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TWISTOR PHASE SPACE DYNAMICS AND THE LORENTZ FORCE EQUATION.

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ABSTRACT

Using Lorentz force equation as an input a Hamiltonian mechanics on the non-projective two twistor phase space TxT is formulated.

Such a construction automatically reproduces dynamics of the intrinsic classical relativistic spin.

The charge appears as a dynamical variable.

It is also shown that if the classical relativistic spin function on TxT vanishes, the natural conformally invariant symplectic structure on TxT reduces to the natural symplectic structure on the cotangent bundle of the Kałuża-Klein space.

1 INTRODUCTION.

The classical motion of a relativistic electrically charged massive and spinning particle exposed to an external electromagnetic field is, in Minkowski space, described by the Lorentz-Dirac (LD) force equation and by the so called Bargmann, Michel, Telegdi (BMT) equation for the intrinsic angular momentum (the spin).
If we by X^a, P_a, S_a, F_{ab}, $m^2 := P^b P_b$, e and g denote the four-position, the four-momentum, the Pauli-Lubański four-vector, the external electromagnetic field tensor, the mass squared, the charge and the gyromagnetic ratio of the particle then these Poincaré covariant equations may be written as follows:

$$\dot{X}^a = P^a, \tag{1.1}$$

$$\dot{P}_a = eF_{ab}P^b + D_a, \tag{1.2}$$

$$\dot{S}_a = \frac{ge}{2}F_{ab}S^b + \frac{ge}{2m^2}(F_{ik}S^iP^k)P_a - \frac{1}{m^2}(\dot{P}_kS^k)P_a \tag{1.3}$$

where

$$P_aS^a = 0 \tag{1.4}$$

and

$$D_aP^a = 0 \qquad (1.5).$$

D_a is a small space-like correction four-vector (small compared with the space-like four-vector $eF_{ab}P^b$) containing higher derivatives of the external electromagnetic field F_{kl}, F^*_{kl} and terms nonlinear in the spin variable S^i [1,2].

When the particle forms (a classical limit of) an electron and the radiation damping effects are neglected the value of g equals 2 (the Dirac value).

The equations (1.1) - (1.5) are such that the mass squared and the spin squared of the particle:

$$m^2 := P_aP^a \qquad s^2 := -\frac{1}{m^2}S_aS^a \tag{1.6}$$

are constants of the motion.

The dot in (1.1) - (1.3) denotes differentiation with respect to a real parameter l which is, by virtue of (1.1), linearly related to the proper time τ of the particle by:

$$\tau = \pm ml + \tau_0 \qquad (1.7).$$

τ_0 is an arbitrary real number representing the freedom of choice of the origin of the proper time.

F_{ab} denotes the value of the external electromagnetic field evaluated at the particle's four-position X^a. Consequently, the four-position coincides with the location of the charge e.

In this paper we assume that $D_a = 0$ in (1.2) and then examine (1.1) and (1.2) using two distinct twistors as variables.

This analysis will automatically produce the BMT equation in (1.3) with $g = 2$ [4].

In the next section we give a physical interpretation to the sixteen variables corresponding to a point in the space of two twistors TxT [3,4].

In section three the free particle symplectic potential on TxT is expressed using these physical variables. The non-uniqeness of choice of the free particle Hamiltonian is disscused.

The free particle equations of motion given as a canonical flow in the phase space of two twistors are presented in twistors' Weyl spinor coordinates as well as in Poincaré covariant physically interpretable coordinates. These has been presented before [3] in a somewhat preliminary shape.

In section four a deformed Poincaré covariant symplectic structure and a deformed Poincaré scalar Hamiltonian function on TxT are presented. The new Poincaré covariant flow in TxT canonical with respect to the deformed symplectic structure and generated by the deformed Hamiltonian reproduces

(1.1) - (1.4) (with $D_a = 0$ and $g = 2$) and also produces certain additional equations of motion. The latter arise because TxT is sixteen dimensional while the number of independent variables describing the particle according to (1.1) - (1.4) is only twelve (the four-position, the four-momentum, the Pauli-Lubański spin four-vector fulfilling (1.4) and the charge).
Our attempt to interpret physically the remaining four variables is presented already in section two. However, a (partial) confirmation of the correctness of these tentative identifications is provided first when the interaction with an external electromagnetic field is "switched" on. This is done in section four.

A first version of the material contained in section four appeared in [5] where the electric charge was not defined as a dynamical variable. This weakness of the model is removed in section four of the present paper.

In the appendix the formal proof of the statements made in section four is presented.

Upper case latin letters with lower case greek indices denote twistors.
Upper case latin letters with lower case latin indices denote four-vectors and four-tensors.
Lower case greek letters with upper case latin indices (either primed or unprimed) denote Weyl spinors.
The Minkowski metric has the signature $+ - - -$.

2 PHYSICAL VARIABLES IDENTIFIED AS FUNCTIONS ON TΔT.

The symbol TxT usually denotes the direct product of two twistor spaces. However, in our investigations, we will not be using the whole of TxT but rather TΔT which from now on will denote the space TxT less its diagonal i.e. TΔT := $\{$TxT - $\{$(t, t) ϵ TxT; t ϵ T$\}\}$.

The twistor coordinates of a point in TΔT will be expressed in terms of two Weyl spinors:

$$Z^\alpha = (\omega^A,\ \pi_{A'}) \qquad and \qquad W^\alpha = (\lambda^A,\ \eta_{A'}) \tag{2.1}$$

or dually (complex conjugation):

$$\overline{Z}_\alpha = (\overline{\pi}_A,\ \overline{\omega}^{A'}) \qquad and \qquad \overline{W}_\alpha = (\overline{\eta}_A,\ \overline{\lambda}^{A'}) \tag{2.2}.$$

Using these two twistors and their twistor conjugates four independent conformally (SU(2,2)) scalar functions may be formed on TΔT [4,7,8]:

$$s_1 = Z^\alpha \overline{Z}_\alpha \qquad and \qquad s_2 = W^\alpha \overline{W}_\alpha \tag{2.3}$$

$$a = Z^\alpha \overline{W}_\alpha \qquad and \qquad \overline{a} = W^\alpha \overline{Z}_\alpha \tag{2.4}.$$

In addition, the following two Poincaré scalar functions may also be defined on TΔT:

$$f = \pi^{A'} \eta_{A'} \qquad and \qquad \overline{f} = \overline{\pi}^A \overline{\eta}_A \tag{2.5}.$$

The scalar functions introduced above may be represented by six real valued functions on TΔT given by:

$$e = s_1 + s_2 \qquad and \qquad k = s_1 - s_2 \tag{2.6}$$

$$\mid a \mid \qquad and \qquad \vartheta = arg a = -arg \overline{a} \tag{2.7}$$

$$\mid f \mid \qquad and \qquad \varphi = arg f = -arg \overline{f} \tag{2.8}.$$

Below, Poincaré covariant functions on TΔT will be identified as physical quantities according to the following recipe [4,5] (we employ here the abstract index notation according to Penrose [6]):

$$P_a := \pi_{A'} \overline{\pi}_A + \eta_{A'} \overline{\eta}_A \tag{2.9}$$

will denote a massive particle's four-momentum expressed as a sum of the four-momenta of its two massless parts.

$$X^a := \frac{1}{2}(Z^a + \overline{Z}^a) \qquad where \qquad Z^a := \frac{i}{f}(\omega^A \eta^{A'} - \lambda^A \pi^{A'}) \tag{2.10}$$

will denote a massive particle's four-position in Minkowski space time.

A massive particle's Pauli-Lubański spin four-vector will be given by:

$$S_a := \frac{k}{2}(\pi_{A'}\overline{\pi}_A - \eta_{A'}\overline{\eta}_A) + a\eta_{A'}\overline{\pi}_A + \overline{a}\pi_{A'}\overline{\eta}_A \qquad (2.11).$$

The definition in (2.11) is dictated by the assumption that a massive particle's four angular momentum should be a sum of the four angular momenta of its two massless parts (see e.g. [3]).

Note that P_a and S_a are automatically orthogonal to each other i.e. we always have $P_a S^a = 0$.

From the above it follows that the imaginary part of Z^a:

$$Y^a = \frac{1}{2i}(Z^a - \overline{Z}^a) = \tag{2.12}$$

may be written as:

$$= \frac{1}{2f\overline{f}}\Big[(a\eta^{A'}\overline{\pi}^A + \overline{a}\pi^{A'}\overline{\eta}^A) - s_1\eta^{A'}\overline{\eta}^A - s_2\pi^{A'}\overline{\pi}^A\Big] = \tag{2.13}$$

or as:

$$= \frac{1}{2f\overline{f}}(S^a - \frac{e}{2}P^a) \qquad (2.14).$$

From the definitions above it also follows that on TΔT the mass function of the particle is given by

$$m = \sqrt{2} \mid f \mid \qquad (2.15)$$

while its spinfunction by:

$$s = \sqrt{\frac{1}{4}k^2 + \mid a \mid^2} \qquad (2.16).$$

A space-like plane spanned by two mutually orthogonal unit four-vector valued functions on TΔT orthogonal to S_a and P_a:

$$E_a := \frac{i}{(m \mid a \mid)}(a\eta_{A'}\overline{\pi}_A - \overline{a}\pi_{A'}\overline{\eta}_A) \qquad (2.17).$$

$$F_a := \frac{1}{(sm \mid a \mid)}[\frac{k}{2}(a\eta_{A'}\overline{\pi}_A + \bar{a}\pi_{A'}\overline{\eta}_A) - \overline{a}a(\pi_{A'}\overline{\pi}_A - \eta_{A'}\overline{\eta}_A)] \qquad (2.18)$$

may be thought of as a polarization plane rigidly attached to the massive particle at its four-position X^a in Minkowski space [3,4].

In effect, all four four-vectors P_a/m, $S_a/(sm)$, E_a and F_a span an orthonormal tetrad rigidly attached to the particle at its four-position X^a in Minkowski space. The number of variables represented by the functions defining this tetrad is six, the number of variables represented by the scalar functions is also six, while the four-position represents four variables; sixteen variables altogether.

With these identifications the inverse relations expressing twistor coordinates in (2.1) - (2.2) as functions of the introduced Poincaré covariant physical variables and the scalars in (2.6) - (2.8) (note that according to (2.15) and (2.16) two of these scalars have a clear physical interpretation) are almost immediate.

The spinor $\pi_{A'}$ up to its phase is given by:

$$\pi_{A'}\overline{\pi}_A = \frac{1}{2}(P_a + \frac{k}{2s^2}S_a - \frac{m \mid a \mid}{s}F_a) \qquad (2.19),$$

the spinor $\eta_{A'}$ up to its phase is given by:

$$\eta_{A'}\overline{\eta}_A = \frac{1}{2}(P_a - \frac{k}{2s^2}S_a + \frac{m\mid a\mid}{s}F_a) \qquad (2.20),$$

while the phase α of the spinor $\pi_{A'}$ is given by:

$$\alpha = \frac{1}{2}(argf + arga) = \frac{1}{2}(\varphi + \vartheta) \qquad (2.21),$$

and the phase β of the spinor $\eta_{A'}$ by:

$$\beta = \frac{1}{2}(argf - arga) = \frac{1}{2}(\varphi - \vartheta) \qquad (2.22).$$

The relations in (2.21) and (2.22) follow from (2.5) and from the fact that the conformal complex valued scalar a in (2.4) may be written as:

$$a = -2Y^{AA'}\overline{\eta}_A\pi_{A'} \qquad (2.23)$$

where Y^a is a real four-vector valued function on TΔT introduced in (2.12).

The remaining spinors are given by (see (2.14) and (2.15)):

$$\omega^A = iX^{AA'}\pi_{A'} - \frac{1}{m^2}(S^{AA'}\pi_{A'} - \frac{e}{2}P^{AA'}\pi_{A'}) \qquad (2.24)$$

and

$$\lambda^A = iX^{AA'}\eta_{A'} - \frac{1}{m^2}(S^{AA'}\eta_{A'} - \frac{e}{2}P^{AA'}\eta_{A'}) \qquad (2.25).$$

3 THE FREE PARTICLE MOTION.

The two twistor space TΔT possesses a natural (free particle) symplectic structure given by [7,8]:

$$\Omega_0 = i(dZ^\alpha \wedge d\overline{Z}_\alpha + dW^\alpha \wedge d\overline{W}_\alpha) \qquad (3.1).$$

Ω_0 may be regarded as exterior derivative of a one-form γ_0 ($\Omega_0 = d\gamma_0$) given by:

$$\gamma_0 = \frac{i}{2}(Z^\alpha d\overline{Z}_\alpha - \overline{Z}_\alpha dZ^\alpha + W^\alpha d\overline{W}_\alpha - \overline{W}_\alpha dW^\alpha) \qquad (3.2).$$

Using the introduced Poincaré covariant physical functions on TΔT, γ_0 may also be written as:

$$\gamma_0 = P_j dX^j + \frac{1}{2}ed\varphi - \frac{1}{2}kd\vartheta + (\frac{k^2}{4s}F_j + \frac{\mid a \mid k}{2ms^2}S_j + \frac{\mid a \mid}{m}P_j)dE^j \qquad (3.3)$$

or equivalently

$$\gamma_0 = P_j dX^j + \frac{1}{2}ed\varphi - \frac{1}{2}kd\vartheta + \frac{k}{2m}(iM_j d\bar{M}^j - i\bar{M}_j dM^j) +$$

$$+ \frac{i\bar{a}}{m^2}M_j dP^j - \frac{ia}{m^2}\bar{M}_j dP^j \qquad (3.4)$$

where M_j is a complex null four-vector valued function on TΔT given by:

$$M_a := \pi_{A'}\bar{\eta}_A \qquad (3.4a).$$

From (3.3) or (3.4) we notice a remarkable fact that for $a = k = 0$ i.e. for the vanishing value of the spin function on TΔT, the conformally invariant symplectic potential γ_0 in (3.2) (and thereby also the symplectic structure Ω_0 in (3.1)) reduces to the natural symplectic potential (while Ω_0 reduces to the natural symplectic structure) on the cotangent bundle of the Kałuża-Klein space. This suggests that e should be identified with the electric charge of the particle.

To generate the free motion of a massive particle built up of the two twistors we used in [3] a Hamiltonian:

$$H_{0_1} = m^2 + s^2 \qquad (3.5)$$

and a somewhat modified one in [4]:

$$H_{0_2} = \frac{1}{2}(m^2 + s^2) \qquad (3.6).$$

Any such a change is of no importance as long as H_0 is a function on TΔT such that:

$$H_0 = H_0(m,\ s) \qquad (3.7).$$

The flow will always correspond to a free particle motion in Minkowski space. In fact any function such that:

$$H_0 = H_0(m,\ s,\ k,\ e) \qquad (3.8)$$

describes a free particle. As m, s, k and e are mutually (Poisson) commuting functions the different choices of H_0 may correspond to different motions of the internal physical variables represented by φ and ϑ.

But in most cases different choices of H_0 simply correspond to a reparametrization of the canonical flow lines.

At this non-quantum level there is thus quite a large freedom of choice of the free particle Hamiltonian H_0. On the quantum level of this approach one should, on the other hand, expect essential differences depending on the choice of $\hat{H}_0$.

In this paper, for simplicity, we choose H_0 as:

$$H_0 := \frac{1}{2}m^2 + (s^2 - \frac{1}{4}e^2) \qquad (3.9)$$

which written out in terms of the introduced scalar functions yields:

$$H_0 := f\overline{f} + \frac{1}{4}k^2 + a\overline{a} - \frac{1}{4}e^2 = f\overline{f} - s_1 s_2 + a\overline{a} \qquad (3.10).$$

The chosen H_0 and Ω_0 in (3.1) generate the following equations of motion in TΔT:

$$\dot{\omega}^A = -if\overline{\eta}^A + ia\lambda^A - is_2\omega^A \tag{3.11}$$

$$\dot{\pi}_{A'} = ia\eta_{A'} - is_2\pi_{A'} \tag{3.12}$$

$$\dot{\lambda}^A = if\overline{\pi}^A + i\overline{a}\omega^A - is_1\lambda^A \tag{3.13}$$

$$\dot{\eta}_{A'} = i\overline{a}\pi_{A'} - is_1\eta_{A'} \tag{3.14}$$

and their complex conjugates (c.c.).

The above equations, written out using functions representing the physical variables as previously identified, read:

$$\dot{e} = 0 \qquad and \qquad \dot{k} = 0 \tag{3.15}$$

$$\dot{\varphi} = -e \qquad and \qquad \dot{\vartheta} = 0 \tag{3.16}$$

$$\dot{X}^a = P^a \tag{3.17}$$

$$\dot{P}_a = 0 \qquad and \qquad \dot{S}_a = 0 \tag{3.18}$$

$$\dot{E}_a = 2sF_a \tag{3.19}$$

$$\dot{F}_a = -2sE_a \qquad (3.20).$$

From (3.19), (3.20) and (1.7) it follows that, with our choice of H_0, the introduced polarization plane rigidly attached to the particle rotates with an angular velocity equal to $(2s/m)$ [3].

4 MOTION IN AN EXTERNAL ELECTRO MAGNETIC FIELD.

In this section we identify the function e on TΔT with the electric charge of the particle. The deformed Poincaré covariant symplectic potential on TΔT we define as:

$$\gamma = \gamma_0 + eA_j dX^j \tag{4.1}$$

where X^a is a four vector-valued function on TΔT given by (2.10) and where $A_j = A_j(X^a)$ denotes an external electromagnetic four-potential. A_j is in this way a four-vector valued function defined on TΔT. γ_0 is given by (3.3) (or equivalently by (3.2) or (3.4)).

The external derivative of γ gives us the deformed symplectic structure on TΔT:

$$\Omega = \Omega_0 + de \wedge dX^j A_j + \frac{1}{2} eF_{jk} dX^j \wedge dX^k \tag{4.2}$$

where $F_{jk} = F_{jk}(X^a)$ denotes the electromagnetic field tensor formed from A_j. $\Omega_0 = d\gamma_0$.

Note that for $a = k = 0$, γ and thereby Ω may be regarded as a deformation of the natural symplectic potential and natural symplectic structure on the cotangent bundle of the Kałuża-Klein space.

As the deformed Hamiltonian function on TΔT we take:

$$H = H_0 + \frac{e}{m^2} F^*_{jk} S^j P^k \tag{4.3}$$

where $F^*_{jk} = F^*_{jk}(X^a)$ represents on TΔT the dual of the external electromagnetic tensor field.

It is shown in the appendix that, with respect to Ω, H generates a Poincaré covariant canonical flow in TΔT provided Maxwell's empty space equations are fulfilled at the location of the particle:

$$F^*_{[jk,n]} = 0 \qquad (4.4).$$

For future reference we note that using (2.12), (2.14), (2.15) and the skew symmetry of the dual of the external electro-magnetic field tensor the generating function H may also be written as:

$$H = H_0 + eF^*_{ik} Y^i P^k \qquad (4.5).$$

Expressed in twistor coordinates the flow canonical with respect to Ω and generated by H is given by the following equations of motion (see proof in the appendix):

$$\dot{\omega}^A = -if\overline{\eta}^A + ia\lambda^A - is_2\omega^A +$$

$$+e\mu^A{}_B Y^{BB'}\pi_{B'} + ieX^{AA'}\overline{\mu}_{A'}{}^{B'}\pi_{B'} + iC\omega^A \qquad (4.6)$$

$$\dot{\pi}_{A'} = ia\eta_{A'} - is_2\pi_{A'} + e\overline{\mu}_{A'}{}^{B'}\pi_{B'} + iC\pi_{A'} \qquad (4.7)$$

$$\dot{\lambda}^A = if\overline{\pi}^A + i\overline{a}\omega^A - is_1\lambda^A +$$

$$+e\mu^A{}_B Y^{BB'}\eta_{B'} + ieX^{AA'}\overline{\mu}_{A'}{}^{B'}\eta_{B'} + iC\lambda^A \qquad (4.8)$$

$$\dot{\eta}_{A'} = i\overline{a}\pi_{A'} - is_1\eta_{A'} + e\overline{\mu}_{A'}{}^{B'}\eta_{B'} + iC\eta_{A'} \qquad (4.9)$$

where

$$C = (F^*_{ik}Y^i P^k - A^i P_i) \qquad (4.10)$$

and where $\mu_{AB} = \mu_{AB}(X^c)$ is a spinor field corresponding to $F_{ab} = F_{ab}(X^c)$ [6]:

$$\mu_{AB} = \frac{1}{2}F_{AA'B}{}^{A'} \qquad (4.11).$$

Conversely one has [6]:

$$F_{ab} = \mu_{AB}\epsilon_{A'B'} + \overline{\mu}_{A'B'}\epsilon_{AB} \tag{4.12}$$

$$F^{*}_{ab} = i\overline{\mu}_{A'B'}\epsilon_{AB} - i\mu_{AB}\epsilon_{A'B'} \tag{4.13}$$

Written out in terms of the introduced Poincaré covariant physical variables the above equations of motion read:

$$\dot{X}^{j} = P^{j} \qquad and \qquad \dot{P}_{j} = eF_{jk}P^{k} \tag{4.14}$$

$$\dot{S}_{j} = eF_{jk}S^{k} \tag{4.15}$$

$$\dot{e} = 0 \qquad and \qquad \dot{k} = 0 \tag{4.16}$$

$$\dot{\vartheta} = 0 \tag{4.17}$$

$$\dot{\varphi} = -e - 2P_{j}A^{j} + \frac{2}{m^{2}}F^{*}_{jk}S^{j}P^{k} \tag{4.18}$$

$$\dot{E}_{j} = 2sF_{j} + eF_{kj}E^{k} \tag{4.19}$$

$$\dot{F}_{j} = -2sE_{j} + eF_{kj}F^{k} \tag{4.20}$$

As may be seen the equations in (4.14) and (4.15) are the same as those in (1.1) - (1.3) (with $D_j = 0$ and $g = 2$) while the relation in (1.4) is automatically fulfilled because of the way S_j and P_j were defined in (2.9) and (2.11).

The charge function e appears as a dynamical variable and according to (4.16) is a constant of motion.

Conformally scalar functions k and ϑ in (4.16) and (4.17) do not yet have any clear physical interpretation. They form two ((Poisson) non-commuting) constants of motion.

The first two terms in (4.18) correspond to the Aharonov-Bohm effect while the third term arises because of the non-vanishing intrinsic spin of the particle.

The motion of the polarization plane is given by (4.19) and (4.20).

Finally we note that the equations of motion in (4.6) - (4.9) may also be written in a twistor covariant way i.e. entirely in terms of Z^α and W^α:

$$\dot{Z}^\alpha = (if + l_1 f) I^{\alpha\beta} \bar{W}_\beta + (ia - \bar{c}_2) W^\alpha - (is_2 + iC + \bar{c}_3) Z^\alpha - b I^{\alpha\beta} \bar{Z}_\beta \quad (4.21)$$

$$\dot{W}^\alpha = (if + l_2 f) I^{\beta\alpha} \bar{Z}_\beta + (i\bar{a} + \bar{c}_1) Z^\alpha - (is_1 + iC - \bar{c}_3) W^\alpha - \bar{b} I^{\beta\alpha} \bar{W}_\beta \quad (4.22)$$

where $I^{\alpha\beta}$ is the so called infinity twistor and where c_1, c_2, c_3 are certain, conveniently chosen, complex valued Poincaré scalar functions on TΔT describing the external electromagnetic field (e and $\bar{f}$ are defined in (2.5) - (2.6)):

$$c_1 = \frac{e\mu^{AB}\bar{\eta}_A\bar{\eta}_B}{\bar{f}} \quad (4.23)$$

$$c_2 = \frac{e\mu^{AB}\bar{\pi}_A\bar{\pi}_B}{\bar{f}} \quad (4.24)$$

$$c_3 = -\frac{e\mu^{AB}\bar{\pi}_A\bar{\eta}_B}{\bar{f}} \quad (4.25).$$

In (4.21) - (4.22) l_1 and l_2 are real valued Poincaré scalar functions on TΔT while b is a complex valued Poincaré scalar function on TΔT given by (see (2.3) and (2.4)):

$$l_1 = -\frac{1}{m^2}[ac_2 + \bar{a}\bar{c}_2 + s_1(c_3 + \bar{c}_3)] \quad (4.26),$$

$$l_2 = \frac{1}{m^2}[\bar{a}c_1 + a\bar{c}_1 + s_2(c_3 + \bar{c}_3)] \quad (4.27),$$

$$b = \frac{1}{m^2}[a(\bar{c}_3 - c_3) - (\bar{c}_2 s_2 + c_1 s_1)] \qquad (4.28).$$

The deformed symplectic potential in (4.1), the corresponding symplectic structure in (4.2), the deformed Hamiltonian in (4.3) (or (4.5)) may all be written in a twistor covariant way i.e. entirely in terms of Z^α and W^α and the infinity twistor $I^{\alpha\beta}$. The arising expressions are, however, quite complicated and not especially illuminating from the physical point of view. In this paper we therefore omit their presentation.

5 CONCLUSIONS AND REMARKS.

In this paper we describe the dynamics of a relativistic charged particle with spin in an external electro-magnetic field using two-twistor phase space TΔT. We have shown that there exists a Hamiltonian dynamics on TΔT which after passing to space-time coordinates reproduces the Lorentz force dynamics and Bargmann-Michel-Telegdi dynamics (with $g = 2$) and also indicates connection to the Kałuża-Klein space dynamics. Conversely, one can say that there exists a sort of the square root of the Lorentz force dynamics which is realized as a Hamiltonian dynamics on TΔT.

It will be interesting to see how the quantized version of the above formalism corresponds to the Dirac equation coupled to an external electromagnetic field. It seems that the approach developed by A. Odzijewicz and his group [9-12] would be of great value here.

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7 APPENDIX; A FORMAL PROOF OF THE MAIN RESULT OF SECT. 4

In order to prove that the Hamiltonian in (4.5) and the symplectic structure Ω in (4.2) generate equations (4.6) - (4.9) which, in turn, imply (4.14) - (4.20) we have to prove that:

$$V \rfloor \Omega = -dH \qquad (A.1)$$

where H is that in (4.5) Ω is that in (4.2) and where

$$V = V_0 + V_1 \qquad (A.2)$$

where the vector-field V_0 according to (3.12)-(3.15) is given by:

$$V_0 = (-if\overline{\eta}^A + ia\lambda^A - is_2\omega^A)\frac{\partial}{\partial\omega^A} + (ia\eta_{A'} - is_2\pi_{A'})\frac{\partial}{\partial\pi_{A'}} +$$

$$+(if\overline{\pi}^A + i\overline{a}\omega^A - is_1\lambda^A)\frac{\partial}{\partial\lambda^A} + (i\overline{a}\pi_{A'} - is_1\eta_{A'})\frac{\partial}{\partial\eta_{A'}} + c.c. \qquad (A.3),$$

or using the introduced four-vector variables (see (3.16)-(3.21)):

$$V_0 = P^j\frac{\partial}{\partial X^j} - e\frac{\partial}{\partial\varphi} - 2sE^j\frac{\partial}{\partial F^j} + 2sF^j\frac{\partial}{\partial E^j} \qquad (A.3a).$$

The vector field V_1 is according to (4.6)-(4.9) given by:

$$V_1 = (e\mu^A{}_B Y^{BB'}\pi_{B'} + ieX^{AA'}\overline{\mu}_{A'}{}^{B'}\pi_{B'} + iC\omega^A)\frac{\partial}{\partial\omega^A} + c.c.+$$

$$+(e\overline{\mu}_{A'}{}^{B'}\pi_{B'} + iC\pi_{A'})\frac{\partial}{\partial\pi_{A'}} + c.c.+$$

$$+(e\mu^A{}_B Y^{BB'}\eta_{B'} + ieX^{AA'}\overline{\mu}_{A'}{}^{B'}\eta_{B'} + iC\lambda^A)\frac{\partial}{\partial\lambda^A} + c.c.+$$

$$+(e\overline{\mu}_{A'}{}^{B'}\eta_{B'} + iC\eta_{A'})\frac{\partial}{\partial\eta_{A'}} + c.c. \qquad (A.4).$$

or using the introduced four-vector variables (see (4.14)-(4.20)):

$$V_1 = eF_{jk}P^k\frac{\partial}{\partial P^j} + 2C\frac{\partial}{\partial\varphi} + F^{kj}F_k\frac{\partial}{\partial F^j} + F^{kj}E_k\frac{\partial}{\partial E^j} \qquad (A.4a).$$

To facilitate the caculations the inner product on the left hand side of (A.1) may be split into a sum of partial inner products:

$$V \rfloor \Omega = V_0 \rfloor \Omega_0 + V_0 \rfloor \Omega_1 + V_1 \rfloor \Omega_0 + V_1 \rfloor \Omega_1 \qquad (A.5)$$

where

$$\Omega = \Omega_0 + \Omega_1 \qquad (A.6)$$

$$\Omega_0 = i(dZ^\alpha \wedge d\overline{Z}_\alpha + dW^\alpha \wedge d\overline{W}_\alpha) \qquad (A.7)$$

and

$$\Omega_1 = de \wedge dX^i A_i + \frac{1}{2} e F_{ik} dX^i \wedge dX^k \qquad (A.8).$$

By assumption, which may be checked by direct calculations, one has:

$$V_0 \rfloor \Omega_0 = -d(\frac{1}{2}m^2 + s^2 - \frac{1}{4}e^2) \qquad (A.9).$$

Using the fact (see (3.16)-(3.21)) that:

$$V_0 = P^i \frac{\partial}{\partial X^i} + \qquad (A.10)$$

$$+ \text{ terms in directions linearly independent of } \tfrac{\partial}{\partial X^i}$$

and that V_0 has no component along $\frac{\partial}{\partial e}$ one obtains by direct calculations:

$$V_0 \rfloor \Omega_1 = -A_i P^i de + e F_{ik} P^i dX^k \qquad (A.11).$$

Further, tedious spinor algebra manipulations yield:

$$V_1 \rfloor \Omega_0 = -C de + e F_{ik} P^k dX^i - e F^*_{ik} d(Y^i P^k) = -C de + e F_{ik} P^k dX^i$$

$$-d(e F^*_{ik} Y^i P^k) + e Y^i P^k dF^*_{ik} + (Y^i P^k F^*_{ik}) de \qquad (A.12).$$

Using the fact that according to the equations of motion the vector components of V_1 in the direction of $\frac{\partial}{\partial X^i}$ and in the direction of $\frac{\partial}{\partial e}$ are equal to zero one gets automatically:

$$V_1 \rfloor \Omega_1 = 0 \qquad (A.13).$$

Putting together $(A.9)$, $(A.11)$, $(A.12)$, $(A.13)$ and inserting $C = (Y^i P^k F^*_{ik}) - A_i P^i$ yields:

$$V \rfloor \Omega = -d(\frac{1}{2}m^2 + s^2 - \frac{1}{4}e^2) - d(e F^*_{ik} Y^i P^k) + e Y^i P^k dF^*_{ik} = \qquad (A.14)$$

$$= -d(\frac{1}{2}m^2 + s^2 - \frac{1}{4}e^2) - d(eF^*_{ik}Y^iP^k) = -dH \qquad (A.15)$$

provided the last term in (A.14) vanishes for all choices of P_i and Y_i. That will always happen if the empty space Maxwell's equations:

$$F^*_{[ik,n]} = 0 \qquad (A.16)$$

are fulfilled at the location of the particle i.e. at its four-position in Minkowski space.

This completes the proof of our assertion.

Note that the first pair of Maxwell's equations is satisfied by virtue of the fact that the external electromagnetic field is given by means of a four-potential A_j in the expression for the symplectic one-form γ. This automatically ensures that the symplectic structure Ω is a closed two-form on TΔT.