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ON QUANTUM COHOMOLOGY AND DYNAMICAL SYSTEMS

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We investigate aspects of quantum cohomology and Floer cohomology in the context of a generic classical Hamiltonian system. In particular, we show that Floer's instanton equation is related to a quantum Euler character in the quantum cohomology defined by topological nonlinear σ-model. This relation is an infinite dimensional analogy with the relation between Poincaré-Hopf and Gauss-Bonnet-Chern formulae in classical Morse theory. By applying localization techniques to functional integrals we then show that for a Kähler manifold this quantum Euler character also coincides with the Euler character determined by the deRham cohomology of the target space. Our results are consistent with the Arnold conjecture which estimates periodic solutions to classical Hamilton's equations in terms of deRham cohomology of the phase space.

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The methods of quantum field theory that were originally developed to understand particle physics, have since proven useful also in statistical physics. Recently it has been noticed that these methods could even be succesfully applied to classical Hamiltonian dynamics. There, one of the intriguing open problems is the Arnold conjecture [1], [2] which states, that on a compact phase space the number of periodic solutions to Hamilton's equations is bounded from below by the sum of Betti numbers.

In this Letter we shall be interested in developing functional integral techniques to address issues such as the Arnold conjecture. In particular, we argue that the methods of topological quantum field theories when combined with functional localization techniques appear quite effective also in the case of classical dynamical systems.

We shall consider Hamilton's equations on a phase space which is a compact symplectic manifold X with local coordinates ϕ^a. We are interested in T-periodic trajectories that solve Hamilton's equations, *i.e.* are critical points of the classical action

$$S_{\mathrm{cl}} = \int_0^T \mathrm{d}\tau (\vartheta_a \dot{\phi}^a - H(\phi, \tau)) \tag{1}$$

Here ϑ_a are components of the symplectic potential corresponding to the symplectic two-form $\omega = \mathrm{d}\vartheta$. We assume that the Hamiltonian depends *explicitly* on time τ in a T-periodic manner $H(\phi, 0) = H(\phi, T)$, so that energy is not necessarily conserved. Hamilton's equations are

$$\dot{\phi}^a - \omega^{ab}\partial_b H(\phi;\tau) \ = \ \dot{\phi}^a - \mathcal{X}_H^a = 0 \tag{2}$$

with T-periodic boundary condition $\phi(0) = \phi(T)$. Without any loss of generality we shall assume that such periodic solutions are nondegenerate.

When energy is conserved so that H does not have explicit dependence on τ each critical point of H generates trivially a T-periodic trajectory. According to the classical Morse theory their number is bounded from below by the sum of Betti numbers on X and consequently Arnold's conjecture is valid:

$$\{\ \#\text{periodic trajectories}\ \} \ \geq \ \sum B_k = \sum \dim H^k(X, R) \tag{3}$$

However, if H depends explicitly on time so that energy is *not* conserved, the critical points of H do not solve (2) and the methods of finite dimensional Morse theory are no longer applicable. In order to show that (3) nevertheless remains valid we need an infinite dimensional generalization of Morse theory. Unfortunately this is not very easy: There is no minimum for (1), and periodic solutions of (2) are saddle points of

(1) with an infinite Morse index. Due to such difficulties, for explicitly τ-dependent Hamiltonians the conjecture has only been proven in certain special cases [2].

In the approach to Arnold conjecture developed by Floer [3], [2] one starts by defining a gradient flow in the space of closed loops $\phi(0) = \phi(T)$

$$\frac{\partial \phi^a}{\partial \sigma} = -g^{ab} \frac{\delta S_{\rm cl}}{\delta \phi^b} \tag{4}$$

where g_{ab} is a Riemannian metric on X. Using this metric and the symplectic two-form ω_{ab} we set

$$I^a{}_b = g^{ac} \omega_{cb}$$

Since $I^a{}_c I^c{}_b = -\delta^a{}_b$ this defines an almost complex structure on the manifold X and (4) becomes

$$\partial_\sigma \phi^a + I^a{}_b \partial_\tau \phi^b = \partial^a H(\phi, \tau). \tag{5}$$

This equation is defined on a cylinder $S^1 \times R$ with local coordinates τ and σ. It describes the flow of loops $\phi(\tau)$ on X, and the bounded orbits tend asymptotically to the periodic solutions of Hamilton's equation (2).

Using (5), Floer constructs a complex with the solutions to (2) being the vertices and the trajectories (5), so-called pseudo-holomorphic instantons, connecting them as the edges. He proves that for a generic Hamiltonian the cohomology of this complex is in fact *independent* of $H(\phi, \tau)$. Subsequently Witten [4] found that Floer's cohomology has connections to a quantum cohomology which is generated by the quantum ground states of a topological σ-model. Using the more general Novikov ring structure Sadov [5] then established that these two cohomologies in fact coincide. According to the Arnold conjecture (3) these cohomologies should also be intimately connected with the standard deRham cohomology of the underlying phase space.

Witten's quantum cohomology is based on solutions of Cauchy-Riemann equations for holomorphic curves

$$\partial_\sigma \phi^a + I^a{}_b \partial_\tau \phi^b = 0 \tag{6}$$

This corresponds to the (denegerate) special case of (5) with $H = 0$, which is not generic from the point of view of Hamiltonian dynamics. Consequently it is not clear how the topological σ-model, even if it describes Floer's cohomology, could be applied to understand Arnold's conjecture. For this, one needs to extend the topological σ-model so that it accounts for an *arbitrary* nontrivial Hamiltonian $H(\phi, \tau)$. Such issues have been addressed by Sadov [5]. In the present Letter we shall continue his work by explaining how functional integrals and localization techniques, when applied to the topological σ-model, can be used to derive Morse-theoretic relations for a generic Hamiltonian system. In particular, we shall explain how the standard, finite dimensional De Rham cohomology relates to quantum cohomology by studying an infinite dimensional version of Poincaré-Hopf and Gauss-Bonnet-Chern formulæ for (5).

Topological σ-model [4] is a theory of maps from a Riemann surface Σ with metric $\eta_{\alpha\beta}$ and almost complex structure $\epsilon_\alpha{}^\beta$ to a manifold X with Riemannian metric g_{ab} and almost complex structure $I^a{}_b$. We assume that the almost complex structures are both compatible with the metrics, so that for example on X we have $g_{ab} = I^c{}_a I^d{}_b g_{cd}$. Moreover, if we have

$$D_c I^a{}_b = \partial_c I^a{}_b + \Gamma^a_{cd} I^d{}_b - \Gamma^d_{cb} I^a{}_d = 0 \tag{7}$$

$I^a{}_b$ is an integrable complex structure and g_{ab} is Kähler. However, in the following we do not necessarily assume (7).

The basic fields are maps $\phi^a : \Sigma \to X,\ a = 1 \ldots \dim X$, which correspond to local coordinates on X. Anticommuting fields ψ^a are sections of $\phi^* TX$, the pullback of the tangent bundle of X. Anticommuting fields ρ^a_α, $(\alpha = 1, 2)$, are one-forms on Σ with values on $\phi^* TX$, so they are sections of the bundle $\mathcal{E} = \phi^* TX \otimes T^*\Sigma$. Commuting auxiliary fields F^a_α are sections of the same bundle as ρ^a_α. Because the rank of $\mathcal{E}$ is infinitely bigger than dimension of the space of maps from Σ to X we restrict to a subbundle, the self-dual part $\mathcal{E}^+$. This means that ρ^a_α and F^a_α both satisfy the self-duality constraint

$$\rho^a_\alpha = \epsilon_\alpha{}^\beta I^a{}_b \rho^b_\beta \tag{8}$$

The fields have a grading which at the classical level corresponds to a bosonic symmetry with charges $0, 1, -1, 0$ for $\phi^a, \psi^a, \rho^a_\alpha$ and F^a_α, respectively.

The action of topological σ-model can be constructed in the following way: Consider a nilpotent fermionic operator $\tilde{Q}$ of degree -1 constructed from the fields of the theory

$$\tilde{Q} = \int_\Sigma d^2x \left[i\psi^a(x) \frac{\delta}{\delta\phi^a(x)} + F^a_\alpha(x) \frac{\delta}{\delta\rho^a_\alpha(x)} \right] \equiv i\psi^a \partial_a + F^a_\alpha \iota^\alpha_a, \tag{9}$$

where summation over a always implies an integration over Σ and

$$\partial_a = \frac{\delta}{\delta\phi^a(x)}, \qquad \iota^\alpha_a = \frac{\delta}{\delta\rho^a_\alpha(x)}.$$

This we identify as a differential operator $\mathrm{d}\otimes 1+1\otimes\delta$ in the superspace defined on the complex $\Omega(\mathcal{E})\otimes\Omega(\Pi\mathcal{E})$. Here $\Pi\mathcal{E}$ means that the coordinates anticommute. Now introduce a canonical conjugation of $\tilde{Q}$

$$Q=e^{-\chi}\tilde{Q}e^{\chi}=\tilde{Q}+\{\tilde{Q},\chi\}+\frac{1}{2}\{\{\tilde{Q},\chi\},\chi\}+\cdots \tag{10}$$

The cohomologies defined by the operators $\tilde{Q}$ and Q are the same, and we still have $Q^2=0$. A suitable conjugation is defined by

$$\chi=i\psi^c\rho^b_\beta\pi^\alpha_a(\delta_\alpha{}^\beta\Gamma^a_{bc}-\frac{1}{2}\epsilon_\alpha{}^\beta D_cI^a{}_b),$$

where

$$\pi^\alpha_a=\frac{\delta}{\delta F^a_\alpha(\sigma)}.$$

In a coordinate free language this can be written as $\chi=-i(\rho,\hat{\Gamma}\pi)$ with

$$\hat{\Gamma}^{a\beta}_{\alpha b}=\hat{\Gamma}^a_b\otimes E_\alpha{}^\beta=\delta_\alpha{}^\beta\Gamma^a{}_b-\frac{1}{2}\epsilon_\alpha{}^\beta DI^a{}_b$$

being a connection 1-form and $\psi^a\sim d\phi^a$ denoting the basis of 1-forms on the space of maps $\Sigma\to X$. Notice that the covariant derivative of the self-dual sub-bundle $\mathcal{E}^+$ can be written using the modified connection

$$D^+=\frac{1}{2}[d+\hat{\Gamma}+\epsilon\circ(d+\hat{\Gamma})\circ I].$$

A straightforward calculation gives

$$\begin{aligned} Q &= i\psi^a\partial_a+\left[F^a_\alpha-i\psi^c\rho^b_\beta(\delta_\alpha{}^\beta\Gamma^a_{bc}-\frac{1}{2}\epsilon_\alpha{}^\beta D_cI^a{}_b)\right]\iota^\alpha_a-iF^b_\beta\psi^c(\delta_\alpha{}^\beta\Gamma^a_{bc}-\frac{1}{2}\epsilon_\alpha{}^\beta D_cI^a{}_b)\pi^\alpha_a \\ &+ \frac{1}{2}\psi^c\psi^d\rho^b_\beta\left[-\delta_\alpha{}^\beta R^a{}_{bcd}+\frac{1}{2}\delta_\alpha{}^\beta D_dI^e{}_bD_cI^a{}_e+\frac{1}{2}\epsilon_\alpha{}^\beta I^e{}_bR^a{}_{ecd}-\frac{1}{2}\epsilon_\alpha{}^\beta I^a{}_eR^e{}_{bcd}\right]\pi^\alpha_a \end{aligned}$$

or in short

$$Q=i(\psi,\partial)+(F+i\rho\hat{\Gamma},\iota)-i(F\hat{\Gamma},\pi)-\frac{1}{2}(\rho\hat{R},\pi).$$

Here

$$\frac{1}{2}\hat{R}=d\hat{\Gamma}+\hat{\Gamma}\wedge\hat{\Gamma}$$

is the Riemann curvature 2-form corresponding to the connection $\hat{\Gamma}^{a\beta}_{\alpha b}$. In components,

$$\frac{1}{2}\hat{R}^{a\alpha}_{\beta b}\;=\;(\frac{1}{2}R^a{}_b-\frac{1}{4}DI^e{}_bDI^a{}_e)\delta_\beta{}^\alpha\;+\;\frac{1}{4}(\;I^a{}_eR^e{}_b-I^e{}_bR^a{}_e\;)\epsilon_\beta{}^\alpha. \tag{11}$$

This operator Q is exactly the same as in [4] when we take into account the self-duality condition (8) for ρ^a_α and F^a_α.

We shall be interested in cohomological actions of the form

$$S = \{Q, \theta\} \tag{12}$$

Such actions are automatically invariant under the BRST-transformation generated by Q and consequently the partition function

$$Z = \int [\,\mathrm{d}\phi^a][\,\mathrm{d}F^a_\alpha][\,\mathrm{d}\psi^a][\rho^a_\alpha]\,\mathrm{e}^{iS} \tag{13}$$

should remain invariant under arbitrary local variations of θ.

If we select

$$\theta = (\rho, s) - \frac{\lambda}{4}(\rho, F) \;=\; \rho^a_\alpha g_{ab}\eta^{\alpha\beta}s^b_\beta - \frac{\lambda}{4}\rho^a_\alpha g_{ab}\eta^{\alpha\beta}F^b_\beta, \tag{14}$$

where $s^a_\alpha[\phi]$ is a section of $\mathcal{E}$ and λ is a parameter, we get

$$\begin{aligned} S \;&=\; \int_\Sigma \Big[-i\rho^a_\alpha D_c(g_{ab}s^{\alpha b})\psi^c + F^a_\alpha g_{ab}s^{\alpha b} - \frac{i}{2}\epsilon^\beta_\alpha \rho^b_\beta D_c I^a{}_b \psi^c g_{ad}s^{\alpha d} - \frac{\lambda}{4}F^a_\alpha F^\alpha_a \\ &+\; \frac{\lambda}{16}D_c I^a{}_e D_d I^e{}_b \psi^c\psi^d\rho^\alpha_a\rho^b_\alpha - \frac{\lambda}{8}R^a{}_{bcd}\psi^c\psi^d\rho^\alpha_a\rho^b_\alpha \Big]. \end{aligned} \tag{15}$$

specializing to $s^a_\alpha[\phi] = \partial_\alpha\phi^a$ and $\lambda = 1$ then gives the usual action [4] of topological σ-model.

Since the partition function (13) is (formally) invariant under local variations of θ we conclude that it must be independent of λ. Indeed, if we eliminate the auxiliary field F^a_α, the partition function yields an infinite dimensional version of the Mathai-Quillen formalism [6], [7]: In this formalism, one has a section Φ of a bundle E over the manifold X, and an (ordinary) integral over X of the so called Thom class

$$Z_{MQ} \;=\; \int\limits_X \int \mathrm{d}\rho \exp -[\frac{1}{4\lambda}(\Phi,\Phi) - i\rho\nabla\Phi - \frac{\lambda}{4}\rho R\rho]. \tag{16}$$

This integral is *independent* of λ, and as $\lambda \to 0$ it localizes to a finite dimensional integral over the moduli space $\mathcal{M}$ of solutions to the equation $\Phi = 0$. On the other hand, as $\lambda \to \infty$ (16) is nothing but the integral of the Pfaffian of the curvature which is the same as the Euler character of the bundle. The integral (16) thus yields an interpolation between the Poincáre-Hopf and Gauss-Bonnet-Chern formulae.

In the present case, elimination of the auxiliary field F^a_α gives an infinite dimensional functional integral version of the Mathai-Quillen formalism: Using (11) the action becomes

$$
\begin{aligned}
S &= \int_\Sigma \Big[\frac{1}{4\lambda}(s^a_\alpha + \epsilon_\alpha{}^\beta I^a{}_b s^b_\beta)(s^\alpha_a + \epsilon^\alpha{}_\beta I_a{}^b s^\beta_b) - \frac{i}{2}\rho^\alpha_a D_c (s^a_\alpha + \epsilon_\alpha{}^\beta I^a{}_b s^b_\beta)\psi^c \\
&- \frac{\lambda}{4}\hat{R}^a{}_{bcd}\psi^c\psi^d\rho^\alpha_a\rho^b_\alpha \Big] \qquad (17)
\end{aligned}
$$

which is clearly of the same functional form as the integrand that appears in (16), the relevant bundle being $\mathcal{E}^+$ and the section

$$
\Phi^a_\alpha = s^a_\alpha + \epsilon_\alpha{}^\beta I^a{}_b s^b_\beta \qquad (18)
$$

Thus we may view (13) as an infinite dimensional version of the integral of the Thom class (16).

Since (13) is (formally) independent of λ, we can consider its $\lambda \to \infty$ limit. According to the finite dimensional Mathai-Quillen formalism, this limit should be related to the Euler character of the functional space. For this, we specialize the world-sheet Σ to be a torus with coordinates σ and τ such that the metric $\eta_{\alpha\beta}$ is a unit matrix with compatible complex structure $\epsilon_\sigma{}^\tau = -\epsilon_\tau{}^\sigma = 1$. In the $\lambda \to \infty$ limit we then find that the partition function evaluates to

$$
Z_{\lambda\to\infty} = \int [\,\mathrm{d}\phi^a][\,\mathrm{d}\psi^a]\,\mathrm{Pfaff}(\hat{R}^a{}_b). \qquad (19)
$$

At least formally, this is the Euler character of the infinite dimensional bundle $\mathcal{E}^+$. In particular, all dependence on $T^*\Sigma$ has vanished from the last integral, it only depends on the Euler character of ϕ^*TX. Formally, this infinite dimensional quantity is a topological invariant and as such does not depend on how we choose the connection. It is the Euler character in the quantum cohomology defined by the quantum ground states of the topological σ-model, and counts the difference in the number of bosonic vacua (even forms) and fermionic vacua (odd forms) in the quantum theory.

In analogy with finite dimensional Morse theory, we next relate the (formal) infinite dimensional Euler character (19) to an alternating sum over critical points of a functional Φ describing the Floer cohomology. For this we consider the limit $\lambda \to 0$, again on a torus Σ with local coordinates σ, τ.

As $\lambda \to 0$, the integral obviously concentrates around the zeroes of

$$
\Phi^a_\alpha \equiv s^a_\alpha + \epsilon_\alpha{}^\beta I^a{}_b s^b_\beta
$$

For simplicity we shall assume that these zeroes are non-degenerate. (A generalization to the degenerate case is straightforward, see for example [8].) Let ϕ_0^a be such that

$$\Phi_\alpha^a[\phi_0] \;=\; 0$$

and write $\phi^a = \phi_0^a + \hat{\phi}^a$. In the absence of degeneracies, the first term in the expansion

$$\Phi_\alpha^a \approx \partial_c(\Phi_\alpha^a)\hat{\phi}^c + \mathcal{O}(\hat{\phi}^2)$$

does not vanish. The corresponding expansion of the action is

$$\begin{aligned} S \;&=\; \int_\Sigma \Big[-\frac{i}{2}\rho_\tau^a g_{ab}\partial_c(\Phi_\tau^a)\psi^c - \frac{i}{2}\rho_\sigma^a g_{ab}\partial_c(\Phi_\sigma^a)\psi^c \\ &+\; \frac{1}{4\lambda}\left(\partial_c(\Phi_\tau^a)g_{ab}\partial_d(\Phi_\tau^b) + \partial_c(\Phi_\sigma^a)g_{ab}\partial_d(\Phi_\sigma^b)\right)\hat{\phi}^c\hat{\phi}^d + \mathcal{O}(\hat{\phi}^3)\Big]. \end{aligned} \tag{20}$$

Using the self-duality of ρ_τ^a and the fact that near ϕ_0 we have $\Phi_\tau^a = -I^a{}_b\Phi_\sigma^b$ this gives

$$S = \int_\tau \left[-i\rho_\sigma^a g_{ab}\partial_c(\Phi_\sigma^a)\psi^c + \frac{1}{2\lambda}\partial_c(\Phi_\sigma^a)g_{ab}\partial_d(\Phi_\sigma^b)\hat{\phi}^c\hat{\phi}^d + \mathcal{O}(\hat{\phi}^3)\right]. \tag{21}$$

As $\lambda \to 0$, we can then evaluate the partition function which yields

$$\begin{aligned} Z_{\lambda\to 0} \;&=\; \int [\,\mathrm{d}\phi_0^a][\,\mathrm{d}\hat{\phi}^a][\,\mathrm{d}\psi^a][\rho_\tau^a][\rho_\sigma^a]\,\det{}^{-\frac{1}{2}}(\frac{i\lambda}{2}g)\exp[iS] \\ &=\; \int [\,\mathrm{d}\phi_0^a]\,\det{}^{-\frac{1}{2}}(g)\,\det{}^{-\frac{1}{2}}\left[\left(\partial_c(\Phi_\sigma^a)g_{ab}\partial_d(\Phi_\sigma^b)\right)\right]\det\left(g_{ab}\partial_c\Phi^b\sigma\right) \\ &=\; \sum_{\Phi_\sigma^a=0} \text{sign det } \;||\partial_b\Phi_\sigma^a|| \end{aligned} \tag{22}$$

In particular, if we select

$$s_\alpha^a = \partial_\alpha\phi^a - \chi_\alpha^a$$

and take χ_α^a to be a self-dual Hamiltonian vector field, *i.e.*

$$\chi_\alpha^a \;=\; \frac{1}{2}\partial^a H_\alpha(\tau,\phi) \tag{23}$$

where $H_\alpha(\phi)$ are two *a priori* arbitrary Hamiltonian functions on X related by the self-duality condition for χ_α^a, we find that

$$\Phi_\sigma^a \;=\; \partial_\sigma\phi^a + I^a{}_b\partial_\tau\phi^b - \partial^a H_\alpha(\tau,\phi) \;=\; 0$$

which is Floer's instanton equation (5).

Note that demanding χ_α^a's to be self-dual together with (23) implies that $I^a{}_b$ must be complex structure so that X is now a Kähler manifold.

This result establishes that the quantum cohomology of the topological σ-model indeed describes the cohomology of Floer's instanton equation, at least in the sense of Poincaré-Hopf and Gauss-Bonnet-Chern formulæ. The underlying idea in Floer's approach to the Arnold conjecture is that this cohomology should also be intimately related to the deRham cohomology of the original symplectic manifold, *i.e.* the target manifold of the σ-model. Such a relation would then explain why an estimate such as (3) makes sense as a Morse inequality. We shall now proceed to evaluate our functional integral using localization methods to establish that the Euler character (19) of quantum cohomology indeed coincides with the Euler character of the deRham cohomology over the symplectic manifold X.

For this, we specialize to a symplectic manifold which is Kähler. We select local coordinates so that $I^a{}_b = i\delta^a{}_b$ and $I^{\bar{a}}{}_{\bar{b}} = -i\delta^{\bar{a}}{}_{\bar{b}}$. Self-duality then implies that $F^a_z = F^{\bar{a}}_{\bar{z}} = 0$ so that the only surviving components are $F^a_{\bar{z}}$ and $F^{\bar{a}}_z$, and similarly for ρ^a_α. Using the (formal) invariance of (13) under local variations of θ, we introduce the functional

$$\theta \ = \ \eta^{\alpha\beta} g_{ab} F^a_\alpha \rho^b_\beta \ + \ \mu g_{ab} \eta^{\alpha\beta} \partial_\alpha \phi^a \rho^b_\beta \tag{24}$$

and consider the pertinent action (12). Explicitly (we set $F^a_{\bar{z}} \equiv F^a$ *etc.*),

$$\begin{aligned} S \ &= \ g_{a\bar{b}} F^a F^{\bar{b}} + R^a{}_{bc\bar{d}} \psi^c \psi^{\bar{d}} \rho^b g_{a\bar{e}} \rho^{\bar{e}} + R^{\bar{a}}{}_{\bar{b}c\bar{d}} \psi^c \psi^{\bar{d}} \rho^{\bar{b}} g_{\bar{a}e} \rho^e + \mu(F^a g_{a\bar{b}} \partial_{\bar{z}} \phi^{\bar{b}} + F^{\bar{a}} g_{\bar{a}b} \partial_z \phi^b) \\ &+ \ \mu \rho^a(-i g_{a\bar{b}} \partial_{\bar{z}} - i g_{a\bar{d}} \partial_{\bar{z}} \phi^{\bar{e}} \Gamma^{\bar{d}}_{\bar{b}\bar{e}}) \psi^{\bar{b}} + \mu \rho^{\bar{a}}(-i g_{\bar{a}b} \partial_z - i g_{\bar{a}d} \partial_z \phi^e \Gamma^d_{be}) \psi^b \end{aligned} \tag{25}$$

We evaluate the corresponding functional integral in the $\mu \to \infty$ limit, by separating the z, $\bar{z}$ independent constant modes (for example in a Fourier decomposition) and scale the non-constant modes by $\frac{1}{\sqrt{\mu}}$,

$$\begin{aligned} \phi^a(z,\bar{z}) \ &= \ \phi^a_o + \hat{\phi}^a(z,\bar{z}) \ \to \ \phi^a_o + \frac{1}{\sqrt{\mu}} \hat{\phi}^a(z,\bar{z}) \\ F^a(z,\bar{z}) \ &= \ F^a_o + \hat{F}^a(z,\bar{z}) \ \to \ F^a_o + \frac{1}{\sqrt{\mu}} \hat{F}^a(z,\bar{z}) \\ \rho^a(z,\bar{z}) \ &= \ \rho^a_o + \hat{\rho}^a(z,\bar{z}) \ \to \ \rho^a_o + \frac{1}{\sqrt{\mu}} \hat{\rho}^a(z,\bar{z}) \\ \psi^a(z,\bar{z}) \ &= \ \psi^a_o + \hat{\psi}^a(z,\bar{z}) \ \to \ \psi^a_o + \frac{1}{\sqrt{\mu}} \hat{\psi}^a(z,\bar{z}) \end{aligned} \tag{26}$$

and similarly for $\phi^{\bar{a}}$, $F^{\bar{a}}$, $\rho^{\bar{a}}$, $\psi^{\bar{a}}$. The Jacobian for this change of variables in (13) is trivial, and evaluating the integrals in the $\mu \to \infty$ limit we end up with the Euler character of the phase space X in the form

$$Z = \int d\phi_o^a d\phi_o^{\bar{a}} d\psi_o^a d\psi_o^{\bar{a}} \ \mathrm{Pfaff}(R^a{}_{bc\bar{d}}\psi_o^c\psi_o^{\bar{d}})\mathrm{Pfaff}(R^{\bar{a}}{}_{\bar{b}c\bar{d}}\psi_o^c\psi_o^{\bar{d}}) \tag{27}$$

which also exhibits the underlying complex structure on X. As a consequence, we have found that the Euler characteres in quantum cohomology and deRham cohomology *coincide*, establishing an intimate relationship between these two cohomologies. In particular, the Floer instanton equation defined over our torus obeys

$$\sum_{\Phi_\sigma^a=0} \mathrm{sign}\,\det \ ||\partial_c \Phi_\sigma^b|| \ = \ \sum_k (-)^k B_k$$

with B_k the Betti numbers of the symplectic manifold X. Obviously this is fully consistent with (3).

In conclusion, we have studied three *a priori* different cohomologies: Floer's cohomology which describes periodic solutions to Hamilton's equations, Witten's quantum cohomology which describes the quantum ground state structure of a topological non-linear σ-model, and standard finite dimensional deRham cohomology. By investigating an infinite dimensional generalization of the familiar Poincaré-Hopf and Gauss-Bonnet-Chern formulæ, we have found that these three cohomologies are intimately related. This result is consistent with the Arnold conjecture. In particular, it indicates that topological field theories and functional localization methods are useful tools also in the study of classical dynamical systems.

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