Abstract

A recently proposed path-integral bosonization scheme for massive fermions in 3 dimensions is extended by keeping the full momentum-dependence of the one-loop vacuum polarization tensor. This makes it possible to discuss both the massive and massless fermion cases on an equal footing, and moreover the results it yields for massless fermions are consistent with the ones of another, seemingly different, canonical quantization approach to the problem of bosonization for a massless fermionic field in 3 dimensions.

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On bosonization in 3 dimensions.

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During the last few years many different proposals have been considered to bosonize fermionic theories in 3 dimensions [1, 2, 3, 4, 5]. In Ref. [2], order-disorder field operators related to a free massless Dirac field were defined. Applying canonical quantization methods, a bosonic, non-local and gauge-invariant action for an Abelian vector field was constructed, the approximate bosonization rules (in Euclidean spacetime) being

$$\begin{aligned} \bar{\psi}\,\not\partial\psi &\leftrightarrow \frac{1}{4}F_{\mu\nu}(-\partial^2)^{-1/2}F_{\mu\nu} + \frac{i}{2}\theta\,\epsilon_{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + nqt \\ \bar{\psi}\gamma_\mu\psi &\leftrightarrow \beta\,\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda - \beta\,\theta\,(-\partial^2)^{-1/2}\partial_\nu F_{\mu\nu} + nqt \end{aligned} \tag{1}$$

where ψ is a two-component Dirac spinor, A_μ is a $U(1)$ gauge field, and nqt means terms non-quadratic in A_μ (the neglecting of non-quadratic terms is what makes this bosonization approximate). The parameter θ is regularization-dependent. This sort of ambiguity, which manifests itself in the bosonization rules, already exists in the fermionic description. It is due to the regularization dependence of the induced Chern-Simons term [7].

In Ref. [4], functional methods were applied to derive bosonization formulae for the free massive Thirring model, and in [5], the Abelian and non-Abelian cases in any dimension $d \geq 2$ were considered. These 'long distance' bosonization rules are reliable for the description of phenomena where the fermionic current is not a strongly varying field, with a typical scale of variation much bigger that the inverse of the fermion mass. In this regime, either the free massive Dirac field or the Thirring model (in 3 dimensions) can be mapped to Chern-Simons theories by using the approximate bosonization rules

$$\begin{aligned} \bar{\psi}(\not\partial + m)\psi &\leftrightarrow \pm\frac{i}{2}\epsilon_{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda \\ \bar{\psi}\gamma_\mu\psi &\leftrightarrow \pm i\sqrt{\frac{1}{4\pi}}\,\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda \end{aligned} \tag{2}$$

This is valid to leading order in $\frac{1}{m}$, while the inclusion of the next-to-leading order term would lead to a Maxwell-Chern-Simons theory instead.

As it was stressed in [4, 5], the possibility of finding exact bosonization rules (in this functional approach), depends on our ability to compute the fermionic determinant in the presence of a background field exactly. Thus in 3 dimensions we must use an approximation scheme. The one presented in [4, 5] amounts to expanding the corresponding effective action in powers of $\frac{\partial}{m}$.

The question presents itself about how to extend this approximation in order to include cases where the derivative expansion is no longer valid, as it is indeed the case for massless fermions. The most obvious attempt to improve the approximation would be to include higher order terms in the derivative expansion. However, when doing so a new problem arises. The resulting theory will present instabilities, which in the Euclidean formulation are manifested in the action being not bounded from below, whereas in Minkowski space the related unitarity problem shows up. It is possible to get rid of this apparent drawback by recalling that the effective higher-order theory is valid only for gauge fields with momenta smaller than a cut-off of the order of the fermion mass m, which is a region free from such unphysical features, as can be easily verified. At any rate, cases where the momentum is larger than the fermion mass remains out of the scope of any (however refined) derivative expansion.

In this letter we attempt to overcome this kind of limitation by including the full momentum dependence in the one-loop quadratic part of the effective action. Whence the results will also be valid for the massless case, without spoiling the proper low-momentum features. As no momentum expansion is performed, there is no instability problem. Keeping the full momentum dependence one introduces a non-locality in the bosonized action, a property shared with the approach of [2]. This non-locality is unavoidable as soon as the derivative expansion, which always produces local terms, is discarded. For massless fermions in particular, one cannot escape the non-locality, since there is a branch cut at zero momentum so the one-loop vacuum-polarization tensor cannot be analytic there.

The above mentioned approaches, canonical and functional, to bosonization in 3 dimensions look *a priori* quite different and their relationship is not at all obvious. We will show that, by keeping the full momentum dependence of the vacuum polarization tensor in the approach of [5], one can reproduce [2] if the mass of the Dirac field is set equal to zero. The result of [5] will survive in the low-momentum (or $m \to \infty$) limit.

We start by constructing a bosonized version of the generating functional of current correlation functions in the case of a free fermionic field in three dimensions, reviewing the procedure followed in [5]. This method builds upon the functional representation of the fermionic generating functional

$$Z(s) = \int [d\psi][d\bar{\psi}] \, \exp\left[- \int d^3x \, \bar{\psi} (\not{\partial} + i \not{s} + m) \, \psi \right] \tag{3}$$

by performing the change of variables

$$\psi(x) \to e^{i\alpha(x)} \, \psi(x) \quad , \quad \bar{\psi}(x) \to e^{-i\alpha(x)} \, \bar{\psi}(x) \tag{4}$$

to obtain

$$Z(s) = \int [d\psi][d\bar{\psi}] \, \exp\left[- \int d^3x \, \bar{\psi} (\not{\partial} + i (\not{s} + \not{\partial}\alpha) + m) \, \psi \right] . \tag{5}$$

Defining $b_\mu = \partial_\mu \alpha$ ($\Rightarrow \; F_{\mu\nu}(b) = \partial_\mu b_\nu - \partial_\nu b_\mu \; = \; 0$), as $Z(s)$ does not depend on b_μ, the pure-gauge field b_μ can be integrated with an arbitrary (non-singular) weight functional $f(b)$, yielding (up to a normalization factor)

$$\begin{aligned} Z(s) = \int [db][d\psi][d\bar{\psi}] \quad f(b) \quad & \delta(F_{\mu\nu}(b)) \exp - \int d^3x \bar{\psi} (\not{\partial} + i(\not{s} + \not{b}) + m) \psi \\ = \quad & \int [db][d\psi][d\bar{\psi}] f(b-s) \delta(F_{\mu\nu}(b-s)) \\ \times \quad & \exp - \int d^3x \bar{\psi} (\not{\partial} + i \; \not{b} + m) \psi \, , \end{aligned} \tag{6}$$

where the last equation follows from the first one by shifting $b \rightarrow b-s$. Introducing a Lagrange multiplier A_μ to exponentiate the δ-functional, integrating over the fermion fields and setting the weight functional equal to one, it yields

$$Z(s) = \int [dA]\,[db]\,\exp\left[-T(b) - i\int d^3x\, A_\mu(\epsilon_{\mu\nu\lambda}\partial_\nu b_\lambda - \epsilon_{\mu\nu\lambda}\partial_\nu s_\lambda)\right] \tag{7}$$

where $T(b)$ denotes the fermionic effective action in the presence of an external vector field

$$T(b) = -\log\det(\not\partial + \not b + m)\,. \tag{8}$$

We now make the approximation of retaining up to quadratic terms in b_μ in (8). This is consistent with the approaches of ref.'s [2] and [5] [1]. The quadratic part of $T(b)$ may be split as

$$\begin{aligned} T(b) &= T_{PC}(b) + T_{PV}(b) \\ T_{PC}(b) &= \int d^3x\, \frac{1}{4}\, F_{\mu\nu}(b)\, F(-\partial^2)\, F_{\mu\nu}(b) \\ T_{PV}(b) &= \int d^3x\, \frac{i}{2}\, b_\mu\, G(-\partial^2)\, \epsilon_{\mu\nu\lambda}\partial_\nu b_\lambda \end{aligned} \tag{9}$$

where T_{PC} and T_{PV} come from the parity-conserving and parity-violating pieces of the vacuum-polarization tensor, respectively [6]. The function F in (9) is regularization-independent, and a standard one-loop calculation yields

$$\tilde{F}(k^2) = \frac{|\,m\,|}{4\pi k^2}\left[1 - \frac{1 - \frac{k^2}{4m^2}}{(\frac{k^2}{4m^2})^{\frac{1}{2}}}\,\arcsin(1 + \frac{4m^2}{k^2})^{-\frac{1}{2}}\right]\,, \tag{10}$$

where here and in what follows we shall always denote momentum-space representation by putting a tilde over the corresponding coordinate-space representation quantity. The function $\tilde{G}$ in (9) is regularization *dependent*, and can be written as

$$\tilde{G}(k^2) = \frac{q}{4\pi} + \frac{m}{2\pi\,|\,k\,|}\,\arcsin(1 + \frac{4m^2}{k^2})^{-\frac{1}{2}}\,, \tag{11}$$

[1] This is equivalent to introducing a 'coupling constant' e by means of the redefinition $b_\mu \rightarrow e\,b_\mu$, and working up to order in e^2.

where q can assume any integer value [7, 8], and may be thought of as the effective number of Pauli-Villars regulators, namely, the number of regulators with positive mass minus the number of negative mass ones. Adding a gauge-fixing term $\frac{\lambda}{2}(\partial \cdot b)^2$, the b - dependent part of the path integral (in momentum-space) reads:

$$I = \int [db] e^{-\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\left(\tilde{b}^\dagger(k)\,\tilde{M}(k)\,\tilde{b}(k) + i\tilde{b}^\dagger(k)|k|\,(\tilde{P}_+(k) - \tilde{P}_-(k))\,\tilde{A}(k)\right)} . \tag{12}$$

We introduced an obvious matrix notation, where the fields are represented by column vectors, the matrix $\tilde{M}$ is given by

$$\tilde{M}(k) \;=\; (\tilde{F}k^2 \,+\, i\,\tilde{G}\mid k\mid)\,\tilde{P}_+ \,+\, (\tilde{F}k^2 \,-\, i\,\tilde{G}\mid k\mid)\,\tilde{P}_- \,+\, \lambda k^2\,\tilde{L}\;, \tag{13}$$

and we introduced a complete set of hermitian orthogonal projectors

$$(\tilde{P}_\pm)_{\mu\nu} \;=\; \frac{1}{2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \,\pm\, i\epsilon_{\mu\lambda\nu}\frac{k_\lambda}{\mid k\mid}\right) \;\;,\;\; \tilde{L}_{\mu\nu} \;=\; \frac{k_\mu k_\nu}{k^2}\;, \tag{14}$$

which verify $\tilde{P}_\pm^2 = \tilde{P}_\pm$, $\tilde{L}^2 = \tilde{L}$; $\tilde{P}_\pm \tilde{L} = 0$, $\tilde{P}_+\tilde{P}_- = 0$; and $\tilde{P}_+ + \tilde{P}_- + \tilde{L} = 1$.

The bosonization formulae are obtained by integrating out the out the $\tilde{b}$-field

$$I \;=\; \exp\left[-\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\,\tilde{A}^\dagger\,(\tilde{P}_+(k) - \tilde{P}_-(k))\,k^2\,\tilde{M}^{-1}(\tilde{P}_+(k) - \tilde{P}_-(k))\,\tilde{A}(k)\right] \tag{15}$$

The inverse of $\tilde{M}$, needed in (15) is computed from (13),

$$\tilde{M}^{-1}(k) \;=\; (\tilde{F}k^2 \,+\, i\,\tilde{G}\mid k\mid)^{-1}\,\tilde{P}_+ \,+\, (\tilde{F}k^2 \,-\, i\,\tilde{G}\mid k\mid)^{-1}\,\tilde{P}_- \,+\, (\lambda k^2)^{-1}\,\tilde{L}\;, \tag{16}$$

and by further use of the projectors' properties, we can write

$$\begin{aligned} Z(s) \;&=\; \int [d\tilde{A}]\exp\int \frac{d^3k}{(2\pi)^3}[\frac{1}{2}k^2\tilde{A}^\dagger \\ &(\frac{1}{\tilde{F}k^2 + i\tilde{G}|k|}\,\tilde{P}_+ \;+\; \frac{1}{\tilde{F}k^2 - i\tilde{G}|k|}\,\tilde{P}_-)\tilde{A} \,-\, i\,\tilde{s}^\dagger\,|k|\,(\tilde{P}_+ \,-\, \tilde{P}_-)\,\tilde{A}]\;. \end{aligned} \tag{17}$$

There is still freedom to write the partition function (17) in different ways, namely, we can always redefine the field $\tilde{A}_\mu$ by performing a non-singular transformation on it. This will, of course, change both the quadratic and linear parts of the action, thus affecting both the bosonized action and the mapping between fermionic currents and bosonic fields, but in such a way that the current correlation functions are not modified, since we are just changing a dummy variable. It is however, necessary to do this in order to show explicitly the connection with the approach of [2]. A general redefinition of $\tilde{A}_\mu$ may be written as $\tilde{A} \to (\tilde{u}_+ P_+ + \tilde{u}_- P_- + \tilde{u}_L L)\tilde{A}$, where the $\tilde{u}$'s are functions of the momentum. Note that the effect of $\tilde{u}_L$ disappears as a consequence of gauge-invariance.

$$
\begin{aligned}
Z(s) &= \int [d\tilde{A}] \exp - \int \frac{d^3k}{(2\pi)^3} (\frac{1}{2} k^2 \tilde{A}^\dagger [\frac{|\tilde{u}_+|^2}{\tilde{F}k^2 + i\tilde{G}|k|} P_+ \\
&+ \frac{|\tilde{u}_-|^2}{\tilde{F}k^2 - i\tilde{G}|k|} P_-] \tilde{A} - i\tilde{s}^\dagger |k| (\tilde{u}_+ P_+ - \tilde{u}_- P_-) \tilde{A}) \;. \qquad (18)
\end{aligned}
$$

In what follows we shall restrict ourselves to the constant-$\tilde{u}_\pm$ case. Expression (18) can be put in coordinate space representation as follows:

$$
\begin{aligned}
Z(s) &= \int [dA] \exp - \int d^3x [\frac{1}{4} F_{\mu\nu}\, C_1\, F_{\mu\nu} - \frac{i}{2} A_\mu\, C_2\, \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda \\
&+ i\,(\frac{u_+ - u_-}{2})\, s_\mu \frac{1}{\sqrt{-\partial^2}} \partial_\nu F_{\nu\mu} - i\,(\frac{u_+ + u_-}{2})\, s_\mu \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda] \qquad (19)
\end{aligned}
$$

where

$$
\begin{aligned}
C_1 &= \frac{1}{2} \frac{|u_+|^2 (F - iG) + |u_-|^2 (F + iG)}{-\partial^2 F^2 + G^2} \\
C_2 &= \frac{i}{2} \frac{|u_+|^2 (F - iG) - |u_-|^2 (F + iG)}{-\partial^2 F^2 + G^2} \qquad (20)
\end{aligned}
$$

Let us discuss now the explicit form adopted by (20) for the cases $m \to \infty$ and $m \to 0$, to make contact with the results of reference [5] (particularized to the Abelian $d = 3$ case) and reference [2], respectively. This is achieved by evaluating C_1 and C_2 in the corresponding limits, and this is in turn determined by the values of F and G. When $m \to \infty$, C_1 tends to a constant which multiplies the Maxwell term. This is neglected to leading order in a derivative expansion, since there is also a Chern-Simons term, multiplied by the constant factor C_2:

$$C_2 \ \rightarrow \ 4\pi \ |u|^2 \ \times \ (q \ + \ \frac{m}{|m|}) \ . \tag{21}$$

C_2 is regularization-dependent, and its ambiguity is reflected here by the undefined constant q. To compare with [5], we partially fix q by the condition $q+\mathrm{sgn}(m) \ = \ \pm 1$, and chosing $u_+ \ = \ u_- \ = \ u \ = \ \frac{1}{2\pi}$, we see that the bosonized action (denoted S_{bos}), in the partition function (19) reduces to

$$S_{bos} \ = \ \int d^3x \ \left(\pm \frac{i}{2} A_\mu \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda \ - \ \frac{i}{\sqrt{4\pi}} s_\mu \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda \right) \ , \tag{22}$$

which agrees with the result of [5].

Now we discuss the limit $m \rightarrow 0$. In this case we have for F and G the behaviours

$$F(k^2) \ \rightarrow \ \frac{e^2}{16} \ |k|^{-1} \quad , \quad G(k^2) \ \rightarrow \ \frac{q}{4\pi} \tag{23}$$

which imply for C_1 and C_2

$$C_1 \rightarrow \frac{16|u|^2}{|k|} \quad C_2 \rightarrow \frac{4\pi|u|^2}{q} \ . \tag{24}$$

By taking then

$$\tilde{u}_+ \ = \ \tilde{u}_- \ = \ \frac{1}{4} \, e^{i\alpha} \ , \tag{25}$$

the bosonized action in coordinate space assumes the form

$$\begin{aligned} S_{bos} \ &= \ \int d^3x \, (\frac{1}{4} \, F_{\mu\nu} \frac{1}{\sqrt{-\partial^2}} F_{\mu\nu} \ - \ \frac{i}{2} \, \frac{\pi}{4q} \, \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \\ &- \ \frac{\sin\alpha}{4} \, s_\mu \, \frac{\partial_\nu F_{\nu\mu}}{\sqrt{-\partial^2}} \ - \ i \frac{\cos\alpha}{4} \, \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda) \ , \end{aligned} \tag{26}$$

thus with the identifications

$$\theta \ = \ \frac{\pi}{4q} \ , \ \alpha \ = \ \arctan \frac{\pi}{4q} \ , \ \beta \ = \ \frac{\cos\alpha}{4} \ , \tag{27}$$

the bosonized action becomes identical with the one of Equation (1), which is the Euclidean version of the one of Ref. [2].

We have thus studied the full bosonized partition function (19) for the low and large momentum regimes. In the general case, the full expression (19) should be retained. It is however, possible to simplify the form of $\tilde{F}$ and $\tilde{G}$, by replacing them by approximate but simpler looking expressions, which may be replaced in (20). With an error smaller than 10 percent over the full range of momenta, we have the approximations:

$$
\begin{aligned}
\tilde{F}(k^2) &= \frac{|\, m \,|}{4\pi k^2} \left[1 - \frac{1 - \frac{k^2}{4m^2}}{(\frac{k^2}{4m^2})^{\frac{1}{2}}} \arcsin(1 + \frac{4m^2}{k^2})^{-\frac{1}{2}} \right] \\
&\simeq \frac{1}{16}[k^2 + (\frac{3\pi m}{4})^2]^{-\frac{1}{2}} \\
\tilde{G}(k^2) &= \frac{q}{4\pi} + \frac{m}{2\pi \,|\, k \,|} \arcsin(1 + \frac{4m^2}{k^2})^{-\frac{1}{2}} \\
&\simeq \frac{q}{4\pi} + \frac{m}{4}[k^2 + \pi^2 m^2]^{-\frac{1}{2}} ,
\end{aligned}
\tag{28}
$$

which are obtained by following the approach of Ref. [10].

In this letter we have extended the method presented in [5], to obtain a bosonization for the free Dirac field, valid over the whole range of distances. A sensible extension is achieved by retaining the complete one-loop quadratic part of the effective action. The (non-local) bosonized theory that is obtained in this way have some advantages with respect to the (local) higher order theory that would have been obtained by considering a finite number of terms, when expanding the effective action in powers of $\frac{\partial}{m}$. On the one hand we can see that the Euclidean action is positive definite leading to a stable bosonized theory. On the other hand it unables us to treat the massive and massless cases in an equal footing, leading to the bosonization formulae for a massless Dirac field (Eq. (1)), obtained by following the canonical method.

In contrast, the higher order Euclidean effective action that results from the approximation of the non-local effective action is not positive definite, which leads to an unstable behaviour, unless a cut-off of the order of the fermion mass is used.

In Minkowski space, a similar situation shows up. A higher order theory leads to the presence of poles in the field propagator which are in conflict with unitarity. However, these poles will be located at a mass scale greater than m and again the imposed cut-off will prevent these poles from producing unphysical effects. Now, if we look Minkowskian version of the non-local Lagrangian Ref. [2] as this equivalence is valid over the whole range of momenta, no unphysical problems should appear. This is precisely the case [9].

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