GRAVITATIONAL PERTURBATIONS OF RELATIVISTIC MEMBRANES AND STRINGS

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Abstract

We consider gravitationally induced perturbations of relativistic Dirac–Goto–Nambu membranes and strings (or p-branes). The dynamics are described by the first and second fundamental tensors, and related curvature tensors in an n-dimensional spacetime. We show how equations of motion can be derived for the perturbations within a general gauge and then discuss how various simple gauge choices can be used to simplify the equations of motion for specific applications. We also show how the same equations of motion can be derived from an effective action by a variational principle. Finally, we compare these equations of motion to those using more familiar notation for brane dynamics, which involves the induced metric on the world-sheet. This work sets up a general formalism for understanding the effects of backreaction on brane dynamics and the background curvature.

1. Introduction

Relativistic membranes and strings (or p-branes) occur as topological defects and other solitonic structures in a variety of physical contexts [1]. Possibly the most exciting of these is the formation of defects during phase transitions in the early universe [2]. The localised energy of these defects is likely to extremely large and therefore their gravitational effects maybe cosmologically significant. In particular, cosmic strings may have been the initial seeds for the formation of galaxies and other large scale structure [3]. Therefore, an understanding of the precise dynamics of branes is of significant interest. Of particular interest is effect of backreaction on the dynamics of the brane and the related effects on background curvature. In this letter we set up a mathematical formalism by which such effects can be studied. The original

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motivation was to study the effects of gravitational radiation backreaction [4,5] on a network of cosmic strings [6] and subsequent stochastic gravitational radiation background [7,8,9]. However, the results presented here are completely general and apply to a p-dimensional brane in an n-dimensional spacetime. (NB. A 0–brane is a point particle, a 1–brane is a string, etc.)

The usual approach to brane dynamics involves specifying the coordinates of the brane $X^\mu = X^\mu(\sigma^a)$, where $a = 0,..,p$ and the σ^a are internal coordinates on the worldsheet. The spacetime interval between two neighbouring points is

$$ds^2 = g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu d\sigma^a d\sigma^b \,, \tag{1}$$

where $g_{\mu\nu}$ is the spacetime metric and $\partial_a = \partial/\partial\sigma^a$. Hence, the induced $p{+}1$ dimensional worldsheet metric is given by

$$\gamma_{ab} = g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \,. \tag{2}$$

The contravariant inverse metric tensor γ^{ab} can be defined as usual by $\gamma^{ab}\gamma_{bc} = \delta^a{}_c$ and $\|\gamma\| = \det(\gamma_{ab})$ is the determinant of the induced metric. This bi-tensorial approach, using quantities such as $\partial_a X^\mu$ that are tensorial with respect to both background and internal indices, is useful for explicit computation in physical applications. However, long calculations using this traditional approach can become extremely cumbersome due to the large number of internal indices involved. For this reason, we will use the more concise pure background tensorial formalism for brane dynamics developed in refs. [10,11,12,13]. In this formalism calculations are simplified by the lack of dependence on gauge and internal coordinate choices. Simple gauge and coordinate choices will allow one to convert the expressions deduced here into more physically usable expressions.

In the application of this formalism it is desirable to organise the tensors governing the dynamics in terms of components that are tangential or perpendicular to the worldsheet. To this end we define the first fundamental tensor, or tangential projection tensor, as

$$\eta^{\mu\nu} = \gamma^{ab}\partial_a X^\mu \partial_b X^\nu \,, \tag{3}$$

and we use the notation $\perp^{\mu\nu} = g^{\mu\nu} - \eta^{\mu\nu}$ for its orthogonal complement. For tensor fields whose support is confined to the worldsheet only the tangentially projected covariant differentiation,

$$\overline{\nabla}_\mu = \eta_\mu{}^\nu \nabla_\nu \,, \tag{4}$$

is well defined. Using this one can define the second fundamental tensor and the curvature vector as

$$K_{\mu\nu}{}^{\rho} = \eta_{\sigma\nu}\overline{\nabla}_{\mu}\eta^{\sigma\rho}\,, \qquad K^{\rho} = g^{\mu\nu}K_{\mu\nu}{}^{\rho}\,. \tag{5}$$

The second fundamental tensor can be shown to have the following elementary properties

$$\perp^{\sigma\mu} K_{\mu\nu}{}^{\rho} = 0\,, \quad \eta_{\sigma\rho}K_{\mu\nu}{}^{\rho} = 0\,, \quad K_{[\mu\nu]}{}^{\rho} = 0\,, \quad \overline{\nabla}_{\mu}\eta_{\nu\rho} = 2K_{\mu(\nu\rho)}\,, \quad \overline{\nabla}_{\mu}\eta^{\mu\rho} = K^{\rho}\,, \tag{6}$$

where round and square brackets denote index symmetrisation and antisymmetrisation, i.e. $A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})$ and $A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu})$. Using the definition of the first fundamental tensor (3) and the properties of the second fundamental tensor (6), one can deduce that

$$\begin{aligned} K^{\rho} &= \|\gamma\|^{-1/2}\partial_a\left(\|\gamma\|^{1/2}\partial^a X^{\rho}\right) + \eta^{\alpha\beta}\Gamma^{\rho}{}_{\alpha\beta}\,,\\ K^{\mu\nu\rho} &= \perp_{\sigma}{}^{\rho}\left(\partial^a X^{\mu}\partial^b X^{\nu}\partial_a\partial_b X^{\sigma} + \eta^{\mu\alpha}\eta^{\mu\beta}\Gamma^{\sigma}{}_{\alpha\beta}\right). \end{aligned} \tag{7}$$

using the obvious abbreviation $\partial^a = \gamma^{ab}\partial_b$. These relations allow one to convert between the traditional bi–tensor formalism and the more concise formalism using only background indices.

For an effective action of the simplest type, as exemplified by the Dirac action for membranes or the Nambu action for strings, the variational principle gives equations of motion for the brane that are expressible [10] simply by

$$K^{\rho} = 0\,, \tag{8}$$

in the absence of any coupling of the worldsheet to any other fields. In this letter, we shall derive the equations of motion for small gravitationally induced perturbations in a general gauge. These equations are shown to simplify in certain special gauge choices. Then we show how the exact same equations of motion can be deduced by introducing perturbations into the effective action before using the variational principle. Finally, we use the properties of the second fundamental tensor (6) and the conversion formulae (7) to compare to those derived using more traditional notation. This allows direct comparison with results derived in ref. [5] for gravitational perturbations of a Goto–Nambu string.

2. Gravitationally induced perturbations

One of the most fascinating aspects of the Einstein field equations is the existence of radiative solutions, similar to those found in electromagnetism and other gauge theories. This similarity may provide a hint to the crucial missing link between general relativity and gauge theories. In order to study such phenomena, one must perturb the Einstein field equations. There are many different ways of doing this, the most common being Lagrangian and Eulerian perturbations. In a Lagrangian scheme, the perturbations are defined with respect to a reference system that is comoving with the relevant displacement, whereas in an Eulerian scheme the reference system remains fixed. Infinitesimal Lagrangian and Eulerian perturbations, denoted by $\delta_{\rm L}$ and $\delta_{\rm E}$ respectively, can be related by the Lie derivative $\mathcal{L}_\xi$,

$$\delta_{\rm L} = \delta_{\rm E} + \mathcal{L}_\xi , , \tag{9}$$

with respect to ξ^μ the Lagrangian perturbation of some arbitrary coordinate system, that is $\xi^\mu = \delta_{\rm L} x^\mu$. For the purposes of this letter, we shall only consider Lagrangian perturbations, since $\delta_{\rm L} X^\mu = 0$. However, it is a simple exercise to deduce the related Eulerian perturbations using (9).

Using the definitions (3) and (5), one can deduce that

$$\delta_{\rm L} \eta^{\mu\nu} = -\eta^{\mu\rho}\eta^{\nu\sigma}\delta_{\rm L} g_{\rho\sigma} \, , \quad \delta_{\rm L}\eta^\mu{}_\nu = \eta^{\mu\rho} \perp^\sigma{}_\nu \delta_{\rm L} g_{\rho\sigma} \, , \tag{10}$$

and

$$\delta_{\rm L} K_{\mu\nu}{}^\rho = \perp^\rho{}_\lambda \, \eta^\sigma{}_\mu \eta^\tau{}_\nu \, \delta_{\rm L}\Gamma^\lambda{}_{\sigma\tau} + \big(2 \perp^\sigma{}_{(\mu} K_{\nu)}{}^{\tau\rho} - K_{\mu\nu}{}^\sigma \eta^{\tau\rho}\big)\delta_{\rm L} g_{\sigma\tau} \, , \tag{11}$$

where the Lagrangian variation of the connection is given by the well known formula

$$\delta_{\rm L}\Gamma^\lambda{}_{\sigma\tau} = g^{\lambda\rho}\big(\nabla_{(\sigma}\delta_{\rm L} g_{\tau)\rho} - \tfrac{1}{2}\nabla_\rho \delta_{\rm L} g_{\sigma\tau}\big) \, . \tag{12}$$

Previous work on this subject [12] was restricted to cases for which the background was fixed in advance, so that there is no Eulerian variation of the metric, that is $\delta_{\rm E} g_{\mu\nu} = 0$. In this case, the Lagrangian variation is just the Lie derivative with respect to ξ^μ, that is $\delta_{\rm L} g_{\sigma\tau} = 2\nabla_{(\sigma}\xi_{\tau)}$. Here, we shall allow also for the possibility that the background spacetime metric is perturbed, so that there will be a non-zero Eulerian variation, $\delta_{\rm E} g_{\sigma\tau} = h_{\sigma\tau}$. Therefore, the total Lagrangian variation of the metric will be given by

$$\delta_{\rm L} g_{\sigma\tau} = 2\nabla_{(\sigma}\xi_{\tau)} + h_{\sigma\tau} \, . \tag{13}$$

As with standard treatments of linearized gravity, we shall ignore terms higher than first order in $h_{\mu\nu}$ and ξ_μ.

Using (10) and (12), the Lagrangian variations of the first fundamental tensor and connection are given by

$$\delta_{\rm L}\eta^{\mu\nu} = -2\eta_\sigma^{(\mu}\overline{\nabla}^{\nu)}\xi^\sigma - \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}\,, \tag{14}$$

and

$$\delta_{\rm L}\Gamma^\lambda{}_{\sigma\tau} = \nabla_{(\sigma}\nabla_{\tau)}\xi^\lambda - \mathcal{R}^\lambda{}_{(\sigma\tau)\rho}\xi^\rho \ + \nabla_{(\sigma}h_{\tau)}{}^\lambda - \tfrac{1}{2}\nabla^\lambda h_{\sigma\tau}\,, \tag{15}$$

where $\mathcal{R}^\lambda{}_{\sigma\tau\rho}$ is the background Riemann curvature tensor, which will be negligible in applications for which the length-scales characterising the geometric features of interest are small compared with those characterising any background spacetime curvature. Substituting (15) into (11) implies that

$$\begin{aligned}\delta_{\rm L}K_{\mu\nu}{}^\rho = {} & \perp^\rho{}_\lambda\left(\overline{\nabla}_{(\mu}\overline{\nabla}_{\nu)}\xi^\lambda - \eta^\sigma{}_{(\mu}\eta^\tau{}_{\nu)}\mathcal{R}^\lambda{}_{\sigma\tau\alpha}\xi^\alpha - K^\sigma{}_{(\mu\nu)}\overline{\nabla}_\sigma\xi^\lambda\right)\\ & + \left(2\perp^\sigma{}_{(\mu}K_{\nu)\tau}{}^\rho - \delta^\rho{}_\tau K_{\mu\nu}{}^\sigma\right)\left(\nabla_\sigma\xi^\tau + \overline{\nabla}^\tau\xi_\sigma\right)\\ & + \perp^\rho{}_\lambda\,\eta^\sigma{}_\mu\eta^\tau{}_\nu\left(\nabla_{(\sigma}h_{\tau)}{}^\lambda - \tfrac{1}{2}\nabla^\lambda\ h_{\sigma\tau}\right) + \left(2\perp^\sigma{}_{(\mu}K_{\nu)}{}^{\tau\rho} - K_{\mu\nu}{}^\sigma\eta^{\tau\rho}\right)h_{\sigma\tau}\,.\end{aligned} \tag{16}$$

The final line is the extra contribution, due to the non-zero Eulerian perturbation of the metric. The corresponding expression for Lagrangian perturbations of the curvature vector is

$$\delta_{\rm L}K^\rho = g^{\mu\nu}\delta_{\rm L}K_{\mu\nu}{}^\rho + K_{\mu\nu}{}^\rho\,\delta_{\rm L}g^{\mu\nu}\,. \tag{17}$$

Substituting from (16), one finally obtains

$$\begin{aligned}\delta_{\rm L}K^\rho = {} & \perp^\rho{}_\lambda\,\eta^{\mu\nu}\left(\overline{\nabla}_\mu\overline{\nabla}_\nu\xi^\lambda - \mathcal{R}^\lambda{}_{\mu\nu\sigma}\xi^\sigma\right) - 2K_\mu{}^{\nu\rho}\,\overline{\nabla}_\nu\xi^\mu - K^\sigma\left(\nabla_\sigma\xi^\rho + \overline{\nabla}^\rho\xi_\sigma\right)\\ & + \perp^\rho{}_\lambda\,\eta^{\mu\nu}\left(\nabla_\mu h_\nu{}^\lambda - \tfrac{1}{2}\nabla^\lambda h_{\mu\nu}\right) - \left(K^{\mu\nu\rho} + K^\mu\eta^{\nu\rho}\right)h_{\mu\nu}\,.\end{aligned} \tag{18}$$

All these Lagrangian variations will be invariant with respect to background coordinate gauge transformations generated by an arbitrary vector field ζ^ρ, according to the specification

$$\xi^\rho \mapsto \xi^\rho - \zeta^\rho\,, \quad h_{\mu\nu} \mapsto h_{\mu\nu} + 2\nabla_{(\mu}\zeta_{\nu)}\,. \tag{19}$$

The worldsheet itself is also invariant with respect to internal coordinate gauge transformations generated by an arbitrary tangential vector field ϵ^ρ according to the specification

$$\xi^\rho \mapsto \xi^\rho + \epsilon^\rho \qquad \perp^\rho{}_\nu\,\epsilon^\nu = 0\,, \tag{20}$$

which can be used to impose the *orthogonal gauge* condition $\eta^\mu{}_\nu\xi^\nu = 0$, without restricting the background gauge freedom (19). In particular, one may also choose the the standard *harmonic* gauge condition, $\nabla^\mu h_{\mu\nu} - \frac{1}{2}\nabla_\nu h = 0$, where $h = g^{\mu\nu}h_{\mu\nu}$, which is usually the most convenient for practical applications, since it greatly simplifies the wave equation for $h_{\mu\nu}$. Another possibility would be to use the comoving gauge condition $\xi^\rho = 0$. However, in this case the wave equation for $h_{\mu\nu}$ would be much more complicated. Since $K^\rho = 0$ for an unperturbed Goto-Nambu string, the equation of motion for the perturbation ξ^μ is

$$\perp^{\rho\lambda} \eta^{\mu\nu}\big(\overline{\nabla}_\mu\overline{\nabla}_\nu\xi_\lambda - \mathcal{R}_{\lambda\mu\nu\sigma}\xi^\sigma\big) - K^{\mu\nu\rho}\big(2\overline{\nabla}_\nu\xi_\mu + h_{\mu\nu}\big) + \perp^{\rho\lambda} \eta^{\mu\nu}\big(\nabla_\mu h_{\nu\lambda} - \tfrac{1}{2}\nabla_\lambda h_{\mu\nu}\big) = 0\,. \quad (21)$$

3. The alternative combined perturbation procedure

The procedure outlines in the proceeding section consists of making successive approximations for the perturbations, whereby one first solves the zeroth order equation for X^μ and then solves the first order equation for ξ^μ. An alternative procedure – which can be used safely when $h_{\mu\nu}$ represents a weak, previously given gravitational wave field, but that leads to runaway solutions for the backreaction problem – is to choose the unperturbed world sheet to coincide with the perturbed world sheet. In this case, the curvature vector non-longer satisfies $K^\rho = 0$, but one now automatically has $\xi^\mu = 0$, independently of the gauge used for $h_{\mu\nu}$. This contrasts with the successive approximation approach in which ξ^ρ could only have been set to zero by fixing the gauge in a manner that would have been incompatible with the harmonic gauge condition. Instead of the separate zeroth and first order equations whereby K^ρ and $\delta_{\rm L} K^\rho$ are set to zero separately, in this alternative procedure one just has a single equation, expressible – neglecting second order corrections – as $K^\rho + \delta_{\rm L} K^\rho = 0$, that is

$$K^\rho - (K^{\mu\nu\rho} + K^\mu\eta^{\nu\rho})h_{\mu\nu} + \perp^{\rho\lambda} \eta^{\mu\nu}\big(\nabla_\mu h_{\nu\lambda} - \tfrac{1}{2}\nabla_\lambda h_{\mu\nu}\big) = 0\,. \quad (22)$$

Remembering that $K^\rho \sim \mathcal{O}(h)$, one sees that – again subject to neglect of second order corrections, $\mathcal{O}(h^2)$ – the tangential projection of (22) is satisfied automatically as a mere identity, while its projection perpendicular to the worldsheet gives the remaining non-trivial part of the dynamical equations of the perturbed worldsheet in the simpler form

$$K^\rho - K^{\mu\nu\rho}h_{\mu\nu} + \perp^{\rho\lambda} \eta^{\mu\nu}\big(\nabla_\mu h_{\nu\lambda} - \tfrac{1}{2}\nabla_\lambda h_{\mu\nu}\big) = 0\,. \quad (23)$$

In this alternative formulation, the inclusion in $h_{\mu\nu}$ of the gravitational self-field leads to the familiar problem of unphysical runaway solutions. This difficulty can be consistently overcome using a local backreaction approximation and resubstituting the equations of motion [4,5].

4. Variational approach to perturbations

One can also obtain the result (23) by considering the variation of an action integral of the form

$$\mathcal{I} = \int \overline{\mathcal{L}} \Vert\gamma\Vert^{1/2} d^{\rm p+1}\sigma = \int \hat{\mathcal{L}} \Vert g\Vert^{1/2} d^{\rm n}x \,, \tag{24}$$

where in standard Dirac notation the distributional background Lagrangian scalar field $\hat{\mathcal{L}}$ is given in terms of the regular worldsheet Lagrangian density $\overline{\mathcal{L}}$ by

$$\hat{\mathcal{L}} = \Vert g\Vert^{-1/2} \int \overline{\mathcal{L}} \delta^{\rm n}[x - x\{\sigma\}] \Vert\gamma\Vert^{1/2} d^{\rm p+1}\sigma \,. \tag{25}$$

In such a formulation, the effect to first order of the Eulerian variation $g_{\mu\nu} \mapsto g_{\mu\nu} + h_{\mu\nu}$ will be expressible simply by $\overline{\mathcal{L}} \mapsto \overline{\mathcal{L}}_{\mathbf{G}}$, where the gravitationally coupled "gross" Lagrangian $\overline{\mathcal{L}}_{\mathbf{G}}$ is given by

$$\overline{\mathcal{L}}_{\mathbf{G}} = \overline{\mathcal{L}} + \tfrac{1}{2} h_{\rho\sigma} \overline{T}^{\rho\sigma} , \tag{26}$$

and the worldsheet energy–momentum density tensor is given by the standard formula

$$\overline{T}^{\mu\nu} = 2\Vert\gamma\Vert^{-1/2} \frac{\delta}{\delta g_{\mu\nu}} \Big(\overline{\mathcal{L}} \Vert\gamma\Vert^{1/2} \Big) = 2 \frac{\delta \overline{\mathcal{L}}}{\delta g_{\mu\nu}} + \overline{\mathcal{L}} \eta^{\mu\nu} \,. \tag{27}$$

The overhead bar is used to distinguish this regular, fworldsheet confined tensor field from the corresponding Dirac distributional background spacetime energy–momentum density tensor field

$$\hat{T}^{\mu\nu} = 2\Vert g\Vert^{-1/2} \frac{\delta}{\delta g_{\mu\nu}} \Big(\hat{\mathcal{L}} \Vert g\Vert^{1/2} \Big) = \Vert g\Vert^{-1/2} \int \overline{T}^{\mu\nu} \, \delta^{\rm n}[x - X\{\sigma^a\}] \, \Vert\gamma\Vert^{1/2} \, d^{\rm p+1}\sigma \,, \tag{28}$$

which will act as the gravitational source for $h_{\mu\nu}$. Using (26) one can define the corresponding "gross" surface energy momentum density tensor as

$$\overline{T}_{\mathbf{G}}{}^{\mu\nu} = \overline{T}^{\mu\nu} + \overline{C}^{\mu\nu\rho\sigma} h_{\rho\sigma} \ , \tag{29}$$

where $\overline{C}^{\mu\nu\rho\sigma} = \overline{C}^{\rho\sigma\mu\nu}$ is the automatically symmetric *hyper-Cauchy tensor* [14],

$$\overline{C}^{\mu\nu\rho\sigma} = \Vert\gamma\Vert^{-1/2} \frac{\delta}{\delta g_{\mu\nu}} \Big(\overline{T}^{\rho\sigma} \Vert\gamma\Vert^{1/2} \Big) = \frac{\delta \overline{T}^{\rho\sigma}}{\delta g_{\mu\nu}} + \tfrac{1}{2} \overline{T}^{\rho\sigma} \eta^{\mu\nu} \,, \tag{30}$$

the relativistic generalisation of the ordinary space projected Cauchy type elasticity tensor [15].

In the application of the variational principle, the worldsheet is supposed to undergo an infinitesimal virtual displacement $x^\mu \mapsto x^\mu + \xi^\mu$ so that the action integrand will undergo a Lagrangian variation given by

$$\Vert\gamma\Vert^{-1/2}\delta_{\rm L}\big(\overline{\mathcal{L}}_{\bf G}\,\Vert\gamma\Vert^{1/2}\big) = \tfrac{1}{2}\overline{T}_{\bf G}{}^{\mu\nu}\delta_{\rm L}g_{\mu\nu} + \tfrac{1}{2}\overline{T}^{\mu\nu}\delta_{\rm L}h_{\mu\nu}\ , \tag{31}$$

where the Lagrangian variations of $g_{\mu\nu}$ and $h_{\mu\nu}$ are given by their Lie derivatives with respect to ξ^ρ,

$$\delta_{\rm L}g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}\ , \quad \delta_{\rm L}h_{\mu\nu} = \xi^\rho\nabla_\rho h_{\mu\nu} + 2h_{\rho(\mu}\nabla_{\nu)}\xi^\rho\ . \tag{32}$$

Therefore, the variation of the action integrand is

$$\Vert\gamma\Vert^{-1/2}\delta_{\rm L}\big(\overline{\mathcal{L}}_{\bf G}\Vert\gamma\Vert^{1/2}\big) = \overline{\nabla}_\mu\Big(\xi^\rho(\overline{T}_{\bf G}{}^\mu{}_\rho + \overline{T}^{\mu\nu}h_{\nu\rho})\Big) - \xi^\rho\Big(\overline{\nabla}_\mu(\overline{T}_{\bf G}{}^\mu{}_\rho + \overline{T}^{\mu\nu}h_{\nu\rho}) - \tfrac{1}{2}\overline{T}^{\mu\nu}\nabla_\rho h_{\mu\nu}\Big)\,. \tag{33}$$

The first term is a surface divergence and can be ignored by using Green's theorem. Therefore, variational invariance reduces to the requirement that the coefficient of ξ^ρ in the second term should vanish, that is

$$\overline{\nabla}_\mu\big(\overline{T}_{\bf G}{}^{\mu\rho} + \overline{T}^{\mu\nu}h_\nu{}^\rho\big) = \tfrac{1}{2}\overline{T}^{\mu\nu}\nabla^\rho h_{\mu\nu}\,. \tag{34}$$

Regrouping the first order terms onto the right side, this dynamical equation can conveniently be rewritten as

$$\overline{\nabla}_\mu\overline{T}^{\mu\rho} = \overline{f}^\rho\ , \tag{35}$$

where the effective surface force density due to the gravitational perturbation is given by

$$\overline{f}^\rho = \tfrac{1}{2}\overline{T}^{\mu\nu}\nabla^\rho h_{\mu\nu} - \overline{\nabla}_\mu\big(\overline{T}^{\mu\nu}h_\nu{}^\rho + \overline{\mathcal{C}}^{\mu\rho\nu\lambda}h_{\nu\lambda}\big)\,. \tag{36}$$

For a Dirac–Goto–Nambu membrane or string, the unperturbed Lagrangian is just a constant and therefore

$$\overline{\mathcal{L}} = -m^{\rm p+1}\ , \quad \overline{T}^{\mu\nu} = -m^{\rm p+1}\eta^{\mu\nu}\ , \tag{37}$$

where m is some fixed parameter having the dimensions of a mass. In general m would be expected to be of the same order of magnitude as the relevant Higgs mass of the underlying field theoretical model. However, in some cases it could be very much larger, for example a global string. Using (37) one can deduce that the hyper-Cauchy tensor is

$$\overline{\mathcal{C}}^{\mu\nu\rho\sigma} = m^{\rm p+1}\big(\eta^{\mu(\rho}\eta^{\sigma)\nu} - \tfrac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma}\big)\,. \tag{38}$$

If one now substitutes (37) and (38) into (26) and (29), the gravitationally coupled "gross" Lagrangian is given by

$$\overline{\mathcal{L}}_{\mathbf{G}} = -m^{\mathrm{p}+1}\big(1 + \tfrac{1}{2}\eta^{\mu\nu}h_{\mu\nu}\big)\,. \tag{39}$$

and the corresponding "gross" surface energy–momentum tensor is given by

$$\overline{T}_{\mathbf{G}}{}^{\mu\nu} = m^{\mathrm{p}+1}\Big[\eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma} - \eta^{\mu\nu}\big(1 + \tfrac{1}{2}\eta^{\rho\sigma}h_{\rho\sigma}\big)\Big]\,. \tag{40}$$

Using the properties the first and second fundamental tensors (6), one can deduce that the force density induced by the gravitational perturbations is

$$\overline{f}^{\rho} = m^{\mathrm{p}+1}\Big[\perp^{\rho\lambda}\eta^{\mu\nu}\big(\nabla_{\mu}h_{\nu\lambda} - \tfrac{1}{2}\nabla_{\lambda}h_{\mu\nu}\big) + \big(\perp^{\rho\nu}K^{\mu} + \tfrac{1}{2}\eta^{\mu\nu}K^{\rho} - K^{\mu\nu\rho}\big)h_{\mu\nu}\Big]\,. \tag{41}$$

This force is orthogonal to the worldsheet, that is $\eta^{\mu}{}_{\rho}\overline{f}^{\rho} = 0$, which is a Noether identity resulting from the lack of internal structure in the Dirac–Goto–Nambu case.

Using (35), the dynamical equations are thus obtained in the final form

$$K^{\rho} + \big(\perp^{\rho\nu}K^{\mu} + \tfrac{1}{2}\eta^{\mu\nu}K^{\rho} - K^{\mu\nu\rho}\big)h_{\mu\nu} + \perp^{\rho\lambda}\eta^{\mu\nu}\big(\nabla_{\mu}h_{\mu\lambda} - \tfrac{1}{2}\nabla_{\lambda}h_{\mu\nu}\big) = 0\,. \tag{42}$$

In order to account for the small,second order discrepancy between this final variational equation (42) and the previous equation (23) that was obtained via a less sophisticated approach by considering $K^{\rho} + \delta_{\mathrm{L}}K^{\rho} = 0$, one must understand that in the variational case we are effectively considering variations of $\|\gamma\|^{1/2}K_{\rho}$, instead of just K^{ρ}. The equation (42) can be seen to be exactly equivalent to

$$K^{\rho} + \delta_{\mathrm{L}}K^{\rho} + \|\gamma\|^{-1/2}\delta_{\mathrm{L}}\big(\|\gamma\|^{1/2}\big)K^{\rho} + h^{\rho}{}_{\sigma}K^{\sigma} = 0 \tag{43}$$

where $\|\gamma\|^{-1/2}\delta_{\mathrm{L}}\big(\|\gamma\|^{1/2}\big) = \frac{1}{2}\eta^{\mu\nu}h_{\mu\nu}$. Another alternative equation could be similarly deduced by considering variations of $\|\gamma\|^{1/2}K^{\rho}$. However all these different dynamical equations can be seen to agree to within corrections $\mathcal{O}(h^2)$, when one uses the fact that $K^{\rho} \sim \mathcal{O}(h)$.

5. Comparison to results in traditional notation

We have shown that the equation (23) describes the dynamics of a perturbed Dirac Goto-Nambu membrane or string. This problem has also been studied for strings using more traditional notation [5]. It should be possible to show that the results obtained by the two approaches are ultimately equivalent. In order to do this we split the forcing terms up into two parts, writing the dynamical equation (23) in the form

$$K^\rho = F_1^\rho + F_2^\rho\,, \tag{44}$$

with

$$F_1^\rho = -\perp^{\rho\lambda}\,\eta^{\alpha\beta}\Big(\nabla_\alpha h_{\lambda\beta} - \tfrac{1}{2}\nabla_\lambda h_{\alpha\beta}\Big)\,, \qquad F_2^\rho = h_{\mu\nu}K^{\mu\nu\rho}\,, \tag{45}$$

in which F_1^ρ depends only on derivatives of $h_{\mu\nu}$ and F_2^ρ depends only on the undifferentiated metric perturbation $h_{\mu\nu}$.

Using the definition of $\perp^{\mu\nu}$, it is a trivial exercise to show that

$$\begin{aligned} F_1^\rho =& \eta^{\rho\lambda}\eta^{\alpha\beta}\Big(\nabla_\alpha h_{\lambda\beta} - \tfrac{1}{2}\nabla_\lambda h_{\alpha\beta}\Big) - \eta^{\alpha\beta}\Big(\nabla_\alpha h^\rho{}_\beta - \tfrac{1}{2}\nabla^\rho h_{\alpha\beta}\Big) \\ =& \Big(\eta^{\alpha\gamma}\eta^{\beta\rho} - \tfrac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\rho}\Big)\nabla_\gamma h_{\alpha\beta} - \tfrac{1}{2}\eta^{\alpha\beta}\Big(- \nabla^\rho h_{\alpha\beta} + \nabla_\alpha h^\rho{}_\beta + \nabla_\beta h_\alpha{}^\rho\Big)\,. \end{aligned} \tag{46}$$

Evaluating the above expression for F_2^ρ is more tricky. Using the formula for $\overline{\nabla}_\alpha\eta^{\mu\nu} = 2K_\alpha{}^{(\mu\nu)}$ with the formula (3) for the first fundamental tensor, and making the chain rule substitution $\partial_a\eta^{\mu\nu} = \partial_a X^\alpha\partial_\alpha\eta^{\mu\nu}$, one can deduce that

$$\begin{aligned} F_2^\rho =& h_{\mu\nu}\Big(\eta^{\mu\alpha}\partial_\alpha\eta^{\rho\nu} - \tfrac{1}{2}\eta^{\alpha\rho}\partial_\alpha\eta^{\mu\nu} + \Gamma^\rho{}_{\alpha\beta}\eta^{\alpha\mu}\eta^{\beta\nu}\Big) \\ =& h_{\mu\nu}\Big(\partial_\alpha\eta^{\rho\nu}\gamma^{ab}\partial_a X^\alpha\partial_b X^\nu - \tfrac{1}{2}\partial_\alpha\eta^{\mu\nu}\gamma^{ab}\partial_a X^\alpha\partial_b X^\rho + \Gamma^\rho{}_{\alpha\beta}\eta^{\alpha\mu}\eta^{\beta\nu}\Big)\,, \end{aligned} \tag{47}$$

from which we finally obtain

$$F_2^\rho = h_{\mu\nu}\Big(\gamma^{ab}\partial_a\big(\gamma^{cd}\partial_c X^\rho\partial_d X^\mu\big)\partial_b X^\nu - \tfrac{1}{2}\gamma^{ab}\partial_a\big(\gamma^{cd}\partial_c X^\mu\partial_d X^\nu\big)\partial_b X^\rho + \Gamma^\rho{}_{\alpha\beta}\eta^{\alpha\mu}\eta^{\beta\nu}\Big)\,. \tag{48}$$

It can be checked that this agrees with what is obtained by the traditional approach [5].

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