise calculation of the spectra cannot be made, so we compromise by using the next-order approximation to generate the spectra from the underlying model.

These models both have quite substantial gravitational waves. They have been chosen to be compatible with present observational data, though they can be regarded as rather extreme cases which maximize the chance of an accurate reconstruction.

The simulated data are constructed by the following procedure.

- The overall normalization reproduces the COBE result.
- The scalar error bars are consistent with cosmic variance limited microwave anisotropy observations up to $l = 200$ (except that for simplicity we have modeled the errors by

a gaussian rather than the formally correct χ^2_{2l+1} distribution). Other cosmological parameters, which affect the microwave anisotropy spectrum, are assumed fixed. The COBRAS/SAMBA satellite can go to much higher l , but of course the other cosmological parameters will be uncertain which limits the estimation of the inflationary parameters. By stopping at $l = 200$, we find that the accuracy we obtain is similar to that suggested by Jungman et al. (1996) for the full problem, so it serves as a reasonable compromise.

- For the tensors, reasonable *a priori* estimates for the error bars are harder to establish. We have assumed data corresponding to l up to 40, which is where the tensor contribution to C_l begins to cut off, and we have chosen error bars so as to reproduce the observational uncertainty in the tensor amplitude suggested by Knox (1995). We then accept whatever uncertainty in the tensor spectral index this gives us, and it happens to be in reasonable agreement with that suggested by Knox.

The simulated data for Model 1 are shown in Figure 2, along with the best fit reconstructions. Since scalar data runs from $l = 2$ to 200, it covers two orders of magnitude in wavenumber, corresponding to $\Delta \ln k \simeq 4.6$. The input and output parameters are shown in Table 4. We performed two fits, the first being a power-law fit and the second also allowing for a variation in the scalar spectral index (though in fact the underlying spectrum has none). The Figures and subsequent discussion use the former.

The results for Models 1 and 2 contain no particular surprises. Although this is intended only to be indicative and certainly falls way short of the sophistication that can be brought into play on realistic data, the error bars are probably fairly reasonable. As expected, the tensor spectral index is the real stumbling block, but at least with these models one obtains a strong handle on A_T^2 , thus allowing a unique reconstruction. For these reconstructions, we find that the lowest-order consistency equation Eq. (4.11) is indeed satisfied

$$0.108 \pm 0.013 = 2 \frac{A_T^2}{A_S^2} = -n_T = 0.25 \pm 0.10, \quad (7.7)$$

for Model 1 and

$$0.14 \pm 0.02 = 2 \frac{A_T^2}{A_S^2} = -n_T = 0.12 \pm 0.11, \quad (7.8)$$

for Model 2. The same is true for the next-order version Eq. (5.7). For Model 1 we obtain

$$0.114 \pm 0.014 = 2 \frac{A_T^2}{A_S^2} \left[1 - \frac{A_T^2}{A_S^2} + (1 - n) \right] = -n_T = 0.25 \pm 0.10, \quad (7.9)$$

whereas for Model 2 we find

$$0.13 \pm 0.02 = 2 \frac{A_T^2}{A_S^2} \left[1 - \frac{A_T^2}{A_S^2} + (1 - n) \right] = -n_T = 0.12 \pm 0.11. \quad (7.10)$$

While encouraging, we see that the test is not particularly strong due to the poorly determined n_T . In models where the tensors are even weaker than considered here, the task of testing the consistency equation will be yet harder.

Proceeding on to the reconstruction, Table 5 shows lowest-order and next-order reconstructions, in comparison to the exact underlying potential for both Models. The consistency equation has been used to eliminate n_T as it is the most poorly determined quantity. A next-order version of $V''(\phi_0)$ cannot be obtained without a value for $dn/d\ln k|_{k_0}$, though the size of the correction could be bounded from the error bars on the null result. The reconstructed potentials, both lowest-order and next-order, for Model 1 are shown in Figure 3 in comparison to the underlying potential. A Taylor series has been used to generate them, and the range of ϕ shown corresponds to the range of observational data (a range of two orders of magnitude in k) determined using Eq. (6.5).

We see that in both models the lowest-order reconstruction has been very successful. The errors are dominated by those in measuring the tensor amplitude. However, in neither case does the next-order result offer a significant improvement, given the observational error bars. The main importance of the next-order result appears therefore to be in bounding the theoretical error, rather than in providing improved accuracy in the overall reconstruction.

Figure 3 can be compared to a similar figure in Liddle and Turner (1994), who investigated reconstruction of a similar exponential potential. However, they did not include any observational errors, concentrating instead on the theoretical errors and on the efficacy of different expansion techniques for the potential. They also assumed reconstruction over a wider range of scales, and had somewhat poorer convergence of the reconstructed potential through expanding about one end of the data (the quadrupole) rather than the center.

VIII Other Ways to Constrain the Potential

Up until now we have concentrated, at least implicitly, on observations connected to large-scale structure in the universe, including microwave background anisotropies. These certainly provide the best source of constraints on the inflationary potential, and one should be very pleased at the prospect of obtaining such constraints. However, they do cover only a small portion of the full inflationary potential. There is of course no way of uncovering information about the potential relevant to larger scales (beyond waiting the relevant number of Hubble times!), but in principle there are a variety of ways of constraining the potential appropriate to smaller scales. We shall discuss such possibilities in this Section. In particular, one may constrain the potential from the fact that inflation must come to an end some 50 e -foldings after the large-scale structure scales pass outside the Hubble radius. Further constraints are associated with the scalar and tensor perturbations on small scales. In principle, laser interferometers could observe the tensor spectrum as a stochastic background, though we shall see that this is not promising. The possible overproduction of primordial black holes (PBHs) immediately after inflation places upper limits on the amplitude of the last scalar fluctuation to cross the Hubble radius just before inflation ends, while distortions to the microwave background spectrum limit scalar fluctuations on mass scales well below large-scale structure scales.

A To the end of inflation and the area law

In traditional inflation models, inflation can come to an end in one of two ways. The first is via some drastic event, such as a quantum tunneling (for example in extended inflation)

or a sudden instability (probably connected to a second field, as in hybrid inflation). If this happened, probably little information can be drawn from the behavior approaching the end of inflation. The second way inflation may come to an end is simply by the potential becoming too (logarithmically) steep to sustain inflation any longer, as in generic chaotic inflation models, so that ϵ reaches unity.

Let us see what one can conclude in the latter case. For definiteness, let us assume that 50 e -foldings are supposed to occur after the scale k_0 , about which reconstruction is attempted, leaves the horizon. The modest dependence of this number on the details of reheating will not be important. By assumption, inflation will end precisely when $\epsilon = 1$. The number of e -foldings which occur between two scalar field values is given exactly by

$$N = \sqrt{\frac{4\pi}{m_{\text{Pl}}^2}} \int_{\phi_1}^{\phi_2} \frac{1}{\sqrt{\epsilon(\phi)}} d\phi. \quad (8.1)$$

For our purposes, this can be neatly written as an integral constraint (Liddle, 1994a)

$$\int_{\phi_0}^{\phi_{\text{end}}} \frac{1}{\sqrt{\epsilon(\phi)}} \frac{d\phi}{m_{\text{Pl}}} = \frac{50}{\sqrt{4\pi}}. \quad (8.2)$$

This can most easily be thought of graphically. We have reconstructed the value of ϵ and its derivative at ϕ_0 , and know $\epsilon(\phi_{\text{end}}) = 1$. As shown in Figure 4, if we plot the curve of $1/\sqrt{\epsilon}$ against ϕ/m_{Pl} , it must be such that it reaches unity just as the area under it reaches $50/\sqrt{4\pi}$. While there remain many ways in which the curve may do this, it does exclude some possibilities such as a sudden flattening of the potential after observable scales leave the horizon.⁹

B Local detection of primordial gravitational waves

A number of authors have examined the possibility that the stochastic background of primordial gravitational waves produced during inflation could be detected locally (Allen, 1988; Grishchuk, 1989; Sahni, 1990; Souradeep and Sahni, 1992; White, 1992; Turner et al., 1993; Liddle, 1994b; Bar-Kana, 1994). In general, the wavenumber of the gravitational waves is related to the value of the inflaton field during inflation via the relation $\ln(k/k_0) = 60 - N$, where N is the number of e -foldings before the end of inflation and $k_0 = a_0 H_0 \approx 3 \times 10^{-18} h$ Hz is the wavenumber of the mode that is just reentering the Hubble radius at the present epoch. Thus, the modes with wavenumbers associated with the maximum sensitivity of typical beam-in-space experiments ($\sim 10^{-3}$ Hz) first crossed the Hubble radius approximately 25 e -foldings before the end of inflation. A direct detection of such waves would therefore provide unique insight into a region of the inflationary potential that cannot be probed by large-scale structure observations. However, we shall see that this is unlikely to be possible.

⁹It appears that this can be used to derive an upper limit, albeit a weak one, on $(\phi_{\text{end}} - \phi_0)$, from the knowledge that $\epsilon \leq 1$. In fact this is not the case, since H starts to exhibit strong variation when ϵ approaches one. The number of e -foldings should then strictly be characterized by the increase in aH rather than a alone (see Liddle et al. (1994) for details). In principle, a yet weaker constraint may be derived by using energy scale arguments to limit how much H can decrease in the late stages of inflation, but such a constraint seems too weak to be worth pursuing.

There are a number of gravitational wave detectors currently under construction or proposal (see e. g. Thorne, 1987, 1995). The ground-based Laser Interferometer Gravitational Wave Observatory (LIGO) should have a peak sensitivity of $\Omega_g \approx 10^{-11} h^{-2}$ at 10 Hz (Christensen, 1992), where Ω_g is the energy density per logarithmic frequency interval. The proposed space-based interferometers, the Laser Gravitational Wave Observatory in Space (Faller et al., 1985; Stebbins et al., 1989) and the Laser Interferometer Space Antenna (Danzmann, 1995) probe lower frequencies, but with a sensitivity to flat spectrum stochastic sources which is less than that of LIGO.

After inflation, the evolution of the gravitational wave perturbation is determined by Eq. (3.43). We have already studied the effect of modes which have wavelengths greater than the Hubble radius by the time of last scattering, which contribute to microwave background anisotropies. However, the scales which can be detected locally will have re-entered the Hubble radius before the onset of matter domination. In this regime they behave as radiation, so their energy density stays fixed during the radiation era but falls during the matter era. This suppression factor is directly measured by the radiation density today, $\Omega_{\text{rad}} = 4 \times 10^{-5} h^{-2}$. Thus the predicted amplitude on scales re-entering before matter-radiation equality is (Allen, 1988; Sahni, 1990; Liddle, 1994b)

$$\Omega_g h^2 = \frac{2}{3\pi} \left(\frac{H}{m_{\text{Pl}}} \right)^2 \times 4 \times 10^{-5}. \quad (8.3)$$

For the inflation models we have been discussing, H always decreases with time, and hence the primordial amplitude on short scales is always less than that on large scales¹⁰. The quadrupole anisotropy already places an extremely stringent limit on the amplitude of the spectrum at large scales, and this immediately translates into a conservative, but robust, constraint across all short scales of (Liddle, 1994b)

$$\Omega_g h^2 \leq 4 \times 10^{-15}. \quad (8.4)$$

This puts the inflationary signal well out of reach of any of the proposed experiments.

C Primordial black holes

It has been conjectured that primordial black holes (PBHs) may form during the reheating phase immediately after inflation (Khlopov, Malomed, and Zel'dovich, 1985; Carr and Lidsey, 1993; Carr, Gilbert, and Lidsey, 1994; Randall, Soljačić, and Guth, 1996; García-Bellido et al., 1996). While there are considerable theoretical uncertainties attached to this possibility, if such formation does occur, it can constrain the scalar spectrum at very short scales. During inflation the first scales to leave the Hubble radius are the last to come back in and this implies that the very last fluctuation to leave will be the first to return. In some regions of the post-inflationary universe, the fluctuation will be so large that one expects that the collapse of a local region into a black hole will become inevitable. The higher the rms amplitude the larger the fraction of the universe forming PBHs. The observational

¹⁰'Superinflation' models have been considered within the context of superstring motivated cosmologies, and it appears that in that case the gravitational wave amplitude could rise sufficiently on short scales to be detectable (Brustein et al., 1995). However, no complete model, demonstrating how superinflation might successfully end, has been constructed thus far (Brustein and Veneziano, 1994; Levin, 1995a).

consequences of the evaporation of these black holes then leads to upper limits on the number that may form and hence on the magnitude of the spectrum on the relevant scales. Thus, one may constrain the amplitude of the density spectrum on scales many orders of magnitude smaller than those probed by large-scale structure observations and microwave background experiments. These constraints lead to an upper limit on the spectral index and may therefore provide insight into features of the inflationary potential towards the end of inflation.

We parametrize the density spectrum in terms of the mass scale M associated with the Hubble radius when a given mode reenters. Hence, $\delta(M) \propto M^{(1-n)/6}$ defines the scalar spectral index. PBHs are never produced in sufficient numbers to be interesting if $n < 1$, but they could be if the spectrum is ‘blue’ with $n > 1$.

When an overdense region with equation of state $p = \gamma\rho$ stops expanding, it must have a size greater than $\sqrt{\gamma}$ times the horizon size in order to collapse against the pressure. The probability of a region of mass M forming a PBH is (Carr, 1975)

$$\beta(M) \approx \delta(M) \exp\left(-\frac{\gamma^2}{2\delta^2(M)}\right). \quad (8.5)$$

The constraints on $\beta(M)$ in the range $10^{10}\text{g} \leq M \leq 10^{17}\text{g}$ have been summarized by Carr and Lidsey (1993). In particular, PBHs with an initial mass $\sim 10^{15}\text{g}$ will be evaporating at the present epoch and may therefore contribute appreciably to the observed gamma-ray and cosmic-ray spectra at 100 MeV (MacGibbon and Carr, 1991). On the other hand, 10^{10}g PBHs have a lifetime ~ 1 sec and, if produced in sufficient numbers, would lead to the photodissociation of deuterium immediately after the nucleosynthesis era (Lindley, 1980). PBHs of mass slightly below 10^{10}g could alter the photon-to-baryon ratio just prior to nucleosynthesis. An upper limit therefore arises by requiring that evaporating PBHs do not generate a photon-to-baryon ratio exceeding the current value $S_0 = 10^9$ (Zel’dovich and Starobinsky, 1976).

Carr et al. (1994) have considered the constraints on $\beta(M)$ below 10^{10}g . In this region the strongest constraint arises if evaporating PBHs leave behind stable Planck mass relics (MacGibbon, 1987; Barrow, Copeland, and Liddle, 1992). The observational constraint from the relics derives from the fact that they cannot have more than the critical density at the present epoch, $\Omega_{\text{rel}} < 1$.

The upshot of this analysis is that the spectral index is typically constrained to be less than about 1.5, depending weakly on assumptions as to the reheat temperature after inflation and whether one takes into account the black hole relic constraint. Because the constraint applies at the end of inflation, on scales greatly separated from the microwave anisotropies, it is independent of the COBE normalization and also of the choice of dark matter. However, in this form it relies on the spectral index being constant right across those scales (which it would be in the hybrid inflation model (Copeland et al., 1994b)). For general inflation models it should be reinterpreted as a specific constraint on the amplitude at the short scales being sampled.

Finally, a constraint on the amplitude of the spectrum at a scale corresponding to an horizon mass $\approx 0.1M_\odot$ can in principle be derived from the recent observations of massive compact halo objects (MACHOs) (Alcock et al., 1993; Aubourg et al., 1993). The estimated mass range of these objects suggests that they constitute about 0.1 per cent of

the critical density. Although the favored explanation for these microlensing events is that they are due to substellar baryonic brown dwarfs, it is quite possible that MACHOs may be primordial black holes and therefore non-baryonic in nature (Nasel'skii and Polnarev; Ivanov, Nasel'skii, and Novikov, 1994; Yokoyama, 1995). Such PBHs could form from vacuum fluctuations in the manner discussed above if the amplitude of spectrum is sufficiently high on the appropriate scale. This may be possible, for example, if the potential has a suitable form (Ivanov et al., 1994). Alternatively, a spike may be imposed on the underlying spectrum by the quantum fluctuations of a second scalar field (Yokoyama, 1995; Randall et al., 1996; García-Bellido et al., 1996). If the amplitude is too high on this particular scale, however, it would lead to the overproduction of MACHO-PBHs. Consistency with the observations therefore constrains both the spectrum and the inflationary potential.

D Spectral distortions

A further constraint on $\delta(M)$ over mass scales considerably smaller than those corresponding to large-scale structure may be derived by considering departures of the microwave spectrum away from a pure blackbody. (For detailed reviews see e. g. Danese and de Zotti (1977) and Sunyaev and Zel'dovich (1980)). Above a redshift of $z_y \approx 2.2 \times 10^4 (\Omega_B h^2)^{-1/2}$, Compton scattering is able to establish local thermodynamic equilibrium whenever there is a sudden redistribution or release of energy into the universe (Burigana, Danese, and de Zotti, 1991). This produces a Bose-Einstein spectrum $n \propto [\exp(x + \mu) - 1]^{-1}$ that is characterized by a chemical potential μ , where $x = h\nu/kT$. (A Planck spectrum corresponds to $\mu = 0$). On the other hand, equilibrium cannot be established for redshifts just below z_y . The distribution of energy at this time could therefore lead to observable spectral distortions ($\mu \neq 0$) in the microwave background at the present epoch. The Far Infrared Absolute Spectrophotometer (FIRAS) aboard COBE has constrained the spectral distortion to be $|\mu| < 3.3 \times 10^{-4}$ (Mather et al., 1994), whilst Hu, Scott, and Silk (1994) have strengthened this limit by considering the COBE measurement of temperature fluctuations on 10° (Bennett et al., 1994). They find that $\mu < 5.0 \times 10^5 (\Delta T/T)_{10^\circ}^2 \approx 6.3 \times 10^{-5}$.

These limits imply that photon diffusion would have been the dominant mechanism for producing spectral distortions (Daly, 1991). Silk (1967) first showed that the damping of adiabatic fluctuations can proceed if their mass scales are below a characteristic mass known as the Silk mass. At sufficiently early times, the photons and baryons in the universe are strongly coupled through Thomson scattering and they therefore behave as a single viscous fluid. When adiabatic fluctuations reenter the Hubble radius, they set up pressure gradients and these result in pressure waves that oscillate as sound waves. As the epoch of recombination approaches, however, the mean-free-path of the photons increases and the photons are able to diffuse out of the overdense regions into underdense regions. Thus, the inhomogeneities in the photon-baryon fluid are damped. The energy stored in the fluctuations is redistributed by the diffusion of photons and it is this transfer of energy during the epoch near to z_y that produces the spectral distortions. The fluctuations that lead to these potentially observable effects have mass scales in the range $10^{-3} < M/M_\odot < 10^3$ (Sunyaev and Zel'dovich, 1970; Barrow and Coles, 1991).

The observational upper limit on μ implies an upper limit on the amplitude of the pressure wave and therefore a limit on $\delta(M)$. The energy density in a linear sound wave is

ρu^2 , where $u \approx c/\sqrt{3}$ is the sound speed. Thus, the dimensionless energy release caused by the damping is $q \approx \delta^2/3$. It can be shown that the spectral distortion is given by $\mu \approx 1.4q$ and it follows, therefore, that $\delta < 1.46\sqrt{\mu} \approx 0.01$.

By normalizing the spectrum at COBE scales ($\sim 10^{22} M_\odot$), an upper limit on the spectral index may be derived. Barrow and Coles (1991) and Daly (1991) assume that the distortion is entirely due to the largest amplitude wave and deduce a limit of $n < 1.8$ for $M \sim 10^{-3} M_\odot$. (The limit becomes weaker for larger scales). Hu et al. (1994) have derived a stronger constraint of $n < 1.5$ by refining these calculations. This is comparable to the PBH constraints we have just discussed (though somewhat weaker if one believes the PBH relic constraint). However, it is probably more reliable because it is based on physics that is relatively well understood and requires a less severe extrapolation to smaller scales.

IX Conclusions

In this paper, we have reviewed the relationship between observations of microwave anisotropies and of large-scale structure and the possibility of connecting them to the potential energy of a scalar field driving inflation. We have argued that, given suitable quality observations, the inflationary idea can be tested and then features of the inflationary potential can be directly measured. In many ways this is remarkable, given that it is impossible, by many orders of magnitude, for an Earth-based accelerator to pursue this task.

It is predicted that inflation produces both gravitational waves and density perturbations. Consequently, the employment of observations may be divided into two main parts. The most challenging is the test of the inflationary consistency relations; if these prove testable and are confirmed, it will provide a powerful vindication of the chaotic inflation paradigm. One could then feel confident in following the less observationally challenging task of employing observations to discern information regarding the inflationary potential, in the form of its value and that of its first few derivatives at a single point.

We have indicated the different approximation schemes that must be invoked. Of paramount importance is the slow-roll expansion, but this must also be coupled to an expansion of the observables. In the simplest instance this latter expansion corresponds to the approximation of power-law spectra. The lowest levels of approximation are certainly able to cope with present-day observations of both microwave anisotropies and large-scale structure. However, in this work we have been forward looking, since the demands that will be imposed on theoretical accuracy by future observations, especially satellite-based microwave background anisotropy measurements, will be high. Indeed, they could in principle threaten the limits of present-day theoretical knowledge regarding the calculation of the spectra.

We must emphasize that our calculations have all been implemented within the standard paradigm for chaotic inflation. The vast majority of known viable models can be expressed within this class, either trivially or by cunning manipulation, but one should bear in mind that there exist some models of inflation for which this is not the case. In some examples, such as old versions of the open inflationary scenario or some multi-field theories, this is because the predictions turn out to be dependent on initial conditions. Although such a situation would be unfortunate it is not logically excluded. Other theories, such as the recently investigated single-bubble open inflationary models, rely on dynamics that are

much more complicated than that of the standard scenario (Sasaki et al., 1993; Bucher et al., 1994; Linde, 1995). They therefore lead to a more complicated relationship between theory and observations. Furthermore, even if the inflationary hypothesis is indeed correct, it may be the case that the actual model produces a very low amplitude of gravitational waves (Lyth 1996). This would make them impossible to measure and such a situation would remove the ability to make a consistency check and thus eliminate most of the potential for reconstruction.

Finally, there remains every possibility that the entire inflationary idea is incorrect; if so, one can at least hope that this is manifested in a failure of the consistency relations. However, it may not prove possible to test the consistency relations; might one then blunder into reconstructing a non-existent object? With sufficiently good observations, such as a CMB satellite will provide, the answer should be no. The C_l spectrum, when it is observed, will contain huge amounts of degenerate information. If the correct underlying theory is topological defects, (see for example Vilenkin and Shellard, 1994), the spectral shape should be very different to any simple inflation model for any values of the cosmological parameters. One can certainly reconstruct a ‘potential’ which would give the observed C_l , but it would probably be of such a complex form as to have little particle physics motivation for it, leaving people to search for other explanations.

In a standard inflation scenario, the C_l give a complete description of the gaussian perturbations generated. This prediction can also be tested against the observations; present observations are compatible with gaussianity though they are not strong enough to give a convincing test. In the future we can expect such tests to be widely applied. While in principle it is possible to construct inflation models giving non-gaussian perturbations, in practice such models are so contrived that again, were such features detected, one would quickly be looking for a more plausible theory for the origin of perturbations. It might well also be that the shape of the power spectrum might be incompatible with the non-gaussian nature, within the general context of inflation.

The bulk of this review has covered work already discussed in the literature. We have given an extensive account of the Stewart and Lyth (1993) calculation of the perturbation spectra, which provides the accuracy needed to discuss anticipated observations. The reconstruction framework has then been described to an accuracy which ought to be sufficient for years to come. However, as well as the review material, we have brought to light a few new results and viewpoints and we summarize these here.

- The consistency equation discussed in the present literature is just one of an infinite hierarchy of consistency equations, each of which can be taken (in principle) to arbitrary accuracy in the slow-roll expansion. Kosowsky and Turner (1995) have written down the form for the second member and we have reproduced it here. However, it is probable that only the first consistency equation will ever be tested.
- We have indicated that since scalar perturbations are much easier to measure than tensor ones, the appropriate form of the first consistency equation to consider is not the lowest-order version, but rather the next-order version. One requires n_T to test the lowest-order version and it is very unlikely that such observations would be available without there also being the appropriate ones to include the next-order version as well. (The only new ingredient in the next-order version over and above those quantities in

the lowest-order version is n).

- We have been more explicit than previous work as to how observations of the primordial spectra should be handled in terms of an expansion in $\ln k$. We discussed how this expansion relates to the slow-roll expansion. A worked example on simulated data has illustrated these ideas in action.

In conclusion, therefore, the relationship between inflationary cosmology and large-scale structure observations is well understood and the theoretical machinery necessary for taking advantage of high accuracy observations is now in place. These promise the possibility of constraining physics at energies inaccessible to any other form of experiment. Such observations are eagerly awaited.

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References

- Abbott, L. F. and M. B. Wise, 1984a, Nucl. Phys. B **244**, 541.
- Abbott, L. F. and M. B. Wise, 1984b, Phys. Lett. B **135**, 279.
- Adams, F. C., J. R. Bond, K. Freese, J. A. Frieman and A. Olinto, 1993, Phys. Rev. D **47**, 426.
- Adams, F. C. and K. Freese, 1995, Phys. Rev. D **51**, 6722.
- Albrecht A. and P. J. Steinhardt, 1982, Phys. Rev. Lett. **48**, 1220.
- Alcock, C. et al., 1993, Nat. **365**, 621.
- Allen, B., 1988, Phys. Rev. D **37**, 2078.
- Allen, B. and S. Koranda, 1994, Phys. Rev. D **50**, 3713.
- Allen, T. J., B. Grinstein, and M. B. Wise, 1987, Phys. Lett. B **197**, 66.
- Arnowitt, R., S. Deser and C. W. Misner, 1962, in Gravitation: An Introduction to Current Research, ed L. Witten (John Wiley, New York).
- Atrio-Barandela, F. and J. Silk, 1994, Phys. Rev. D **49**, 1126.
- Aubourg, E. et al., 1993, Nat. **365**, 623.

- Bar-Kana, R., 1994, Phys. Rev. D **50**, 1157.
- Bardeen, J. M., 1980, Phys. Rev. D **22**, 1882.
- Bardeen, J. M., P. J. Steinhardt and M. S. Turner, 1983, Phys. Rev. D **28**, 679.
- Barrow, J. D. and R. A. Matzner, 1977, Mon. Not. R. astr. Soc. **181**, 719.
- Barrow, J. D. and S. Cotsakis, 1988, Phys. Lett. B **214**, 515.
- Barrow, J. D. and P. Coles, 1991, Mon. Not. R. astr. Soc. **248**, 52.
- Barrow, J. D., E. J. Copeland and A. R. Liddle, 1992, Phys. Rev. D **46**, 645.
- Barrow, J. D. and A. R. Liddle, 1993, Phys. Rev. D **47**, R5219
- Bennett, C. L. et al., 1994, Astrophys. J. **436**, 423.
- Bennett, C. L. et al., 1996, Astrophys. J. **464**, L1.
- Bergmann, P. G., 1968, Int. J. Theor. Phys. **1**, 25.
- Birrell, N. and P. C. W. Davies, 1982, *Quantum Fields in Curved Space* (Cambridge Univ. Press, Cambridge).
- Bond, J. R., 1992, in Highlights in Astronomy Vol 9, Proc of the IAU Joint Discussion, ed J. Bergeron.
- Bond, J. R. and G. Efstathiou, 1987, Mon. Not. R. astr. Soc. **226**, 665.
- Bond, J. R., R. Crittenden, R. L. Davies, G. Efstathiou and P. J. Steinhardt, 1994, Phys. Rev. Lett. **72**, 13.
- Brustein, R. and G. Veneziano, 1994, Phys. Lett. B **329**, 429.
- Brustein, R., M. Gasperini, M. Giovannini and G. Veneziano, 1995, Phys. Lett. B **361**, 45.
- Bucher M., A. S. Goldhaber and N. Turok, 1995, Phys. Rev. D **52**, 3314.
- Bucher M., A. S. Goldhaber and N. Turok, 1995, Nucl. Phys. S **43** 173.
- Burigana, C., L. Danese and G. de Zotti, 1991, Astr. Astrophys. **246** 49.
- Carr, B. J., 1975, Astrophys. J. **205**, 1.
- Carr, B. J. and J. E. Lidsey, 1993, Phys. Rev. D **48**, 543.
- Carr, B. J., J. H. Gilbert and J. E. Lidsey, 1994, Phys. Rev. D **50**, 4853.
- Cen, R., N. Y. Gnedin, L. A. Kofman and J. P. Ostriker, 1992, Astrophys. J. Lett. **399**, L11.
- Christensen, N., 1992, Phys. Rev. D **46**, 5250.

- Copeland, E. J., E. W. Kolb, A. R. Liddle and J. E. Lidsey, 1993a, *Phys. Rev. Lett.* **71**, 219.
- Copeland, E. J., E. W. Kolb, A. R. Liddle and J. E. Lidsey, 1993b, *Phys. Rev. D* **48**, 2529 [CKLL1].
- Copeland, E. J., E. W. Kolb, A. R. Liddle and J. E. Lidsey, 1994a, *Phys. Rev. D* **49**, 1840 [CKLL2].
- Copeland, E. J., A. R. Liddle, D. H. Lyth, E. D. Stewart and D. Wands, 1994b, *Phys. Rev. D* **49**, 6410.
- Copi, C. J., D. N. Schramm and M. S. Turner, 1995, *Science* **267**, 192.
- Crittenden, R., J. R. Bond, R. L. Davis, G. Efstathiou and P. J. Steinhardt, 1993a, *Phys. Rev. Lett.* **71**, 324.
- Crittenden, R., R. L. Davis and P. J. Steinhardt, 1993b, *Astrophys. J. Lett.* **417**, L13.
- Crittenden, R., D. Coulson and N. Turok, 1994, *Phys. Rev. Lett.* **73**, 2390.
- Daly, R. A., 1991, *Astrophys. J.* **371**, 14.
- Danese, L. and G. de Zotti, 1977, *Rivista del Nuovo Cimento* **7**, 277.
- Danzmann, K., 1995, *Ann. N. Y. Acad. Sci.* **759**, 481.
- Davis, R. L., H. M. Hodges, G. F. Smoot, P. J. Steinhardt and M. S. Turner, 1992, *Phys. Rev. Lett.* **69**, 1856.
- Dodelson, S., L. Knox and E. W. Kolb, 1994, *Phys. Rev. Lett.* **72**, 3444.
- Dodelson, S., E. Gates and A. Stebbins, 1996, *Astrophys. J.* **467**, 10.
- Dolgov, A. and J. Silk, 1993, *Phys. Rev. D* **47**, 2619.
- Durrer, R. and M. Sakellariadou, 1994, *Phys. Rev. D* **50**, 6115.
- Easther, R., 1996, *Class. Quant. Grav.* **13**, 1775.
- Easther, R., K. Maeda and D. Wands, 1996, *Phys. Rev. D* **53**, 4247.
- Efstathiou, G. P., 1990, in *Physics of the Early Universe*, eds A. T. Davies, A. Heavens and J. Peacock, SUSSP publications (Edinburgh).
- Faller, J. E., P. L. Bender, J. L. Hall, D. Hils and M. A. Vincent., 1985, in *Proceedings of the Colloquium Kilometric Optical Arrays in Space* (European Space Agency, Noordwijk).
- Freedman, W. L. et al., 1994, *Nat.* **371**, 757.
- Frewin, R. A., A. G. Polnarev and P. Coles, 1994, *Mon. Not. R. astr. Soc.* **266**, L21.
- García-Bellido, J. and D. Wands, 1995, *Phys. Rev. D* **52**, 6739.

- Garcí-Bellido, J. and D. Wands, 1996, Phys. Rev. D **53**, 5437.
- Garcí-Bellido, J., Linde A. D. and D. Wands, 1996, “Density Perturbations and Black Hole Formation in Hybrid Inflation”, Phys. Rev. D, to appear, astro-ph/9605094.
- Gasperini, M. and G. Veneziano, 1993a, Astropart. Phys. **1**, 317.
- Gasperini, M. and G. Veneziano, 1993b, Mod. Phys. Lett. A **8**, 3701.
- Gasperini, M. and G. Veneziano, 1994, Phys. Rev. D **50**, 2519.
- Gasperini, M., M. Giovannini and G. Veneziano, 1995, Phys. Rev. D **52**, R6651.
- Gott, J. R., 1982, Nature **295**, 304.
- Gott, J. R. and T. S. Statler, 1984, Phys. Lett. B **136**, 157.
- Grib, A. A., S. Mamaev and V. Mostepanenko, 1980, *Quantum Effects in Strong External Fields* (Atomizdat, Moscow).
- Grishchuk, L. P., 1974, Zh. Eksp. Teor. Fiz. **67**, 825 (Sov. Phys. JETP **40**, 409).
- Grishchuk, L. P., 1977, Ann. N. Y. Acad. Sci. **302**, 439.
- Grishchuk, L. P., 1989, Sov. Phys. Usp. **31**, 940.
- Grishchuk, L. P. and Yu. V. Sidorav, 1988, in Fourth Seminar on Quantum Gravity, eds M. A. Markov, V. A. Berezin and V. P. Frolov (World Scientific, Singapore).
- Grishchuk, L. P. and M. Solokhin, 1991, Phys. Rev. D **43**, 2566.
- Guth, A. H., 1981, Phys. Rev. D **23**, 347.
- Guth, A. H. and S. -Y. Pi, 1982, Phys. Rev. Lett. **49**, 1110.
- Harrison, E. R., 1970, Phys. Rev. D **1**, 2726.
- Hawking, S. W. and G. F. R. Ellis, 1973, *The Large Scale Structure of Space-time* (Cambridge University Press, Cambridge).
- Hawking, S. W., 1982, Phys. Lett. B **115**, 295.
- Hawking, S. W. and I. G. Moss, 1982, Phys. Lett. B **110**, 35.
- Higgs, P. W., 1959, Nuovo Cimento **11**, 816.
- Hodges, H. M. and G. R. Blumenthal, 1990, Phys. Rev. D **42**, 3329.
- Hu, W., D. Scott and J. Silk, 1994, Astrophys. J. Lett. **430**, L5.
- Hu, W. and N. Sugiyama, 1995, Phys. Rev. D **51**, 2599.
- Hu, W., D. Scott, N. Sugiyama and M. White, 1995, Phys. Rev. D **52**, 5498.

Ivanov, P., P. Nasel'skii and I. Novikov, 1994, Phys. Rev. D **50**, 7173.

Jungman, G., M. Kamionkowski, A. Kosowsky and D. N. Spergel, 1996, Phys. Rev. D **54**, 1332.

Kalara, S., N. Kaloper and K. A. Olive, 1990, Nucl. Phys. B **341**, 252.

Kaloper, N., R. Madden and K. A. Olive, 1995, Nucl. Phys. B **452**, 677.

Kaloper, N., R. Madden and K. A. Olive, 1996, Phys. Lett. B **371**, 34.

Kamionkowski M., Kosowsky A., Stebbins, A., 1996, "A probe of primordial gravitational waves and vorticity", Columbia preprint, astro-ph/9609132.

Khlopov, M. Yu., B. A. Malomed and Ya. B. Zel'dovich, 1985, Mon. Not. R. astr. Soc. **215**, 575.

Knox, L. and M. S. Turner, 1994, Phys. Rev. Lett. **73**, 3347.

Knox, L., 1995, Phys. Rev. D **52**, 4307.

Kodama, H. and M. Sasaki, 1984, Prog. Theor. Phys. Supp. **78**, 1.

Kolb, E. W. and M. S. Turner, 1990, *The Early Universe*, (Addison-Wesley, Redwood City, California).

Kolb, E. W. and S. L. Vadas, 1994, Phys. Rev. D **50**, 2479.

Koranda, S. and B. Allen, 1994, Phys. Rev. D **50**, 3713.

Kosowsky, A., 1996, Annals. Phys. **246**, 49.

Kosowsky, A. and M. S. Turner, 1995, Phys. Rev. D **52**, R1739.

Krauss, L. M. and M. White, 1992, Phys. Rev. Lett. **69**, 869.

Levin, J. J., 1995a, Phys. Rev. D **51**, 462.

Levin, J. J. 1995b, Phys. Rev. D **51**, 1536.

Liddle, A. R., 1994a, Phys. Rev. D **49**, 739.

Liddle, A. R., 1994b, Phys. Rev. D **49**, 3805; Phys. Rev. D **51**, 4603 (E).

Liddle, A. R. and D. H. Lyth, 1992, Phys. Lett. B **291**, 391.

Liddle, A. R. and D. H. Lyth, 1993a, Phys. Rept. **231**, 1.

Liddle, A. R. and D. H. Lyth, 1993b, Mon. Not. R. astr. Soc. **265**, 379.

Liddle, A. R., D. H. Lyth and W. Sutherland, 1992, Phys. Lett. B **279**, 244.

Liddle, A. R. and M. S. Turner, 1994, Phys. Rev. D **50**, 758.

- Liddle, A. R., P. Parsons and J. D. Barrow, 1994, Phys. Rev. D **50**, 7222.
- Liddle, A. R., D. H. Lyth, R. K. Schaefer, Q. Shafi and P. T. V. Viana, 1996, Mon. Not. R. astr. Soc. **281**, 531.
- Lidsey, J. E., 1991a, Class. Quant. Grav. **8**, 923.
- Lidsey, J. E., 1991b, Phys. Lett. B **273**, 42.
- Lidsey, J. E., 1992, Class. Quant. Grav. **9**, 149.
- Lidsey, J. E., 1993, Gen. Rel. Grav. **25**, 399.
- Lidsey, J. E. and P. Coles, 1992, Mon. Not. R. astr. Soc. **258**, 57P.
- Lifshitz, E. M., 1946, Zh. Eksp. Teor. Phys. **16**, 587.
- Linde, A. D., 1982a, Phys. Lett. B **108**, 389.
- Linde, A. D., 1982b, Phys. Lett. B **116**, 335.
- Linde, A. D., 1983, Phys. Lett. B **129**, 177.
- Linde, A. D., 1990a, Phys. Lett. B **249**, 18.
- Linde, A. D., 1990b, *Particle Physics and Inflationary Cosmology* (Harwood Academic, Chur, Switzerland).
- Linde, A. D., 1991, Phys. Lett. B **259**, 38.
- Linde, A. D., 1994, Phys. Rev. D **49**, 748.
- Linde, A. D., 1995, Phys. Lett. B **351**, 99.
- Linde, A. D. and A. Mezhlumian, 1995, Phys. Rev. D **52**, 6789.
- Lindley, D., 1980, Mon. Not. R. astr. Soc. **193**, 593.
- Lucchin, F. and S. Matarrese, 1985a, Phys. Rev. D **32**, 1316.
- Lucchin, F. and S. Matarrese, 1985b, Phys. Lett. B **164**, 282.
- Lucchin, F., S. Matarrese and S. Mollerach, 1992, Astrophys. J. Lett. **401**, 49.
- Lyth, D. H., 1985, Phys. Rev. D **31**, 1792.
- Lyth, D. H. and E. D. Stewart, 1992, Phys. Lett. B **274**, 168.
- Lyth, D. H., 1996, "What would we learn by detecting a gravitational wave signal in the cosmic microwave background anisotropy?", Lancaster preprint, hep-ph/9606387.
- Ma, C.-P. and E. Bertschinger, 1995, Astrophys. J. **455**, 7.
- MacGibbon, J. H., 1987, Nature **320**, 308.

- MacGibbon, J. H. and B. J. Carr, 1991, *Astrophys. J.* **371**, 447.
- Maeda, K., 1989, *Phys. Rev. D* **39**, 3159.
- Makino, N. and M. Sasaki, 1991, *Prog. Theor. Phys.* **86**, 103.
- Mangano, G., G. Miele and C. Stornaiolo, 1995, *Mod. Phys. Lett. A* **10**, 1977.
- Mather, J. C. et al., 1994, *Astrophys. J.* **420**, 439.
- Mielke, E. W. and F. E. Schunck, 1995, *Phys. Rev. D* **52**, 672.
- Misner, C. W., K. S. Thorne and J. A. Wheeler, 1973, *Gravitation* (Freeman, San Francisco).
- Mollerach, S., S. Matarrese and F. Lucchin, 1994, *Phys. Rev. D* **50**, 4835.
- Mukhanov, V. F., 1985, *Pis'ma. Zh. Eksp. Teor. Fiz.* **41**, 402 [*JETP Lett.* **41**, 493].
- Mukhanov, V. F., 1988, *Zh. Eksp. Teor. Fiz.* **94**, 1 [*Sov. Phys. JETP* **41**, 493].
- Mukhanov, V. F., 1989, *Phys. Lett. B* **218**, 17.
- Mukhanov, V. F., H. A. Feldman and R. H. Brandenberger, 1992, *Phys. Rept.* **215**, 203.
- Muslimov, A. G., 1990, *Class. Quant. Grav.* **7**, 231.
- Nakamura, T. T. and E. D. Stewart, 1996, *Phys. Lett. B* **381**, 413.
- Nasel'skii, P. D. and A. G. Polnarev, 1985, *Astron. Zh.* **62**, 833 (*Sov. Astron.* **29**, 487).
- Olive, K. A., 1990, *Phys. Rept.* **190**, 307.
- Olive, K. A., D. N. Schramm, G. Steigman and T. Walker, 1990, *Phys. Lett. B* **236**, 454.
- Peebles, P. J. E. and R. J. Dicke, 1979, in *General Relativity*, eds S. W. Hawking and W. Israel (Cambridge University Press, Cambridge).
- Pogosyan, D. Yu. and A. A. Starobinsky, 1995, *Astrophys. J.* **447**, 465.
- Polarski D. and A. A. Starobinsky, 1995, *Phys. Lett. B* **356**, 196.
- Pollock, M. D. and D. Sahdev, 1989, *Phys. Lett. B* **222**, 12.
- Randall, L., M. Soljačić and A. H. Guth., 1996, *Nucl. Phys. B* **472**, 377.
- Sachs, R. K. and A. M. Wolfe, 1967, *Astrophys. J.* **147**, 73.
- Sahni, V., 1990, *Phys. Rev. D* **42**, 453.
- Salopek, D. S., 1992, *Phys. Rev. Lett.* **69**, 3602.
- Salopek, D. S, J. R. Bond and J. M. Bardeen, 1989, *Phys. Rev. D* **40**, 1753.

- Salopek, D. S. and J. R. Bond, 1990, *Phys. Rev. D* **42**, 3936.
- Salopek, D. S. and J. R. Bond, 1991, *Phys. Rev. D* **43**, 1005.
- Sasaki, M., 1986, *Prog. Theor. Phys.* **76**, 1036.
- Sasaki, M., T. Tanaka, K. Yamamoto and J. Yokoyama, 1993, *Phys. Lett. B* **317**, 510.
- Sasaki, M. and E. D. Stewart, 1996, *Prog. Theor. Phys.* **95**, 71.
- Sato, K., 1981, *Mon. Not. R. astr. Soc.* **195**, 467.
- Schaefer, R. K. and Q. Shafi, 1994, *Phys. Rev. D* **49**, 4990.
- Seljak, U. and E. Bertschinger, 1994, in “Present and Future of the Cosmic Microwave Background”, eds. J.L. Sanz, E. Martinez-Gonzalez and L. Cajon. Springer Verlag (Berlin).
- Seljak, U. and Zaldarriaga, M., 1996a, *Astrophys. J.* **469**, 437.
- Seljak, U. and Zaldarriaga, M., 1996b, “Signature of gravity waves in polarization of the microwave background”, CfA preprint, astro-ph/9609169.
- Silk, J., 1967, *Astrophys. J.* **151**, 459.
- Smoot, G. F. et al., 1992, *Astrophys. J. Lett.* **396**, L1.
- Souradeep, T. and V. Sahni, 1992, *Mod. Phys. Lett. A* **7**, 3541.
- Starobinsky, A. A., 1979, *Pis'ma Zh. Eksp. Teor. Fiz.* **30**, 719 (*JETP Letters* **30**, 682).
- Starobinsky, A. A., 1980, *Phys. Lett. B* **91**, 99.
- Starobinsky, A. A., 1982 *Phys. Lett. B* **117**, 175.
- Starobinsky, A. A., 1985, *Pis'ma Astron. Zh.* **11**, 323 (*Sov. Astron. Lett.* **11**, 133).
- Starobinsky, A. A. and J. Yokoyama, 1995, “Density Perturbations in Brans–Dicke theory”, Kyoto preprint, gr-qc/9502002.
- Stebbins, R. T., P. L. Bender, J. E. Faller, J. L. Hall, D. Hils, and M. A. Vincent., 1989, in Fifth Marcel Grossman Meeting (World Scientific, Singapore).
- Steinhardt, P. J. and M. S. Turner, 1984, *Phys. Rev. D* **29**, 2162.
- Steinhardt, P. J., 1994, in “Anisotropies two years after COBE”, ed L. M. Krauss (World Scientific, Singapore).
- Stewart, E. D. and D. H. Lyth, 1993, *Phys. Lett. B* **302**, 171.
- Stompor, R., 1994, *Astron. Astrophys.* **287**, 693
- Sunyaev, R. A. and Ya. B. Zel'dovich, 1970, *Astrophys. Sp. Sci.* **7**, 20.

- Sunyaev, R. A. and Ya. B. Zel'dovich, 1980, *Ann. Rev. Astr. Astrophys.* **18**, 537.
- Tegmark, M. and G. Efstathiou, 1996, *Mon. Not. R. astr. Soc.* **281**, 1297.
- Thorne, K. S., 1987, in *300 years of Gravitation*, eds S. W. Hawking and W. Israel (Cambridge Univ. Press, Cambridge).
- Thorne, K. S., 1995, "Gravitational Waves", CalTech preprint, gr-qc/9506086.
- Turner, M. S., 1993a, *Phys. Rev. D* **48**, 3502.
- Turner, M. S., 1993b, *Phys. Rev. D* **48**, 5539.
- Turner, M. S., M. White and J. E. Lidsey, 1993, *Phys. Rev. D* **48**, 4613.
- Turner, M. S. and M. White, 1996, *Phys. Rev. D* **53**, 6822.
- Vilenkin, A. and Shellard, E. P. S., 1994, *Cosmic Strings and other topological defects* (Cambridge University Press).
- Wagoner, R. V., 1970, *Phys. Rev. D* **1**, 3204.
- Wands, D., 1994, *Class. Quant. Grav.* **11**, 269.
- Wang, Y., 1996, *Phys. Rev. D* **53**, 639.
- White, M., 1992, *Phys. Rev. D* **42**, 4198.
- White, M., D. Scott and J. Silk, 1994, *Ann. Rev. Astron. Astrophys.* **32**, 319.
- White, M., D. Scott, J. Silk and M. Davis, 1995, *Mon. Not. R. astr. Soc.* **276**, L69.
- Whitt, B., 1984, *Phys. Lett. B* **145**, 176.
- Wright, E. et al., 1992, *Astrophys. J. Lett.* **396**, L13.
- Yokoyama, J., 1995, "Formation of MACHO-primordial black holes in inflationary cosmology", YITP preprint.
- Zaldarriaga, M. and Seljak, U., 1996, "An all-sky analysis of polarization in the microwave background", CfA preprint, astro-ph/9609170
- Zel'dovich, Ya. B., 1972, *Mon. Not. R. astr. Soc.* **160**, 1P.
- Zel'dovich, Ya. B. and A. A. Starobinsky, 1976, *Pis'ma Zh. Eksp. Teor. Fiz.* **24**, 616 (1976) (*JETP Lett.* **24**, 571).

Tables

	Gravitational Waves Important	Gravitational Waves Negligible
$n < 1$	ϵ large, $\eta < 2\epsilon$ Power-Law Inflation	ϵ small, $\eta < -2\epsilon$ Natural Inflation
$n \simeq 1$	ϵ large, $\eta \simeq 2\epsilon$ Intermediate Inflation	$\epsilon, \eta $ small Hybrid Inflation
$n > 1$	ϵ large, $\eta > 2\epsilon$	ϵ small, $\eta > 2\epsilon$ Hybrid Inflation

Table 1: This table illustrates the different possible inflationary behaviors, and quotes a specific inflation model which gives each (except the bottom left case, which while possible in principle has not had any specific inflationary model devised). The description ‘large’ implies significantly larger than zero (but still less than unity).

	lowest-order	next-order (exact)
$V(\phi_0)$	$H(\phi_0)$	$H(\phi_0), \epsilon(\phi_0)$
$V'(\phi_0)$	$H(\phi_0), \epsilon(\phi_0)$	$H(\phi_0), \epsilon(\phi_0), \eta(\phi_0)$
$V''(\phi_0)$	$H(\phi_0), \epsilon(\phi_0), \eta(\phi_0)$	$H(\phi_0), \epsilon(\phi_0), \eta(\phi_0), \xi(\phi_0)$
$V'''(\phi_0)$	$H, \epsilon(\phi_0), \eta(\phi_0), \xi(\phi_0)$	————

Table 2: A summary of the inflationary parameters [H and the slow-roll parameters ϵ , η , and ξ defined in Eqs. (2.27)–(2.29)] needed to reconstruct a given derivative of the potential to a certain order. See Eqs. (5.10)–(5.12). Note that the next-order result is *exact*.

observable	lowest-order	next-order
$A_T^2(k_0)$	$H(\phi_0)$	$H(\phi_0), \epsilon(\phi_0)$
$A_S^2(k_0)$	$H(\phi_0), \epsilon(\phi_0)$	$H(\phi_0), \epsilon(\phi_0), \eta(\phi_0)$
$n(k_0)$	$\epsilon(\phi_0), \eta(\phi_0)$	$\epsilon(\phi_0), \eta(\phi_0), \xi(\phi_0)$
$dn/d \ln k _{k_0}$	$\epsilon(\phi_0), \eta(\phi_0), \xi(\phi_0)$	————

Table 3: The observables, A_T^2 , A_S^2 , n , and $dn/d \ln k$ at the point k_0 may be expressed in terms of H and the slow-roll parameters at the point ϕ_0 . Table 3 lists the inflation parameters required to predict the observable to the indicated order. (See Section 5.1.)

parameter	lowest-order	next-order
H	A_T^2	A_T^2, A_S^2
ϵ	A_T^2, A_S^2	A_T^2, A_S^2, n
η	A_T^2, A_S^2, n	$A_T^2, A_S^2, n, dn/d \ln k$
ξ	$A_T^2, A_S^2, n, dn/d \ln k$	————

Table 4: The inflation parameters may be expressed in terms of the observables, A_T^2 , A_S^2 , n , and $dn/d \ln k$ (see Section 5.1). Through judicious use of the consistency relations one may employ different combinations of observables than listed here, e.g., use of n_T rather than A_T^2/A_S^2 .

	lowest-order	next-order	next-to-next-order
V	A_T^2	A_T^2, A_S^2	A_T^2, A_S^2, n
V'	A_T^2, A_S^2	A_T^2, A_S^2, n	$A_T^2, A_S^2, n, dn/d \ln k$
V''	A_T^2, A_S^2, n	$A_T^2, A_S^2, n, dn/d \ln k$	————
V'''	$A_T^2, A_S^2, n, dn/d \ln k$	————	————

Table 5: A summary of the observables needed to reconstruct a given derivative of the potential to a certain order. The potential and its derivatives are given at a point ϕ_0 , and A_T^2 , A_S^2 , n , and dn/dk are to be evaluated at the point k_0 .

Model 1	Input	Output (power-law fit)	Output (including $dn/d \ln k _{k_0}$)
A_S^2	2.5×10^{-10}	$(2.45 \pm 0.09) \times 10^{-10}$	$(2.45 \pm 0.10) \times 10^{-10}$
A_T^2	0.12×10^{-10}	$(0.132 \pm 0.015) \times 10^{-10}$	$(0.132 \pm 0.015) \times 10^{-10}$
$n - 1$	-0.1	-0.11 ± 0.02	-0.115 ± 0.035
n_T	-0.1	-0.25 ± 0.10	-0.25 ± 0.10
$dn/d \ln k _{k_0}$	0	—	0.003 ± 0.018

Model 2	Input	Output (power-law fit)	Output (including $dn/d \ln k _{k_0}$)
A_S^2	1.34×10^{-10}	$(1.27 \pm 0.04) \times 10^{-10}$	$(1.28 \pm 0.04) \times 10^{-10}$
A_T^2	0.094×10^{-10}	$(0.09 \pm 0.01) \times 10^{-10}$	$(0.09 \pm 0.01) \times 10^{-10}$
$n - 1$	0.00	0.04 ± 0.02	0.06 ± 0.03
n_T	-0.2	-0.12 ± 0.11	-0.12 ± 0.11
$dn/d \ln k _{k_0}$	0	—	-0.01 ± 0.02

Table 6: Input and output values from the two simulated data sets. The amplitudes are given at the central k value (in log units) for the scalars, notionally corresponding to the 20-th multipole.

Model 1	Underlying potential	Lowest-order reconstruction	Next-order reconstruction
$10^{12} V(\phi_0)/m_{\text{Pl}}^4$	28.2	31 ± 4	31 ± 4
$10^{12} V'(\phi_0)/m_{\text{Pl}}^3$	-43.6	-51 ± 9	-52 ± 9
$10^{12} V''(\phi_0)/m_{\text{Pl}}^2$	67.5	83 ± 25	—
Model 2	Underlying potential	Lowest-order reconstruction	Next-order reconstruction
$10^{12} V(\phi_0)/m_{\text{Pl}}^4$	22.4	21 ± 2	21 ± 2
$10^{12} V'(\phi_0)/m_{\text{Pl}}^3$	-38.9	-40 ± 7	-37 ± 6
$10^{12} V''(\phi_0)/m_{\text{Pl}}^2$	94.5	123 ± 27	—

Table 7: Input potential compared with reconstructions for the two models.

Figure Captions

Figure 1

A schematic illustration of the reconstruction strategy. The spectra A_S of the density perturbations and A_T of the gravitational waves are measured over some range of scales which corresponds to some interval of the underlying potential $V(\phi)$.

Figure 2

The simulated data of Model 1, with error bars. The circles are A_S^2 and squares are A_T^2 . The horizontal axis is in $h \text{Mpc}^{-1}$. The lines show the best power-law fits to the simulated data, as given in Table 2. Showing the data in the form of the spectra is schematic; an analysis of true observations would directly fit the amplitude and spectral index to measured quantities.

Figure 3

The reconstructed potentials compared to the underlying one, from the data in Model 1 in Table 4. The dashed line shows the true underlying exponential potential. The two solid lines, which nearly overlap, are Taylor series reconstructions, one using just lowest-order information and the other using the available next-order information. The length of these lines corresponds to the range of k for which the simulated data is available. The observational errors (not shown) dominate the theoretical errors, and of course when taken into account the reconstructions are consistent with the true potential.

Figure 4

An illustration of the area law. Reconstruction finds ϵ and perhaps its derivative, between 60 and 50 e -foldings from the end of inflation, illustrated by the solid part of the curve which ends at a scalar field value indicated by ϕ_{50} . After large-scale structure scales leave the horizon, ϵ (now shown as a dotted curve) must behave so that it reaches unity just as the shaded area under the curve of $\epsilon^{-1/2}$ against ϕ/m_{Pl} reaches $50/\sqrt{4\pi}$.