Perturbative Static Four-Quark Potentials

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Abstract

A first attempt to understand hadron dynamics at low energies in terms of the fundamental quark and gluon degrees of freedom incorporates the effects of the gluonic field into a potential depending only on the spatial positions of the quarks, which are considered in the infinite mass limit. A suitable framework for calculating such potentials between static quarks, i.e. a generalization of the Wilson loop will be discussed.

Making a connection with recent Monte Carlo lattice simulations for the lowest two energies of a system of two quarks and two antiquarks, the static $qq\bar{q}\bar{q}$-potential will be calculated in perturbation theory to fourth order. The result will be shown to be exactly equal to the prediction of a straightforward two-body approach, which in Monte Carlo lattice simulations has been found to be a reasonable approximation for very small interquark distances.

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1 Introduction

As QCD is the accepted theory of the strong interactions, it is no doubt desirable to understand all hadronic phenomena directly in terms of the fundamental fields of QCD. However, QCD being asymptotically free, perturbation theory is applicable only for very short distances and cannot cover the complete range of interest. At present lattice gauge simulations are the only way to study such systems. In a first approach, the static approximation is the natural choice, where the gluonic degrees of freedom are integrated out, and quark loops are ignored (the quenched approximation), giving rise to a potential between the stationary quarks. The potential of the quark-antiquark system, where this approach—leading to the familiar Wilson Loop—is very well known, has been calculated extensively in Monte Carlo lattice simulations. (For recent data, see e.g. [**?**].) The ground state potential of this static system has also been calculated in perturbation theory upto sixth order [**?**].

Here we shall describe how to generalize this procedure to multi-quark systems, especially to $(q\bar{q})^k$ systems. However, even with present-day computers, $q\bar{q}$ lattice simulations are still very demanding, and the amount of computations needed increases rapidly with the number of interacting quarks. Reliable models for multi-quark systems expressing their potentials e.g. in terms of the well known $q\bar{q}$-systems would therefore be of great help. Such two-body approximations have proven successful in many areas of physics, and these models can be formulated without difficulty. For the $qq\bar{q}\bar{q}$-system, which is the simplest one that can be considered consisting of two colour singlets, this model has been tested against numerical data from a Monte Carlo simulation [**?**]. For small distances the agreement has been found reasonable. It has also been observed [**?**] that the two-body model corresponds to lowest order perturbation theory. We shall be able to show that it is correct even to fourth order. To sixth order, however, three- and four-body forces begin to appear.

2 The Generalized Wilson Loop

While the concepts discussed below are of course well known in the context of the Wilson Loop for the $q\bar{q}$-system, we find it useful to start with rephrasing these concepts in the case of an arbitrary number of quarks, leading to a

study of more complicated systems.

When we have assembled a system of several quarks (and antiquarks), gluons will mediate a force between them. Treating this system in an approximation as a quantum mechanical system of several static quarks, the interactions between the quarks are incorporated into a potential. This assembly of quarks is then expected to propagate in time with the usual factor of e^{-itH}, where the interesting piece of the Hamilton operator H is the potential energy. Thus, by calculating appropriate Green functions, the potentials of eigenstates of H can be extracted.

2.1 Setting up Gauge Invariant States

Because of confinement, it makes sense only to talk about systems of quarks where the overall states have colour singlet quantum numbers. The problem with setting up say a $q\bar{q}$-system in a singlet is that the quark and antiquark are located a distance apart. This problem can be overcome by inserting the path ordered exponential $U(x,y,A) = \mathcal{P}e^{ig\int_y^x T^a A_a^\mu(z)dz_\mu}$ between the locations x and y of the quarks in the presence of the gauge potential A. Here g denotes the coupling constant and T^a the representation matrices. Thus $\bar{\psi}(x)U(x,y)\psi(y)|0\rangle$ will serve as a basis state in this case. We must also know how many basis states there are. When dealing with Green functions coming from Monte Carlo lattice simulations, they will have contributions from excited states of the gluonic field, and there are infinitely many of them even in the $q\bar{q}$-case. With suitable methods, the lowest potentials can be extracted, and several have been calculated for the quenched $q\bar{q}$-system – see for example [?], [?]. The situation is different for Green functions calculated in perturbation theory. Here, unlike the lattice simulations, we can and must go to the infinite time limit. We do not expect to reach excited states of the gluon field in finite order perturbation theory, and thus the number of basis states for a system of several quarks is given by the usual arguments of group representation theory, e.g. one for the $q\bar{q}$-system and two for the $qq\bar{q}\bar{q}$-system. In the large time limit we expect that the effects of 'introducing' the quarks into the vacuum will be irrelevant in comparison to their time evolution, and the notion of a potential makes sense.

2.2 Diagonalization

It may be shown that the state

$$\begin{aligned} &|\text{ quarks } q_i \text{ and antiquarks } \bar{q}_j \text{ at time } t\rangle \\ &= \bar{\psi}_i(-t, x_i) \ldots U(-t, x_i, y_i, A) \ldots \psi(-t, y_i)|0\rangle \end{aligned} \tag{1}$$

satisfies Schrödinger's equation. Forming the overlap of states at time $-t$ and t, we get an equation between Green functions and expressions of the form $\mathbf{A}_{ij} \stackrel{def}{=} \langle A_i|e^{-itH}|A_j\rangle$, where $|A_i\rangle$ stands for some basis state and we have introduced the matrix $\mathbf{A}$. By assuming a decomposition of these basis states into eigenstates of H, a diagonalization procedure will yield the potentials. In the case of the Green functions coming from lattice simulations, one considers a practical number of basis states, expands them in energy eigenstates and drops contributions with e^{-itE_i} for energies E_i above a certain limit. Of course we implicitly assume Wick-rotation. In perturbation theory, where a power expansion of $e^{-itE_i(g)}$ in the coupling g will not be exponentially damped, we need to consider all linearly independent basis states, a number that is finite, as remarked in the last section. Because of this finiteness, we can find an invertible transformation to energy eigenstates, and the diagonalization is straightforward. In fact, given a matrix $\mathbf{A}$ satisfying certain consistency relations, we can perturbatively prove [**?**] the existence of a time-independent basis transformation such that in this new basis $\mathbf{A}$ is not only diagonal, but its eigenvalues are of the form $e^{-itE_i(g)}$. Here the energy $E_i(g)$ of the i-th basis state, which can be calculated perturbatively, is for static quarks equal to the i-th potential (apart from an irrelevant constant, the rest mass).

2.3 Loops

What remains to be done is to bring the Green functions of the last paragraph to more familiar forms. Since we work within the static approximation, the full quark propagator in the presence of gauge fields can be calculated [**?**]:

$$\begin{aligned} S_0(x, y, A) &= -i[\mathcal{P}e^{ig\int_y^x T^a A_a^\mu(z)dz_\mu}]e^{-im|x^0-y^0|}\delta(\vec{x}-\vec{y}) \times \\ &\quad [\frac{1+\gamma^0}{2}\Theta(x^0-y^0) + \frac{1-\gamma^0}{2}\Theta(y^0-x^0)] \end{aligned} \tag{2}$$

We shall now outline how various contour integrations, i.e. loops arise. Considering the well-known $q\bar{q}$-case, we find a path-ordered line integral from antiquark to quark arising from the U in eq. (1), then the path-ordered line integral propagating the quark forward in time from eq. (2). Another U and the antiquark propagating backwards in time close the rectangle of the familiar Wilson loop. Starting with the Green functions described below eq. (1) and evaluating them for propagation from $-t/2$ to $t/2$, the following diagrammatic rule for calculating the Green function dealing with an arbitrary number $k/2$ of quark-antiquark pairs (i.e. k quarks and antiquarks) partitioned into $q\bar{q}$ singlets is seen to hold:

1. Draw two horizontal lines, the lower denoting time $-t/2$, the upper $t/2$. Mark the position of every quark and antiquark on the lower line and once again vertically above it on the upper line.

2. At the $-t/2$ level connect every quark-antiquark pair that is set up as a singlet at $-t/2$ with a line, having an arrow pointing from antiquark to quark.

3. At the $t/2$ level connect every quark-antiquark pair that is set up as a singlet at $t/2$ with a line, the arrow in which points from quark to antiquark.

4. Join the quarks at the $-t/2$ level with quarks at the same position at the $t/2$ level, arrow pointing upwards, i.e. forward in time.

5. Join the antiquarks at the $t/2$ level with the antiquarks at the $-t/2$ level, arrow pointing downwards, i.e. backwards in time.

6. Associate a path-ordered exponential of $e^{ig\oint_C T^a A_a^\mu(z)dz_\mu}$ together with a trace for every closed loop C occurring.

7. Determine the overall sign: If the pairings at the $-t/2$ level are the same as those on the $t/2$ level, there must be a $+$ sign. (This follows from the positivity of the norm on a Hilbert space if one lets $t \to 0$.) If this is not so, determine the sign of the permutation of antiquarks on the upper line that is necessary to give the same pairings as on the lower line. This is the overall sign.

8. Multiply by $(\delta(\vec{0})e^{-imt})^k$, where k is the total number of quarks and antiquarks.

9. Insert the factor so obtained in the numerator[1] of $\frac{\int [DA^a_\mu][D\eta^*_a][D\eta_a]e^{i\int d^4x[\mathcal{L}]}}{\int [DA^a_\mu][D\eta^*_a][D\eta_a]e^{i\int d^4x[\mathcal{L}]}}$

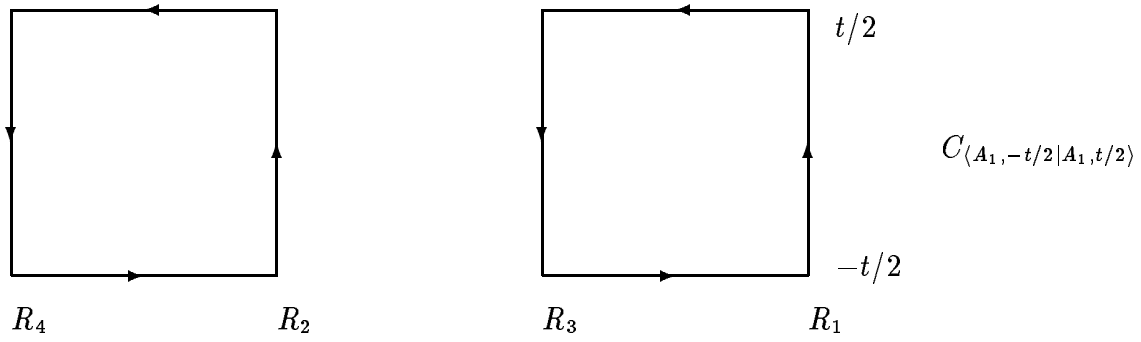
This gives the Green function in the chosen singlet structure.

3 The $qq\bar{q}\bar{q}$-Potentials

In $SU(\mathrm{N})$ gauge theory with quarks in the fundamental representation, we want to calculate the $qq\bar{q}\bar{q}$-potential in perturbation theory to fourth order. It has been remarked in subsection 2.1 that there are two independent basis states for this system, and one easily recognises a choice of these in the two possible ways of pairing the system into two quark-antiquark singlets. Assuming the first static quark at position R_1, the second at R_2, and the antiquarks at R_3 and R_4, we will label the two states $|A_1\rangle = \mathbf{1}_{13}\mathbf{1}_{24}$ and $|A_2\rangle = \mathbf{1}_{14}\mathbf{1}_{23}$.

3.1 Calculating the Green Functions

According to subsection 2.3, we encounter the following types of loops:



and

[1] With η we denote the ghost fields, with $\mathcal{L}$ the Lagrangian without fermions