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The Hamiltonian Structure of Soliton Equations and Deformed W-Algebras

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ABSTRACT

The Poisson bracket algebra corresponding to the second Hamiltonian structure of a large class of generalized KdV and mKdV integrable hierarchies is carefully analysed. These algebras are known to have conformal properties, and their relation to $\mathcal{W}$-algebras has been previously investigated in some particular cases. The class of equations that is considered includes practically all the generalizations of the Drinfel'd-Sokolov hierarchies constructed in the literature. In particular, it has been recently shown that it includes matrix generalizations of the Gelfand-Dickey and the constrained KP hierarchies. Therefore, our results provide a unified description of the relation between the Hamiltonian structure of soliton equations and $\mathcal{W}$-algebras, and it comprises almost all the results formerly obtained by other authors. The main result of this paper is an explicit general equation showing that the second Poisson bracket algebra is a deformation of the Dirac bracket algebra corresponding to the $\mathcal{W}$-algebras obtained through Hamiltonian reduction.

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1. Introduction

This paper completes the results of Ref. [1], where we investigated the connection between the $\mathcal{W}$-algebras obtained through Hamiltonian reduction of affine algebras [2,3,4,5,6], and the Hamiltonian structure of the generalized Drinfel'd-Sokolov integrable hierarchies constructed in [7,8].

The different hierarchies of [7,8] are characterised by the data $\{g, [w], \mathbf{s}_w, \mathbf{s}, \Lambda\}$ where g is a finite Lie algebra and $[w]$ indicates a conjugacy class of the Weyl group of g that specifies a Heisenberg subalgebra $\mathcal{H}[w]$ of the affine algebra $\widehat{g}$ of g. The two vectors $\mathbf{s}_w$ and $\mathbf{s}$, whose components are $\mathrm{rank}(g) + 1$ non-negative integers, define two gradations of $\widehat{g}$ such that the gradation $\mathbf{s}_w$ is also a gradation of $\mathcal{H}[w]$ and $\mathbf{s} \preceq \mathbf{s}_w$ (see [7,8]). Finally, Λ is a constant element of $\mathcal{H}[w]$ with positive $\mathbf{s}_w$-grade. The original Drinfel'd-Sokolov hierarchies [9] are recovered with the principal Heisenberg subalgebra, and a remarkable property of the class of integrable hierarchies of [7,8] is that it includes practically all the generalizations of the Drinfel'd-Sokolov construction so far proposed in the literature by several authors.

Firstly, it includes all the hierarchies considered in [10,11], which are recovered when the $\mathbf{s}_w$-grade of Λ equals 1. Secondly, when the affine algebra $\widehat{g}$ is simply laced, it has been proved in [12] that the integrable hierarchies of [7,8] coincide with those constructed by Kac and Wakimoto in [13] following the tau-function approach. Thirdly, the interpretation of the integrable hierarchies of [7,8] by means of pseudo-differential operators has been recently worked out in [14,15]. To be specific, the integrable hierarchies investigated there correspond to the conjugacy classes of the Weyl group of $g = sl(n, \mathbb{C})$ of the form

$$[w] = [\underbrace{r, \ldots, r}_{p \text{ times}}, \underbrace{1, \ldots, 1}_{v \text{ times}}], \quad n = p\, r + v\,,$$

and to the elements $\Lambda \in \mathcal{H}[w]$ of minimal positive $\mathbf{s}_w$-grade. Then, when $v = 0$, it is shown that the corresponding integrable hierarchies provide $p \times p$ matrix generalizations of the Gelfand-Dickey r-KdV hierarchy [16] associated to the Lax operator

$$L_{\mathrm{GD}} = \partial^r + u_2\, \partial^{r-2} + \cdots + u_{r-1}\, \partial + u_r\,.$$

Moreover, when $v > 0$, they lead to $p \times p$ matrix generalizations of the constrained KP (cKP) hierarchy [17], which is recovered for $p = v = 1$ and corresponds to the pseudo-differential Lax operator

$$L_{\mathrm{cKP}} = \partial^r + u_2\, \partial^{r-2} + \cdots + u_{r-1}\, \partial + u_r + \phi\, \partial^{-1} \varphi\,.$$

Finally, the link between the approach used in [7,8] and the Adler-Kostant-Symes (AKS) construction has been explained in [14,18,19], where it is shown that the former corresponds to the "nice reductions" of the AKS system that exhibit local monodromy invariants.

All this ensures that the class of integrable equations of [7,8] is large enough to provide a meaningful unified pattern of the relation between $\mathcal{W}$-algebras and the Hamiltonian structure of soliton equations, and the purpose of this paper is to complete the results of [1] about the precise form of this relation.

In [1], the relation between the second Hamiltonian structure of the integrable hierarchies of [7,8] and $\mathcal{W}$-algebras was investigated in the particular case when certain non-degeneracy condition is satisfied. Then, it was shown that the second Poisson bracket algebra contains a $\mathcal{W}$-algebra as a subalgebra, but that only a small subset of the $\mathcal{W}$-algebras obtained through Hamiltonian reduction are recovered from the class of integrable hierarchies constrained by the non-degeneracy condition. These $\mathcal{W}$-algebras are characterized by two conditions [1]. First, the J_+ that defines the associated embedding of $sl(2,\mathbb{C})$ has to be related in a precise way to some $\mathbf{s}_w$-graded element Λ of a Heisenberg subalgebra of $\widehat{g}$, and, second, J_+ has to satisfy the non-degeneracy condition. For the different hierarchies constructed from the Lie algebra $g = sl(n,\mathbb{C})$, these results are summarized in the Theorem 3 of [1], which shows that the non-degeneracy condition constrains the $\mathbf{s}_w$-grade of Λ to be 1 or 2. It is remarkable that, using the notation of this theorem, the cases involving $\Lambda = \Lambda^{(1)}$ coincide with the generalizations of the constrained KP hierarchy investigated in [14,15]. Therefore, in particular, the results of [1] allow a definite identification of the $\mathcal{W}$-algebras that correspond with the Poisson bracket algebras giving their Hamiltonian structure.

However, the non-degeneracy condition is not essential to the approach of [7,8]. Indeed, it is possible to construct an integrable hierarchy of equations from any $\mathbf{s}_w$-graded element Λ of a Heisenberg subalgebra of $\widehat{g}$ that satisfies a weaker version of that condition, which has to be required to ensure that the relevant Poisson bracket algebra is polynomial. In this paper, we will study the Hamiltonian structure of the integrable hierarchies obtained in the most general case when only the weaker version of the non-degeneracy condition is satisfied. Nevertheless, the results of [1] suggest that the resulting Poisson bracket algebras can be related either to a $\mathcal{W}$-algebra or just to an affine Kac Moody algebra (see Theorem 2). Since we are only interested in the former case, we will still restrict our study to those hierarchies where Λ has a non-vanishing component with zero $\mathbf{s}$-grade and the maximal $\mathbf{s}$-grade of Λ and of the potential q equals 1 (see eq. (5.3)). Apart from that, the case considered in this paper will be completely general. Then, the relation with

$\mathcal{W}$-algebras will be established through the comparison of the (second) Poisson bracket of the hierarchy with the Dirac brackets defining the $\mathcal{W}$-algebras.

Since our study relies on the description of the Poisson bracket of the hierarchies as the outcome of the reduction of a Poisson algebra, it overlaps with some results of [14,18]. However, our method is closer to the spirit of the original references, namely [7,8,9,10,11], and, at the end of the day, it provides an explicit expression for the (second) Poisson bracket algebra that relates it with certain deformations of the $\mathcal{W}$-algebras whose properties are still to be studied. In any case, the specialisation of our results to the cases studied in [1] comfirms and clarifies the results obtained in that reference when the non-degeneracy condition is satisfied. In particular, it shows that the $\mathcal{W}$-subalgebra is actually decoupled from the rest of the (second) Poisson bracket algebra, and it allows one to investigate the existence of an energy momentum tensor.

The paper is organized as follows. In Section 2, we briefly summarize the required features of the integrable hierarchies of [7,8] and of their second Hamiltonian structure. In particular, we point out that it is invariant with respect to a family of conformal transformations, an important property that is not mentioned in the original papers but which is implicit in [1]. Section 3 contains an elementary review of the reduction of Poisson manifolds, just to fix our notation.

The next three sections constitute the body of the paper. In Section 4, we show that the form of the second Poisson bracket follows from the reduction of a Poisson manifold by first-class constraints. Nevertheless, in general, the phase-space of the integrable hierarchies is only a subset of the resulting reduced Poisson manifold, which means that a (non-Hamiltonian in general) additional reduction is required to obtain the second Poisson bracket algebra. This additional reduction can be specified by selecting a gauge slice, and a convenient choice is proposed in Section 5. This choice generalizes the gauge slice used in [1] and it allows one to relate the set of generators of the second Poisson bracket algebra with the generators of one of the $\mathcal{W}$-algebras obtained through Hamiltonian reduction. Then, in Section 6, we will obtain a formula, eq. (6.18), which shows that the second Poisson bracket algebra is given by a modification of the Dirac bracket defining the $\mathcal{W}$-algebra; this is the main result of the paper. Whilst Section 4 is general, in Sections 5 and 6 we only consider the restricted set of integrable hierarchies whose Hamiltonian structures are expected to be related to $\mathcal{W}$-algebras (see Theorem 2 of [1]).

In Section 7 we consider some examples that illustrate the use of eq. (6.17), and, in particular, our previous results in [1] are corroborated and clarified. Finally, we present our conclusions in Section 8.

2. Hamiltonian structure of generalized integrable hierarchies.

In [7,8], generalized integrable hierarchies of partial differential equations were associated with the loop algebra $\widehat{g} = g \otimes \mathbb{C}[z, z^{-1}]$ of a finite simple Lie algebra g. Their construction requires the use of the Heisenberg subalgebras of $\widehat{g}$, which are classified by the conjugacy classes of the Weyl group of g [20] (see also the appendix of [1]). Moreover, if $[w]$ is a conjugacy class and $\mathcal{H}[w]$ is the corresponding Heisenberg subalgebra, there exists a (non-unique) $\mathbb{Z}$-gradation of $\widehat{g}$, denoted by $\mathbf{s}_w$, such that $\mathcal{H}[w]$ is graded by $\mathbf{s}_w$.

Let us remind that, up to conjugation, the different $\mathbb{Z}$-gradations of $\widehat{g}$ can be defined by a set of non-negative integers $\mathbf{s} = (s_0, s_1, \ldots, s_{\mathrm{rank}(g)})$ via the derivation [21, chapter 8]

$$d_{\mathbf{s}} = N_{\mathbf{s}}\, z \frac{d}{dz} + h_{\mathbf{s}}\,, \tag{2.1}$$

where

$$h_{\mathbf{s}} = \sum_{j=1}^{\mathrm{rank}(g)} \left(\frac{2}{\boldsymbol{\alpha}_j^2}\right) s_j\, \boldsymbol{\omega}_j \cdot \boldsymbol{h}\,, \quad \text{and} \quad N_{\mathbf{s}} = \sum_{j=0}^{\mathrm{rank}(g)} k_j\, s_j\,; \tag{2.2}$$

in these equations, $k_0 = 1$ and the k_j's are the labels of the Dynkin diagram of g, $\boldsymbol{h}$ denotes a generic element of its Cartan subalgebra, the $\boldsymbol{\alpha}_j$'s are the simple roots, and the $\boldsymbol{\omega}_j$'s are the fundamental weights. Then, $[h_{\mathbf{s}}, e_j^{\pm}] = \pm s_j e_j^{\pm}$, where $e_j^{\pm}$, $j = 1, \ldots, \mathrm{rank}(g)$, are the raising $(+)$ and lowering $(-)$ operators associated to the simple roots of g. Under the $\mathbb{Z}$-gradation $\mathbf{s}$, $\widehat{g}$ decomposes as

$$\widehat{g} = \bigoplus_{j\in\mathbb{Z}} \widehat{g}_j(\mathbf{s})\,, \qquad [\widehat{g}_j(\mathbf{s}), \widehat{g}_k(\mathbf{s})] \subset \widehat{g}_{j+k}(\mathbf{s})\,,$$

with $\widehat{g}_j(\mathbf{s}) = \{v \in \widehat{g} \mid [d_{\mathbf{s}}, v] = j\,v\}$. Within the set of gradations that can be conjugated to the previous form in terms of the same basis for the simple roots, one can introduce the following partial ordering: $\mathbf{s} \preceq \mathbf{s}'$ if $s'_j \neq 0$ whenever $s_j \neq 0$, which translates into a set of inclusion relations among the corresponding graded subspaces [7].

The integrable hierarchies of [7,8] are sets of zero-curvature equations for a Lax operator constructed from the data $\{[w], \mathbf{s}_w, \mathbf{s}, \Lambda\}$, where $\mathbf{s}$ is a $\mathbb{Z}$-gradation of $\widehat{g}$ such that $\mathbf{s} \preceq \mathbf{s}_w$, and Λ is a constant element of $\mathcal{H}[w]$ with well defined positive $\mathbf{s}_w$-grade i, $\Lambda \in \mathcal{H}[w] \cap \widehat{g}_i(\mathbf{s}_w)$[1]. The Lax operator takes the form

$$L = \partial_x + \Lambda + q(x)\,, \tag{2.3}$$

[1] In this paper, we will not distinguish between regular and non-regular Λ's, *i.e.*, using the terminology of [7], between type I and type II hierarchies.

where the potential $q(x)$ is an element of $C^\infty(\mathbf{S}^1, Q)$, *i.e.*, a periodic function[2] of $x \in \mathbf{S}^1$ taking values on the subspace of $\widehat{g}$

$$Q \; = \; \widehat{g}_{\,\geq 0}(\mathbf{s}) \cap \widehat{g}_{\,<i}(\mathbf{s}_w)\,; \tag{2.4}$$

in this last equation, we have introduced the notation $\widehat{g}_{\,>n}(\mathbf{s}) = \bigoplus_{j>n} \widehat{g}_{\,j}(\mathbf{s})$ and so on.

The function $q(x)$ plays the role of the phase-space coordinate in this system. However, there exist symmetries corresponding to the gauge transformations

$$q(x) \to \widetilde{q}(x) \; = \; \exp\left(\mathrm{ad}\, S(x)\right)\, \left(\partial_x \, + \, \Lambda \, + \, q(x)\right) \, - \, \partial_x \, - \, \Lambda \tag{2.5}$$

generated by any $S(x) \in C^\infty(\mathbf{S}^1, P)$, where P is the nilpotent subalgebra

$$P \; = \; \widehat{g}_{\,0}(\mathbf{s}) \cap \widehat{g}_{\,<0}(\mathbf{s}_w)\,; \tag{2.6}$$

the infinitesimal form of (2.5) is

$$\widetilde{q}(x) \, - \, q(x) \; = \; \left[S(x)\,,\, \partial_x \, + \, \Lambda \, + \, q(x)\right]\,, \qquad S(x) \ll 1\,. \tag{2.7}$$

From now on, the group formed by these transformations will be called G, and, consequently, the actual phas- space of the system is $C^\infty(\mathbf{S}^1, Q)/G$, *i.e.*, the set of gauge equivalence classes of Lax operators of the form (2.3).

The construction of [7,8] is restricted to the case when the condition[3]

$$\mathrm{Ker}(\mathrm{ad}\, \Lambda) \cap P \; = \; \{0\} \tag{2.8}$$

is satisfied. Actually, (2.8) ensures the existence of a basis for the gauge invariant functionals that is polynomial in the components of $q(x)$ and their x-derivatives. That basis can be easily constructed by following the Drinfel'd-Sokolov procedure, which can be summarized as follows [5,7,9,22,23]. First, one has to choose some complementary space Q^{can} of $[\Lambda, P]$ in Q, *i.e.*,

$$Q \; = \; [\Lambda\,,\, P] \, + \, Q^{\mathrm{can}}\,, \tag{2.9}$$

and, second, one simply performs a non-singular gauge transformation to take $q(x)$ to $q^{\mathrm{can}}(x) \in C^\infty\left(S^1, Q^{\mathrm{can}}\right)$. Then, the desired basis is provided by the components of $q^{\mathrm{can}}(x)$

[2] The choice of the domain where the potential is defined is not crucial. The only constraint is that its boundary conditions ensure that the integrals involved in the definition of the Poisson brackets exist.

[3] This condition is automatically satisfied if Λ is a regular element of $\mathcal{H}[w]$ [1,7].

and their derivatives, understood as functionals of $q(x)$, which implies that $C^\infty(\mathbf{S}^1, Q)/G$ can be identified with $C^\infty(\mathbf{S}^1, Q^{\rm can})$.

It is convenient to distinguish the set $\mathrm{Pol}(Q)$ of local functionals of $q(x)$ of the form $f[q(x)] = f\left(x, q(x), q'(x), \ldots, q^{(n)}(x), \ldots\right)$ for any (differential) polynomial f, and the set $\mathrm{Fun}(Q)$ of functionals of the form

$$\varphi[q] \;=\; \int_{\mathbf{S}^1} dx\, f[q(x)]\,. \tag{2.10}$$

The condition (2.8) ensures that the phase-space of the hierarchy corresponds just to the subset of local gauge invariant functionals $\mathrm{Pol}_0(Q)$ or $\mathrm{Fun}_0(Q)$, *e.g.*,

$$\mathrm{Fun}_0(Q) \;=\; \left\{\, \varphi \in \mathrm{Fun}(Q) \;\big|\; \varphi[\widetilde{q}] \;=\; \varphi[q] \,\right\}, \tag{2.11}$$

whose elements are constant on each gauge equivalence class.

One of the main properties of these integrable hierarchies is that they are Hamiltonian with respect to the Poisson bracket

$$\{\varphi\,,\,\psi\}_2[q] \;=\; \Big((d_q\varphi)_0\,,\,[(d_q\psi)_0, L]\Big) \;-\; \Big((d_q\varphi)_{<0}\,,\,[(d_q\psi)_{<0}, L]\Big)\,, \tag{2.12}$$

where $\varphi, \psi \in \mathrm{Fun}_0(Q)$, $(A)_0$ and $(A)_{<0}$ are the components of $A \in \widehat{g}$ whose $\mathbf{s}$-grade is 0 and < 0, respectively, and $d_q\varphi$ is the functional derivative of φ. Even more, for the homogeneous gradation $\mathbf{s} = (1, 0, \ldots, 0)$, the hierarchy is of the KdV type and it admits another coordinated Poisson bracket [8]

$$\{\varphi\,,\,\psi\}_1[q] \;=\; -\Big(d_q\varphi\,,\, z^{-1}\,[d_q\psi, L]\Big)\,; \tag{2.13}$$

i.e., the hierarchy has a bi-Hamiltonian structure.

In (2.12) and (2.13), we have used the natural invariant non-degenerate bilinear form on $C^\infty(\mathbf{S}^1, \widehat{g}\,)$

$$(A, B) \;=\; \int_{\mathbf{S}^1} dx \langle A(x)\,,\, B(x)\rangle_{\widehat{g}}\,, \tag{2.14}$$

where $\langle\cdot,\cdot\rangle_{\widehat{g}}$ is defined in terms of the Cartan-Killing form $\langle\cdot,\cdot\rangle$ of g as $\langle a\otimes z^n, b\otimes z^m\rangle_{\widehat{g}} = \langle a, b\rangle\, \delta_{n+m,0}$. Then, $d_q\varphi$ is specified by the equation[4]

$$\frac{d}{d\epsilon}\, \varphi[q \,+\, \epsilon r]\bigg|_{\epsilon=0} \;=\; (d_q\varphi\,,\, r) \quad \text{for all} \quad r(x) \in C^\infty(\mathbf{S}^1, Q)\,. \tag{2.15}$$

[4] $d_q\varphi$ is a function of $x \in \mathbf{S}^1$ taking values in the subalgebra $\widehat{g}_{\leq 0}(\mathbf{s})$, and it is uniquely defined by this equation only up to terms in $\widehat{g}_{\leq -i}(\mathbf{s}_w)$ [8].

In this paper, we will only be interested in the properties of the, so called, "second" Poisson bracket $\{\cdot,\cdot\}_2$. First, let us mention that both Poisson brackets are preserved by the action of G, the group of gauge transformations on $C^\infty(\mathbf{S}^1, Q)$. Let us consider the transformation (2.5), and the corresponding pullback $\widetilde{\varphi}[q] = \varphi[\widetilde{q}]$ of a generic functional φ. Then, it is straightforward to prove that $\widetilde{\{\varphi,\psi\}}_{1,2} = \{\widetilde{\varphi},\widetilde{\psi}\}_{1,2}$, which, in particular, ensures that the two Poisson brackets are well defined on the set $\mathrm{Fun}_0(Q)$ of gauge invariant functionals [8]. The precise definition of the second Poisson bracket within the framework of Hamiltonian reduction is discussed in Section 4 where, in particular, it is shown that the restriction to $\mathrm{Fun}_0(Q)$ is actually essential to ensure that this bracket is well defined.

The second and distinctive property of the second Poisson bracket is that it is invariant under the conformal transformation $x \mapsto y(x)$ together with

$$\begin{aligned} q(x) &= \sum_{j<i} q^j(x)\, e_j \\ \longmapsto \breve{q}(y) &= \sum_{j<i} (y'(x))^{j/i-1}\, q^j(x)\, e_j \;-\; \frac{1}{i}\, \frac{y''(x)}{y'^2(x)}\, h_{\mathbf{s}_w}\,, \end{aligned} \tag{2.16}$$

where $\{e_j \mid e_j \in Q \cap \widehat{g}_j(\mathbf{s}_w)\}$ is a $\mathbf{s}_w$-graded basis of Q [8][5]. This conformal transformation induces a corresponding transformation on $\mathrm{Fun}_0(Q)$, the space of gauge invariant functionals on Q, which suggests that the second Poisson bracket algebra could be a classical extended (chiral) conformal algebra. Notice that this possibility at least requires the existence of an energy-momentum tensor $T(x) \in \mathrm{Pol}_0(Q)$, such that the infinitesimal transformation of $\varphi \in \mathrm{Fun}_0(Q)$ corresponding to $y(x) = x + \epsilon(x)$ is given by $\delta_\epsilon \varphi = \{T_\epsilon, \varphi\}_2$ with $T_\epsilon = \int_{S^1} dx\, \epsilon(x)\, T(x)$ [1].

Another important property is that the two Poisson brackets admit non-trivial centres. In fact, by construction, the $\mathbf{s}$-grade of the components of $q(x)$ is bounded, and there exists a non-negative integer n such that

$$\Lambda \,+\, q(x) \,\in\, \bigoplus_{j=0}^{n} \widehat{g}_j(\mathbf{s})\,. \tag{2.17}$$

Then, it follows from (2.12), and (2.13) that the components of $q(x)$ whose $\mathbf{s}$-grade equals n are centres of the two Poisson bracket algebras. However, in general they are not gauge invariant centres The existence of gauge invariant centres has been established in [1,8]. For $\{\cdot,\cdot\}_2$, there may be centres only if the $\mathbf{s}_w$-grade of Λ is $i > 1$ and, then, they are in

[5] The equations of the integrable hierarchy are not invariant under this conformal transformation, but only under the scale transformations corresponding to the particular case $y(x) = \lambda x$.

one-to-one correspondence with the elements of [1]

$$\mathcal{Z} \;=\; \Big[\mathrm{Ker}(\mathrm{ad}\,\Lambda) \cap \widehat{g}_{\,i-1}(\mathbf{s}_w)\Big] \;\cup\; \Big[\mathrm{Cent}\big(\mathrm{Ker}(\mathrm{ad}\,\Lambda)\big) \cap \Big[\bigoplus_{j=1}^{i-1} \widehat{g}_{\,j}(\mathbf{s}_w)\Big]\Big]\,. \tag{2.18}$$

When Λ is regular, notice that $\mathcal{Z}$ consists just of those elements of $\mathcal{H}[w]$ whose $\mathbf{s}_w$-grade is > 0 and $< i$.

In the next sections, we will investigate the relation between the second Poisson bracket algebra and $\mathcal{W}$-algebras restricting ourselves to the case when $n = 1$ in (2.17). According to the Theorem 2 of [1], this restriction is required to ensure that the second Poisson bracket algebra can be related to a $\mathcal{W}$-algebra and not just to a Kac-Moody algebra; apart from this constraint, our results will be completely general. The non-vanishing component of Λ with zero $\mathbf{s}$-grade, $(\Lambda)_0$, is a nilpotent element of $\widehat{g}_{\,0}(\mathbf{s})$, and there exists $J_- \in \widehat{g}_{\,-i}(\mathbf{s}_w) \cap \widehat{g}_{\,0}(\mathbf{s})$ such that $J_+ = (\Lambda)_0$, J_-, and $J_0 = [J_+, J_-]$ close an $A_1 = sl(2,\mathbb{C})$ subalgebra of $\widehat{g}_{\,0}(\mathbf{s})$ [1]:

$$[J_0\,,\, J_\pm] \;=\; \pm\, J_\pm\,, \quad [J_+\,,\, J_-] \;=\; J_0\,; \tag{2.19}$$

J_0 and $h_{\mathbf{s}}$ live in the same Cartan subalgebra of $g \simeq g \otimes 1$ for any $\mathbf{s}$. Actually, this $sl(2,\mathbb{C})$ subalgebra specifies the $\mathcal{W}$-algebra that will be related to the second Poisson bracket.

Within the above mentioned restriction, the second Poisson bracket becomes

$$\begin{aligned}\{\varphi\,,\,\psi\}_2[q] \;&=\; \Big((d_q\varphi)_0\,,\, [(d_q\psi)_0\,, \partial_x + J_+ + (q(x))_0]\Big)\\ &=\; \Big([(d_q\varphi)_0\,, (d_q\psi)_0]\,,\; J_+ + (q(x))_0\Big) \;+\; \Big(\partial_x (d_q\varphi)_0\,, (d_q\psi)_0\Big)\,,\end{aligned} \tag{2.20}$$

which, now, is invariant under a more general class of conformal transformations than (2.16). Let us consider a generic $\mathbb{Z}$-gradation of $\widehat{g}$, $\mathbf{s}^*$, such that $[J_+, h_{\mathbf{s}^*} - iJ_0] = 0$, and a $\mathbf{s}^*$-graded basis $\{e_j^* \mid e_j^* \in \widehat{g}_{\,j}(\mathbf{s}^*)\}$ of Q. Then, the second Poisson bracket is invariant under the conformal transformation $x \mapsto y(x)$ together with

$$\begin{aligned}(q(x))_0 \;&=\; \sum_j q^{*j}(x)\, e_j^*\\ \longmapsto (\check{q}(y))_0 \;&=\; \sum_j (y'(x))^{j/i-1}\; q^{*j}(x)\, e_j^* \;-\; \frac{1}{i}\, \frac{y''(x)}{y'^2(x)}\, h_{\mathbf{s}^*}\,.\end{aligned} \tag{2.21}$$

This can be proved by following Sec. 4 of [8] and using that

$$\partial_x \,+\, J_+ \,+\, (q(x))_0 \;=\; y'(x)\, U[y]\Big(\partial_y \,+\, J_+ \,+\, (\check{q}(y))_0\Big)\, U^{-1}[y]\,, \tag{2.22}$$

with $U[y] = (y'(x))^{-\frac{1}{i}h_{s^*}}$. Notice that, now, the components of $q_1(x)$ are just centres of the second Poisson bracket, and this is the reason why their conformal transformation is not specified by (2.21). Actually, these centres pose the main problem in proving the existence of an energy-momentum tensor for (2.21) [1].

Therefore, the second Poisson bracket algebra has conformal properties with respect to each of those conformal transformations and the crucial question is whether it is a genuine $\mathcal{W}$-algebra with respect to any of them. This means that it has to be generated by an energy-momentum tensor and primary fields for some particular choice of $\mathbf{s}^*$, which, according to the already known results [1,5,23], should be the gradation induced by $h_{\mathbf{s}^*} = iJ_0$.

3. Elementary aspects of the reduction of Poisson manifolds.

As explained in the previous section, the condition (2.8) ensures the possibility of choosing a gauge slice $Q^{\rm can} \subset Q$ such that the phase-space of the integrable hierarchies of [7,8] is identified with $C^\infty(\mathbf{S}^1, Q^{\rm can})$. The characteristic properties of $Q^{\rm can}$ are the following: $Q = [\Lambda, P] + Q^{\rm can}$ and, moreover, for any potential $q(x) \in C^\infty(\mathbf{S}^1, Q)$ there exists a unique $q(x)$-dependent gauge transformation that takes $q(x)$ to a unique element $q^{\rm can}[q(x)] \in C^\infty(\mathbf{S}^1, Q^{\rm can})$,

$$q(x) \;\mapsto\; q^{\rm can}[q(x)] \;=\; \exp\left({\rm ad}\, S^{\rm can}[q(x)]\right) \Big(\partial_x + \Lambda + q(x)\Big) \;-\; \partial_x \;-\; \Lambda\,; \tag{3.1}$$

obviously, $S^{\rm can}[q(x)] = 0$ and $q^{\rm can}[q(x)] = q(x)$ for $q(x) \in C^\infty(\mathbf{S}^1, Q^{\rm can})$. Then, if (2.8) is satisfied, the use of the Drinfel'd-Sokolov procedure allows one to choose $Q^{\rm can}$ such that $S^{\rm can}[q(x)]$ and $q^{\rm can}[q(x)]$ are local functionals of $q(x)$, *i.e.*, elements of ${\rm Pol}(Q)$. Since $q^{\rm can}[q(x)]$ is gauge invariant, its components and their derivatives provide a basis for ${\rm Pol}_0(Q)$ and, correspondingly, for ${\rm Fun}_0(Q)$.

All this shows that there is a one-to-one map between the set ${\rm Fun}_0(Q)$ of gauge invariant functionals of $q(x)$ and the set of functionals ${\rm Fun}(Q^{\rm can})$, and, therefore, that the second Poisson bracket (2.12) induces a new bracket $\{\cdot,\cdot\}^*$ on ${\rm Fun}(Q^{\rm can})$ such that the Poisson algebras $\big({\rm Fun}(Q^{\rm can})\,,\;\{\cdot,\cdot\}^*\big)$ and $\big({\rm Fun}_0(Q)\,,\;\{\cdot,\cdot\}_2\big)$ are isomorphic. This way, all the gauge invariant information of the original phase-space, and, in particular, its conformal properties, is transferred to the new Poisson algebra $\big({\rm Fun}(Q^{\rm can})\,,\;\{\cdot,\cdot\}^*\big)$.

In the next sections, it will be useful to have a formal general definition of the induced Poisson bracket. Let M be a manifold and let G be a Lie group such that each element

$g \in G$ defines a smooth action $\Phi_g : M \to M$ on the manifold M; for the sake of clarity, we will assume that M is finite dimensional, even though the resulting expressions are valid in the general case too. We will be interested in the case when $\big(\mathcal{O}, \{\cdot,\cdot\}\big)$ is a Poisson algebra, where $\mathcal{O}$ is the set of G-invariant C^∞ functions on M, and $\{\cdot,\cdot\}$ is a Poisson bracket in $\mathcal{O}$. Then, let us consider the canonical projection $\pi : M \to M/G$ on the set of equivalence classes M/G. Since the elements of $\mathcal{O}$ are constant on each equivalence class, its pullback map $\pi^* : C^\infty(M/G) \to \mathcal{O}$ is actually a one-to-one map that assigns to any function $\hat{\phi}$ on M/G the G-invariant function $\pi^*(\hat{\phi}) = \hat{\phi} \circ \pi$. This leads to the definition of the induced Poisson bracket

$$\{\hat{\phi}\,,\, \hat{\psi}\}^* \;=\; (\pi^*)^{-1} \left(\left\{\pi^*(\hat{\phi})\,,\, \pi^*(\hat{\psi})\right\}\right) \tag{3.2}$$

on $C^\infty(M/G)$, which has the property that the Poisson algebras $(\mathcal{O}, \{\cdot,\cdot\})$ and $(C^\infty(M/G), \{\cdot,\cdot\}^*)$ are isomorphic.

Notice that, in our case, the roles of M, G, and $\mathcal{O}$ are played by $C^\infty(\mathbf{S}^1, Q)$, the group of gauge transformations, and $\mathrm{Fun}_0(Q)$, respectively. In addition, since $C^\infty(\mathbf{S}^1, Q)/G$ can be identified with $C^\infty(\mathbf{S}^1, Q^{\mathrm{can}})$, the canonical projection is specified by $\pi\left(q(x)\right) = q^{(\mathrm{can})}[q(x)]$, and its pullback map is the isomorphism

$$\hat{\varphi}[q^{\mathrm{can}}] \in \mathrm{Fun}(Q^{\mathrm{can}}) \;\longrightarrow\; \pi^*(\hat{\varphi})[q] \;=\; \hat{\varphi}[q^{\mathrm{can}}[q]] \in \mathrm{Fun}_0(Q)\,. \tag{3.3}$$

In Section 5, it will be shown that Q^{can} can be chosen such that it contains the set of generators of the $\mathcal{W}$-algebra associated to the $sl(2,\mathbb{C})$ subalgebra of $\widehat{g}_{\,0}(\mathbf{s})$ characterised by $J_+ = (\Lambda)_0$, *i.e.*, all the lowest weights corresponding to the adjoint action of the $sl(2,\mathbb{C})$ subalgebra on $\widehat{g}_{\,0}(\mathbf{s})$. Then, it will be possible to define another Poisson bracket on $\mathrm{Fun}(Q^{\mathrm{can}})$: the Dirac bracket that especifies the $\mathcal{W}$-algebra. Since our main objective is to relate those two, a priori different, Poisson brackets, it will be convenient to briefly summarize the main features of the Dirac bracket. Again, for the sake of clarity, we will restrict ourselves to the case of finite dimensional manifolds, and we will follow the nice review included in [22]; more detailed discussions can be found, for instance, in [24].

Let us consider a Poisson manifold $\big(\mathcal{M}, \{\cdot,\cdot\}\big)$ and a submanifold $\widetilde{M}$ that is the zero-set of a collection of constraints $\{\phi^\mu\} \subset C^\infty(\mathcal{M})$, *i.e.*,

$$\widetilde{M} \;=\; \big\{p \in \mathcal{M} \;\big|\; \phi^\mu(p) = 0 \quad \mathrm{for\ \ all} \quad \mu\big\}\,. \tag{3.4}$$

When $\det\big(\overline{\{\phi^\mu, \phi^\nu\}}\big) \neq 0$, where the bar indicates the restriction to $\widetilde{M}$, one can define the following Poisson bracket on $C^\infty(\widetilde{M})$

$$\{\hat{\varphi}\,,\, \hat{\psi}\}^{\mathrm{D}} \;=\; \overline{\{\varphi\,,\, \psi\} \;-\; \{\varphi\,,\, \phi^\mu\}\, \Delta_{\mu\nu}\, \{\phi^\nu\,,\, \psi\}}\,, \tag{3.5}$$

which was originally considered by Dirac [25] and, therefore, it is referred to as the Dirac bracket. In (3.5), $\Delta_{\mu\,\nu}$ is the inverse matrix of $\Delta^{\mu\,\nu} = \{\phi^\mu, \phi^\nu\}$, $\hat{\varphi}, \hat{\psi} \in C^\infty(\widetilde{M})$, and φ and ψ are two arbitrary C^∞ functions on M such that $\overline{\varphi} = \hat{\varphi}$ and $\overline{\psi} = \hat{\psi}$. Notice that the characteristic property of the right-hand-side of (3.5) is that $\{\cdot, \phi^\sigma\} - \{\cdot, \phi^\mu\}\Delta_{\mu\,\nu}\{\phi^\nu, \phi^\sigma\} = 0$ for any constraint ϕ^σ.

The relation between the Dirac bracket and eq. (3.2) arises when the reduction from $\mathcal{M}$ to $\widetilde{M}$ is a Hamiltonian reduction by imposing first-class constraints. Consider a submanifold $M \subset \mathcal{M}$ that can be given as the zero-set of a collection of first-class constraints $\{\phi^i\} \subset C^\infty(\mathcal{M})$, *i.e.*,

$$M \;=\; \{p \in \mathcal{M} \bigm| \phi^i(p) \;= 0 \;\text{ for all } i\} \quad \text{and} \quad \{\phi^i \;,\; \phi^j\}\Big|_{\phi^k=0} \;=\; 0\,. \tag{3.6}$$

Then, it is well known that the Poisson structure on $\mathcal{M}$ does not induce a Poisson structure on M, but on the set of equivalence classes M/G, where G is the group of transformations generated by the Hamiltonian vector fields associated to the constraints, *i.e.*, the infinitesimal G-transformation generated by ϕ^i is $\delta^i\,\varphi \;=\; \{\phi^i\;,\;\varphi\}$, for any $\varphi \in C^\infty(M)$. These transformations are usually called gauge transformations, and the restriction to $C^\infty(M/G)$ is required to ensure that $\phi^i = 0$ is consistent with $\{\phi^i, \cdot\} = 0$ for all ϕ^i. Finally, let us assume that it is possible to find a submanifold $\widetilde{M} \subset M$, the gauge slice, such that it has exactly one common point with every equivalence class; then, M/G can be identified with $\widetilde{M}$.

The gauge slice $\widetilde{M}$ can be completely fixed by adding some additional gauge fixing constraints $\{\chi^i\}$, *i.e.*,

$$\widetilde{M} \;=\; \{p \in \mathcal{M} \bigm| \phi^i(p) \;=\; \chi^j(p) \;=\; 0 \;\text{ for all } i, j\}\,. \tag{3.7}$$

Then, denoting by $\{\phi^\mu\} = \{\phi^i\} \cup \{\chi^j\}$ the total set of constraints, the reduced Poisson structure on $C^\infty(\widetilde{M})$ corresponds precisely to the Dirac bracket (3.5)[6]. One can easily check that the Dirac bracket is actually the Poisson bracket defined by (3.2) in this case. Since the original constraints ϕ^i are first-class, $\Delta^{\mu\,\nu}$ has the block form

$$\overline{\Delta}^{\mu\,\nu} \;=\; \overline{\{\phi^\mu, \phi^\nu\}} \;=\; \begin{array}{c c} & \begin{array}{cc} \phi & \chi \end{array} \\ \begin{array}{c} \phi \\ \chi \end{array} & \begin{pmatrix} 0 & A \\ B & C \end{pmatrix} \end{array}\,, \tag{3.8}$$

[6] Properly speaking, this is the result when $\Delta^{\mu\,\nu}$ is an invertible matrix, which is true if all the first-class constraints generate transformations on M, and, then, the number of first-class constrains ϕ^i equals the number of gauge fixing constraints χ^j. Otherwise, the first-class constraints that do not generate any transformation on $C^\infty(M)$ can be imposed directly, and one says that the result is a Poisson submanifold of $(\mathcal{M}, \{\cdot,\cdot\})$; after doing that, the previously described procedure can be applied to the remaining constraints.

which means that

$$\overline{\Delta}_{\mu\nu} \;=\; \begin{array}{c c} & \begin{array}{cc} \phi & \chi \end{array} \\ \begin{array}{c} \phi \\ \chi \end{array} & \begin{pmatrix} -B^{-1}CA^{-1} & B^{-1} \\ A^{-1} & 0 \end{pmatrix} \end{array} . \tag{3.9}$$

Now, for any $\hat{\varphi}, \hat{\psi} \in C^{\infty}(\widetilde{M})$, let us consider the gauge invariant functions $\varphi = \pi^*(\hat{\varphi})$ and $\psi = \pi^*(\hat{\psi})$, which obviously satisfy that $\{\varphi, \phi^i\} = \{\psi, \phi^i\} = 0$ for all i. This way, taking into account (3.9), the second term involved in the definition of the Dirac bracket vanishes and

$$\{\hat{\varphi}\,,\,\hat{\psi}\}^{\mathrm{D}} \;=\; \overline{\{\pi^*(\hat{\varphi})\,,\,\pi^*(\hat{\psi})\}} \;=\; \overline{\{\hat{\varphi}\,,\,\hat{\psi}\}^* \circ \pi} \;=\; \{\hat{\varphi}\,,\,\hat{\psi}\}^*\,, \tag{3.10}$$

where we have used (3.2), and that the restriction of π to $\widetilde{M}$ is the identity map.

4. The second Poisson bracket as a reduced bracket.

It is generally recognized that the most effective and systematic available method to construct $\mathcal{W}$-algebras consists in the Hamiltonian reduction of current algebras [2,3,4,5,6]. Within this method, there is a $\mathcal{W}$-algebra associated to each embedding of $A_1 = sl(2,\mathbb{C})$ into a simple Lie algebra g. The components of the reduced current correspond to the lowest weights in the decomposition of g under the adjoint action of the $sl(2,\mathbb{C})$ subalgebra, and the $\mathcal{W}$-algebra is the Dirac bracket algebra associated to the reduction. In the next sections, we will establish the relation between the second Poisson bracket algebra of the integrable hierarchies of [7,8] and certain $\mathcal{W}$-algebras constructed from $\widehat{g}_0(\mathbf{s})$. Notice that, in general, $\widehat{g}_0(\mathbf{s})$ is a finite reductive Lie algebra, but the restriction of the bilinear form $\langle\cdot,\cdot\rangle_{\widehat{g}}$ of $\widehat{g}$ provides a non-degenerate invariant bilinear form for $\widehat{g}_0(\mathbf{s})$.

We start our proof by showing that the form of the second Poisson bracket follows from the reduction of certain Poisson manifold by imposing first-class constraints, which implies that the set of invariant functionals with respect to the group of transformations generated by those constraints is a Poisson manifold with respect to the second Poisson bracket. Nevertheless, when the $\mathbf{s}_w$-grade of Λ is $i > 1$, it will be apparent that the second Poisson bracket algebra is defined only on a subset of those invariant functionals, or, equivalently, that an additional reduction is generally required to obtain $\big(\mathrm{Fun}_0(Q), \{\cdot,\cdot\}_2\big) \simeq \big(\mathrm{Fun}(Q^{\mathrm{can}}), \{\cdot,\cdot\}^*\big)$.

Let $C^{\infty}(\mathbf{S}^1, \widehat{g}\,)$ be the set of periodic functions $J(x)$ of $x \in \mathbf{S}^1$ taking values on the loop algebra $\widehat{g}$, and let us consider the endomorphism $R_{\mathbf{s}} : \widehat{g} \to \widehat{g}$ defined by

$$R_{\mathbf{s}}(A) \;=\; \frac{1}{2}\Big((A)_{\geq 0} - (A)_{<0}\Big)\,, \qquad A \in \widehat{g}\,. \tag{4.1}$$

$R_{\mathbf{s}}$ satisfies the modified Yang-Baxter equation [8,19], which means that it is a classical r-matrix [26]. Therefore, it defines a different Lie algebra structure on $\widehat{g}$ whose Lie bracket is

$$\begin{aligned} [A\,,\,B]_{R_{\mathbf{s}}} &= [R_{\mathbf{s}}(A)\,,\,B] + [A\,,\,R_{\mathbf{s}}(B)] \\ &= [(A)_{\geq 0}\,,\,(B)_{\geq 0}] - [(A)_{<0}\,,\,(B)_{<0}]\,, \end{aligned} \tag{4.2}$$

which satisfies the Jacobi identities as a consequence of the modified Yang-Baxter equation.

Next, we will show that the form of the second Poisson bracket (2.12) follows from the reduction of the Kirillov-Poisson bracket

$$\{\varphi\,,\,\psi\}_{R_{\mathbf{s}}}[J] = \Big([d_J\varphi\,,\,d_J\psi]_{R_{\mathbf{s}}}\,,\,J(x)\Big) + \Big(\partial_x(d_J\varphi)_0\,,\,(d_J\psi)_0\Big) \tag{4.3}$$

by imposing first-class constraints. This Poisson bracket corresponds to the untwisted affinization —in x— of the infinite Lie algebra specified by $[\cdot,\cdot]_{R_{\mathbf{s}}}$ on $\widehat{g} = g\otimes\mathbb{C}[z,z^{-1}]$, and it is defined on the set $\mathrm{Fun}(\widehat{g})$ of functionals of the form $\varphi[J] = \int_{\mathbf{S}^1} f(x,J(x),J'(x),\ldots)$, for any differential polynomial $f(x,J(x),J'(x),\ldots)\in\mathrm{Pol}(\widehat{g})$. The last term on the right hand side of (4.3) is a central extension, and it is worth noticing that it can also be written as [19,26]

$$\Big(\partial_x(d_J\varphi)_0\,,\,(d_J\psi)_0\Big) = \Big(\partial_x R_{\mathbf{s}}(d_J\varphi)\,,\,d_J\psi\Big) + \Big(\partial_x d_J\varphi\,,\,R_{\mathbf{s}}(d_J\psi)\Big)\,. \tag{4.4}$$

To compare with the brief review of the previous section, notice that $C^\infty(\mathbf{S}^1,\,\widehat{g})$ now plays the role of $\mathcal{M}$. Then, the space of potentials of the integrable hierarchy, $C^\infty(\mathbf{S}^1,Q)$, is the zero-set of the constraints

$$\phi_\theta[J] = (\theta(x)\,,\,J(x)-\Lambda) = \int_{\mathbf{S}^1} dx\,\langle\theta(x)\,,\,J(x)-\Lambda\rangle_{\widehat{g}} \in \mathrm{Fun}(\widehat{g})\,, \tag{4.5}$$

for any $\theta(x)\in C^\infty(\mathbf{S}^1,\Gamma)$, where Γ is the subspace

$$\begin{aligned} \Gamma &= \widehat{g}_{>0}(\mathbf{s})\cup\widehat{g}_{\leq -i}(\mathbf{s}_w) \\ &= \widehat{g}_{>0}(\mathbf{s})\cup\Big[\widehat{g}_{<0}(\mathbf{s})\cap\widehat{g}_{\leq -i}(\mathbf{s}_w)\Big]\cup\Big[\widehat{g}_{0}(\mathbf{s})\cap\widehat{g}_{\leq -i}(\mathbf{s}_w)\Big]\,, \end{aligned} \tag{4.6}$$

i.e., $C^\infty(\mathbf{S}^1,Q)$ is the set of functions $q(x) = J(x)-\Lambda$ such that $\phi_\theta[\Lambda+q]=0$ for any $\theta(x)\in C^\infty(\mathbf{S}^1,\Gamma)$.

Let us check that those constraints are first-class. Using that $d_J\phi_\theta = \theta(x)$, the Poisson bracket of two constraints is

$$\{\phi_\theta\,,\,\phi_\gamma\}_{R_{\mathbf{s}}}[J] = \big([\theta(x),\gamma(x)]_{R_{\mathbf{s}}}\,,\,J(x)\big) + \big(\partial_x(\theta(x))_0\,,\,(\gamma(x))_0\big)\,. \tag{4.7}$$

But, taking into account (4.6), one can check that $\big([\theta(x),\gamma(x)]_{R_{\mathbf{s}}}\big)_{\leq 0} \in \widehat{g}_{\leq -2i}(\mathbf{s}_w)$, which ensures that $\big([\theta(x),\gamma(x)]_{R_{\mathbf{s}}},\Lambda\big) = 0$. Moreover, the last term, $\big(\partial_x(\theta(x))_0,(\gamma(x))_0\big)$ also vanishes because both $(\theta(x))_0$ and $(\gamma(x))_0$ have $\mathbf{s}_w$-grade $\leq -i$. In conclusion,

$$\{\phi_\theta\, ,\, \phi_\gamma\}_{R_{\mathbf{s}}} \;=\; \phi_{[\theta,\gamma]_{R_{\mathbf{s}}}}\, , \quad \text{and} \quad [\theta(x)\, ,\, \gamma(x)]_{R_{\mathbf{s}}} \in C^\infty(\mathbf{S}^1,\Gamma)\, , \tag{4.8}$$

which proves that the constraints imposed by ϕ_θ with $\theta \in C^\infty(\mathbf{S}^1,\Gamma)$ are actually first-class.

The restriction of the Poisson-Lie bracket (4.3) to functions of the form $J(x) = \Lambda + q(x)$, with $q(x) \in C^\infty(\mathbf{S}^1, Q)$, is precisely the second Poisson bracket (2.12):

$$\{\varphi\, ,\, \psi\}_{R_{\mathbf{s}}}[J] \Bigg|_{J(x)=\Lambda+q(x)} \;=\; \{\varphi\, ,\, \psi\}_2[q]\, , \tag{4.9}$$

which, as we have explained in the previous section, is only well defined on the set of invariant functionals under the group of transformations generated by the first-class constraints. Let $\varphi \in \mathrm{Fun}(Q)$ and $\theta(x) \in C^\infty(\mathbf{S}^1,\Gamma)$, then, the infinitesimal transformation of φ generated by the constraint ϕ_θ is

$$\begin{aligned} \delta_\theta\varphi[q] \;&=\; \{\phi_\theta\, ,\, \varphi\}_2[q] \\ &=\; -\Big((d_q\varphi)_0\, ,\, [(\theta(x))_0\, ,\, \partial_x + \Lambda + q(x)]\Big) \\ &\qquad -\; \Big((d_q\varphi)_{<0}\, ,\, [\Lambda + q(x)\, ,\, (\theta(x))_{<0}]\Big)\, . \end{aligned} \tag{4.10}$$

However, taking into account (4.6), it follows that $[\Lambda + q(x), (\theta(x))_{<0}] \in \widehat{g}_{\leq 0}(\mathbf{s}_w) \subset \widehat{g}_{\leq 0}(\mathbf{s})$, and, hence, the last term on the right hand side of (4.10) vanishes.

It is convenient to split the subspace Γ in the following two disjoint subsets

$$\Gamma \;=\; \Gamma_0 \,\cup\, \Gamma_{\neq 0}\, , \tag{4.11}$$

where

$$\Gamma_0 \;=\; \widehat{g}_{0}(\mathbf{s}) \cap \widehat{g}_{\leq -i}(\mathbf{s}_w) \quad \text{and} \quad \Gamma_{\neq 0} \;=\; \widehat{g}_{>0}(\mathbf{s}) \cup \Big[\widehat{g}_{<0}(\mathbf{s}) \cap \widehat{g}_{\leq -i}(\mathbf{s}_w)\Big]\, ; \tag{4.12}$$

Γ_0 is either empty or a nilpotent subalgebra of $\widehat{g}_{0}(\mathbf{s})$. Then, (4.10) proves that $\mathrm{Fun}(Q)$ is already invariant with respect to the transformations generated by all the constraints associated to $\Gamma_{\neq 0}$. On the contrary, when Γ_0 is not empty, the vector fields $\{\phi_\theta, \cdot\}_2$ with $\theta(x) \in C^\infty(\mathbf{S}^1, \Gamma_0)$ generate non-trivial transformations on $\mathrm{Fun}(Q)$ that can be understood as the infinitesimal form of

$$q(x) \to \widetilde{q}(x) \;=\; \exp\left(\mathrm{ad}\, U(x)\right)\, \left(\partial_x \,+\, \Lambda \,+\, q(x)\right) \,-\, \partial_x - , \Lambda \tag{4.13}$$

for any $U(x) \in C^\infty(\mathbf{S}^1, \Gamma_0)$. Then, the second Poisson bracket $\{\cdot,\cdot\}_2$ is only well defined on those functionals which are invariant under these transformations.

An important observation is that Γ_0 is a subset of P (see (2.6)), but $\Gamma_0 \neq P$ whenever $i > 1$. This means that, in general, the group of transformations generated by the first-class constraints (4.5) is only a subgroup of the group of gauge transformations of the integrable hierarchy, and, consequently, the phase-space of the integrable hierarchy is actually a subset of the set of functionals which are left invariant by those first-class constraints. Therefore, our discussion in the previous paragraph ensures that the second Poisson bracket algebra is well defined[7]. Nevertheless, when $i > 1$ and, consequently, $P \neq \Gamma_0$, it also shows that the second Poisson bracket algebra is the restriction of $\{\cdot,\cdot\}_2$ to $\mathrm{Fun}_0(Q)$, which means that an additional reduction is required to obtain $\{\cdot,\cdot\}^*$.

The different roles of the two subsets of constraints associated to $\Gamma_{\neq 0}$ and Γ_0 suggest to describe the reduction leading from (4.3) to (2.12) as a two steps process. The first step would be the reduction of (4.3) with the first-class constraints associated to $\Gamma_{\neq 0}$, which leads to a reduced current of the form

$$J(x) \;-\; \Lambda \in \widehat{g}_{\,0}(\mathbf{s}) \cup \big[\, \widehat{g}_{\,>0}(\mathbf{s}) \cap \widehat{g}_{\,<i}(\mathbf{s}_w)\big]\;. \tag{4.14}$$

Notice that there is no restriction at all on the components of $J(x)$ in the subalgebra $\widehat{g}_{\,0}(\mathbf{s})$. Therefore, only the component of Λ whose $\mathbf{s}$-grade is positive is relevant at this step, and (4.14) is equivalent to $J(x) = (\Lambda)_{>0} + \kappa(x)$, where $\kappa(x) \in C^\infty(\mathbf{S}^1, Q^\bullet)$ with

$$Q^\bullet \;=\; \widehat{g}_{\,0}(\mathbf{s}) \cup \big[\, \widehat{g}_{\,>0}(\mathbf{s}) \cap \widehat{g}_{\,<i}(\mathbf{s}_w)\big]\;. \tag{4.15}$$

Since the first-class constraints associated to $\Gamma_{\neq 0}$ do not generate any transformation on the reduced manifold, the bracket (4.3) induces a well defined Poisson structure on $C^\infty(\mathbf{S}^1, Q^\bullet)$ and, correspondingly, on the set of functionals $\mathrm{Fun}(Q^\bullet)$:

$$\begin{aligned} \{\varphi,\psi\}^\bullet[\kappa] \;=\; \{\varphi,\psi\}_{R_{\mathbf{s}}}[J]\Big|_{J(x)=(\Lambda)_{>0}+\kappa(x)} &= \Big((d_\kappa\varphi)_0\;,\; \big[(d_\kappa\psi)_0, \partial_x + (\kappa(x))_0\big]\Big) \\ &- \Big((d_\kappa\varphi)_{<0}\;,\; \big[(d_\kappa\psi)_{<0}, (\Lambda + \kappa(x))_{>0}\big]\Big)\;; \end{aligned} \tag{4.16}$$

in other words, $\big(\mathrm{Fun}(Q^\bullet), \{\cdot,\cdot\}^\bullet\big)$ is a Poisson submanifold of $\big(\mathrm{Fun}(\,\widehat{g}\,), \{\cdot,\cdot\}_{R_{\mathbf{s}}}\big)$.

Once we have imposed the constraints associated to $\Gamma_{\neq 0}$, the second step is the reduction of (4.16) with the first-class constraints induced by Γ_0. Actually, this reduction is independent of the previous one, which suggests that $(\Lambda)_0$ could be generally chosen

[7] This result is equivalent to the lemma 3.2 of [8], and it is interesting to compare the proof presented there with ours.

independently of $(\Lambda)_{>0}$. Instead, in order to recover the second Poisson bracket of the integrable hierarchies of [7,8], it is constrained by the condition that $\Lambda = (\Lambda)_0 + (\Lambda)_{>0}$ is a constant graded element of $\mathcal{H}[w] \cap \widehat{g}_i(\mathbf{s}_w)$. Then, $\kappa(x) = (\Lambda)_0 + q(x)$, and the resulting reduced bracket is the second Poisson bracket (2.12), which is only well defined on the set of invariant functionals with respect to the transformations (4.13). Since $\widehat{g}_{\leq 0}(\mathbf{s}_w) \subset \widehat{g}_{\leq 0}(\mathbf{s})$, this group of transformations has the characteristic property that

$$\exp\bigl(\mathrm{ad}\, U(x)\bigr)\ \left((\Lambda)_{>0}\right) \;=\; (\Lambda)_{>0} \tag{4.17}$$

for any $U(x) \in C^\infty(\mathbf{S}^1, \Gamma_0)$. This has already been realized by the authors of [14,18], where they suggest that new reductions of affine algebras could be obtained by enlarging this group of transformations while keeping the condition (4.17).

5. A convenient choice for the gauge slice Q^{can}.

So far, we have exhibited the relation between the second Poisson bracket of the integrable hierarchies of [7,8] and the reduction of the Kirillov-Poisson bracket (4.3) by imposing first-class constraints. The result is that the second Poisson bracket is actually well defined for those functionals of $q(x)$ that are invariant under the gauge transformations generated by the elements of $C^\infty(S^1, \Gamma_0)$; nevertheless, the phase-space of the integrable hierarchies consists only of the gauge invariant functionals under the transformations generated by the elements of $C^\infty(S^1, P)$, and

$$\Gamma_0 \;=\; \widehat{g}_0(\mathbf{s}) \cap \widehat{g}_{\leq -i}(\mathbf{s}_w) \;\subseteq\; P \;=\; \widehat{g}_0(\mathbf{s}) \cap \widehat{g}_{<0}(\mathbf{s}_w)\,. \tag{5.1}$$

In the particular case when the $\mathbf{s}_w$-grade of Λ is $i=1$, Γ_0 and P coincide. Then, the second Poisson bracket algebra can be entirely understood in terms of first-class constraints and, according to Section 3, the new bracket induced on the set $\mathrm{Fun}(Q^{\mathrm{can}})$ is the associated Dirac bracket; *i.e.*, the new bracket corresponds to a $\mathcal{W}$-algebra. Actually, when $i=1$ notice that $[\Lambda, P] \in \widehat{g}_0(\mathbf{s})$ and $P = \Gamma_0$, which implies that

$$[\Lambda\,,\, P] \;=\; [(\Lambda)_0\,,\, P]\,, \tag{5.2}$$

Therefore, the condition (2.8) is equivalent to the non-degeneracy condition and, hence, this case is one of those considered in [1]. It is important to remark that most of the results of [14,15,18] are restricted to integrable hierarchies constructed from elements Λ whose $\mathbf{s}_w$-grade equals 1.

In contrast, our objective is to investigate the structure of the new bracket $\{\cdot,\cdot\}^*$ in the general situation when $i>1$. Nevertheless, we are only interested in those integrable hierarchies whose second Poisson bracket algebra is expected to be related to a $\mathcal{W}$-algebra and not just to a Kac-Moody algebra; consequently, and according to the Theorem 2 of [1], we will only consider integrable hierarchies where

$$\Lambda + q(x) \in \widehat{g}_0(\mathbf{s}) \oplus \widehat{g}_1(\mathbf{s})\,, \tag{5.3}$$

and $(\Lambda)_0 \neq 0$, which, in particular, implies that Γ_0 is not empty.

Then, $(\Lambda)_0$ is a nilpotent element of $\widehat{g}_0(\mathbf{s})$, and the Jacobson-Morozov theorem [27] affirms that there exists an element $J_- \in \widehat{g}_0(\mathbf{s}) \cap \widehat{g}_{-i}(\mathbf{s}_w)$ such that $J_+ = (\Lambda)_0$, J_-, and $J_0 = [J_+, J_-]$ generate an $sl(2,\mathbb{C})$ subalgebra of $\widehat{g}_0(\mathbf{s})$. Moreover, J_0 and $h_{\mathbf{s}_w}$ live in the same Cartan subalgebra of g (properly speaking, of $g\otimes 1$), and they are related in such a way that $Y = h_{\mathbf{s}_w} - iJ_0$ commutes with $J_\pm$ [28]; actually, J_0, $J_\pm$, and Y generate a $sl(2,\mathbb{C})\oplus u(1)$ subalgebra of $\widehat{g}_0(\mathbf{s})$. Under the adjoint action of this subalgebra, $\widehat{g}_0(\mathbf{s})$ decomposes as the direct sum of a finite number of irreducible representations

$$\widehat{g}_0(\mathbf{s}) \;=\; \bigoplus_{k=1}^{n} D_{j_k}(y_k)\,, \tag{5.4}$$

where j_k and y_k are the $sl(2,\mathbb{C})$ spin and the eigenvalue of Y that label each representation. In addition, the decomposition (5.4) ensures that $\widehat{g}_0(\mathbf{s})$ has the following two orthogonal decompositions with respect to the bilinear form $\langle\cdot,\cdot\rangle_{\widehat{g}}$ [8]

$$\begin{aligned} \widehat{g}_0(\mathbf{s}) \;&=\; \mathrm{Ker}(\mathrm{ad}\, J_+) \oplus \mathrm{Im}(\mathrm{ad}\, J_-) \\ &=\; \mathrm{Ker}(\mathrm{ad}\, J_-) \oplus \mathrm{Im}(\mathrm{ad}\, J_+)\,, \end{aligned} \tag{5.5}$$

where $\mathrm{Ker}(\mathrm{ad}\, J_\pm)$ is the subset of highest $(+)$ or lowest $(-)$ weights with respect to the $sl(2,\mathbb{C})$ subalgebra. Since $Y = h_{\mathbf{s}_w} - iJ_0$ commutes with $J_\pm$, all the subspaces $\mathrm{Ker}(J_\pm)$ and $\mathrm{Im}(J_\pm)$ are stable under the adjoint action of $h_{\mathbf{s}_w}$ and, hence, they can be decomposed under the $\mathbb{Z}$-gradation $\mathbf{s}_w$.

Recall that J_0 also induces a $\mathbb{Z}/2$-gradation of $\widehat{g}_0(s)$ where the elements of $D_{j_k}(y_k)$ have grades $\{-j_k, -j_k+1, \ldots, j_k-1, j_k\}$, while their corresponding $\mathbf{s}_w$-grades are $\{-ij_k+y_k,\ -i(j_k-1)+y_k,\ \ldots,\ i(j_k-1)+y_k,\ ij_k+y_k\}$. However, in general, it is not possible to compare the graded subspaces of $\widehat{g}_0(\mathbf{s})$ corresponding to this gradation and to $\mathbf{s}_w$.

[8] From now on, it will be assumed that the operators $(\mathrm{ad}\, J_\pm)$ are restricted to $\widehat{g}_0(\mathbf{s})$, even though we will not explicitly indicate it.

Let $i_1, \ldots, i_p$ be all the indices for which $s_{i_1} = \cdots = s_{i_p} = 0$, *i.e.*, the set of vanishing components of $\mathbf{s}$. Then, $\widehat{g}_0(\mathbf{s})$ is the reductive finite Lie algebra

$$\widehat{g}_0(\mathbf{s}) \;=\; h_1 \oplus \cdots \oplus h_q \oplus u(1)^{\oplus(\mathrm{rank}(g)-p)}\,, \tag{5.6}$$

where $h_1 \oplus \cdots \oplus h_q$ is the semisimple Lie algebra whose Dynkin diagram is the subdiagram of the extended Dynkin diagram of g consisting of the vertices $i_1, \ldots, i_p$. Then, the problem is that the raising (lowering) operators associated to the vertices $i_1, \ldots, i_p$ do not always have non-positive (non-negative) grade in the gradation induced by J_0. For example, this means that one cannot ensure that $\mathrm{Ker}(\mathrm{ad}\, J_-)$ is a subset of $\widehat{g}_{\leq 0}(\mathbf{s}_w)$, even though the grade of the elements of $\mathrm{Ker}(\mathrm{ad}\, J_-)$ is ≤ 0 in the gradation induced by J_0.

Concerning the potential, eq. (5.3) implies that it can have non vanishing components in $C^\infty(\mathbf{S}^1, Q \cap \widehat{g}_1(\mathbf{s}))$ when $i > 1$, *i.e.*,

$$q(x) \;=\; (q(x))_0 \;+\; (q(x))_1\,, \tag{5.7}$$

Nevertheless, even if $i > 1$, there will be cases when this component is absent. For instance, if $\mathbf{s} = (1, 0, \ldots, 0)$ is the homogeneous gradation and $i \leq (\mathbf{s}_w)_0$ then $(q(x))_1 = 0$; actually, in these cases, $i = (\mathbf{s}_w)_0$ is the lowest possible positive $\mathbf{s}_w$-grade of the elements of $\mathcal{H}[w]$ [1]. From now on, we will also use the notation

$$\Lambda \;=\; J_+ \;+\; \lambda_1\,, \quad \text{where } \lambda_1 \in \widehat{g}_1(\mathbf{s}) \cap \widehat{g}_i(\mathbf{s}_w)\,. \tag{5.8}$$

When (5.3) is fulfilled, the second Poisson bracket simplifies to (2.20), which resembles the Kirillov-Poisson bracket corresponding to the untwisted affinization —in x— of the finite Lie algebra $\widehat{g}_0(\mathbf{s})$, where the affine $\widehat{g}_0(\mathbf{s})$ current has been reduced to $J(x) = J_+ + (q(x))_0$. The difference is that, in (2.20), φ and ψ are gauge invariant functionals not only of $(q(x))_0$, but also of $(q(x))_1$ when $i > 1$. Then, even though the components of $(q(x))_1$ are centres of $\{\cdot,\cdot\}_2$, the (infinitesimal) gauge invariance condition for $\varphi \in \mathrm{Fun}_0(Q)$ (see (2.7))

$$\begin{aligned}\delta_S\varphi \;&=\; \varphi[\widetilde{q}] - \varphi[q] \;=\; (d_q\varphi\,,\, [S(x), \partial_x + \Lambda + q(x)])\\ &=\; ((d_q\varphi)_0\,,\, [S(x), \partial_x + J_+ + (q(x))_0])\\ &\qquad +\; ((d_q\varphi)_{-1}\,,\, [S(x), \lambda_1 + (q(x))_1]) \;=\; 0\end{aligned} \tag{5.9}$$

relates the dependence on $(q(x))_0$ and on $(q(x))_1$.

As we have explained in Section 2, the phase-space of the integrable hierarchies of [7,8] is the set $\mathrm{Fun}_0(Q)$ of gauge invariant functionals. In this case, the gauge transformations (2.5) can be expressed as

$$\begin{aligned}(q(x))_0 \to (\widetilde{q}(x))_0 \;&=\; \exp\left(\mathrm{ad}\, S(x)\right)\,(\partial_x \;+\; J_+ \;+\; (q(x))_0) \;-\; \partial_x \;-\; J_+\,,\\ (q(x))_1 \to (\widetilde{q}(x))_1 \;&=\; \exp\left(\mathrm{ad}\, S(x)\right)\,(\lambda_1 \;+\; (q(x))_1) \;-\; \lambda_1\,,\end{aligned} \tag{5.10}$$

for any $S(x) \in C^\infty(\mathbf{S}^1, P)$. Moreover, Λ is restricted by the condition (2.8), which ensures the possibility of choosing a gauge slice $Q^{\rm can} \subset Q$ such that the set of equivalence classes $C^\infty(\mathbf{S}^1, Q)/G$ and the set of gauge invariant functionals $\mathrm{Fun}_0(Q)$ can be identified with $C^\infty(\mathbf{S}^1, Q^{\rm can})$ and $\mathrm{Fun}(Q^{\rm can})$, respectively (see eq. (3.3)).

The condition (2.8) is a weak version of the non-degeneracy condition

$$\mathrm{Ker}(\mathrm{ad}\, J_+) \cap P \;=\; \{0\} \tag{5.11}$$

that is required in the Hamiltonian reduction approach to $\mathcal{W}$-algebras to ensure polynomiality [5,23], and it plays exactly the same role here. Actually, consider the subset $\Gamma_0 = P \cap \widehat{g}_{\leq -i}(\mathbf{s}_w)$ of P. Since $\widehat{g}_{\leq 0}(\mathbf{s}_w) \subset \widehat{g}_{\leq 0}(\mathbf{s})$, the condition (2.8) implies that $\mathrm{Ker}(\mathrm{ad}\, J_+) \cap \Gamma_0 = \{0\}$, which means that all the elements of Γ_0 satisfy the non-degeneracy condition (5.11). Then, the weaker condition (2.8) permits that some elements of $P \cap \widehat{g}_{>-i}(\mathbf{s}_w)$ commute with J_+ only if they do not also commute with λ_1; therefore, (2.8) can also be expressed as

$$\big[\mathrm{Ker}(\mathrm{ad}\, J_+) \cap P\big] \cap \big[\mathrm{Ker}(\mathrm{ad}\, \lambda_1) \cap P\big] \;=\; \{0\}\,. \tag{5.12}$$

Another important consequence of (2.8) is the following. Since the non-degenerate invariant bilinear form of $\widehat{g}$ provides a one-to-one map between $\mathrm{Ker}(\mathrm{ad}\, J_+)$ and $\mathrm{Ker}(\mathrm{ad}\, J_-)$, and between $\widehat{g}_{\leq -i}$ and $\widehat{g}_{\geq i}$, the condition (2.8) implies that $\mathrm{Ker}(\mathrm{ad}\, J_-) \cap \widehat{g}_{\geq i} = \{0\}$, which ensures that all the lowest weights in the decomposition (5.4) are included in the subspace Q:

$$\mathrm{Ker}(\mathrm{ad}\, J_-) \;\subset\; Q\,. \tag{5.13}$$

The study of the Poisson bracket induced on $\mathrm{Fun}(Q^{\rm can})$ requires a convenient choice of the gauge slice, which will be achieved through the Drinfel'd-Sokolov procedure [5,7,9,22,23]. In the following, we will show that the gauge slice can be chosen such that $Q^{\rm can} \cap \widehat{g}_0(\mathbf{s}) = \mathrm{Ker}(\mathrm{ad}\, J_-)$, which is very convenient since $C^\infty(\mathbf{S}^1, \mathrm{Ker}(\mathrm{ad}\, J_-))$ is the phase-space of one of the $\mathcal{W}$-algebras obtained through the Hamiltonian reduction of the affine Lie algebra of $\widehat{g}_0(\mathbf{s})$ [2,3,4,5,6].

We start by decomposing the subalgebra P of gauge transformation generators as

$$P \;=\; \overline{P} \cup P^\star\,, \tag{5.14}$$

such that

$$P^\star \;=\; \mathrm{Ker}(\mathrm{ad}\, J_+) \cap P\,, \qquad \text{and} \qquad \mathrm{Ker}(\mathrm{ad}\, J_+) \cap \overline{P} \;=\; \{0\}\,; \tag{5.15}$$

obviously, $\overline{P} \cap P^\star = \{0\}$. The subset P^* contains those elements that do not satisfy the non-degeneracy condition (5.11), and it is always empty for $i=1$; moreover, the set Γ_0 is always a subset of $\overline{P}$. Now, let us consider the gauge transformation generated by an element $\alpha_{(-j)}(x) \in C^\infty(\mathbf{S}^1, P \cap \widehat{g}_{-j}(\mathbf{s}_w))$. According to (5.14), this element can be decomposed as $\alpha_{(-j)}(x) = \overline{\alpha}_{(-j)}(x) + \alpha^\star_{(-j)}(x)$, where $\overline{\alpha}_{(-j)}(x) \in C^\infty(\mathbf{S}^1, \overline{P})$ and $\alpha^\star_{(-j)}(x) \in C^\infty(\mathbf{S}^1, P^\star)$, and the gauge transformation is just

$$(\widetilde{q}(x))_0 \;=\; (q(x))_0 \;-\; [J_+, \overline{\alpha}_{(-j)}(x)] \;+\; \cdots \,, \tag{5.16}$$

$$(\widetilde{q}(x))_1 \;=\; \Big((q(x))_1 \;-\; [\lambda_1, \overline{\alpha}_{(-j)}(x)] \Big) \;-\; [\lambda_1, \alpha^\star_{(-j)}(x)] \;+\; \cdots \,, \tag{5.17}$$

where the dots indicate terms whose $\mathbf{s}_w$-grade is $< i-j$. Notice that this transformation does not change the components of $q(x)$ whose grade is $> i-j$. Then, considering (5.5) and (5.13), it immediately follows that $\overline{\alpha}_{(-j)}(x)$ can be fixed uniquely with eq. (5.16) by the condition that the component of $(\widetilde{q}(x))_0$ whose $\mathbf{s}_w$-grade equals $i-j$ lives in $\mathrm{Ker}(\mathrm{ad}\, J_-) \cap \widehat{g}_{i-j}(\mathbf{s}_w)$. Once $\overline{\alpha}_{(-j)}(x)$ is known, $\alpha^\star_{(-j)}(x)$ can be fixed using eq. (5.17) and some appropriate choice of $Q^{\mathrm{can}} \cap \widehat{g}_1(\mathbf{s})$.

If we denote by $-p$ the lowest $\mathbf{s}_w$-grade of the elements of $\widehat{g}_0(\mathbf{s})$, all this shows that there exist unique elements $\alpha_{(-1)}(x), \ldots, \alpha_{(-p)}(x) \in C^\infty(\mathbf{S}^1, P)$ such that

$$\exp\big(\mathrm{ad}\; \alpha_{(-p)}(x)\big) \circ \cdots \circ \exp\big(\mathrm{ad}\; \alpha_{(-1)}(x)\big) \;\equiv\; \exp\left(\mathrm{ad}\; S^{\mathrm{can}}(x)\right) \tag{5.18}$$

generates a gauge transformation $q(x) \rightarrow \widetilde{q}(x) \;=\; q^{\mathrm{can}}(x)$, where $q^{\mathrm{can}}(x) \in C^\infty(\mathbf{S}^1, Q^{\mathrm{can}})$ and (see (5.13))

$$Q^{\mathrm{can}} \cap \widehat{g}_0(\mathbf{s}) \;=\; \mathrm{Ker}(\mathrm{ad}\; J_-)\,. \tag{5.19}$$

The consequence of using the Drinfel'd-Sokolov procedure is that the components of $S^{\mathrm{can}}(x)$ and $q^{\mathrm{can}}(x)$ are local functionals of $q(x)$ and, in particular, $q^{\mathrm{can}}(x)$ is a local gauge invariant functional, *i.e.*,

$$S^{\mathrm{can}}(x) \;=\; S^{\mathrm{can}}[q(x)] \in \mathrm{Pol}(Q)\,, \qquad q^{\mathrm{can}}(x) \;=\; q^{\mathrm{can}}[q(x)] \in \mathrm{Pol}_0(Q)\,. \tag{5.20}$$

A very important feature of this choice of the gauge slice is that $Q^{\mathrm{can}} \cap \widehat{g}_0(\mathbf{s})$ is completely specified by the $sl(2,\mathbb{C})$ subalgebra $(J_0, J_\pm)$, *i.e.*, by $(\Lambda)_0$; hence, it is somehow independent of the gradation $\mathbf{s}_w$ we have started with. This is important since, at the end of Section 2, we have already pointed out that there are different conformal transformations compatible with the second Poisson bracket, and that they are associated with those gradations $\mathbf{s}^*$ such that $[(\Lambda)_0, h_{\mathbf{s}^*} - iJ_0] = 0$; from this point of view, $\mathbf{s}_w$ is just a particular choice of $\mathbf{s}^*$.

6. The second Poisson bracket as a modified Dirac bracket.

As explained in Section 3, the restriction of the second Poisson bracket to the gauge invariant functionals on Q specifies a new bracket $\{\cdot,\cdot\}^*$ in $\mathrm{Fun}(Q^{\mathrm{can}})$ such that the Poisson manifolds $(\mathrm{Fun}_0(Q), \{\cdot,\cdot\}_2)$ and $(\mathrm{Fun}(Q^{\mathrm{can}}), \{\cdot,\cdot\}^*)$ are isomorphic. The new bracket is formally defined in (3.2) where, in this case, the map is

$$\pi : C^\infty(\mathbf{S}^1, Q) \to C^\infty(\mathbf{S}^1, Q^{\mathrm{can}}) \,, \qquad \pi(q(x)) \;=\; q^{\mathrm{can}}[q(x)] \,, \tag{6.1}$$

whose pullback is given by eq. (3.3). In the previous section, we have shown that the gauge slice Q^{can} can be chosen such that it contains the phase-space of the $\mathcal{W}$-algebra specified by the embedding of $sl(2,\mathbb{C})$ into $\widehat{g}_{\,0}(\mathbf{s})$ corresponding to $J_+ = (\Lambda)_0$,

$$Q^{\mathrm{can}} \;=\; \mathrm{Ker}(\mathrm{ad}\; J_-) \,\oplus\, [Q^{\mathrm{can}} \cap \widehat{g}_{\,1}(\mathbf{s})] \,. \tag{6.2}$$

Our next objective is to establish a precise relation between the bracket $\{\cdot,\cdot\}^*$ and this $\mathcal{W}$-algebra.

According to Section 4, the Poisson manifold $(\mathrm{Fun}(Q^{\mathrm{can}}), \{\cdot,\cdot\}^*)$ is a reduction of $(\mathrm{Fun}(Q^\bullet), \{\cdot,\cdot\}^\bullet)$ (see eq. (4.16)), with

$$\{\varphi,\psi\}^\bullet[\kappa] \;=\; \Big((d_\kappa\varphi)_0 \,,\, \big[(d_\kappa\psi)_0, \partial_x + (\kappa(x))_0\big] \Big) \,, \tag{6.3}$$

and $Q^\bullet = \widehat{g}_{\,0}(\mathbf{s}) \cup [\,\widehat{g}_{\,1}(\mathbf{s}) \cap \widehat{g}_{\,<i}(\mathbf{s}_w)]$. The set of functions $C^\infty(\mathbf{S}^1, Q^{\mathrm{can}})$ can be expressed as the zero-set of the following functionals

$$\phi_\theta[\kappa] \;=\; \Big(\theta(x), \kappa(x) - (\Lambda)_0\Big) \in \mathrm{Fun}(Q^\bullet) \,, \tag{6.4}$$

where, according to (5.5), $\theta(x)$ is a function of x taking values on

$$\Gamma^{\mathrm{can}} \;=\; \mathrm{Im}(\mathrm{ad}\; J_-) \cup \Upsilon \,, \tag{6.5}$$

and Υ is a subset of $\widehat{g}_{\,-1}(\mathbf{s}_w) \cap \widehat{g}_{\,\geq -i}(\mathbf{s}_w)$ that specifies the form of $Q^{\mathrm{can}} \cap \widehat{g}_{\,1}(\mathbf{s})$.

The constraints associated to Υ can be imposed directly because they do not generate any transformation on the reduced manifold, which is always a Poisson submanifold of $(\mathrm{Fun}(Q^\bullet), \{\cdot,\cdot\}^\bullet)$; this is the reason why we have not indicated any particular choice for

$Q^{\rm can} \cap \widehat{g}_1(\mathbf{s})$. Then, we are left only with the constraints associated to $\Gamma^{\rm can} \cap \widehat{g}_0(\mathbf{s})$, which can be split in the following three disjoint subsets

$$
\begin{aligned}
\Gamma^{\rm can} \cap \widehat{g}_0(\mathbf{s}) &= \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \,, \\
\Gamma_0 &= {\rm Im}({\rm ad}\ J_-) \cap \widehat{g}_{\leq -i}(\mathbf{s}_w) \\
\Gamma_1 &= {\rm Im}({\rm ad}\ J_-) \cap \Big[\widehat{g}_{>-i}(\mathbf{s}_w) \cap \widehat{g}_{<0}(\mathbf{s}_w) \Big] \\
\Gamma_2 &= {\rm Im}({\rm ad}\ J_-) \cap \widehat{g}_{\geq 0}(\mathbf{s}_w) \,;
\end{aligned}
\tag{6.6}
$$

notice that the two definitions of Γ_0 in eqs. (4.12) and (6.6) coincide because its elements satisfy the non-degeneracy condition (5.11). Moreover, the non-degenerate invariant bilinear form of $\widehat{g}$ provides a one-to-one map between Γ_2 and ${\rm Im}({\rm ad}\ J_+) \cap \widehat{g}_{\leq 0}(\mathbf{s}_w) = [J_+, \Gamma_0]$. Using again the fact that the elements of Γ_0 satisfy the non-degeneracy condition, this identification provides a one-to-one map between Γ_0 and Γ_2, which, in particular, shows that they have the same dimension as vector subspaces.

Within the Hamiltonian reduction construction of $\mathcal{W}$-algebras, they are given in terms of Dirac brackets. In our case, we can construct the Dirac bracket corresponding to the Hamiltonian reduction from $({\rm Fun}(Q^\bullet), \{\cdot,\cdot\}^\bullet)$ to $({\rm Fun}(Q^{\rm can}), \{\cdot,\cdot\}^D)$. To do that, it will be convenient to introduce x-dependent constraints. Let us choose a basis $\{\theta^j\}$ for $\Gamma^{\rm can} \cap \widehat{g}_0(\mathbf{s}_w)$ and consider the constraints

$$
\phi^i(x) \;=\; \Big\langle \theta^i \,,\, (\kappa(x))_0 - (\Lambda)_0 \Big\rangle_{\widehat{g}} \in {\rm Pol}(Q^\bullet) \,, \quad \text{for any} \quad x \in S^1 \,. \tag{6.7}
$$

Then, the Dirac bracket is given by the field theoretical version of (3.5)

$$
\begin{aligned}
\{\hat{\varphi} \,,\, \hat{\psi}\}^D &= \overline{\{\varphi \,,\, \psi\}_2} \\
&\quad - \sum_{i,j} \int_{S^1} dx\, dy\, \overline{\{\varphi \,,\, \phi^i(x)\}_2}\ \Delta_{i,j}(x,y)\ \overline{\{\phi^j(y) \,,\, \psi\}_2} \,,
\end{aligned}
\tag{6.8}
$$

where $\Delta_{i,j}(x,y)$ is the inverse of

$$
\begin{aligned}
\Delta^{i,j}(x,y)\ [q^{\rm can}] &= \overline{\{\phi^i(x) \,,\, \phi^j(y)\}_2[q]} \\
&= \Big\langle [\theta^i, \theta^j] \,,\, J_+ + (q^{\rm can}(x))_0 \Big\rangle_{\widehat{g}}\ \delta(x-y) + \Big\langle \theta^i \,,\, \theta^j \Big\rangle_{\widehat{g}} \partial_x \delta(x-y) \,,
\end{aligned}
\tag{6.9}
$$

i.e.,

$$
\sum_k \int_{S^1} dw\ \Delta_{i,k}(x,w)\ \Delta^{k,j}(w,y) \;=\; \delta_i^{\,j}\ \delta(x-y) \,, \tag{6.10}
$$

and, in general, it is a matrix differential operator. In the last equations, the bar denotes the restriction from $Q^\bullet$ to the gauge slice $Q^{\rm can}$. This restriction can be done in two steps.

First, from $Q^\bullet$ to Q, which means that $\kappa(x) - (\Lambda)_0 = q(x) \in C^\infty(S^1, Q)$, and, second, from Q to Q^{can}, *i.e.*, $q(x) \in C^\infty(S^1, Q^{(\mathrm{can})})$. This has been taken into account in (6.8) and (6.9), where we have also used that

$$\{\cdot,\cdot\}^\bullet[\kappa]\,\Big|_{\kappa=(\Lambda)_0+q} = \{\cdot,\cdot\}_2[q]\,. \tag{6.11}$$

Moreover, $\hat\varphi, \hat\psi \in \mathrm{Fun}(Q^{\mathrm{can}})$, and φ and ψ are any two functionals in $\mathrm{Fun}(Q)$ such that $\overline{\varphi} = \hat\varphi$ and $\overline{\psi} = \hat\psi$.

In particular, to calculate the Dirac bracket, one can choose $\varphi = \hat\varphi$ and $\psi = \hat\psi$. Then, since the components of $(q(x))_1$ are centres of both $\{\cdot,\cdot\}^\bullet$ and $\{\cdot,\cdot\}_2$, and since Q^{can} is a subset of Q, it follows that all the components of $(q^{\mathrm{can}}(x))_1$ are just centres of $\{\cdot,\cdot\}^D$. Therefore, the Dirac bracket (6.8) is non-degenerate only when it is restricted to $\mathrm{Fun}(Q^{\mathrm{can}}) \cap \widehat{g}_0(\mathbf{s})$, and it corresponds just to the reduction of $\big(\mathrm{Fun}(\widehat{g}_0(\mathbf{s})), \{\cdot,\cdot\}^\bullet\big)$, which is a classical realization of the centrally extended affine current algebra of $\widehat{g}_0(\mathbf{s})$, to $\big(\mathrm{Fun}(\mathrm{Ker}(\mathrm{ad}\, J_-)), \{\cdot,\cdot\}^D\big)$. All this proves that the later is the $\mathcal{W}$-algebra corresponding to $\widehat{g}_0(\mathbf{s})$ and to its $sl(2,\mathbb{C})$ subalgebra specified by $J_+ = (\Lambda)_0$.

We have defined two brackets $\{\cdot,\cdot\}^D$ and $\{\cdot,\cdot\}^*$ in the set of functionals $\mathrm{Fun}(Q^{(\mathrm{can})})$, and the most important question is how to compare them. In [1,8], it was shown that the second Poisson bracket of the integrable hierarchies of [7,8] have non-trivial centres that are in one-to-one correspondence with the elements of the set $\mathcal{Z}$ defined in (2.18). Nevertheless, the explicit example presented in [1] shows that it is not generally true that all the components of $(q^{\mathrm{can}}(x))_1$ are centres of $\{\cdot,\cdot\}^*$, which means that, in general, this bracket can be different to the Dirac bracket.

The defining property of $\{\cdot,\cdot\}^*$ is that the Poisson algebras $(\mathrm{Fun}_0(Q), \{\cdot,\cdot\}_2)$ and $(\mathrm{Fun}(Q^{\mathrm{can}}), \{\cdot,\cdot\}^*)$ are isomorphic. In contrast, the Poisson structure induced by the Dirac bracket involves the restriction to the invariant functionals with respect to the transformations generated by the Hamiltonian vector fields associated to some set of first-class constraints (see Section 3). Taking into account this, we can get another indication that the two Poisson brackets will be generally different by analysing the infinitesimal gauge invariance conditions satisfied by the elements of $\mathrm{Fun}_0(Q)$. The condition associated to $S(x) \in C^\infty(S^1, P)$ is given by eq. (5.9), and it can be expressed as

$$0 = \delta_S\varphi = \{\varphi, \phi_S\}_2[q] + \Big((d_q\varphi)_{-1}\,,\, [S(x), \lambda_1 + (q(x))_1]\Big)\,, \tag{6.12}$$

where $\phi_S[q] = \big(S(x), q(x)\big)$, which shows that all these constraints will be Hamiltonian only when the choice of Λ, $\mathbf{s}_w$ and $\mathbf{s}$ ensure that $Q \cap \widehat{g}_1(\mathbf{s}) = \{0\}$, *i.e.*, when all the components of $q(x)$ have zero $\mathbf{s}$-grade.

The explicit relation between the two Poisson brackets $\{\cdot,\cdot\}^D$ and $\{\cdot,\cdot\}^*$ can be obtained by generalizing the proof of eq. (3.10) in Section 3. The starting point will be the observation that, taking into account (6.6), $\Delta^{i,j}(x,y)$, defined in eq. (6.9), has the block form

$$\Delta^{i,j}(x,y)\,[q^{\rm can}] \;=\; \begin{array}{c|ccc} & \Gamma_0 & \Gamma_1 & \Gamma_2 \\ \hline \Gamma_0 & & & \\ \Gamma_1 & & & \\ \Gamma_2 & & & \end{array} \begin{array}{c} \\ \begin{pmatrix} 0 & 0 & A \\ 0 & B & C \\ D & E & F \end{pmatrix} \end{array} \,, \tag{6.13}$$

which implies that

$$\Delta_{i,j}(x,y)\,[q^{\rm can}] \;=\; \begin{array}{c} \\ \Gamma_0 \\ \Gamma_1 \\ \Gamma_2 \end{array} \begin{array}{c} \begin{array}{ccc} \Gamma_0 & \Gamma_1 & \Gamma_2 \end{array} \\ \begin{pmatrix} \tilde{F} & \tilde{E} & D^{-1} \\ \tilde{C} & B^{-1} & 0 \\ A^{-1} & 0 & 0 \end{pmatrix} \end{array} \,, \tag{6.14}$$

where

$$\begin{aligned} \tilde{C} &= -B^{-1}\,C\,A^{-1} \\ \tilde{E} &= -D^{-1}\,E\,B^{-1} \\ \tilde{F} &= -D^{-1}\,F\,A^{-1} \;+\; D^{-1}\,E\,B^{-1}\,C\,A^{-1} \end{aligned} \tag{6.15}$$

are matrix differential operators. In the following, it will be important that

$$\begin{aligned} B &= \langle [\theta^i,\theta^j]\,,\, J_+ + (q^{\rm can}(x))_0\rangle_{\widehat{g}}\; \delta(x-y) \\ &\equiv M^{i,j}[(q^{(\rm can)}(x))_0]\,\delta(x-y)\,, \end{aligned} \tag{6.16}$$

and, hence, that $M^{i,j}$ is just an antisymmetric $(q^{(\rm can)}(x))_0$-dependent matrix for any $\theta^i,\theta^j\in\Gamma_1$.

Now, for any $\hat{\varphi},\hat{\psi}\in C^\infty(S^1,Q^{\rm can})$, let us consider the gauge invariant functions $\varphi[q] = \pi^*(\hat{\varphi})[q] = \hat{\varphi}[q^{\rm can}[q]]$ and $\psi[q] = \pi^*(\hat{\psi})[q] = \hat{\psi}[q^{\rm can}[q]]$, which, obviously, satisfy eq. (6.12). Then

$$\{\varphi,\phi^i(x)\}_2[q] \;=\; -\Big\langle (d_q\varphi)_{-1}\,,\,[\theta^i,\lambda_1 + (q(x))_1]\Big\rangle_{\widehat{g}}\,, \tag{6.17}$$

for any $\theta^i\in\Gamma_0\cup\Gamma_1\subset P$, and, in particular, the right hand side of (6.17) vanishes when $\theta^i\in\Gamma_0$. Taking into account all this, eq. (6.8) becomes

$$\begin{aligned} \{\hat{\varphi},\hat{\psi}\}^*[q^{\rm can}] &= \overline{\{\varphi,\psi\}_2[q]} \\ &= \{\hat{\varphi},\hat{\psi}\}^D[q^{\rm can}] \;+\; \mathcal{C}(\hat{\varphi},\hat{\psi})\,[q^{\rm can}]\,, \qquad \text{with} \end{aligned}$$

$$\begin{aligned} \mathcal{C}(\hat{\varphi},\hat{\psi})\,[q^{\rm can}] \;=\; \sum_{\theta^i,\theta^j\in\Gamma_1} \int_{S^1} dx\, \Big\langle\, \overline{(d_q\varphi)_{-1}}\,,\,[\theta^i,\lambda_1 + (q^{\rm can}(x))_1]\Big\rangle_{\widehat{g}} \\ M_{i,j}[(q^{\rm can}(x))_0]\,\Big\langle\, \overline{(d_q\psi)_{-1}}\,,\,[\theta^j,\lambda_1 + (q^{\rm can}(x))_1]\Big\rangle_{\widehat{g}}\,, \end{aligned} \tag{6.18}$$

where the antisymmetric matrix $M_{i,j}[(q^{\rm can}(x))_0]$ is the inverse of $M^{i,j}[(q^{(\rm can)}(x))_0]$ (see eq. (6.16)).

Since the Dirac bracket $\{\cdot,\cdot\}^D$ defines a $\mathcal{W}$-algebra, eq. (6.18) explicitly shows that the second Poisson bracket of the integrable hierarchies of [7,8] is a $\mathcal{W}$-algebra modified by the (polynomial) term $\mathcal{C}(\cdot,\cdot)$, which it is the main result of this paper. Because of its importance, let us remind that we have restricted ourselves to those integrable hierarchies where $\Lambda + q(x) \in \widehat{g}_0(\mathbf{s}) \oplus \widehat{g}_1(\mathbf{s})$ and $(\Lambda)_0 \neq 0$.

6.1 The centres of $\{\cdot,\cdot\}^*$.

The components of $q^{\rm can}(x)$ generate the algebra defined by $\{\cdot,\cdot\}^*$, *i.e.*, the second Poisson bracket algebra of the integrable hierarchy. In general, we can split them in two sets. First, the components of $(q^{\rm can}(x))_0$, which also generate the $\mathcal{W}$-algebra given by the Dirac bracket $\{\cdot,\cdot\}^D$. The second set consists of the components of $(q^{\rm can}(x))_1$, which are centres of the Dirac bracket, but, in general, not of $\{\cdot,\cdot\}^*$.

The centres of $\{\cdot,\cdot\}^*$ are in one-to-one relation with the elements of the set $\mathcal{Z}$ defined in eq. (2.18). Since they have vanishing brackets with all the other generators of the second Poisson bracket algebra, they can be chosen to be zero. This is equivalent to an additional trivial reduction, which means that the resulting subset of generators form a Poisson subalgebra. Then, taking into account our choice of $Q^{\rm can}$, we conclude that the number of non-trivial generators of the second Poisson bracket algebra equals the number of generators of the associated $\mathcal{W}$-algebra plus

$$\dim(Q \cap \widehat{g}_1(\mathbf{s})) \;-\; \dim(P^*) \;-\; \dim(\mathcal{Z})\,. \tag{6.19}$$

The constraints that the centres of $\{\cdot,\cdot\}^*$ have to be zero are equivalent to impose relations among the components of $q^{\rm can}(x)$ that allow one to express some of them as polynomial functions of the other ones. To be more precise, let us recall the definition of the centres [1,7,8]. Consider the transformation

$$\begin{aligned}\mathcal{L} \;&=\; \exp\left(\mathrm{ad}\, V\right)\,(L) \;=\; L \;+\; [V,L] \;+\; \frac{1}{2}[V,[V,L]] \;+\; \cdots \\ &=\; \partial_x + \Lambda \;+\; h(x)\end{aligned} \tag{6.20}$$

that "abelianizes" the Lax operator, where $V(x) \in C^\infty\big(S^1, \mathrm{Im}(\mathrm{ad}\,\Lambda) \cap \widehat{g}_{<0}(\mathbf{s}_w)\big)$ and $h(x) \in C^\infty\big(S^1, \mathrm{Ker}(\mathrm{ad}\,\Lambda) \cap \widehat{g}_{<i}\big)$ are polynomial functionals of the components of $q(x)$ and their x-derivatives. Then, when $i>1$, the functionals of the form $\Theta_b(x) = \overline{\langle b, h(x)\rangle}_{\widehat{g}}$, for any

$$b \in \mathcal{Z}^{\vee} \;=\; \Big[\mathrm{Ker}(\mathrm{ad}\,\Lambda) \cap \widehat{g}_{\,1-i}(\mathbf{s}_w)\Big] \;\cup\; \Big[\mathrm{Cent}\big(\mathrm{Ker}(\mathrm{ad}\,\Lambda)\big) \cap \Big[\bigoplus_{j=1-i}^{-1} \widehat{g}_{\,j}(\mathbf{s}_w)\Big]\Big]\,, \qquad (6.21)$$

are gauge invariant centres of $\{\cdot,\cdot\}^*$; *i.e.*, the Θ_b's are the components of $h(x)$ along $\mathcal{Z}$. Moreover, when $b \in \mathcal{Z}^{\vee} \cap \widehat{g}_{\,-j}(\mathbf{s}_w)$, it is quite easy to check that $\Theta_b(x)$ is a polynomial functional only of the components of $q(x)$ whose $\mathbf{s}_w$ grade is $\geq j$, and that it is linear in the components of $q(x)$ whose $\mathbf{s}_w$-grade equals j. From now on, we will use the notation $\Theta_j(x) = \Theta_b(x)$ when $b \in \mathcal{Z}^{\vee} \cap \widehat{g}_{\,-j}(\mathbf{s}_w)$, even though one should introduce an additional index to indicate that $\mathcal{Z}^{\vee}$ could have more than one linearly independent element in $\widehat{g}_{\,-j}(\mathbf{s}_w)$ [9].

Since the centres are gauge invariant functionals, one can construct them directly in terms of the gauge-fixed Lax operator $L^{\mathrm{can}} = \partial_x + \Lambda + q^{\mathrm{can}}(x)$; then,

$$\Theta_j(x) \;=\; \overline{\langle b\,,\, h(x)\rangle}_{\,\widehat{g}} \;=\; \langle b\,,\, q^{\mathrm{can}}(x)\rangle_{\,\widehat{g}} \;+\; \cdots\,, \qquad (6.22)$$

where $\cdots$ indicate non-linear terms that are polynomial in the components of $q^{\mathrm{can}}(x)$ whose $\mathbf{s}_w$-grade is $> j$.

Let us now show that $\langle b\,,\, q^{\mathrm{can}}(x)\rangle_{\,\widehat{g}} \neq 0$. Notice that $\mathcal{Z} \subset Q$, which, since the bilinear form is non-degenerate, implies that b cannot be orthogonal to all the elements of Q. Moreover, since $\mathcal{Z}^{\vee} \subset \mathrm{Ker}(\mathrm{ad}\,\Lambda)$, the invariance of the bilinear form also implies that b is orthogonal to all the elements of $[\Lambda, P]$, and, according to (2.9), we conclude that $\langle b\,,\, q^{\mathrm{can}}(x)\rangle_{\,\widehat{g}} \neq 0$.

Even more, whenever $(b_{-1}) \neq 0$, one can also prove that $\langle (b)_{-1}, (q^{\mathrm{can}}(x))_1\rangle_{\,\widehat{g}} \neq 0$. Notice that $b \in \mathcal{Z}^{\vee} \subset \mathrm{Ker}(\mathrm{ad}\,\Lambda)$; then, $[b,\Lambda]=0$, which implies that $[(b)_0, J_+] + [b_{-1}, \lambda_1] = 0$. Again, let us consider eq. (2.9) and our particular choice of gauge slice:

$$Q \cap \widehat{g}_{\,1}(\mathbf{s}) \;=\; [\lambda_1\,,\, P^*] \;+\; Q_{\mathrm{can}} \cap \widehat{g}_{\,1}(\mathbf{s})\,. \qquad (6.23)$$

Since the bilinear form is non-degenerate, $(b)_{-1} \neq 0$ cannot be orthogonal to all the elements of $Q \cap \widehat{g}_{\,1}(\mathbf{s})$. But, using that the bilinear form is invariant and that $P^* \subset$

[9] In addition to these centres, the functionals $\Theta_b(x)$ where either $b \in \mathrm{Cent}\big(\mathrm{Ker}(\mathrm{ad}\,\Lambda)\big) \cap \widehat{g}_{\,0}(\mathbf{s}_w)$, if $i > 1$, or $b \in \mathrm{Ker}(\mathrm{ad}\,\Lambda) \cap \widehat{g}_{\,0}(\mathbf{s}_w)$, if $i = 1$, are constant along all the flows of the integrable hierarchy [7]. In the second reference of [14] and in [15], these constant functionals are related to the existence of "residual" gauge symmetries, which is used there to induce an additional reduction of the second Poisson bracket algebra. However, this additional reduction cannot be done by following the Drinfel'd-Sokolov procedure and, therefore, the result is a non-polynomial algebra.

Ker(ad J_+), the identity $[b_{-1}, \lambda_1] = -[(b)_0, J_+]$ implies that $(b)_{-1}$ is orthogonal to all the elements of $[\lambda_1, P^*]$ and, therefore, that $\langle (b)_{-1}, (q^{\rm can}(x))_1 \rangle_{\widehat{g}} \neq 0$

Consequently, when the centres are taken to be zero, eq. (6.22) can be iteratively used, in order of descending $\mathbf{s}_w$-grades starting with Θ_{i-1}, to eliminate all the generators of the second Poisson bracket algebra of the form $\langle b\, ,\, q^{\rm can}(x) \rangle_{\widehat{g}}$, for any $b \in \mathcal{Z}^\vee$.

A logical question is whether this procedure can always be used to express certain components of $(q^{\rm can}(x))_1$ in terms of the components of $(q^{\rm can}(x))_0$. Since $Q \subset \widehat{g}_0(\mathbf{s}) \oplus \widehat{g}_1(\mathbf{s})$, any $b \in \mathcal{Z}^\vee$ has the decomposition $b = (b)_0 + (b)_{-1}$, and, then,

$$\langle b\, ,\, q^{\rm can}(x) \rangle_{\widehat{g}} \;=\; \langle (b)_0\, ,\, (q^{\rm can}(x))_0 \rangle_{\widehat{g}} \;+\; \langle (b)_{-1}\, ,\, (q^{\rm can}(x))_1 \rangle_{\widehat{g}} \;.$$

Therefore, when the centre $\Theta_j(x)$ is associated to an element b such that $(b)_{-1} \neq 0$, it can indeed be used to eliminate that component as

$$\langle (b)_{-1}\, ,\, (q^{\rm can}(x))_1 \rangle_{\widehat{g}} \;=\; -\, \langle (b)_0\, ,\, (q^{\rm can}(x))_0 \rangle_{\widehat{g}} \;+\; \cdots \;, \tag{6.24}$$

where $\cdots$ indicate non-linear polynomial functionals of the components of $q^{\rm can}(x)$ whose $\mathbf{s}_w$-grade is $> j$.

On the contrary, whenever $(b)_{-1} = 0$, the condition $\Theta_j(x) = 0$ allows one to eliminate the component $\langle (b)_0, (q^{\rm can}(x))_0 \rangle_{\widehat{g}}$, which is one of the generators of the $\mathcal{W}$-algebra given by the corresponding Dirac bracket. This possibility only occurs if Λ is non-regular because, only in this case, Ker(ad Λ) $\cap\, \widehat{g}_{i-1}(\mathbf{s}_w)$ can contain nilpotent elements.

7. Examples.

In order to clarify the structure of the second Poisson bracket algebra, it will be convenient to distinguish the two sets of generators $W_a(y)$ and $B_a(y)$ associated to the components of $q^{\rm can}[q(x)]$ whose $\mathbf{s}$-grade equals 0 and 1, respectively. Then, for any $\omega_a \in Q^{\rm can} \cap \widehat{g}_0(\mathbf{s}) = {\rm Ker}({\rm ad}\, J_-)$ and $\beta_a \in Q^{\rm can} \cap \widehat{g}_{-1}(\mathbf{s})$, we define the gauge invariant functionals

$$W_a(y) \;=\; \Big\langle \omega_a\, ,\, q^{\rm (can)}[q(y)] \Big\rangle_{\widehat{g}} \;, \tag{7.1}$$

and

$$B_a(y) \;=\; \Big\langle \beta_a\, ,\, q^{\rm (can)}[q(y)] \Big\rangle_{\widehat{g}} \;; \tag{7.2}$$

we will also use $\hat{W}_a(y) = \overline{W_a(y)}$ and $\hat{B}_a(y) = \overline{B_a(y)}$.

This way, according to (6.18), the second Poisson bracket algebra can be expressed as

$$\begin{aligned}\{\hat{W}_a(y),\hat{W}_b(z)\}^* &= \{\hat{W}_a(y),\hat{W}_b(z)\}^D + \mathcal{C}\left(\hat{W}_a(y),\hat{W}_b(z)\right)\\ \{\hat{W}_a(y),\hat{B}_b(z)\}^* &= \mathcal{C}\left(\hat{W}_a(y),\hat{B}_b(z)\right)\\ \{\hat{B}_a(y),\hat{B}_b(z)\}^* &= \mathcal{C}\left(\hat{B}_a(y),\hat{B}_b(z)\right)\ ,\end{aligned} \tag{7.3}$$

where

$$\{\hat{W}_a(y),\hat{W}_b(z)\}^D = \sum_j P^j_{a,b}(\hat{W}_1(y),\hat{W}_2(y),\ldots)\partial_y^{(j)}(y-z) \tag{7.4}$$

is the $\mathcal{W}$-algebra associated to the $sl(2,\mathbb{C})$ subalgebra of $\widehat{g}_0(\mathbf{s})$ specified by $J_+=(\Lambda)_0$, and $P^j_{a,b}$ is a differential polynomial.

In (6.18), $\mathcal{C}(\hat{\varphi},\hat{\psi})$ arises as a consequence of the existence of non-vanishing components of $q(x)$ whose $\mathbf{s}$-grade equals 1. Actually, the factor $\mathcal{C}\left(\hat{W}_a(y),\hat{W}_b(z)\right)$ is particularly interesting since it corresponds to a possible deformation of the $\mathcal{W}$-algebra that would be induced by the dependence of $(q^{\rm can}[q(x)])_0$ on the components of $(q(x))_1$. According to eq. (3.1), $q^{\rm can}[q(x)]$ is given by

$$\begin{aligned}(q^{\rm can}[q(x)])_0 &= \exp\left(\mathrm{ad}\, S^{(\rm can)}[q(x)]\right)\left(\partial_x + (\Lambda)_0 + (q(x))_0\right)\\ &\qquad - \partial_x - (\Lambda)_0\\ (q^{\rm can}[q(x)])_1 &= \exp\left(\mathrm{ad}\, S^{(\rm can)}[q(x)]\right)\left((\Lambda)_1 + (q(x))_1\right) - (\Lambda)_1\ .\end{aligned} \tag{7.5}$$

which makes clear that $(q^{\rm can}[q(x)])_0$ can only depend on $(q(x))_1$ through $S^{\rm can}[q(x)]$. Now, considering the choice of $Q^{\rm can}$ made in Section 5, it follows that $S^{\rm can}[q(x)]$ depends on $(q(x))_1$ only when $P^*=\mathrm{Ker}(\mathrm{ad}\,J_+)\cap P\neq\{0\}$.

Therefore, $\mathcal{C}(\hat{W}_a(y),\hat{W}_b(z))$ vanishes when $P^*=\{0\}$, and, in this case, the restriction of $\{\cdot,\cdot\}^*$ to the $\hat{W}_a$'s is just the $\mathcal{W}$-algebra given by the Dirac bracket. $P^*=\{0\}$ is equivalent to the non-degeneracy condition (5.11), and this is precisely the case discussed in [1]. Now, using eq. (6.18), we can clarify the results of that paper, and this will constitute our first example.

Nevertheless, the main motivation of this work was to investigate the form of the second Poisson bracket algebras precisely when the non-degeneracy condition is not satisfied, which is illustrated by the other examples. They correspond to some integrable hierarchies associated to A_{N-1} and to the following data: $[w]=[N]$ is the conjugacy class of the Coxeter element, which means that $\mathcal{H}[w]=\mathcal{H}[N]$ is the principal Heisenberg subalgebra, $\mathbf{s}_w=(1,1,\ldots,1)$ is the principal gradation, $\mathbf{s}=(1,0,\ldots,0)$ is the homogeneous gradation, and $\Lambda\in\mathcal{H}[N]$ has grade $1<i<N$. Since these hierarchies have flow equations with

fractional scaling dimension, and following the terminology of [29], we will refer to them as fractional A_{N-1} KdV hierarchies, and we will use the notation $[N]^i$.

Even though in all these examples the non-degeneracy condition is not satisfied, their second Poisson bracket algebra is given just by the Dirac bracket, which means that all the components of $(q^{\rm can}(x))_1$ can be expressed as functionals of the components of $(q^{\rm can}(x))_0$, and that $\mathcal{C}(\cdot,\cdot)$ vanishes. We present them here because they clarify the results obtained by other authors. However, it is clear that they can hardly be used to gain intuition about the general case. The reason is that they exhibit very special features that should not be expected to hold in general. For instance, consider the semisimple element J_0 of the $sl(2,\mathbb{C})$ subalgebra of A_{N-1} specified by $J_+ = (\Lambda)_0$, which has been obtained in [5,30]; for $[N]^i$, if $N = mi + r$ with $m = [N/i]$ and $0 \leq r < i$, the result is (see eq. (3.68) of [5])

$$J_0 = \mathrm{diag}\Bigl(\underbrace{m/2,\ldots,m/2}_{r\ \text{times}}\,,\ \underbrace{(m-1)/2,\ldots,(m-1)/2}_{(i-r)\ \text{times}}\,,\ \ldots\ ,\ \underbrace{-m/2,\ldots,-m/2}_{r\ \text{times}}\Bigr)\,, \tag{7.6}$$

which means that the gradation induced by J_0 on $\widehat{g}_0(\mathbf{s}) \simeq A_{N-1}\otimes 1$ assigns non-negative grade to all the raising operators $e_1^+,\ldots,e_r^+$ corresponding to the same basis for the simple roots used in eq. (2.1). Then, the gradation induced by J_0 is comparable with the gradation $\mathbf{s}_w$ through the partial ordering defined in Section 2 and, since the principal gradation is always maximal, it follows that (see eq. (5.4))

$$\begin{aligned}
&\widehat{g}_0(\mathbf{s}_w) \subset \bigoplus_{k=1}^{n} \bigl\{X_0^{(j_k)}(y_k)\bigr\}\,,\\
&\bigoplus_{k=1}^{n} \bigl\{X_m^{(j_k)}(y_k)\ \big|\ m>0\bigr\} \subset \widehat{g}_0(\mathbf{s}) \cap \widehat{g}_{>0}(\mathbf{s}_w)\\
&\bigoplus_{k=1}^{n} \bigl\{X_m^{(j_k)}(y_k)\ \big|\ m<0\bigr\} \subset \widehat{g}_0(\mathbf{s}) \cap \widehat{g}_{<0}(\mathbf{s}_w)
\end{aligned} \tag{7.7}$$

as a very particular characteristic of all the fractional KdV A_{N-1} hierarchies. In (7.7), we have labelled the elements of the irreducible representation $D_{j_k}(y_k)$ by the eigenvalue of J_0, *i.e.*, $[J_0, X_m^{(j_k)}(y_k)] = m X_m^{(j_k)}(y_k)$. An straightforward consequence of (7.7) is that all the elements of $P^* = \mathrm{Ker}(\mathrm{ad}\,J_+) \cap \widehat{g}_{<0}(\mathbf{s}_w)$ are spin-singlets, which enables a very convenient simplification of the gauge fixing procedure that will be illustrated by our last example.

Finally, let us also point out another two important particular features exhibited by our examples. First, the number of centres equals the dimension of $Q^{\rm can} \cap \widehat{g}_1(\mathbf{s})$, and, since Λ is regular, all the components of $(q^{\rm can}(x))_1$ can be expressed as polynomial functionals of the components of $(q^{\rm can}(x))_0$. The second is that $\overline{P} = \Gamma_0 \cup \Gamma_1$ is a subalgebra of P (see eq. (5.14)), which, in eq. (6.16), implies that $\langle [\theta^i, \theta^j], (q^{\rm can}(x))_0 \rangle_{\widehat{g}} = 0$ and, hence, $M_{i,j}[(q^{\rm can}(x))_0]$ is just a constant matrix.

Going beyond these particular cases would require a detailed analysis of eqs. (6.18) and (7.3). In particular, it would be important to know if there exists an energy-momentum tensor that generates the conformal symmetry of the second Poisson bracket algebra. If this happens together with a non-trivial $\mathcal{C}(\cdot,\cdot)$, it might lead to the construction of new extended conformal algebras different that those associated to the $sl(2,\mathbb{C})$ subalgebras of a Lie algebra through Drinfel'd-Sokolov Hamiltonian reduction. Regretfully, those very interesting cases must involve large rank Lie algebras, and we have not yet been able to find any example.

7.1 *The second Poisson bracket algebra when* $\mathrm{Ker}(\mathrm{ad}\,(\Lambda)_0) \cap P = \{0\}$.

Let us start by studying the form of the matrix in eq. (6.16). Since the bilinear form of $\widehat{g}$ provides a one-to-one map between $\mathrm{Ker}(\mathrm{ad}\, J_+)$ and $\widehat{g}_{<0}(\mathbf{s}_w)$ and $\mathrm{Ker}(\mathrm{ad}\, J_-)$ and $\widehat{g}_{>0}(\mathbf{s}_w)$, respectively, the non-degeneracy condition (5.11) is equivalent to

$$\mathrm{Ker}(\mathrm{ad}\, J_-) \cap \Big[\widehat{g}_0(\mathbf{s}) \cap \widehat{g}_{>0}(\mathbf{s}_w)\Big] \;=\; \{0\}\,. \tag{7.8}$$

Then, with our choice of $Q^{\rm can}$, it follows that $Q^{\rm can} \cap \widehat{g}_0(\mathbf{s}) \subset \widehat{g}_{\leq 0}(\mathbf{s}_w)$, and, hence, eq. (6.16) simplifies to

$$M^{i,j}[(q^{\rm can}(x))_0] \;=\; \big\langle [\theta^i, \theta^j]\,,\, J_+ \big\rangle_{\widehat{g}}\;, \tag{7.9}$$

which shows that $M_{i,j}$ is simply a constant antisymmetric matrix.

The condition $P^* = \{0\}$ ensures that $S^{\rm can}[q(x)]$ is independent of $(q(x))_1$, and, hence, that $(d_q W_a(y))_{-1} = 0$. Moreover, it also allows the calculation of $(d_q B_a(y))_{-1}$. According to eq. (7.5), $B_a(y)$ is given by

$$B_a(y) \;=\; \Big\langle \beta_a\,,\, (q(y))_1 \;+\; \big[S^{\rm can}[(q(x))_0]\,,\, (q(x))_1\big] \;+\; \cdots \Big\rangle_{\widehat{g}}\;. \tag{7.10}$$

Then, using that $S^{\rm can}[q(x)]$ vanishes when $q(x) \in C^\infty(S^1, Q^{\rm can})$, and that $S^{\rm can}[q(x)]$ is independent of $(q(x))_1$, it follows that

$$\overline{(d_q B_a(y))_{-1}} \;=\; \beta_a\, \delta(x-y)\,. \tag{7.11}$$

All this shows that, when the non-degeneracy condition (5.11) is satisfied, the second Poisson bracket algebra (7.3) becomes

$$
\begin{aligned}
\{\hat{W}_a(y), \hat{W}_b(z)\}^* &= \{\hat{W}_a(y), \hat{W}_b(z)\}^D \\
\{\hat{W}_a(y), \hat{B}_b(z)\}^* &= 0 \\
\{\hat{B}_a(y), \hat{B}_b(z)\}^* &= -\sum_{\theta^i,\theta^j\in\Gamma_1} \Big\langle \beta_a \, , \, [\theta^i, \lambda_1 + (q^{\rm can}(y))_1] \Big\rangle_{\widehat{g}} \; M_{i,j} \\
& \qquad \Big\langle \beta_b \, , \, [\theta^i, \lambda_1 + (q^{\rm can}(y))_1] \Big\rangle_{\widehat{g}} \; \delta(y-z) \, ,
\end{aligned}
\tag{7.12}
$$

which shows that the $\hat{W}_a(y)$'s generate the $\mathcal{W}$-algebra corresponding to the Dirac bracket, and that the two sets of generators $\hat{W}_a(y)$ and $\hat{B}_b(y)$ are actually decoupled.

The explicit form of the second Poisson bracket given by eq. (7.12) allows one to investigate the existence of an energy-momentum tensor in this case. Since the two sets of generators $\hat{W}_a(y)$ and $\hat{B}_b(y)$ are decoupled, it has to be of the form $T(x) = \mathcal{T}(x) + \Delta T(x)$, where $\mathcal{T}(x)$ is the energy-momentum tensor of the $\mathcal{W}$-algebra generated by the $\hat{W}_a(y)$'s and $\Delta T(x)$ generates the conformal transformation of the $\hat{B}_b(y)$'s. Since $\Delta T(x)$ has to be a differential polynomial functional of $\hat{B}_b(y)$, the form of $\{\hat{B}_a(y), \hat{B}_b(z)\}^*$ shows that it will be impossible to find $\Delta T(x)$ unless all the terms that depend on $(q^{\rm can}(x))_1$ in (7.12) vanish, which, for instance, happens if the $\mathbf{s}_w$-grade of Λ is $i=2$. In such case, the relevant bracket simplifies to

$$
\begin{aligned}
\{\hat{B}_a(y), \hat{B}_b(z)\}^* &= -\sum_{\theta^i,\theta^j\in\Gamma_1} \langle \beta_a \, , \, [\theta^i, \lambda_1] \rangle_{\widehat{g}} \; M_{i,j} \, \langle \beta_b \, , \, [\theta^i, \lambda_1] \rangle_{\widehat{g}} \; \delta(y-z) \\
&\equiv \Omega_{a,b} \, \delta(y-z) \, ,
\end{aligned}
\tag{7.13}
$$

where $\Omega_{a,b}$ is an antisymmetric non-degenerate matrix, if we assume that all the centres of the second Poisson bracket have already been removed. Since Ω is non-degenerate, it is possible to choose a basis $\hat{B}_b^{\pm}(y)$ such that the only non-vanishing Poisson brackets are

$$
\{\hat{B}_a^+(y), \hat{B}_b^-(z)\}^* = \delta_{a,b} \, \delta(y-z) \, , \tag{7.14}
$$

which means that the restriction of the second Poisson bracket algebra to the $\hat{B}_b(y)$'s is just a set of decoupled "b–c" algebras. Therefore, if the total number of generators is $2N$, the corresponding energy-momentum tensor is just

$$
\Delta T(x) = -\sum_{a=1}^{N} \Big[\Delta_a \, \big(\hat{B}_a^+(x)\big)' \, \hat{B}_a^-(x) \; + \; (\Delta_a - 1) \, \hat{B}_a^+(x) \, \big(\hat{B}_a^-(x)\big)' \Big] \, , \tag{7.15}
$$

and it assigns conformal dimensions Δ_a and $1-\Delta_a$ to the generators $\hat{B}_a^+(x)$ and $\hat{B}_a^-(x)$, respectively, where the Δ_a's are completely arbitrary real numbers. As a particular example of this construction, let us refer to the KdV-hierarchy associated to $\mathcal{H}[3,3] \subset A_5^{(1)}$ discussed in [1].

7.2 The fractional $[N]^3$ hierarchies.

The fractional $[N]^2$ hierarchies have already been discussed in [1]. In these cases, Λ satisfies the condition (2.8) only when N is odd, and, then, it also satisfies the non-degeneracy condition (5.11) [30]. Therefore, its second Poisson bracket algebra is the $\mathcal{W}$-algebra associated to the $sl(2,\mathbb{C})$ subalgebra labelled by the partition

$$N = \left(\frac{N+1}{2}\right) + \left(\frac{N-1}{2}\right) , \tag{7.16}$$

which is nothing else than the fractional $\mathcal{W}_N^{(2)}$ algebra of [30,31].

Let us now consider the fractional $[N]^3$ hierarchies, where[10]

$$\Lambda = \begin{pmatrix} 0 & \mathbb{I}_{N-3} \\ z\mathbb{I}_3 & 0 \end{pmatrix} , \qquad N \geq 4 , \tag{7.17}$$

which satisfies the condition (2.8) only if $N \not\in 3\mathbb{Z}$ [1]; therefore, we restrict ourselves to this case when, moreover, Λ is regular, *i.e.*, $\mathrm{Ker}(\mathrm{ad}\,\Lambda) = \mathcal{H}[N]$. In contrast, $(\Lambda)_0$ does not satisfy the condition (5.11), and P^* is the one-dimensional subspace generated by [1]

$$\sum_{j=1}^{(N-1)/3} E_{3j,3j-1} , \tag{7.18}$$

when $N \in 1 + 3\mathbb{Z}$, and by

$$\sum_{j=1}^{(N+1)/3} E_{3j-1,3j-2} , \tag{7.19}$$

when $N \in 2 + 3\mathbb{Z}$.

Since P^* is one-dimensional, $S^{\mathrm{can}}[q(x)]$ will be a functional of the components of $(q(x))_0$ and of a single component of $(q(x))_1$. This means that $\overline{(d_q W_a)_{-1}}$ is a function of x taking values on certain one-dimensional subspace of $\widehat{g}_{\,-1}(\mathbf{s})$. Then, since $M_{i,j}[(q^{(\mathrm{can})}(x))_0]$ is antisymmetric, $\mathcal{C}\big(\hat{W}_a(y), \hat{W}_b(z)\big)$ vanishes and the restriction of the second Poisson bracket to the $\hat{W}_a(y)$'s is given just by the Dirac bracket in this case.

In particular, this means that all the components of $(q^{\mathrm{can}}(x))_1$ can be eliminated by taking the centres to be zero, which can be explicitly checked by using eq. (6.19). Since Λ is regular, $\mathcal{Z}$ is generated by the elements of the principal Heisenberg subalgebra

[10] From now on, we will use the defining representation of A_{N-1} in terms of traceless $N \times N$ matrices, and $\mathbb{I}_k$ will be the $k \times k$ identity matrix.

whose principal grade is 1 and 2, *i.e.*, $\dim(\mathcal{Z}) = 2$. Moreover, $\dim(Q \cap \widehat{g}_{\,1}(\mathbf{s})) = 3$, and $\dim(P^*) = 1$.

Therefore, the second Poisson bracket algebra corresponding to these fractional KdV hierarchies is the $\mathcal{W}$-algebra associated to the $A_1 = sl(2,\mathbb{C})$ subalgebra of A_{N-1} labelled by the partition [1]

$$N \;=\; \left(\frac{N+2}{3}\right) \;+\; \left(\frac{N-1}{3}\right) \;+\; \left(\frac{N-1}{3}\right)\,, \tag{7.20}$$

when $N \in 1 + 3\mathbb{Z}$, and

$$N \;=\; \left(\frac{N+1}{3}\right) \;+\; \left(\frac{N+1}{3}\right) \;+\; \left(\frac{N-2}{3}\right)\,, \tag{7.21}$$

when $N \in 2 + 3\mathbb{Z}$ [11].

According to [1], and using the same notation as in the Theorem 3 of that reference, the $\mathcal{W}$-algebra corresponding to $[4]^3$ is also the second Poisson bracket algebra of the integrable hierarchy associated to the conjugacy class $[w] = [2,1,1]$ of the Weyl group of A_3, with $\Lambda = \Lambda^{(1)}$, or to $[w] = [3,1]$, with $\Lambda = \Lambda^{(2)}$. Also, the $\mathcal{W}$-algebra corresponding to $[5]^3$ is the second Poisson bracket algebra of the integrable hierarchy constructed from $[w] = [2,2,1]$ and $\Lambda = \Lambda^{(1)}$. In contrast, all the $\mathcal{W}$-algebras corresponding to $[N]^3$ with $N \geq 7$ are not the second Poisson bracket algebra of any generalized integrable hierarchy constructed from a Λ that satisfies the non-degeneracy condition (5.11) (see Theorem 3 of [1]).

Finally, let us point out that, although these examples consider only some particular cases involving the A_n algebras, in general, they show that whenever the subspace P^* is one-dimensional, the restriction of the second Poisson bracket to the $\hat{W}_a(y)$'s will be given just by the Dirac bracket and, hence, that it will be the $\mathcal{W}$-algebra associated to the $sl(2,\mathbb{C})$ subalgebra of g specified by $J_+ = (\Lambda)_0$.

7.3 The fractional $[5]^4$ *hierarchy.*

In our last example, the non-degeneracy condition (5.11) is not satisfied either, but, now, $\dim(P^*) > 1$.

---

[11] After the work of [31], the name fractional $\mathcal{W}$-algebra, or $\mathcal{W}_N^l$ has been used in a quite confusing way to denote different extensions of the conformal algebra obtained by reduction of the current algebra of A_{N-1}. Since all the cases where the resulting algebra is polynomial can be associated to some $sl(2,\mathbb{C})$ subalgebra of A_{N-1} [30], we will use it to label the different $\mathcal{W}$-algebras that are related to the fractional KdV hierarchies. Nevertheless, let us notice that the $\mathcal{W}$-algebra corresponding to $[4]^3$, which, according to (7.20), is associated to the partition $4 = 2+1+1$, is just the "$\mathcal{W}_4^{(3)}$" algebra of [32].

Let us consider the fractional A_4 hierarchy corresponding to the principal Heisenberg subalgebra and to the element

$$\Lambda = \begin{pmatrix} & & & & 1 \\ z & & & & \\ & z & & & \\ & & z & & \\ & & & z & \end{pmatrix}$$

with principal grade $i = 4$, which is regular. Then, Q is of the form

$$\begin{pmatrix} * & * & * & * & \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} + z \begin{pmatrix} & & & & \\ & & & & \\ * & & & & \\ * & * & & & \\ * & * & * & & \end{pmatrix} , \tag{7.22}$$

while P is the set of lower triangular matrices.

In this case, $(\Lambda)_0$ specifies the $sl(2, \mathbb{C})$ subalgebra of A_5 generated by

$$J_+ = (\Lambda)_0 = E_{1,5}\, , \quad J_- = \frac{1}{2} E_{5,1}\, , \quad \text{and} \quad J_0 = \frac{1}{2}\left(E_{1,1} - E_{5,5}\right) , \tag{7.23}$$

which is labelled by the partition $5 = 2 + 1 + 1 + 1$, and, according to Section 5,

$$Q^{\rm can} \cap \widehat{g}_0(\mathbf{s}) = \mathrm{Ker}(\mathrm{ad}\, J_-) = \begin{pmatrix} a & & & & \\ * & * & * & * & \\ * & * & * & * & \\ * & * & * & * & \\ * & * & * & * & a \end{pmatrix} , \tag{7.24}$$

and $P^* = P \cap \mathrm{Ker}(\mathrm{ad}\, J_+)$ is generated by the three elements $E_{3,2}$, $E_{4,3}$, and $E_{4,2}$.

Let us consider eq. (6.19). Since Λ is regular, $\mathcal{Z}$ is generated by the elements of the principal Heisenberg subalgebra of $A_4^{(1)}$ whose principal grade is 1, 2, and 3; hence, $\dim(\mathcal{Z}) = 3$. Moreover, $\dim(Q \cap \widehat{g}_1(\mathbf{s})) = 6$, and $\dim(P^*) = 3$. Therefore, since $b_{-1} \neq 0$ for any $b \in \mathcal{Z}^\vee$, eq. (6.24) ensures that all the components of $(q^{\rm can}(x))_1$ can be expressed as polynomial functionals of the components of $(q^{\rm can}(x))_0$ when the centres are chosen to be zero.

In order to use eq. (6.18), we need

$$\Gamma_1 = \mathrm{Im}(\mathrm{ad}\, J_-) \cap \left[\widehat{g}_{>-4} \cap \widehat{g}_{<0}\right] = \begin{pmatrix} & & & & \\ * & & & & \\ * & & & & \\ * & & & & \\ & * & * & * & \end{pmatrix} , \tag{7.25}$$

for which we choose the following basis

$$\{\theta^1, \ldots, \theta^6\} \; = \; \{E_{2,1}, E_{3,1}, E_{4,1}, E_{5,2}, E_{5,3}, E_{5,4}\}\,. \tag{7.26}$$

Then,

$$[\theta^j, \theta^k] \; = \; \big(\delta_{j,k+3} \; - \; \delta_{j+3,k}\big)\, E_{5,1} \tag{7.27}$$

is orthogonal to all the elements of $Q^{\rm can}$, and, therefore, according to (6.16), the matrix

$$M_{j,k}[(q^{\rm can}(x))_0] \; = M_{j,k} \; = \; \delta_{j+3,k} \; - \; \delta_{j,k+3} \tag{7.28}$$

is just a constant matrix as expected; eq. (7.27) ensures that, also in this case, $\overline{P}$ is a subalgebra. Consequently, the additional term in (6.18) becomes

$$\mathcal{C}(\hat{\varphi}, \hat{\psi})[q^{\rm can}] \; = \; \sum_{j=1}^{3} \int_{S^1} dx \bigg(\big\langle \overline{(d_q\varphi)_{-1}}\,,\, [\theta^j, \lambda_1 + (q^{\rm can}(x))_1] \big\rangle_{\widehat{g}}$$

$$\big\langle \overline{(d_q\psi)_{-1}}\,,\, [\theta^{j+3}, \lambda_1 + (q^{\rm can}(x))_1] \big\rangle_{\widehat{g}} \; - \; (\varphi \leftrightarrow \psi) \bigg)\,. \tag{7.29}$$

So far, we have only specified the choice of $Q^{\rm can} \cap \widehat{g}_{\,0}(\mathbf{s})$, but, to calculate the additional term (7.29), we also need $Q^{\rm can} \cap \widehat{g}_{\,1}(\mathbf{s})$ in order to obtain the dependence of $S^{\rm can}$ in the components of $(q(x))_1$, and the form of $(q^{\rm can}(x))_1$. In this particular example this can be easily done because, as a particular feature that we have already announced,

$$P^* \; \subset \; {\rm Ker}({\rm ad}\ J_-)\,. \tag{7.30}$$

Then, since ${\rm Ker}({\rm ad}\ J_-)$ is a subalgebra of $\widehat{g}_{\,0}$, the $q(x)$-dependent gauge transformation that takes $q(x)$ to $q^{\rm can}[q(x)]$ can be constructed in the following simple way. First, let us consider a gauge transformation generated by $\overline{S}(x) \in C^\infty(S^1, \overline{P})$ such that

$$(\overline{q}(x))_0 \; = \; \exp\big({\rm ad}\, \overline{S}(x)\big) \Big(\partial_x \; + \; J_+ + \; (q(x))_0\Big) \; - \; \partial_x \; - \; J_+ \tag{7.31}$$

is an element of $C^\infty\big(S^1, {\rm Ker}({\rm ad}\ J_-)\big)$; obviously, $\overline{S}(x)$ and $(\overline{q}(x))_0$ are local functionals only of $(q(x))_0$, and, correspondingly,

$$(\overline{q}(x))_1 \; = \; \exp\big({\rm ad}\, \overline{S}(x)\big) \Big(\lambda_1 \; + \; (q(x))_1\Big) \; - \; \lambda_1\,. \tag{7.32}$$

The second step is to specify the gauge transformation generated by $S^*(x) \in C^\infty(S^1, P^*)$ that fixes the form of $Q^{\rm can} \cap \widehat{g}_{\,1}(\mathbf{s})$:

$$(q^{\rm can}(x))_1 \; = \; \exp\big({\rm ad}\ S^*(x)\big) \Big(\lambda_1 \; + \; (\overline{q}(x))_1\Big) \; - \; \lambda_1\,, \tag{7.33}$$

which makes sense because eq. (7.30) ensures that

$$(q^{\rm can}(x))_0 \;=\; \exp\bigl({\rm ad}\, S^*(x)\bigr)\Bigl(\partial_x \,+\, J_+ + \,(\overline{q}(x))_0\Bigr) \;-\; \partial_x \;-\; J_+ \tag{7.34}$$

is in $C^\infty\bigl(S^1, {\rm Ker}({\rm ad}\, J_-)\bigr)$ too. Therefore, $S^{\rm can}[q(x)]$ is given by

$$\exp\left({\rm ad}\, S^{\rm can}[q(x)]\right) \;=\; \exp\left({\rm ad}\, S^*[q(x)]\right) \circ \exp\left({\rm ad}\, \overline{S}[(q(x))_0]\right) \;, \tag{7.35}$$

and one has to remember that $\overline{S}[(q(x))_0]$, $S^*[q(x)]$, and $S^{\rm can}[q(x)]$ vanish for $q(x) \in C^\infty(S^1, Q^{\rm can})$.

Since we only need the dependence on $(q(x))_1$, we only have to obtain $S^*(x)$, which can be written as

$$S^*(x) \;=\; \alpha(x)\, E_{3,2} \;+\; \beta(x)\, E_{4,3} \;+\; \gamma(x)\, E_{4,2}\;; \tag{7.36}$$

recall that $\alpha(x)$, $\beta(x)$, and $\gamma(x)$ will be local functionals of $(q(x))_0$ and $(q(x))_1$ that vanish when $q(x) \in C^\infty(S^1, Q^{\rm can})$. According to (7.24), $(q(x))_1$ can be written as

$$(q(x))_1 \;=\; z \begin{pmatrix} & & & \\ a(x) & & & \\ d(x) & b(x) & & \\ f(x) & e(x) & c(x) & \end{pmatrix}\;; \tag{7.37}$$

then, eqs. (7.33) and (7.34) become

$$\begin{aligned}
a^{\rm can}(x) \;&=\; \overline{a}(x) \;+\; \alpha(x) \;=\; a(x) \;+\; \alpha(x) \;+\; \cdots \\
b^{\rm can}(x) \;&=\; \overline{b}(x) \;+\; \beta(x) \;-\; \alpha(x) \;=\; b(x) \;+\; \beta(x) \;-\; \alpha(x) \;+\; \cdots \\
c^{\rm can}(x) \;&=\; \overline{c}(x) \;-\; \beta(x) \;=\; c(x) \;-\; \beta(x) \;+\; \cdots \\
d^{\rm can}(x) \;&=\; \overline{d}(x) \;+\; \gamma(x) \;+\; \overline{a}(x)\, \beta(x) \;+\; \frac{1}{2}\alpha(x)\, \beta(x) \;=\; d(x) \;+\; \gamma(x) \;+\; \cdots \\
e^{\rm can}(x) \;&=\; \overline{e}(x) \;-\; \gamma(x) \;-\; \overline{c}(x)\, \alpha(x) \;+\; \frac{1}{2}\alpha(x)\, \beta(x) \;=\; e(x) \;-\; \gamma(x) \;+\; \cdots \\
f^{\rm can}(x) \;&=\; f(x) \;+\; \cdots \;,
\end{aligned} \tag{7.38}$$

where the dots correspond to products of two or more components of $\overline{S}(x)$ and $S^*(x)$ that vanish when restricted to the gauge slice and, hence, they will not contribute to $\overline{(d_q\varphi)_{-1}}$. Then, a possible choice of the gauge slice, compatible with Section 5, is specified by

$$Q^{\rm can} \cap \widehat{g}_{\,1}(\mathbf{s}) \;=\; z \begin{pmatrix} & & & \\ 0 & & & \\ 0 & 0 & & \\ * & * & * & \end{pmatrix}\;, \tag{7.39}$$

which means that

$$\begin{aligned} a^{\rm can}(x) = 0 \ \Rightarrow \alpha(x) \ &= \ - \, a(x) \ + \ \cdots \\ b^{\rm can}(x) = 0 \ \Rightarrow \beta(x) \ &= \ - \, a(x) \ - \ b(x) \ + \ \cdots \\ d^{\rm can}(x) = 0 \ \Rightarrow \gamma(x) \ &= \ - \, d(x) \ + \cdots \ , \end{aligned} \tag{7.40}$$

and that $\overline{(d_q W_a)_{-1}}$ will be a function of x taking values in the subspace of $\widehat{g}_{\,-1}(\mathbf{s})$ generated by $\{z^{-1}E_{1,3}\ ,\ \ z^{-1}E_{2,4}\ ,\ \ z^{-1}E_{1,4}\}$.

Consequently, using (7.26) and (7.29), one concludes that the additional term $\mathcal{C}(\hat{W}_a(y), \hat{W}_b(z))$ vanishes identically, and that, again, the restriction of the second Poisson bracket to the $\hat{W}_a$'s is just the $\mathcal{W}$-algebra associated to the $sl(2,\mathbb{C})$ subalgebra of A_4 labelled by the partition $5 = 2+1+1+1$. Notice that, according to [1], this integrable hierarchy has the same second Poisson bracket algebra as the hierarchies associated to the conjugacy classes $[w] = [2,1,1,1]$ and $[w] = [3,1,1]$ of the Weyl group of A_4, and to $\Lambda = \Lambda^{(1)}$ and $\Lambda^{(2)}$, respectively.

Finally, let us point out that the previous analysis can be extended for the generic fractional $[N]^{N-1}$ hierarchy, which shows that its second Poisson bracket algebra is just the $\mathcal{W}$-algebra associated to the partition $N = 2+1+\cdots+1$.

8. Conclusions.

The main result of this paper is summarized by eq. (6.18), which shows that the second Poisson bracket algebra of the integrable hierarchies of [7,8] corresponds to a deformation of one of the $\mathcal{W}$-algebras obtained by (classical) Drinfel'd-Sokolov reduction. Those integrable hierarchies of partial differential equations are associated to the non-conjugate Heisenberg subalgebras of the loop algebra $\widehat{g}$ of a finite simple Lie algebra. Their construction involves, first, the choice of a $\mathbb{Z}$-gradation $\mathbf{s}_w$ of $\widehat{g}$ that is compatible with the Heisenberg subalgebra; next, a second gradation $\mathbf{s}$ that is $\preceq \mathbf{s}_w$ with respect to a partial ordering, and, finally, a constant graded element Λ of the Heisenberg subalgebra. In this paper, we have considered the most general case that, according to [1], is expected to be related to the $\mathcal{W}$-algebras obtained by Drinfeld-Sokolov reduction. Thus, the $\mathcal{W}$-algebra that is related to the second Poisson bracket algebra is specified by the zero $\mathbf{s}$-graded (reductive) subalgebra of $\widehat{g}$ and by the $sl(2,\mathbb{C})$ subalgebra whose J_+ is the zero $\mathbf{s}$-graded component of Λ.

Although our results suggest that second Poisson bracket algebras could lead to non-trivial deformations of the already known $\mathcal{W}$-algebras, and, maybe, to different extended conformal algebras, we have not succeeded yet in finding an example exhibiting that feature, but we find no reason to exclude that possibility when considering the integrable hierarchies corresponding to large rank Lie algebras.

Instead, eq. (6.18) allows one to characterize two general families of hierarchies where the deformation is trivial, *i.e.*, where the second Poisson bracket algebra consists of a $\mathcal{W}$-algebra plus some additional generators. The first family consists of those integrable hierarchies for which the subalgebra $P^* = \mathrm{Ker}(\mathrm{ad}\, J_+) \cap \left[\widehat{g}_{\,0}(\mathbf{s}) \cap \widehat{g}_{\,<0}(\mathbf{s}_w) \right]$ is empty, and, in this case, the additional generators are not coupled to the $\mathcal{W}$-algebra. This is the only case actually considered in [1], even though the decoupling was not shown there. Notice that all the particular cases where Λ is the element of the Heisenberg subalgebra with minimum positive $\mathbf{s}_w$-grade and this grade equals $i = (\mathbf{s}_w)_0 \geq 1$ are included in this family; consequently, their second Poisson bracket algebra is just the $\mathcal{W}$-algebra associated to J_+. The second family includes all those hierarchies for wich P^* is one-dimensional, and their simple analysis by means of eq. (6.18) is a nice example of how useful that equation is. In relation to [1], this second family provide new examples of $\mathcal{W}$-algebras that can be indentified with the second Poisson bracket algebra of the integrable hierarchies of [7,8], even though we are still far from proving that this relation works for all the $\mathcal{W}$-algebras obtained through Hamiltonian reduction. Actually, it is known that the second Poisson bracket algebra of the integrable hierarchies specified by the non-degeneracy condition (5.11) can be identified with only a rather limited subset of classical $\mathcal{W}$-algebras [1,14,18]. However, a detailed analysis of the new cases covered by this paper, along the same lines of Section 6 of [1], would be required to establish more definite conclusions on this relevant question.

Finally, let us point out that, above all, the use of eq. (6.18) largely simplifies the study of the second Poisson bracket algebra of the integrable hierarchies of [7,8], and it should make possible to unveil some of their still unknown properties. For instance, in the general case, it is still unclear whether the second Poisson bracket algebra contains an energy-momentum tensor for some of the conformal transformations described in Section 2, which is a necessary condition to properly think of them as extended conformal algebras. Actually, in Sec. 7.1, we have investigated the existence of an energy-momentum tensor for the hierarchies constrained by the non-degeneracy condition, which illustrates that it is indeed a very restrictive constraint. In any case, a detailed general study of eq. (6.18), and, in particular, of the term $\mathcal{C}\{\cdot,\cdot\}$, should shed some light on this and other relevant related questions.

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