Boundary Yang-Baxter equation in the RSOS representation

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Abstract

The boundary Yang-Baxter equation in the RSOS representation is found to posses two classes of trigonometric solution; diagonal and off-diagonal. The diagonal solution is not a special limit of the off-diagonal one and is unique to the RSOS representation as it contains $p-3$ parameters where $p-1$ is the number of allowed height. The corresponding commuting transfer matrix is also constructed.

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1 Introduction

The boundary Yang-Baxter equation (BYBE) is relevant to integrable statistical models[1, 2] and quantum field theories[3] in the present of a boundary; it is the necessary condition for the integrability of these models. The equation is also interesting in its own right; it is known for example that for some special solution, the BYBE in the rational limit can be regarded as a defining relation for the twisted Yangian[4] and is therefore related to deformation of the orthogonal and simplectic Lie algebra.

To date, several solutions of the BYBE have appeared in the form of vertex representation. Far less is known however for the solution in the SOS or RSOS form[5, 6]. From a mathematical point of view, finding solution in the RSOS form will reveal the special mathematical structure associated with the algebra defined by the boundary Yang-Baxter equation when the deformation parameter is a root of unity. From a physical point view, the solutions have applications in statistical mechanics and field theory. In the context of statistical mechanics, the solutions give rise to integrable SOS/RSOS models with boundary, whose simplest case includes the Ising model, and the study of integrable statistical model with boundary will shed light on the issue of the dependence of the Casimir energy on the boundary and surface properties[7, 8]. From the field theory point of view, the solutions are relevant to the study of the restricted sine-Gordon model[13] and the perturbed (coset[9, 10]) conformal field theory[11] with boundary. Besides, it is known that the Kondo problem in the overscreened case has boundary scattering matrix given in the RSOS form[12], hence our solutions may find application in this problem.

2 Solutions to the boundary Yang-Baxter equation

2.1 Generalities

In this section we solve the boundary Yang-Baxter equation for the RSOS(p) ; $p = 3, 4\ldots$ scattering theory. The RSOS(p) scattering theory is based on a $p-1$ - fold degenerate vacuum structure, vacua can be associated with nodes of the $\mathcal{A}_{p-1}$ Dynkin diagram. The quasi particles in the scattering theory are kinks that interpolate neighboring vacua, they can be denoted by non-commutative symbols $K_{ab}(\theta)$ where $|a-b| = 1$ and θ is the kink rapidity. Formally, scattering between two kinks can be represented by the following equation (see Fig.(1))

$$K_{da}(\theta)K_{ab}(\theta') = \sum_c S_{dc}^{ab}(\theta - \theta')K_{dc}(\theta')K_{cb}(\theta) \tag{2.1.1}$$

where the S-matrix is given by

$$S_{dc}^{ab}(\theta) = u(\theta)\left(\frac{[a][c]}{[d][b]}\right)^{-\theta/2\pi i} W_{dc}^{ab}(\theta) \tag{2.1.2}$$

and

$$W_{dc}^{ab}(\theta) = \left(\sinh(\theta/p)\delta_{bd}\left(\frac{[a][c]}{[d][b]}\right)^{1/2} + \sinh((i\pi - \theta)/p)\delta_{ac}\right) \tag{2.1.3}$$

satisfies the Yang-Baxter equation in the RSOS representation. Here $[a]$ denotes the usual q-number given by

$$[a] = \frac{\sin(a\pi/p)}{\sin(\pi/p)}$$

and the overall factor $u(\theta)$ is a product of Gamma functions which can be found in [13] and satisfies the relations

$$\begin{aligned} u(\theta)u(-\theta)\sinh((i\pi - \theta)/p)\sinh((i\pi + \theta)/p) &= 1 \\ u(i\pi - \theta) &= u(\theta)\ . \end{aligned}$$

This factor, together with the overall q-number factor, ensures that the S-matrix satisfies both crossing and unitarity constraints:

$$S_{dc}^{ab}(\theta) = S_{ad}^{bc}(i\pi - \theta) \tag{2.1.4}$$

$$\sum_{c'} S_{dc'}^{ab}(\theta)S_{dc}^{c'b}(-\theta) = \delta_{ac}\ . \tag{2.1.5}$$

Consider now the above scattering theory in the presence of a boundary denoted formally by B, then the scattering between the a kink and the boundary is described by the equation

$$K_{ab}(\theta)B = \sum_c R_{ac}^b(\theta)K_{bc}(-\theta)B \tag{2.1.6}$$

which can be given a graphical representation shown in Fig.(2).

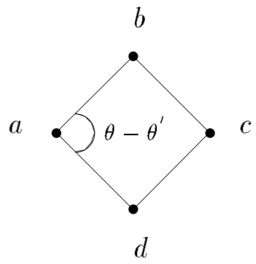

Figure 1: The bulk RSOS scattering matrix

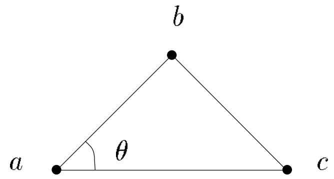

Figure 2: The boundary RSOS scattering matrix

The function R^b_{ac} is called the boundary scattering matrix and satisfies the Boundary Yang-Baxter (BYB) equation, which in the RSOS representation takes the form

$$\sum_{a',b'} R^a_{bb'}(\theta) S^{ac}_{b'a'}(\theta'+\theta) R^{a'}_{b'b''}(\theta') S^{a'c}_{b''a''}(\theta'-\theta) =$$
$$\sum_{a',b'} S^{ac}_{ba'}(\theta'-\theta) R^{a'}_{bb'}(\theta') S^{a'c}_{b'a''}(\theta'+\theta) R^{a''}_{b'b''}(\theta) \ . \tag{2.1.7}$$

In general, the function $R^a_{bc}(\theta)$ can be written as

$$R^a_{bc}(\theta) = R(\theta) \left(\frac{[b][c]}{[a][a]}\right)^{-\theta/2\pi i} \left(\delta_{b\neq c} X^a_{bc}(\theta) + \delta_{bc}(\delta_{b,a+1} U_a(\theta) + \delta_{b,a-1} D_a(\theta))\right) \tag{2.1.8}$$

where $R(\theta)$ has to be determined from the boundary crossing and unitarity constraints, while X^a_{bc} and U_a, D_a have to be determined from the BYB equation. An overall q-number factor has also been multiplied to the above to cancel that from the bulk S-matrix in order to simplifies the BYB equation. Note that due to the restriction that vacuum assumes value $1, \ldots, p-1$, these unknown functions are not defined for $X^1_{bc}, X^{p-1}_{bc}, D_1, U_{p-1}$. The case $p=3$ has only diagonal scattering, so X^a_{bc} is taken to be zero.

2.2 Non-diagonal scattering

We consider the scattering where the off-diagonal component X^a_{bc} is non-vanishing. To start, consider the case $b \neq c \neq b^{''}$ in eqn.(2.1.7) where the BYB equation gives

$$X^a_{a-1,a+1}(\theta^{'})X^{a+2}_{a+1,a+3}(\theta) = X^a_{a-1,a+1}(\theta)X^{a+2}_{a+1,a+3}(\theta^{'}) \;; 2 \leq a \leq p-4 \quad (2.2.1)$$

which implies that $X^a_{a\pm1,a\mp1}$ can be written as

$$X^a_{a\pm1,a\mp1}(\theta) = h_\pm(\theta)X^a_\pm \quad (2.2.2)$$

where the unknown functions $h_\pm$ only depends on θ and $X^a_\pm$ only on a.

On the other hand, the case $c = b = b^{''}$, $a = a^{''}$ gives

$$X^a_{a-1,a+1}(\theta^{'})X^a_{a+1,a-1}(\theta) = X^a_{a-1,a+1}(\theta)X^a_{a+1,a-1}(\theta^{'}) \;; 2 \leq a \leq p-2 \quad (2.2.3)$$

which implies that

$$h_+(\theta^{'})h_-(\theta) = h_+(\theta)h_-(\theta^{'}) \;, \quad (2.2.4)$$

from which we conclude that

$$h_+(\theta) \propto h_-(\theta) \;. \quad (2.2.5)$$

Absorbing the constant of proportionality into X^a_- or X^a_+, we can regard h_+ and h_- to be identical, so we can absorb the common $h(\theta)_\pm$ into the overall $R(\theta)$ factor and treat X^a_{bc} as θ independent from now on.

With this simplification, the boundary Yang-Baxter equation can be broken down into the following independent equations in addition to the above two equations:

$$\begin{aligned} U^{'}_a D_{a+2} f_+ \left(1 + f_-[a]/[a+1]\right) + D^{'}_{a+2} D_{a+2} f_- \left(1 + f_+[a+2]/[a+1]\right) \\ + X^{a+2}_{a+1,a+3} X^{a+2}_{a+3,a+1} f_- = U_a D^{'}_{a+2} f_+ \left(1 + f_-[a+2]/[a+1]\right) \\ + U^{'}_a U_a f_- \left(1 + f_+[a]/[a+1]\right) + X^a_{a-1,a+1} X^a_{a+1,a-1} f_- \end{aligned} \quad (2.2.6)$$

for $1 \leq a \leq p-3$, and

$$\begin{aligned} D^{'}_{a+1} f_- \left(1 + f_+[a+1]/[a]\right) + U^{'}_{a-1} f_+ \left(1 + f_-[a-1]/[a]\right) \\ = U_{a-1} f_+ - U_{a+1} f_- \end{aligned} \quad (2.2.7)$$

$$U^{'}_a f_- \left(1 + f_+[a]/[a+1]\right) + D^{'}_{a+2} f_+ \left(1 + f_-[a+2]/[a+1]\right)$$

$$= D_{a+2} f_+ - D_a f_- \tag{2.2.8}$$

$$\begin{aligned} U'_{a-1} f_+ f_- ([a+1][a-1])/([a]^2) + U_{a+1}\,(1 + f_-[a+1]/[a]) \\ + D'_{a+1}\,(1 + f_-[a+1]/[a])\,(1 + f_+[a+1]/[a]) \\ = U'_{a+1} + D_{a+1}\,(1 + f_+[a+1]/[a])] \end{aligned} \tag{2.2.9}$$

$$\begin{aligned} U'_{a+2} f_+ f_- ([a][a+2])/([a+1]^2) + D_a\,(1 + f_-[a]/[a+1]) \\ + U'_a\,(1 + f_-[a]/[a+1])\,(1 + f_+[a]/[a+1]) \\ = D'_a + U_a\,(1 + f_+[a]/[a+1]) \end{aligned} \tag{2.2.10}$$

which are defined for $2 \le a \le p-3$, and are valid *only if* the off-diagonal weight X^a_{bc} is nonvanishing. In the above equations, we used a more compact notation defined below

$$\begin{array}{lclclcl} U_a & = & U_a(\theta) & \qquad & D_a & = & D_a(\theta) \\ U'_a & = & U_a(\theta') & \qquad & D'_a & = & D_a(\theta') \end{array}$$

and

$$f_\pm = \sinh((\theta' \pm \theta)/p)/\sinh((i\pi - \theta' \mp \theta)/p)\ .$$

It should also be mentioned that the last term in the rhs (lhs) of eqn.(2.2.6) is present only when $a \ne 1(p-3)$. So one has to impose the conditions

$$\begin{array}{lcl} X^1_{0,2} X^1_{2,0} & = & 0 \\ X^{p-1}_{p-2,p} X^{p-1}_{p,p-2} & = & 0\ , \end{array} \tag{2.2.11}$$

while no such "boundary" conditions have to be satisfied by U_a, D_a. Notice also that most of the above equations do not apply to the case $p = 4$, so we shall consider this case separately.

From the above it is clear that these equations can be divided into two sets; eqns.(2.2.7)-(2.2.10) determine U_a and D_a, while eqns.(2.2.6),(2.2.11) determines X^a_{bc}. To proceed, we try to construct from these equations some recursion relations for the unknown functions. Indeed, comparing eqn.(2.2.7) with eqn.(2.2.9) and similarly eqn.(2.2.8) with eqn.(2.2.10), we deduce that

$$\begin{aligned} (D'_a - D_a)(1 + f_+[a]/[a-1]) &= (U'_{a-2} - U_{a-2}) f_+[a]/[a-1] \\ &\quad + (U'_a - U_a) \end{aligned} \tag{2.2.12}$$

$$\begin{aligned} (U'_a - U_a)(1 + f_+[a]/[a+1]) &= (D'_{a+2} - D_{a+2}) f_+[a]/[a+1] \\ &\quad + (D'_a - D_a)\ . \end{aligned} \tag{2.2.13}$$

Substituting one into another, we get

$$\begin{aligned}\frac{(U'_{a+2}-U_{a+2})-(U'_a-U_a)}{(U'_a-U_a)-(U'_{a-2}-U_{a-2})} &= \frac{\sinh(((a+1)i\pi+\theta'+\theta)/p)}{\sinh(((a-1)i\pi+\theta'+\theta)/p)} &(2.2.14)\\ \frac{(D'_{a+2}-D_{a+2})-(D'_a-D_a)}{(D'_a-D_a)-(D'_{a-2}-D_{a-2})} &= \frac{\sinh(((a+1)i\pi-\theta'-\theta)/p)}{\sinh(((a-1)i\pi-\theta'-\theta)/p)} &(2.2.15)\end{aligned}$$

and writing the rhs respectively as

$$\frac{\cosh((2\theta'+(a+1)i\pi)/p)-\cosh((2\theta+(a+1)i\pi)/p)}{\cosh((2\theta'+(a-1)i\pi)/p)-\cosh((2\theta+(a-1)i\pi)/p)}$$
$$\frac{\cosh((2\theta'-(a+1)i\pi)/p)-\cosh((2\theta-(a+1)i\pi)/p)}{\cosh((2\theta'-(a-1)i\pi)/p)-\cosh((2\theta-(a-1)i\pi)/p)} \quad ,$$

it is clear that

$$\begin{aligned}U_{a+2}(\theta)-U_a(\theta) &= \cosh((2\theta+(a+1)i\pi)/p)+\beta'_a\\ D_{a+2}(\theta)-D_a(\theta) &= \cosh((2\theta-(a+1)i\pi)/p)+\phi'_a\end{aligned}$$

where β'_a, ϕ'_a are unknown functions that depend only on a. Iterating the above, one finds

$$\begin{aligned}U_a(\theta) &\propto \sinh((2\theta+ai\pi)/p)+\alpha(\theta)+\beta_a &(2.2.16)\\ D_a(\theta) &\propto \sinh((2\theta-ai\pi)/p)+\gamma(\theta)+\phi_a &(2.2.17)\end{aligned}$$

where $\alpha(\theta), \gamma(\theta)$ are unknown functions of θ, and β_a, ϕ_a depend only on a. Furthermore, from eqns.(2.2.6)-(2.2.10), one can establish the following symmetry property

$$U_a(\theta) = -D_a(-\theta)\ , \qquad (2.2.18)$$

which reduces the number of unknown functions to two; $\alpha(\theta), \beta_a$. To determine them, we have to substitute the above expressions for U_a, D_a back into eqns.(2.2.7)-(2.2.10). Notice, however that these equations are linear in U_a, D_a, so it suffices to consider each unknown $\alpha(\theta), \beta_a$ separately. Doing this amounts to finding special solutions to eqns.(2.2.7)-(2.2.10) where U_a, D_a have only θ or a dependence. The solutions are given by

$$\alpha(\theta) = 0 \qquad\qquad \beta_a \propto 1/\sinh(ai\pi/p) \qquad (2.2.19)$$

respectively.

In summary, the general non-diagonal solution to the four linear equations is given by

$$\begin{aligned} U_a(\theta) &= \frac{k\sinh((2\theta+ai\pi)/p)}{2} - \frac{1}{2k\sinh(ai\pi/p)} \\ D_{a+1}(\theta) &= \frac{k\sinh((2\theta-(a+1)i\pi)/p)}{2} + \frac{1}{2k\sinh((a+1)i\pi/p)} \end{aligned} \tag{2.2.20}$$

where k is a free parameter and $1 \leq a \leq p-2$. Having found U_a, D_a, the unknown function X^a_{bc} can be easily obtained from eqn.(2.2.6), which can be further simplified with the symmetry properties given in eqn.(2.2.18) and taking θ' to be $-\theta$ since X^a_{bc} does not depend on the rapidity. This gives

$$X^{a+2}_{a+1,a+3}X^{a+2}_{a+3,a+1} - X^a_{a-1,a+1}X^a_{a+1,a-1} = D_{a+2}(\theta)U_{a+2}(\theta) - D_a(\theta)U_a(\theta)\ . \tag{2.2.21}$$

Substituting U_a, D_a into the rhs, we get

$$\begin{aligned} &\left(\frac{1}{2k\sinh(ai\pi/p)} - \frac{k\sinh(ai\pi/p)}{2}\right)^2 \\ &-\left(\frac{1}{2k\sinh((a+2)i\pi/p)} - \frac{k\sinh((a+2)i\pi/p)}{2}\right)^2 \quad . \end{aligned}$$

Clearly, we have

$$\begin{aligned} X^a_{a-1,a+1}X^a_{a+1,a-1} &= \left(\frac{1}{2k\sinh(i\pi/p)} - \frac{k\sinh(i\pi/p)}{2}\right)^2 \\ &\quad -\left(\frac{1}{2k\sinh(ai\pi/p)} - \frac{k\sinh(ai\pi/p)}{2}\right)^2 \end{aligned} \tag{2.2.22}$$

where used has been made of eqn.(2.2.11) to determine the first term. Hence, X^a_{bc} is determined up to the above product, but we shall see later that in general the difference between $X^a_{a-1,a+1}$ and $X^a_{a+1,a-1}$ has no physical consequence in the sense that only the above product affects the determination of the overall $R(\theta)$ factor. These solutions have the property that

$$U_{p-a}(\theta) = -D_a(\theta)\ , \qquad X^{p-a}_{p-a-1,p-a+1}X^{p-a}_{p-a+1,p-a-1} = X^a_{a-1,a+1}X^a_{a+1,a-1}\ . \tag{2.2.23}$$

Next, we consider the boundary Yang-Baxter equation for $p=4$. The functions U_1, D_3 are diagonal scattering components and do not couple to the rest of the unknown functions, so we shall defer to next section for their computation. While the rest of the unknown functions U_2, D_2 and X^2_{13}, X^2_{31}

satisfy the following equations

$$X^2_{13}(\theta')X^2_{31}(\theta) = X^2_{13}(\theta)X^2_{31}(\theta') \qquad (2.2.24)$$

$$U_2\left(1+\sqrt{2}f_-\right) + D_2'\left(1+\sqrt{2}f_+\right)\left(1+\sqrt{2}f_-\right)$$
$$= U_2' + D_2\left(1+\sqrt{2}f_+\right) \qquad (2.2.25)$$

$$D_2\left(1+\sqrt{2}f_-\right) + U_2'\left(1+\sqrt{2}f_+\right)\left(1+\sqrt{2}f_-\right)$$
$$= D_2' + U_2\left(1+\sqrt{2}f_+\right) \; . \qquad (2.2.26)$$

Here we have used the compact notation introduced earlier for the unknown U_a, D_a, and written out explicitly the rapidity dependence of X^a_{bc}. As before, the last two equations have been derived based on the assumption that X^2_{13}, X^2_{31} are nonvanishing. From eqn.(2.2.24), we deduce that

$$X^2_{13}(\theta) \propto X^2_{31}(\theta) \; . \qquad (2.2.27)$$

The rest of the equations can be turned into ordinary differential equations in the limit $\theta' \to \theta$, giving

$$\left(\dot{U_2}(\theta) + \dot{D_2}(\theta)\right)\tanh(\theta/2) + 2\left(U_2(\theta) + D_2(\theta)\right) = 0 \qquad (2.2.28)$$
$$\left(\dot{U_2}(\theta) - \dot{D_2}(\theta)\right)\tanh(\theta/2) - 2\left(U_2(\theta) - D_2(\theta)\right) = 0 \; , \qquad (2.2.29)$$

which can be integrated to give

$$U_2(\theta) = B/\sinh(\theta/2) + C\cosh(\theta/2) \qquad (2.2.30)$$
$$D_2(\theta) = B/\sinh(\theta/2) - C\cosh(\theta/2) \; , \qquad (2.2.31)$$

with B, C being the free parameters.

This completes the determination of the non-diagonal solutions to the BYBE.

2.3 Diagonal scattering

For the diagonal scattering, we take

$$R^a_{bc}(\theta) = ([b]/[a])^{-\theta/\pi i} R(\theta)\delta_{bc}\left(\delta_{b,a+1}U_a(\theta) + \delta_{b,a-1}D_a(\theta)\right) \; , \qquad (2.3.1)$$

and the BYB equation is equivalent to a single equation

$$U_a(\theta')D_{a+2}(\theta)\sinh((\theta'+\theta)/p)\sinh(((a+1)i\pi - \theta' + \theta)/p)$$
$$+ D_{a+2}(\theta')D_{a+2}(\theta)\sinh((\theta'-\theta)/p)\sinh(((a+1)i\pi + \theta' + \theta)/p) =$$
$$U_a(\theta)D_{a+2}(\theta')\sinh((\theta'+\theta)/p)\sinh(((a+1)i\pi + \theta' - \theta)/p)$$
$$+ U_a(\theta')U_a(\theta)\sinh((\theta'-\theta)/p)\sinh(((a+1)i\pi - \theta' - \theta)/p) \; , \qquad (2.3.2)$$

which holds only for $1 \leq a \leq p-3$. So the functions U_{p-2} and D_2 can not be determined from the BYB relation. The above can be recast into an ordinary differential equation as

$$
\begin{aligned}
\dot{\mathcal{R}}(\theta)\, &(\cosh((2\theta + (a+1)i\pi)/p) - \cosh((2\theta - (a+1)i\pi)/p)) \\
&+ 2\mathcal{R}(\theta)\, (\sinh((2\theta + (a+1)i\pi)/p) + \sinh((2\theta - (a+1)i\pi)/p)) \\
&= 2\mathcal{R}(\theta)^2 \sinh((2\theta + (a+1)i\pi)/p) + \sinh((2\theta - (a+1)i\pi)/p) \qquad (2.3.3)
\end{aligned}
$$

where

$$\mathcal{R}(\theta) \equiv D_{a+2}(\theta)/U_a(\theta)$$

and $\dot{}$ denotes differentiation with respect to θ. This differential equation can be integrated to give

$$\frac{D_{a+2}(\theta)}{U_a(\theta)} = \frac{(\cos\xi_a - \cosh((2\theta - (a+1)i\pi)/p))}{(\cos\xi_a - \cosh((2\theta + (a+1)i\pi)/p))} \qquad (2.3.4)$$

where ξ_a is a free parameter.

Thus for the diagonal solution, there are $p-3$ parameters ξ_a. This solution includes a particular case of $p = 4$ which has been omitted earlier.

Further relations from boundary unitarity and crossing symmetry will be required to disentangle U_a and D_{a+2}, and determine U_2 and D_{p-2}, see later.

2.4 Boundary unitarity and crossing symmetry

The boundary unitarity and crossing symmetry conditions of the scattering matrix $R^a_{bc}(\theta)$ determine to some extend the overall factor $R(\theta)$. These conditions can be written respectively as

$$
\begin{aligned}
R^a_{bc}(\theta) R^a_{cd}(-\theta) &= \delta_{bd} &(2.4.1)\\
\sum_d S^{ac}_{bd}(2\theta) R^d_{bc}(i\pi - \theta) &= R^a_{bc}(\theta)\ . &(2.4.2)
\end{aligned}
$$

As before, we treat the general case of the non-diagonal scattering ($p > 4$) first. Substituting the expression for R^a_{bc} into the unitarity condition, we get the following

$$R(\theta)R(-\theta)\, (X^a_{bd} X^a_{db} \delta_{b\neq d} - U_a(\theta) D_a(\theta)) = 1$$

where used has been made of the symmetry eqn.(2.2.18). Applying the results eqns.(2.2.20),(2.2.22) to the above leads to

$$
\begin{aligned}
R(\theta)R(-\theta)\, &((1+k^2)\sinh^2(\theta/p) + k^2 \sinh^4(\theta/p) \\
&- \left(\frac{1}{2k\sinh(i\pi/p)} - \frac{k\sinh(i\pi/p)}{2}\right)^2) = 1\ . \qquad (2.4.3)
\end{aligned}
$$

While for crossing symmetry condition we get

$$u(2\theta)R(i\pi-\theta)\sinh((2\theta)/p) = R(\theta)$$

where used has been made of the relations

$$\begin{aligned} D_{a+2}(i\pi-\theta)[a+2]/[a+1] - U_a(i\pi-\theta)[a]/[a+1] \\ = f(2\theta)(U_a(\theta) - U_a(i\pi-\theta)) \end{aligned} \tag{2.4.4}$$

$$\begin{aligned} U_a(i\pi-\theta)[a]/[a+1] - D_{a+2}(i\pi-\theta)[a+2]/[a+1] \\ = f(2\theta)(D_a(\theta) - D_a(i\pi-\theta)) \end{aligned} \tag{2.4.5}$$

which can be obtained from taking the limit $\theta' \to i\pi-\theta$ in eqns.(2.2.7),(2.2.8). Here $f(\theta) = \sinh(\theta/p)/\sinh((i\pi-\theta)/p)$.

The factor $R(\theta)$ can be determined from eqns.(2.4.3),(2.4) up to the usual CDD ambiguity following the method used in [3] by writing $R(\theta) = R_0(\theta)R_1(\theta)$ where R_0 satisfies

$$\begin{array}{rcl} R_0(\theta)R_0(-\theta) & = & 1 \\ u(2\theta)R_0(i\pi-\theta)\sinh((2\theta)/p) & = & R_0(\theta)\ , \end{array} \tag{2.4.6}$$

whose minimal solution can be found in eqn.(8.3) of [14], and R_1 satisfies

$$\begin{aligned} R_1(\theta)R_1(-\theta)((1+k^2)\sinh^2(\theta/p) + k^2\sinh^4(\theta/p) \\ - (1/2k\sinh(i\pi/p) - k\sinh(i\pi/p)/2)^2) = 1 \end{aligned} \tag{2.4.7}$$

$$R_1(\theta) = R_1(i\pi-\theta) \tag{2.4.8}$$

which can be mapped into eqn.(5.20) of [3] with λ and $\cos(\xi)$ of [3] correspond respectively to

$$i/p \qquad \text{and} \qquad \left(\frac{1}{2k\sinh(i\pi/p)} - \frac{k\sinh(i\pi/p)}{2}\right)^2$$

here. The solution can then be read off easily.

It is intriguing that the RSOS restriction imposes a relation between the two free parameters ξ, k (or ϕ_o, M) in [3], it would be interesting to explore the implication of this relation.

For $p=4$, the crossing symmetry condition is the same as eqn.(2.4), but unitarity now requires that

$$R(\theta)R(-\theta)(X_{13}^2(\theta)X_{31}^2(\theta) + C^2\cosh^2(\theta/2) - B^2/\sinh^2(\theta/2)) = 1 \tag{2.4.9}$$

The factor $X^2_{13}(\theta)X^2_{31}(\theta)$ is actually redundant and can be absorbed into $R(\theta)$. So setting it as -1, we can rewrite the above as

$$\frac{R(\theta)}{\sinh(\theta/2)}\frac{R(\theta)}{\sinh(-\theta/2)}(B^2+(1-C'^2)\sinh^2(\theta/2)-C'^2\sinh^4(\theta/2))=1\,, \tag{2.4.10}$$

which can again be mapped into eqn.(5.20) of [3] and read off the solution. It should be remarked that in this case, there are two parameters instead of one for the higher p cases.

Finally, we consider the diagonal case. Unitarity relation gives

$$\left.\begin{array}{lcl} R(\theta)R(-\theta)U_a(\theta)U_a(-\theta) & = & 1 \\ R(\theta)R(-\theta)D_{a+1}(\theta)D_{a+1}(-\theta) & = & 1 \end{array}\right. ;1\leq a\leq p-2 \tag{2.4.11}$$

The crossing symmetry on the other hand gives

$$\begin{aligned} & u(2\theta)R(i\pi-\theta)(U_a(i\pi-\theta)\sinh(i\pi/p)\sinh((2\theta+ai\pi)/p) \\ & \quad +D_{a+2}(i\pi-\theta)\sinh((i\pi-2\theta)/p)\sinh((a+2)i\pi)) \\ & \quad =R(\theta)U_a(\theta)\sinh(2\theta/p)\sinh((a+1)i\pi/p) \qquad (2.4.12) \\ & u(2\theta)R(i\pi-\theta)(D_{a+2}(i\pi-\theta)\sinh(i\pi/p)\sinh((ai\pi-2\theta)/p) \\ & \quad +U_a(i\pi-\theta)\sinh((i\pi-2\theta)/p)\sinh(ai\pi)) \\ & \quad =R(\theta)D_{a+2}(\theta)\sinh(2\theta/p)\sinh((a+1)i\pi/p) \qquad (2.4.13) \end{aligned}$$

for $1\leq a\leq p-3$, and

$$\begin{aligned} u(2\theta)R(i\pi-\theta)U_{p-2}(i\pi-\theta)\sinh((2i\pi-2\theta)/p) &= R(\theta)U_{p-2}(\theta) \\ & \qquad (2.4.14) \\ u(2\theta)R(i\pi-\theta)D_2(i\pi-\theta)\sinh((2i\pi-2\theta)/p) &= R(\theta)D_2(\theta) \end{aligned}$$

for the remaining D_2, U_{p-2}.

These equations can be solved separately, we set

$$\begin{array}{lcl} R(\theta)R(-\theta) & = & 1 \\ u(2\theta)R(i\pi-\theta)\sinh(2\theta/p) & = & R(\theta)\,, \end{array} \tag{2.4.15}$$

so that $R(\theta)$ has exactly the same solution as that of $R_0(\theta)$ considered earlier.

While U_a, D_a satisfy

$$\left.\begin{array}{lcl} U_a(\theta)U_a(-\theta) & = & 1 \\ D_{a+1}(\theta)D_{a+1}(-\theta) & = & 1 \end{array}\right. ;1\leq a\leq p-2 \tag{2.4.16}$$

and

$$
\begin{aligned}
&U_a(i\pi-\theta)\sinh(i\pi/p)\sinh((2\theta+ai\pi)/p)\\
&\quad +D_{a+2}(i\pi-\theta)\sinh((i\pi-2\theta)/p)\sinh((a+2)i\pi/p)\\
&\quad = U_a(\theta)\sinh(2\theta/p)\sinh((a+1)i\pi/p) &(2.4.17)\\
&D_{a+2}(i\pi-\theta)\sinh(i\pi/p)\sinh((ai\pi-2\theta)/p)\\
&\quad +U_a(i\pi-\theta)\sinh((i\pi-2\theta)/p)\sinh(ai\pi/p)\\
&\quad = D_{a+2}(\theta)\sinh(2\theta/p)\sinh((a+1)i\pi/p)\ . &(2.4.18)
\end{aligned}
$$

for $1\le a\le p-3$,

$$
\begin{array}{rcl}
D_2(i\pi-\theta)\sinh(2(i\pi-\theta)/p) & = & D_2(\theta)\sinh(2\theta/p)\\
U_{p-2}(i\pi-\theta)\sinh(2(i\pi-\theta)/p) & = & U_{p-2}(\theta)\sinh(2\theta/p)
\end{array}\ . \qquad (2.4.19)
$$

Substituting eqn.(2.3.4) into the above we get a relation between $U_a(\theta)$ $((D_a\theta))$ and $U_a(i\pi-\theta)$ $(U_a(i\pi-\theta))$;

$$
\begin{aligned}
\frac{U_a(\theta)}{U_a(i\pi-\theta)} &= \frac{\sinh(2(i\pi-\theta)/p)(\cos\xi_a-\cosh((2\theta+(a+1)i\pi)/p))}{\sinh(2\theta/p)(\cos\xi_a-\cosh((2(i\pi-\theta)+(a+1)i\pi)/p))}\\
\frac{D_{a+2}(\theta)}{D_{a+2}(i\pi-\theta)} &= \frac{\sinh(2(i\pi-\theta)/p)(\cos\xi_a-\cosh((2\theta-(a+1)i\pi)/p))}{\sinh(2\theta/p)(\cos\xi_a-\cosh((2(i\pi-\theta)-(a+1)i\pi)/p))}
\end{aligned} \qquad (2.4.20)
$$

for $1\le a\le p-3$. These relations together with eqns.(2.4.16),(2.4.19) should allow the determination of $U_a(\theta)$ $(D_a(\theta))$ up to CDD factor.

To summarize, there are two classes of solutions to the BYB equation; diagonal and non-diagonal. Unlike the solution in vertex representation, the former is not a special limit of the later. In fact, the diagonal solution carries $p-3$ parameters and is unique to the RSOS representation.

3 Commuting transfer matrix

Following the technique proposed in [1], one can similarly construct a family of commuting transfer matrix for the RSOS model with boundary.

To start, it can be shown that if R^a_{bc} is a solution to the BYB in the RSOS form then the following

$$
\sum_a S^{fe}_{ba}(\theta-\theta_1)S^{ae}_{cd}(\theta+\theta_1)R^a_{bc}(\theta) \qquad (3.1)
$$

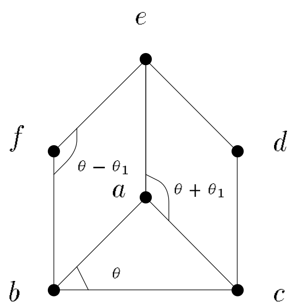

Figure 3: “Decorated” boundary scattering matrix I

also satisfies the BYBE, where $S^{fe}_{ba}(\theta)$ is the bulk YBE solution given in (2.1.2) and θ_1 is an arbitrary parameter. The proof is essentially the same as that of the vertex representation given in [1] and we shall not repeat it here. It is convenient to think of the BYBE as defining relation of some associative algebra generated by the symbol R^a_{bc}. So the solutions given in the previous section correspond to particular representations of this algebra where the "quantum space" is trivial and the auxiliary space is the space of a one step path $\mathcal{P}_1$ on a truncated Bratteli diagram with ab and ac being respectively the in and out state. In this context, the above "decorated" solution then corresponds to a representation whose quantum space is isomorphic to $\mathcal{P}_1$ that is formed by the nodes f, b (d, c). Clearly, the above construction can be repeated for an arbitrary number of time (say $N+1$) giving a boundary R-matrix that acts on $\mathcal{P}_N$ the collection of N-step paths on a truncated Bratteli diagram. We shall denote such solution as $\check{R}^a_{bc}$ which should be regarded as an operator on $\mathcal{P}_N$. Explicitly, it's matrix element is given by

$$\begin{aligned}\check{R}^a_{bc}(\theta)_{\alpha_1,\cdots,\alpha_{N+1};\alpha''_1,\cdots,\alpha''_{N+1}} &= \delta_{a\alpha'_1}\delta_{b\alpha_1}\delta_{c\alpha''_1}\prod_{i=1}^{N}\sum_{\alpha'_{i+1}}\Big(S^{\alpha_i\alpha'_i}_{\alpha_{i+1}\alpha'_{i+1}}(\theta-\theta_i) \\ &\quad \times S^{\alpha'_{i+1}\alpha'_i}_{\alpha''_{i+1}\alpha''_i}(\theta+\theta_i)\Big)\, R^{\alpha'_{N+1}}_{\alpha_{N+1}\alpha''_{N+1}}(\theta) \qquad (3.2)\end{aligned}$$

which has the graphical representation given in Fig.(4). It carries N parameters θ_i from the bulk S matrix and additional parameters from R^a_{bc}. To form a commuting transfer matrix out of $\check{R}^a_{bc}$, like in the vertex case, one has

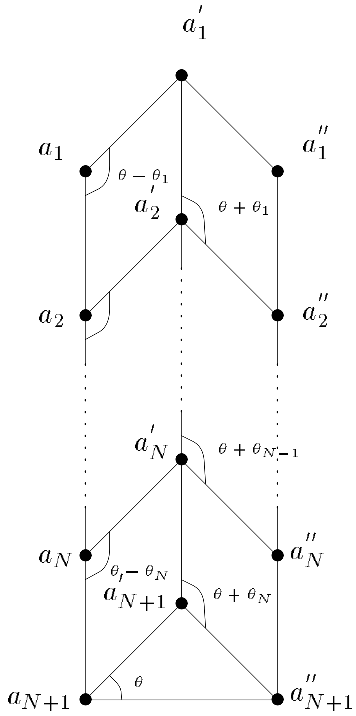

Figure 4: "Decorated" boundary scattering matrix II

to combine it with another BYBE solution (denoted here as $\tilde{R}$) as follows

$$T(\theta)_{\alpha_1,\cdots,\alpha_{N+1};\alpha_1'',\cdots,\alpha_{N+1}''} \equiv \sum_{a,b,c} \tilde{R}^a_{cb}(\theta)\check{R}^a_{b,c}(\theta)_{\alpha_1,\cdots,\alpha_{N+1};\alpha_1'',\cdots,\alpha_{N+1}''} . \tag{3.3}$$

Hence the transfer matrix $T(\theta)$ is again an operator that acts on $\mathcal{P}_N$.

Since the bulk S-matrix is symmetric and satisfies the unitarity condition

$$\sum_\alpha S^{ac}_{b\alpha}(\theta)S^{\alpha c}_{bd}(-\theta) \propto \delta_{ac} , \tag{3.4}$$

as well as the crossing unitarity condition given by

$$\sum_\alpha S^{b\alpha}_{cd}(\theta)S^{d\alpha}_{ab}(2i\pi-\theta) \propto \delta_{ac} , \tag{3.5}$$

it suffices to choose

$$\tilde{R}^a_{bc}(\theta) \equiv R^a_{bc}(i\pi-\theta) . \tag{3.6}$$

The proof has been given in [1]in the vertex language. Essentially, to show that

$$[T(\theta),T(\theta')] = 0 ,$$

one inserts four bulk S-matrices using eqns.(3.4),(3.5) (where the θ's are replaced respectively by $\theta'-\theta$ and $\theta'+\theta$) into $T(\theta)T(\theta')$ and then uses the BYBE to permute the $\check{R}$'s, and the $\tilde{R}$'s. Because of the argument $2i\pi-\theta-\theta'$ that appears in eqn.(3.5), the corresponding BYBE to permute the two $\tilde{R}$'s contains spectral parameters $i\pi-\theta$ and $i\pi-\theta'$, hence one can take $\tilde{R}$ to be given in eqn.(3.6).

4 Discussion and open problems

So far, we managed to obtain solutions to the BYBE and the corresponding commuting transfer matrix. It would be necessary to diagonalize the transfer matrix in order to study the statistical models given by these solutions. For application to field theory, diagonalization of the transfer matrix is also needed in order to write down the Bethe anatz equation. For this purpose, a systematic approach generalizing the algebraic Bethe anatz for the periodic boundary condition has been devised in [1]. However, the method relies upon the conservation of the S^z in the vertex language and is thus applicable only to diagonal boundary scattering theory. Moreover, the method also needs to be modified to be used for the RSOS models[15]. An alternative approach perhaps is given by functional method[16]. We hope to report on this in the near future.

Note added in proof

After we finished this work, we learned that some of our results have also been independently obtained in [6] in the trigonometric limit.

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