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Spin on the 4-ball

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Abstract

Using known mode properties, the functional determinant for massless spin-half fields on the Euclidean 4-ball is calculated and shown to be different for spectral (nonlocal) and mixed (local) boundary conditions. The local result agrees with that from a conformal argument. Some higher-spin results and a sum rule are also given.

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1. Introduction

The theory of spinors in spaces with boundaries is of interest physically in connection with quantum cosmology and supergravity. (See D'Eath and Esposito [1] and Esposito [2] for some history of these questions.) In mathematics it is encountered in the spin-index theorem and the Atiyah, Patodi and Singer η spectral asymmetry function, the standard reference being Gilkey's book, [3].

As explained in [1], for self-adjointness of the Dirac operator, there is a choice between spectral and local (mixed) boundary conditions, the former being of relevance for the spin-index and the latter having more physical significance in connection with supersymmetry, string theory and quantum gravity, [4,5], although in the guise of relative conditions they do have a cohomological importance, [3,6].

In the special case of the Euclidean 4-ball, it was shown [7–9] that the value of $\zeta(0)$, which determines the scaling of the theory, was the same for both sets of conditions. In this note we report on the same question for the one-loop effective action, which is, up to factors, $\zeta'(0)$. Our method will be that explained in [10].

2. Mode properties and calculation

The analysis of the modes of the massless Dirac equation on the 4-ball was carried out by D'Eath and Esposito [1,7] and we will do no more here on this matter than use their results. For local boundary conditions they found that the eigenvalues, α^2, are the roots of the equation

$$F_p^L(\alpha) = J_{p-1}^2(\alpha) - J_p^2(\alpha) = 0 \tag{1}$$

with a degeneracy, for a given p, of $p^2 - p$, $p = 1, 2, \ldots$. For spectral conditions, there is the simpler, scalar-like condition,

$$F_p^S(\alpha) = J_p(\alpha) = 0 \tag{2}$$

with degeneracy $2(p^2 + p)$, $p = 1, 2, \ldots$.

Our approach is based on the Mittag-Leffler decomposition,

$$z^{-\beta} F_p(z) = \gamma \prod_\alpha \left(1 - \frac{z^2}{\alpha^2}\right), \tag{3}$$

where

$$\beta = p, \quad \gamma = \frac{1}{2^p p!}, \quad \text{spectral,}$$

$$\beta = 2(p-1), \quad \gamma = \frac{1}{\big(2^{p-1}(p-1)!\big)^2}, \quad \text{local.}$$

This standard decomposition was earlier employed by Moss [11] and by D'Eath and Esposito [1] when looking at the heat-kernel expansion and $\zeta(0)$. Here, when finding $\zeta'(0)$, we need the normalising factor, γ, which follows from the small-z behaviour of $F_p(z)$.

A few details of the calculation will be given but, for brevity, some of our previous work must be utilised.

Bypassing a number of steps, which are fully explained in [10,12], we define the quantities

$$G_N \sim \sum_{p=1}^{\infty} p^N \left[(p-\frac{1}{2}) \ln \frac{2p}{p+\epsilon} + (\epsilon - p) + \sum_{n=1}^{N+1} \left(\frac{E_n(t)}{\epsilon^n} - \frac{E_n(1)}{p^n} \right) + I_N(p) \right] \quad (4)$$

and

$$H_N \sim \sum_{p=1}^{\infty} p^N \left[p \ln \frac{2p}{p+\epsilon} + \epsilon - p - \frac{1}{2} \ln \frac{\epsilon}{p} + \sum_{n=1}^{N+1} \left(\frac{T_n(t)}{\epsilon^n} - \frac{T_n(1)}{p^n} \right) + I_N(p) \right], \quad (5)$$

with

$$I_N(p) = \int_0^\infty \left(\frac{1}{2} - \frac{1}{\tau} + \sum_{k=1}^{[N/2]+1} (-1)^k B_{2k} \frac{\tau^{2k-1}}{(2k)!} + \frac{1}{e^\tau - 1} \right) \frac{e^{-\tau p}}{\tau} \, d\tau,$$

in terms of which we can write the spin-half quantities,

$$\begin{aligned} {\zeta^L_{1/2}}'(0) &= 2(G_2 - G_1), \\ {\zeta^S_{1/2}}'(0) &= 2(H_2 + H_1). \end{aligned} \quad (6)$$

The labels S and L refer to spectral and local boundary conditions respectively.

In equations (4) and (5) the $\sim$ symbol signifies that the mass-independent part of the large-mass asymptotic limit is to be taken. The $E_n(t)$ are the polynomials in $t = p/\epsilon$, $\epsilon = (m^2 + p^2)^{1/2}$, that occur in the asymptotic expansion of $F_p^L(im)$ of (1) derived by D'Eath and Esposito (they call them $A_n/2$) from Olver's series. The $T_n(t)$ are the corresponding polynomials for the scalar case, [11,12]. The condition that makes equation (4) possible is $E_n(1) = T_n(1)$ which can be proved from the explicit definition of the E_n. We note that $T_n(1)$ is zero for n even and that $T_{2k-1}(1) = (-1)^k B_{2k}/2k(2k-1)$ in terms of Bernoulli numbers.

We have made use of the algebraic results of D'Eath and Esposito, [1] section IV, in deriving (4).

Expression (5) is identical to one occurring for scalar fields on the even ball, except that N, there being the power of p in the expansion of the degeneracy, is even. Hence for $N=2$, our previous result in [12,13] for the 4-ball (see also [14]) could be used without change.

From the technique outlined in [10] the following useful limits can be deduced,

$$
\begin{aligned}
&\sum_{p=1}^{\infty} p^N(\epsilon-p) \sim -\zeta_R(-N-1)+O(\ln m),\\
&\sum_{p=1}^{\infty} p^N \ln\big(\frac{2p}{p+\epsilon}\big) \sim -\zeta_R'(-N)+\ln 2\,\zeta_R(-N)+O(\ln m),\\
&\sum_{p=1}^{\infty} p^N \ln\big(\frac{\epsilon}{p}\big) \sim \zeta_R'(-N)+O(\ln m).
\end{aligned}
\tag{7}
$$

It is necessary to state that a hidden regularisation has been employed to render the summations finite. This consists of removing sufficient of the Taylor expansion of the summand and will not be indicated. Since the entire expression is finite, the divergent terms so introduced must all cancel.

These limits enable some of the terms in (4) and (5) to be dealt with quickly. The rest, *i.e.* the polynomial and integral contributions, need a little more work. We write them as in [12,10],

$$
\begin{aligned}
&\sum_{p=1}^{\infty} p^N \Bigg[\sum_{n=1,3,\ldots}^{N+1} P_n(1)\Big(\frac{1}{\epsilon^n}-\frac{1}{p^n}\Big)+\sum_{n=1}^{N+1}\frac{P_n'(t)}{\epsilon^n}\Bigg]\\
&\quad+\lim_{s\to 0}\int_0^{\infty}\Bigg(\frac{1}{2}-\frac{1}{\tau}+\sum_{k=1}^{[N/2]+1}(-1)^k B_{2k}\frac{\tau^{2k-1}}{(2k)!}+\frac{1}{e^\tau-1}\Bigg)\tau^{s-1}(-1)^N\frac{d^N}{d\tau^N}\frac{1}{e^\tau-1}d\tau,
\end{aligned}
\tag{8}
$$

where P_n stands for either E_n or T_n and $P_n'(t)=P_n(t)-P_n(1)$.

A recursion is developed for the multiple derivative in (8) and the contribution from the integral found to be, after some algebra,

$$
\zeta_R'(-N-1)+\frac{1}{2}\zeta_R'(-N)+\zeta_R(-N-1)+\sum_{k=1}^{N+1}M_k^{(N)}\zeta_R'(-k),
\tag{9}
$$

where the coefficient matrix M is defined by

$$
M_k^{(N)}=\sum_{l=k}^{N+1}A_l^{(N)}\frac{S_{l+1}^{(k+1)}}{l!}
$$

in terms of easily evaluated recursion constants $A_l^{(j)}$ and Stirling numbers $S_l^{(k)}$, [12,10].

Assembling the various pieces, and using special values for the $M_k^{(N)}$, we find

$$\begin{aligned} G_N = {} & \frac{\zeta_R'(-N)}{2} + \frac{\zeta_R'(-N-1)}{N+1} + \sum_{k=1}^{N-1} M_k^{(N)} \zeta_R'(-k) \\ & + \big(\frac{1}{2}\zeta_R(-N) - \zeta_R(-N-1)\big) \ln 2 + \int_0^1 t^N E''_{N+1}(t)\, dt + L_N, \quad N \geq 1, \end{aligned} \tag{10}$$

where $P_n''(t) = P_n'(t)/(1-t^2)$ and

$$L_N = T_{N+1}(1) \bigg(\ln 2 + \sum_{k=1}^{N} \frac{1}{k} + \sum_{q=1}^{N/2} \frac{(-1)^q \sqrt{\pi}\,(N/2)!}{2q(N/2-q)!\Gamma(q+1/2)} \bigg).$$

The last two terms in (10) come from the first line of (8).

Explicitly for $N = 0$, a case needed later,

$$G_0 = -\frac{1}{24} + \frac{1}{12} \ln 2 + \zeta_R'(-1). \tag{11}$$

For spectral conditions,

$$\begin{aligned} H_N = {} & -\frac{\zeta_R'(-N)}{2} + \frac{\zeta_R'(-N-1)}{N+1} + \sum_{k=1}^{N-1} M_k^{(N)} \zeta_R'(-k) \\ & + \zeta_R(-N-1) \ln 2 + \int_0^1 t^N T''_{N+1}(t)\, dt + L_N, \quad N \geq 1, \end{aligned} \tag{12}$$

and

$$H_0 = \frac{5}{24} - \frac{1}{6} \ln 2 + \zeta_R'(-1) - \zeta_R'(0). \tag{13}$$

Making the constructions (6), one finds for local spin-half,

$$\zeta^{L\,\prime}_{1/2}(0) = \frac{251}{15120} - \frac{11}{180} \ln 2 + \frac{2}{3}\big(\zeta_R'(-3) - \zeta_R'(-1)\big) \approx 0.088108 \tag{14}$$

and for spectral,

$$\zeta^{S\,\prime}_{1/2}(0) = -\frac{2489}{30240} + \frac{1}{45} \ln 2 + \frac{2}{3}\big(\zeta_R'(-3) - \zeta_R'(-1)\big) \approx 0.046962 \tag{15}$$

which are the main results of this note.

The specific forms of the E_n polynomials given in [1], have been used to evaluate the integrals in (10). We remark that in the corresponding evaluation of $\zeta(0)$ (=11/360), one needs only the particular value $P_N(1)$, which equals $\zeta_R(-N)/N$, a non-transcendental, local quantity.

3. Higher spins

The eigenvalue conditions for some higher-spin theories are summarised in [7] section VI. A mechanical application of the present technique yields the following results.

For real spin-0 with Dirichlet conditions,

$$
\begin{aligned}
2\zeta_0^{D\prime}(0) &= 2H_2 \\
&= \frac{173}{15120} + \frac{1}{45}\ln 2 + \frac{2}{3}\zeta_R'(-3) - \zeta_R'(-2) + \frac{1}{3}\zeta_R'(-1) \\
&\approx 0.005738.
\end{aligned}
$$

For spin-1 (Maxwell) with Dirichlet (magnetic) conditions,

$$
\begin{aligned}
\zeta_{\rm TV}'(0) &= 2(H_2 - 2H_0) \\
&= -\frac{6127}{15120} + \frac{16}{45}\ln 2 + \frac{2}{3}\zeta_R'(-3) - \zeta_R'(-2) - \frac{5}{3}\zeta_R'(-1) + 2\zeta_R'(0) \\
&\approx -1.68691.
\end{aligned} \tag{16}
$$

For spin-3/2 physical degrees of freedom with spectral conditions,

$$
\begin{aligned}
\zeta_{3/2}^{S}{}'(0) &= 2(H_2 + H_1 - H_0) \\
&= -\frac{27689}{30240} + \frac{31}{45}\ln 2 + \frac{2}{3}\zeta_R'(-3) - \frac{14}{3}\zeta_R'(-1) + 4\zeta_R'(0) \\
&\approx -3.33834.
\end{aligned} \tag{17}
$$

These results imply, rather trivially, the sum rule,

$$
\zeta_{3/2}^{S}{}'(0) - \zeta_{1/2}^{S}{}'(0) = 2\big(\zeta_{\rm TV}'(0) - 2\zeta_0^{D\prime}(0)\big). \tag{18}
$$

The same relation holds also for $\zeta(0)$,

$$
\zeta_{3/2}^{S}(0) - \zeta_{1/2}^{S}(0) = 2\big(\zeta_{\rm TV}(0) - 2\zeta_0^{D}(0)\big), \tag{19}
$$

and, in fact, for all coefficients in the heat-kernel expansion, as can be checked numerically from the tables provided in [15] and [16].

The specific values,

$$
\zeta_{3/2}^{S}(0) = -\frac{289}{360}, \quad \zeta_{1/2}^{S}(0) = \frac{11}{360}, \quad \zeta_{\rm TV}(0) = -\frac{77}{180}, \quad \zeta_0^{D}(0) = -\frac{1}{180}, \tag{20}
$$

were computed in references [1,7,17], see also [18,16,19]. The spectral label, S, can be replaced by the local one, L, in (20).

The sum rules are only special cases of the general relation

$$\zeta^S_{3/2}(s)-\zeta^S_{1/2}(s)=2\big(\zeta_{\rm TV}(s)-2\zeta^D_0(s)\big), \tag{21}$$

which is a consequence of the eigenvalue condition, (2), and the various quadratic degeneracies.

For spin-2 transverse-traceless modes with Dirichlet conditions, [20], *i.e.*

$$F^{\rm TT}_p=J_p(\alpha)=0$$

and degeneracy $2(p^2-4)$, $p\geq 3$, we find

$$\begin{aligned}\zeta'_{\rm TT}(0)&=2(\overline{H}_2-4\overline{H}_0)=2(H_2-H_0)+6\big(\zeta'_R(0)+\ln 2\big)\\&=-\frac{25027}{15120}+\frac{331}{45}\ln 2+\frac{2}{3}\zeta'_R(-3)-\zeta'_R(-2)-\frac{23}{3}\zeta'_R(-1)+14\zeta'_R(0)\\&\approx -8.119619,\end{aligned} \tag{22}$$

where the bar signifies that the $p=1$ term has been left out in (5). (The easiest way of doing this is to remove the overall $p=1$ term at the outset.)

For the record, the local spin-3/2 expression is

$$\begin{aligned}{\zeta^L_{3/2}}'(0)&=2(\overline{G}_2-\overline{G}_1-2\overline{G}_0)\\&=\frac{2771}{15120}+\frac{289}{180}\ln 2+\frac{2}{3}\zeta'_R(-3)-\frac{14}{3}\zeta'_R(-1)+4\zeta'_R(0)\\&\approx -1.60405,\end{aligned} \tag{23}$$

which exhibits the anomaly value of $-289/360$.

Arbitrary-spin fields can be treated in exactly the same way, most easily using the mode analysis given in [21,22], and will be discussed in a later communication.

4. Comments

The above expressions for the $\zeta'(0)$ have also been obtained by Kirsten and Cognola [16] using the method of Bordag *et al*, [14].

The local result, (14), agrees with that of Apps, reported in [13] and found using a conformal transformation from the 4-hemisphere. In fact, the final expression in

(14) is $\zeta'_S(0)$ on the hemisphere, the rest coming from the cocycle function obtained from an integration of the conformal anomaly, as in [23,24] for example.

Spectral conditions are also conformally invariant and it seems that (15) can be interpreted in a similar way. The same structure is also apparent in (17) and (23) for spin 3/2.

This suggests that the eigenvalue problem on the hemisphere is the same, or is equivalent, for spectral and local boundary conditions. This is confirmed by, and may explain, the equality of $\zeta(0)$ for these conditions found by D'Eath and Esposito in flat space and by Kamenshchik and Mishakov on the bounded sphere. To the author's knowledge, the cocycle function has not been calculated for spectral conditions.

The extension to higher, even-dimensional spaces is straightforward and simply consists of substituting (10) or (12) into the appropriate polynomial form of the spinor degeneracy. For odd dimensions the major difference is that the p-sums run over half odd-integers and presents no problem [10]. For example, the Maxwell modes on the 3-ball are classic, *e.g.* [25], and it is soon shown that the magnetic determinant is obtained by doubling the scalar Dirichlet value and subtracting $-2\ln 2$ to allow for the different starting point of the mode sum. Similarly, the electric determinant is the double the scalar Robin one, with $\beta = 1/2$, again minus $-2\ln 2$.

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