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Using Schwinger's quantum action principle, dispersion relations are obtained for neutral scalar mesons interacting with bi-local sources. These relations are used as the basis of a method for representing the effect of interactions in the Gaussian approximation to field theory, and it is argued that a marked inhomogeneity, in space-time dependence of the sources, forces a discrete spectrum on the field. The development of such a system is characterized by features commonly associated with chaos and self-organization (localization by domain or cell formation). The Green functions play the role of an iterative map in phase space. Stable systems reside at the fixed points of the map. The present work can be applied to self-interacting theories by choosing suitable properties for the sources. Rapid transport leads to a second order phase transition and anomalous dispersion. Finally, it is shown that there is a compact representation of the non-equilibrium dynamics in terms of generalized chemical potentials, or equivalently as a pseudo-gauge theory, with an imaginary charge. This analogy shows, more clearly, how dissipation and entropy production are related to the source picture and transform a flip-flop like behaviour between two reservoirs into the Landau problem in a constant 'magnetic field'. A summary of conventions and formalism is provided as a basis for future work.

I. INTRODUCTION

An increasing number of actual problems in physics find their natural expression not in the the static (equilibrium) aspects of quantum systems, but in the kinematical (non-equilibrium) development of their average properties. Examples include studies of early universe expansion [1–3], heavy ion collisions and the postulated quark-gluon plasma [4], lasers and other driven systems [5–7], and particle creation in changing fields of force [8,9].

The migration from equilibrium to non-equilibrium involves a shift of paradigm. In common with zero-temperature field theory, particle systems at equilibrium are often treated by a scattering formalism, with an initial (in) state and a final (out) state; this is only sensible if both are known and are at equilibrium with the same thermodynamic reservoir. The physics of a non-equilibrium system demands different boundary conditions. The initial and final states are (by definition) not characterized by the same ensemble and it is more appropriate to define the state (spectral profile or density matrix) of the system at some initial time $\overline{t}_i$ and compute the final state and its consequence at a later time $\overline{t}$. This describes to an initial value problem which is deftly handled by Schwinger's closed time path (CTP) action principle. The new picture also implies a concern with probabilities, or expectation values rather than amplitudes.

In equilibrium, one is used to the notion of translational invariance in space and time, implying that physical quantities only depend on the differences of coordinates $x - x'$. When the field is driven into disequilibrium, it acquires an additional dependence on the average position and time $\overline{x} = \frac{1}{2}(x + x')$. This is measured relative to an initial point of reference $\overline{x}_i$. In practical applications it is usually necessary to assume that the dependence on the average coordinate is *quasi-static* or of low adiabatic order in order to make computations tractable. The dependence on the average coordinate has important features: the preservation of unitarity demands that the statistical state of the field only depend on $\overline{x}$ and not $x - x'$. Since the state of the field can only be altered by the intervention of sources or sinks (hereafter referred to collectively as sources), the sources must also develop with respect to the average coordinate. Since one is interested both in fluctuations and the average kinematics, it is convenient to work with variables and sources which are bi-local objects rather than working with the field itself. This is in contrast to the pure field approach used by Schwinger [10]. Self-interacting theories are a special case in which the field is its own source; they pose mainly calculational problems—conceptually no new issues are introduced other than self-consistency.

Since the external sources affect the average state, they can be regarded as thermodynamical reservoirs, with the caveat that they must suffer a 'recoil' or back-reaction as a result of their effect on the system. This is not negligeable off equilibrium. Many quantum systems (the laser, for example) can be treated as two-reservoir systems in which the 'external' reservoir is of comparable magnitude to the local one.

The nomenclature 'open system' is used to describe a system coupled to independent sources. The name 'closed system' is given to a system without sources or one in which the field is its own source; in the latter case, the source must become effectively impotent with regard to the further development of the average state, as equilibrium is approached. Equilibrium is only achieved when, either the whole system approaches some driven limit cycle, or the contact with external sources is effectively terminated. In a closed system, the final equilibrium is a thermal state, or a state of maximum entropy. Some authors define equilibrium to mean a thermal state rather than merely a static one—and non-equilibrium to mean anything else. This is somewhat misleading since a non-thermal but static state is still final in the absence of new perturbations and must therefore be considered a point of equilibrium for the system.

The purpose of the present paper is to extend Schwinger's method of analysis to treat non-equilibrium ensembles of bosons. This touches on and extends a number of apparently different approaches to non-equilibrium [1,3,11]. Since Schwinger's original work [10] on the initial-value problem, most authors have been seduced by the functional integral and have therefore missed the often subtle advantages of Schwinger's methodology. It is intended that the present work should convey a pedagogical flavour of the suggested approach, which overlaps with the existing literature in strategic places without actually following any of them. In particular, conventions and definitions (which differ from most other accounts) have been chosen rather carefully for practical purposes. Some well known results are rederived in order to make the present work as self-contained as possible. The paper begins with a summarial discussion of the formalism, paying special attention to the action principle and unitarity; later the most general quadratic theory which maintains unitarity is presented and the Green functions are calculated for prescribed sources. Particular attention is paid to the effects of non-locality in the sources—an issue which has been largely neglected in previous work, and turns out to place strong requirements on the behaviour of stable systems. Finally a brief comparison is made between the present work and other approaches.

II. FORMALISM AND CONVENTIONS

The conceptual framework for the decription of non-equilibrium processes will include operator field theory, the method of sources and the local momentum space Green functions. In addition it proves convenient to use Schwinger's quantum action principle. This is a statement about the unitary development of the field with respect to the variation of certain variables. Since it embodies the equations of motion and the fundamental commutation relations for the field, it is both compact and elegant. One begins with the action operator, which is defined to be the classical action with the classical field replaced by the field operator, together with a suitable ordering prescription for the fields. Here the ordering will be the usual time-ordering and the action that for a real scalar field without self-interactions. The Minkowski metric-signature is $(-+++\cdots)$ which allows straightforward comparison with the Euclidean theory.

$$S = \int dV_x \{ \frac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) + \frac{1}{2} m^2 \phi^2 - J\phi \} \tag{1}$$

where dV_x is the Minkowski volume element. The operator equations of motion now follow from the quantum action principle [12]

$$\delta \langle \phi | \phi' \rangle = i \langle \phi | \delta S | \phi' \rangle \tag{2}$$

giving

$$\left(-\Box + m^2 \right) \phi(x) = J(x) \tag{3}$$

Given that $\phi(x) = \phi(x_i)$ at initial time t_i (or, more generally, on the the space-like hypersurface σ_i), the solution to (3) may be written

$$\phi(x) = \phi(x_i) + \int_{\sigma_i}^{\sigma} dV_{x'} G_c(x, x') J(x') \tag{4}$$

where σ_i and σ are the initial and final hypersurfaces and $G_c(x, x')$ is a Green function which satisfies

$$(-\overset{x}{\Box} + m^2) G_c(x, x') = \delta(x, x'). \tag{5}$$

Both the Feynman propagator and the retarded Green function have this property.

The surface integral under the variation of the action vanishes independently implying that the generator of infinitesimal unitary transformations on the field is [12]

$$\chi_\phi = \int d\sigma^\mu \phi \partial_\mu \phi. \tag{6}$$

Since it is easily established [12] that the unitary variation of any operator A is

$$\delta A = -i[A, \chi_A] \tag{7}$$

it follows that, on any spacelike hypersurface with orthogonal vector $\hat{n}^\mu$, one has

$$[\phi, \Pi_\mu]\hat{n}^\mu = i\delta(\mathbf{x}, \mathbf{x}') \tag{8}$$

with $\Pi_\mu = \partial_\mu \phi$. This is the covariant statement of the canonical commutation relations for the field and its conjugate momentum. To avoid unnecessary notation it is convenient to write this simply as

$$[\phi(\mathbf{x}, t), \partial_t \phi(\mathbf{x}', t)] = i\delta(\mathbf{x}, \mathbf{x}') \tag{9}$$

with the understanding that general covariance is easily restored by introducing a suitable time-like vector.

From the action principle (2) it can be shown by repeated functional differentiation with respect to the source that

$$\frac{\delta^n \langle \phi_2 | \phi_1 \rangle_J}{\delta J(x_1) \dots \delta J(x_n)} = i^n \langle \phi_2 | T(\phi(x_1) \dots \phi(x_n)) | \phi_1 \rangle, \tag{10}$$

thus the Taylor expansion of the amplitude may be written in the shorthand form

$$\langle \phi_2 | \phi_1 \rangle_J = \langle \phi_2 | T e^{i J \phi} | \phi_1 \rangle \tag{11}$$

where T denotes time-ordering (latest time to the left). This formula may be regarded as a generating functional for the n-point functions of the theory. The complex conjugate of this quantity is

$$\langle \phi_2 | \phi_1 \rangle_J^\dagger = \langle \phi_1 | \phi_2 \rangle_J = \langle \phi_1 | T^\dagger e^{-i J \phi} | \phi_2 \rangle \tag{12}$$

where $T^\dagger$ stands for anti-time-ordering (latest field to the right). This reverse-ordering is necessary to ensure the cancellation of intermediate fields in the identity:

$$\langle \phi_2 | \phi_1 \rangle_J \times \langle \phi_1 | \phi_2 \rangle_J = \langle \phi_2 | T^\dagger e^{-i J \phi} T e^{i J \phi} | \phi_2 \rangle = 1. \tag{13}$$

This is the key observation for the construction of the expectation values. Notice how the operator ordering in (13) starts from an early time, increases to a final time (at the centre of the operator product) and then reverses back to the initial time. Each field, at each instant along the closed time path has a mirror counterpart required for the cancellation of the intermediate operators in (13). This property can now be used to advantage to construct a 'closed time path' action principle [10,13].

Consider an expectation value of the form

$$\langle t | X(t') | t \rangle = \sum_{i,i'} \langle t | i \rangle \langle i | X(t') | i' \rangle \langle i' | t \rangle \tag{14}$$

where the sum over intermediate states i, i' is a sum over all states and $\langle t|$ is a shorthand which refers to either a pure state of the system, or a mixed state, specified at time t. The expectation value specifies the average value of the operator X at the time t' given the state of the system at time t. It involves conjugate amplitudes and hence the conjugate forms of the action principle:

$$\delta \langle t | i \rangle = i \langle t | \delta S_{ti} | i \rangle \tag{15}$$

$$\delta \langle i | t \rangle = -i \langle i | \delta S_{it}^\dagger | t \rangle \tag{16}$$

$$S_{ab} = \int_a^b dt L. \tag{17}$$

To obtain (14) from an action principle one would therefore like to introduce the operator X by functional differentiation with respect to an appropriate source (or combination of sources) between an amplitude and its conjugate. This is achieved in the following way. First one observes that

$$\delta\langle t|t\rangle = \delta(\sum_i \langle t|i\rangle \times \langle i|t\rangle) = \delta(1) = 0, \tag{18}$$

so differentiation of this object is to no avail. However, if we make an artificial distinction between the amplitude and its conjugate by labelling all objects in the former with a $+$ symbol and all objects in the latter with a $-$ symbol,

$$\delta\langle t|t\rangle = \lim_{+\to-} i\langle t|\delta S_+ - \delta S_-^\dagger|t\rangle \tag{19}$$

then we can use the solution of this quantity as a generating functional for (14) since X can be expanded in terms of either $\frac{\delta}{\delta J_+}$ or $\frac{\delta}{\delta J_-}$. This breaks the symmetry of symbols in (18). At the end of a variational calculation one removes the $+$ and $-$ symbols restoring the conjugate relationship between the two amplitudes, having inserted the appropriate operators by differentiation with respect to the source of only one of them. Note in (19) that, for any unitary field theory, the action is self-adjoint, thus we may drop the dagger symbol in future. Also, in treating the $+$ and $-$ parts of the field as being artificially independent, the condition

$$\phi_+(t_\infty) = \phi_-(t_\infty) \tag{20}$$

is required to ensure that the limit $+ \to -$ restores the single identity of the field operators, and additionally one must have that all ϕ_- fields (at any time) must stand to the left of all ϕ_+ fields (at any time). Since $-$ fields are anti-time-ordered and $+$ fields are time ordered, this condition arises naturally and ensures the triviality of (13).

The meaning of the above procedure can be illustrated by noting that the solution to (19) may be written

$$\langle t_i|t_i\rangle_{J_\pm} = \langle t_i|T^\dagger e^{-iJ_i\phi_-} T e^{iJ_+\phi_+}|t_i\rangle. \tag{21}$$

The expectation value of the field is found using the ordered expression

$$-i\frac{\delta}{\delta J_+(x)}\langle t_i|t_i\rangle\Big|_{+=-} = \langle\phi(x)\rangle \tag{22}$$

$$= \lim_{+\to-}\langle t_i|\exp\left(-i\int_{t_i}^{\infty} J_-\phi_- + i\int_t^\infty J_+\phi_+\right)\phi_+(x)\exp\left(i\int_{t_i}^{t}\right)|t_i\rangle. \tag{23}$$

Taking the limit $+ \to -$, one has

$$\langle\phi(x)\rangle = \langle t_i|\exp\left(-i\int_{t_i}^{t} J\phi\right)\phi(x)\exp\left(i\int_{t_i}^{t} J\phi\right)|t_i\rangle. \tag{24}$$

This shows that the average value of the operator depends only on the past (retarded) history of the system beginning from the initial time t_i. It can be shown (see appendix A) that the closed time path generating functional is closely related to the generator for the retarded n-point functions. The acausal (advanced) pieces cancel in the limit $+ \to -$.

So far, the discussion has used the slightly trivial example of pure states $\langle t|$. As noted implicitly by Schwinger [10], the same action principle holds when $\langle t|\ldots|t\rangle$ is replaced by $\langle t|\rho(t)\ldots|t\rangle$ (a mixture of states) since this does not affect the conjugate relationship between amplitudes. The nature of the expectation value can therefore be left out of the discussion for the most part. Indeed, in practice, the effect of a non-trivial density matrix in the expectation value can be mimicked by the introduction of suitable sources [10,14]—a procedure which will be adopted in the next section. To present the formalism in a way conducive to generalization, the next step is to present the Green functions for the case of pure-state vacuum expectation values and then introduce the finite temperature (mixed state) modifications which will be the starting point for writing down an ansatz for non-equilibrium.

The above use of generating functionals is closely related to the path integral approaches of Calzetta and Hu [1], and Lawrie [3]. It proves useful not to pass directly to the path integral however, but to follow Schwinger's approach. For the remainder of the paper, equation (19) will be considered the starting point for the discussion of non-equilibrium field theory.

From equations (1) and (19) one obtains the operator equations of motion for the field. Taking the initial time to be t_i, the furthermost future time to be t_∞ and the final time at which expectation values are to be computed as t_f, then using the boundary condition in equation (20),

$$\phi_+(x) = \phi(x_i) + \int_{t_i}^{t} G_c(x,x')J_+(x')dV_{x'}$$

$$\phi_-(x) = \phi_+(t_\infty) + \int_{t_\infty}^{t_f} G_c(x,x')J_-(x')dV_{x'}$$

$$= \phi(x_i) + \int_{t_i}^{t_\infty} G_c(x,x')J_+(x')dV_{x'} + \int_{t_\infty}^{t_f} G_c(x,x')J_-(x')dV_{x'} \tag{25}$$

where $G_c(x,x')$ is a retarded (causal) Green function. Notice that, as the distinction between + and − is removed, these equations reduce to (4). Substituting these into the exponential solution to (19) and defining a vector and its transverse by $J^T = (J_+ J_-)$, one may write

$$\ln\langle 0,t_i|0,t_i\rangle = -i\int_{t_f}^{t_\infty} J_-(x')dV_{x'}\phi(x_i) + i\int_{t_i}^{t_\infty} J^T(x)G(x,x')J(x')dV_x dV_{x'} \tag{26}$$

where

$$G(x,x') = \begin{pmatrix} \theta(x-x')G_c(x,x') & 0 \\ -G_c(x,x') & \theta(x'-x)G_c(x,x') \end{pmatrix}, \tag{27}$$

and $\theta(x-x')$ is the step function which satisfies

$$\theta(x) + \theta(-x) = 1. \tag{28}$$

As a result of this property, the sum of rows and columns in (27) is zero. This is a reflection of the triviality of equation (13). It further implies the causality of expectation values derived from this generating functional. While (27) has a simple physical derivation in terms of the equations of motion, a more symmetrical form can be obtained by attaching a variational interpretation to $G_c(x,x')$ directly. Again, following Schwinger, and varying with respect to the sources

$$\delta_2\delta_1\langle t_i|t_i\rangle = (i)^2\langle t|(\delta_2 S_+ - \delta_2 S_-)(\delta_1 S_+ - \delta_1 S_-)|\rangle t> \tag{29}$$

where, according to the ordering rule, this equals

$$\delta_2\delta_1\langle t_i,t_i\rangle = (i)^2\langle t|\phi_+(x_2)\phi_+(x_1) + \phi_-(x_2)\phi_-(x_1) - \phi_-(x_2)\phi_+(x_1) - \phi_+(x_2)\phi_-(x_1)|\rangle t>. \tag{30}$$

Comparing the solution of this to

$$\exp\left(\frac{i}{2}\int dV_x dV_{x'} J^T G(x,x')J(x')\right) \tag{31}$$

one has

$$G(x,x') = \begin{pmatrix} G_{++} & G_{+-} \\ G_{-+} & G_{--} \end{pmatrix}, \tag{32}$$

where

$$\langle\phi_+(x)\phi_+(x')\rangle = -iG_{++}(x,x') \tag{33}$$
$$\langle\phi_+(x)\phi_-(x')\rangle = iG_{+-}(x,x') \tag{34}$$
$$\langle\phi_-(x)\phi_+(x')\rangle = iG_{-+}(x,x') \tag{35}$$
$$\langle\phi_-(x)\phi_-(x')\rangle = -iG_{--}(x,x'). \tag{36}$$

As the distinction between + and − is lifted, the assumed ordering implies that

$$G_{++}(x,x') = i\langle T\phi(x)\phi(x')\rangle = G_F(x,x') \tag{37}$$
$$G_{+-}(x,x') = -i\langle\phi(x)\phi(x')\rangle = -G^{(-)}(x,x') \tag{38}$$
$$G_{-+}(x,x') = -i\langle\phi(x')\phi(x)\rangle = G^{(+)}(x,x') \tag{39}$$
$$G_{--}(x,x') = i\langle T^\dagger\phi(x)\phi(x')\rangle = G_{AF}(x,x') \tag{40}$$
$$\tag{41}$$

where G_F is the Feynman propagator, $G^{(\pm)}$ are the positive and negative frequency Wightman functions and G_{AF} is the anti-time ordered propagator. In the limit of zero source, these quantities satisfy the equations

$$(-\Box + m^2)G_F(x,x') = \delta(x,x') \tag{42}$$
$$(-\Box + m^2)G^{(\pm)}(x,x') = 0 \tag{43}$$
$$(-\Box + m^2)G_{AF}(x,x') = -\delta(x,x') \tag{44}$$
$$\tag{45}$$

by virtue of equation (5) for the field operator. The non-zero right hand side of (42) and (43) are due to the time ordering. From the time-ordering (37) it follows that

$$-iG_F(x,x') = i\theta(t-t')G^{(+)}(x,x') - i\theta(t'-t)G^{(-)}(x,x'). \tag{46}$$

Substituting this relation into (37) and using the commutation relations for the field (9) proves (37). Similarly (44) follows from the relation

$$G_{AF}(x,x') = -G_F^*(x,x'). \tag{47}$$

Using (28), it is now straightforward to see that the sum of the rows and columns in (32) is vanishing, as required for causality. A number of additional relations between the Green functions can be proven. The retarded and advanced Green functions satisfy:

$$\begin{aligned} G_{ret}(x,x') &= -\theta(t-t')[\phi(x),\phi(x')] \\ G_{adv}(x,x') &= \theta(t'-t)[\phi(x),\phi(x')]. \end{aligned} \tag{48}$$

Also, in virtue of (28) it is easy to see that

$$G_F = G_{ret} + G^{(-)} = G_{adv} - G^{(+)} \tag{49}$$

$$G_{AF} = = G_{ret} + G^{(+)} = G_{adv} - G^{(}(-)). \tag{50}$$

The unequal-time commutator and anti-commutator Green-functions are defined by

$$\begin{aligned} \tilde{G}(x,x') &= [\phi(x),\phi(x')] = G^{(+)} + G^{(-)} \\ \overline{G}(x,x') &= \{\phi(x),\phi(x')\} = G^{(+)} - G^{(-)} \end{aligned} \tag{51}$$

These will be useful later and serve to pin-point the conventions used in this work. Before considering the momentum-space representation of these functions it is useful to note that $G(x,x')$ can be written entirely in terms of the formal quantity

$$H(x,x') \equiv i\langle t|\phi(x)\phi(x')|t\rangle \tag{52}$$

as

$$G(x,x') = \theta(t-t')\begin{pmatrix} H(x,x') & -H(x',x) \\ -H(x,x') & H(x',x) \end{pmatrix} + \theta(t'-t)\begin{pmatrix} H(x',x) & -H(x',x) \\ -H(x,x') & H(x,x') \end{pmatrix}, \tag{53}$$

where

$$H(x',x) = H(x,x')^* \tag{54}$$

and the sum of rows and columns is manifestly zero. Since the spectrum of the operator $-\Box + m^2$ on the complex wave e^{ikx} is solved for any k satisfying a dispersion relation, the solution to (43) is the most general linear combination of plane waves satisfying the dispersion relation $k^2 + m^2 = 0$. This implies that the vacuum positive and negative frequency Wightman functions can be written, in n spacetime dimensions,

$$G^{(+)} = -2\pi i \int \frac{d^n k}{(2\pi)^n} e^{ik_\mu (x-x')^\mu} \theta(k_0)\delta(k^2+m^2) \tag{55}$$

$$G^{(-)} = 2\pi i \int \frac{d^n k}{(2\pi)^n} e^{ik_\mu (x-x')^\mu} \theta(-k_0)\delta(k^2+m^2). \tag{56}$$

Defining the fourier transform of $G(x,x')$ by

$$\int \frac{d^n k}{(2\pi)^n} e^{ik_\mu (x-x')^\mu} G(k), \tag{57}$$

and using the integral representation

$$\theta(t-t') = i\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega + i\epsilon} \tag{58}$$

it is straightforward to show from (46) that

$$G_F(k) = \frac{1}{k^2 + m^2 - i\epsilon}, \tag{59}$$

which is fully consistent with (42). $G_{AF}(k)$ is easily obtained from (47).

Note that, had the symmetrical form of $G(x, x')$ not been used, similar results could still have been obtained. It is possible, in the manner of a symmetry transformation to redefine the Wightman functions so that positive and negative frequencies are mixed. This simply mixes up the Feynman and anti-Feynman propagator also. For instance, if one defines

$$G^{(+)}(k) = 2\pi i\delta(k^2 + m^2)[\theta(k_0) + \alpha\theta(k_0) + \beta\theta(-k_0)], \tag{60}$$

then the corresponding Feynman propagator becomes

$$G_F(k) = \frac{1 + \alpha}{k^2 + m^2 - i\epsilon} - \frac{\beta}{k^2 + m^2 + i\epsilon} \tag{61}$$

where the last term is evidently a piece from G_{AF}. Since this only complicates the matter, such redefinitions will not be pursued further.

So far, this summary of the action principle has only explicitly encompassed pure state expectation values, which are comparatively trivial. A statistical system with real particle densities, as well as perhaps a temperature and entropy, is described by a mixture of such vacuum expectation values (since the character of the actual pure state is not usually knowable), with the the statistical weight given by the density matrix ρ. The simplest example of such is a system in thermal equilibrium ($\rho = \exp(-\beta H)$). Although a thermal system is quite extraordinary as many particle systems go, it serves as a useful reference point, both from the viewpoint of formalism and from a physical perspective, since very many physical systems can be characterized by a temperature of sorts. A statistical expectation value for some operator X may be written

$$\langle t|X(t')|t\rangle_s \equiv \frac{\mathrm{Tr}\langle t|\rho(t)X(t')|t\rangle}{\mathrm{Tr}\langle t|\rho(t)|t\rangle} \tag{62}$$

and characterizes the average value of X at the time t' given the state of the system at time t. Notice that the trace is over *probabilities* of the form $\langle t|t\rangle$ rather than amplitudes $\langle t|t'\rangle$. The latter would be meaningless. The structure of the expectation value is therefore simply that in equation (14) and the closed time path action principle applies. Indeed, it is noteworthy that the density matrix itself is merely an operator which can effectively by introduced into the pure state generating functional by functional differentiation with respect to an appropriate source. There is therefore no loss of generality in taking the closed time path action principle at face value and making no special reference to ρ..

The cyclic property of the trace in (62) has noteworthy implications for the Green functions and sources in the CTP formalism. Consider the expectation value in (62). This may be rewritten as

$$\begin{aligned}\langle X(t')\rangle_s &= \frac{\mathrm{Tr}\langle \rho(t)e^{iH(t'-t)}X(t)e^{-iH(t'-t)}\rangle}{\mathrm{Tr}\langle t|\rho(t)|t\rangle} \\ &= \frac{\mathrm{Tr}\langle e^{-iH(t'-t)}\rho(t)e^{iH(t'-t)}X(t)\rangle}{\mathrm{Tr}\langle t|\rho(t)|t\rangle}\end{aligned} \tag{63}$$

where H is the Hamiltonian of the system. Using this 'relativity' between the time-dependence of ρ and X, it is possible to place all of the dynamical development of the system in either one or the other. An example of the use of density matrix time-development is given in ref. [11]. In the CTP formalism, the distinction between forward moving times and backward moving times makes equation (63) effectively

$$\langle X(t')\rangle_s = \frac{\langle e^{-iH_-(t'-t)}\rho(t)e^{iH_+(t'-t)}X(t)\rangle}{\mathrm{Tr}\langle t|\rho(t)|t\rangle}. \tag{64}$$

The cyclic property of the trace therefore implies that the density matrix ρ always sits between the $+$ and $-$ branches of the operator product and hence it must always be reflected by the *off-diagonal* terms in $\pm$-space. In the special case of a thermal density matrix, the same observation leads to the well-known KMS condition [15], by identifying the inverse-temperature β with imaginary time. This is seen by considering the thermal Wightman function

$$
\begin{aligned}
G^{(+)}_{\beta}(x,x') &= \frac{\mathrm{Tr}\langle t|e^{-\beta H}\phi(x)\phi(x')|t\rangle}{\mathrm{Tr}\langle t|e^{-\beta H}|t\rangle} \\
&= \frac{\mathrm{Tr}\langle t|e^{-\beta H}\phi(x)e^{\beta H}e^{-\beta H}\phi(x')|t\rangle}{\mathrm{Tr}\langle t|e^{-\beta H}|t\rangle} \\
&= \frac{\mathrm{Tr}\langle t|e^{-\beta H}\phi(x')\phi(\mathbf{x},t+i\beta)|t\rangle}{\mathrm{Tr}\langle t|e^{-\beta H}|t\rangle} \\
&= -G^{(-)}(\mathbf{x},t+i\beta,x') \qquad (65)
\end{aligned}
$$

where the cyclic property of the trace has been used. The left and right hand sides are precisely the elements of the off-diagonal G_{+-} and G_{-+}. This property is, in fact, sufficient to determine the thermal Green functions.

To determine these, in a form which manifestly reduced to the vacuum case, one writes

$$
\begin{aligned}
G^{(+)}(k) &= -2\pi i[\theta(k_0)+X]\delta(k^2+m^2) \\
G^{(-)}(k) &= 2\pi i[\theta(-k_0)+Y]\delta(k^2+m^2) \qquad (66)
\end{aligned}
$$

with X and Y to be determined. Since the commutator function $\tilde{G}(x,x')$ must be independent of the state of the system (in order to preserve the canonical commutation relations), it follows immediately that $X=Y$. If one then employs the KMS condition which, in momentum space, becomes the definite relation

$$
G^{(+)}(k) = -e^{\beta k_0}G^{(-)}(k) \qquad (67)
$$

it follows that

$$
\theta(k_0)+X = e^{|k_0|\beta}[\theta(-k_0)+X] \qquad (68)
$$

and hence

$$
X = \theta(k_0)f(|k_0|) = f_{>0}(k_0) \qquad (69)
$$

where

$$
f(|k_0|) = \frac{1}{e^{\beta|k_0|}-1}. \qquad (70)
$$

By considering (amongst other things) $\overline{G}(x,x')$, it follows that $f(|k_0|)$ is an even function of $|k_0|$ thus $f(k_0)\theta(k_0) = f(-k_0)\theta(-k_0)$, whereupon it is trivial to show the unitarity relation

$$
G^{(+)}(x,x') = G^{(-)}(x,x')^*. \qquad (71)
$$

Note that the fact that $G^{(+)}$ consists only of positive frequencies k_0 is pivotal in this derivation. The Feynman propagator can now be obtained from equation (46) by using the integral representation of the step-function (58). The thermal Green functions are therefore summarized by

$$
\begin{aligned}
G^{(+)}(k) &= -2\pi i\theta(k_0)[1+f(|k_0|)]\delta(k^2+m^2) \qquad (72) \\
G_F(k) &= \frac{1}{k^2+m^2-i\epsilon} + 2\pi i f(|k_0|)\delta(k^2+m^2)\theta(k_0). \qquad (73)
\end{aligned}
$$

Another important form of $G^{(\pm)}$ is obtained by performing the integral over k_0, thereby enforcing the role of the delta-function in (66). This gives a result which will prove more useful for calculations later and is more closely related to the ansatz used by Lawrie in ref. [3]:

$$
\begin{aligned}
G^{(+)}(x,x') &= -2\pi i\int \frac{d^{n-1}k}{(2\pi)^{n-1}} e^{i(\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-\omega(t-t'))}\frac{(1+2f_{>0}(|\omega|))}{2|\omega|} \\
&= -2\pi i\int \frac{d^{n-1}k}{(2\pi)^{n-1}} e^{i(\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-\omega(t-t'))}\frac{(1+f(|\omega|))}{2|\omega|} \qquad (74)
\end{aligned}
$$

where $f_{>0}(k_0)$ is the function composed of only positive frequencies. It is now straightforward to verify that the canonical commutation relations are satisfied, by differentiating $\tilde{G}(x,x')$ with respect to t' (see equation (51)).

This completes the presentation of conventions to be used in the remainder of the paper. It is convenient to add here that a bar (e.g. $\overline{a}$) represents an object which is even, while an object with a tilde (e.g. $\tilde{a}$) represents one which is odd with respect to its arguments.

III. INTERACTION WITH SOURCES

The formalism demonstrated so far has been for free fields. Free fields are always in a state of equilibrium and therefore the discussion needs to be widened to incorporate collisions or interactions. The present work will deal with interactions which can be mediated by sources of the type $\phi(x)A(x,x')\phi(x')$. This includes a variety of self interactions, contact with external forces and noise or impurity scattering, depending on the nature of $A(x,x')$. The self-energy of an interacting field theory has this form, for instance, thus sources of the quadratic type can also be a representation of the lowest order, self-consistent 'particle dressings'. In Feynman diagram language, these represent the re-summed one-loop approximation, Hartree approximation and so on. Lawrie [3] uses the notion of such sources to effect a renormalization (resummation) of perturbation theory in a real scalar field theory. The same idea is expressed in a different language in the work of Calzetta and Hu [1]. Since it is not the aim of this paper to discuss specific models, the specific nature of the source terms will not be specified here. Rather the discussion will centre around what general properties such a system might have and a discussion of possible applications will follow.

In an interacting theory, one normally perturbs about the free field theory. Unfortunately, the dispersion relation (or 'mass shell' constraint) for free particles is no longer appropriate, since it reflects none of the interactions which 'dress' the particles. A more satisfactory starting point would be a 'quasi-particle' mass shell, including some of the interaction effects as the basis for a perturbation theory. This is the essence of a renormalization and can be effected by the use of sources [16].

The starting point for the investigation of non-equilibrium fields will therefore be the closed time path (CTP) action principle, taking the action S for a neutral scalar boson and supplementing it by quadratic sources. Observing the CTP operator ordering, one has

$$\begin{aligned} S_+ - S_- \rightarrow S_+ - S_- + \frac{1}{2}\int dV_x dV_{x'} [& T(\phi_+(x)A_{++}(x,x')\phi_+(x')) \\ & + \phi_-(x)A_{-+}(x,x')\phi_+(x') \\ & + \phi_-(x')A_{+-}(x,x')\phi_+(x) \\ & + T^\dagger(\phi_-(x)A_{--}(x,x')\phi_-(x'))] \end{aligned} \tag{75}$$

It should be clear that no fundamental field theory may contain off-diagonal terms in $\pm$-space. The CTP action principle is, by construction, diagonal, being the difference between S_+ and S_- (see equation (19)). However, it was remarked earlier that the effect of a density matrix must be reflected in off-diagonal terms so, while such terms are certainly not fundamental, they can exist as off-diagonal self-energies representing the dynamics of a density matrix. Moreover, since off-diagonal terms represent a point of contact between fields moving forward in time and fields moving backward in time, one might anticipate that off-diagonal sources would be at least partly responsible for choosing an arrow of time (the generation of entropy). The explicit coupling will therefore play an important role in both non-equilibrium kinematics and dynamics.

The essential unitarity of the CTP formalism is seen, from equations (15) and (16), to be summarized by the following property of the transformation function:

$$\langle t_i|t_i\rangle_\pm^* = \langle t_i|t_i\rangle_\mp \tag{76}$$

namely that complex conjugation merely exchanges + labels with − labels and vice versa. If one defines indices $a,b = +,-$, then the operator defined by the second variation of (75) with respect to the field ϕ_a, $S_{ab} = \delta_a\delta_b(S_+ - S_-)$, with $S_{ab} = S_{++}, S_{+-}\ldots$, satisfies the relations

$$\begin{aligned} S_{++}^*(x,x') &= -S_{--}(x,x') \\ S_{+-}^*(x,x') &= -S{-+}(x,x'). \end{aligned} \tag{77}$$

This, in turn, implies that S_{ab} may be written in terms of real constants A, B, C and γ_μ.

$$S_{ab}(x,x') = \begin{pmatrix} (-\Box + m^2)\delta(x,x') + A(x,x') + iC(x,x') & B(x,x') + \gamma^\mu(x,x')\overleftrightarrow{D}_\mu^{\gamma\,\prime} - iC(x,x') \\ -B(x,x') - \gamma^\mu(x,x')\overleftrightarrow{D}_\mu^{\gamma\,\prime} - iC(x,x') & (\Box - m^2)\delta(x,x') - A(x,x') + iC(x,x') \end{pmatrix} \tag{78}$$

where a new derivative has been defined to commute with the function $\gamma^\mu(x,x')$:

$$\overset{x}{D}{}_\mu^\gamma \equiv \overset{x}{\partial}_\mu + \frac{1}{2}\gamma_\nu^{-1}(x,x')\overset{x}{\partial}_\mu \gamma_\nu(x,x'). \tag{79}$$

Note, first of all, that the sum of rows and columns in this operator is zero, as required for unitarity and subsequent causality. Derivatives higher than first order in the sources could be rewritten using the field equations (still to be found) and absorbed into other terms, thus such terms are redundant. There can be no non-vanishing terms of the form $\phi_+ \partial \phi_+$ without violating time reversal invariance or merely adding total derivatives to the action. Finally $C \neq 0$ is clearly disallowed in a fundamental theory on the grounds of unitarity. It turns out, by considering the field equations, that the only fully consistent choice is $C = 0$, even though such a term does not violate equation (76). Equation (78) agrees with the form given by Lawrie, up to differences in conventions and the inclusion here of $B(x, x')$.

The significance of the off-diagonal terms involving γ^μ can be seen by writing out the coupling fully:

$$\gamma^\mu(x, x') \cdot \left(\phi_1 D^\gamma_\mu \phi_2 - \phi_2 D^\gamma_\mu \phi_1\right). \tag{80}$$

The term in parentheses has the form of a current between components ϕ_1 (the the forward moving field) and ϕ_2 (the backward moving field). When these two are in equilibrium there will be no dissipation to the external reservoir and these off-diagonal terms will vanish. This indicates that these off-diagonal components (which are related to off-diagonal density matrix elements, as noted earlier) can be understood as the mediators of a detailed balance condition for the field. A similar conclusion was reached in reference [1] by a different argument for the quantity referred to here as $\tilde{B}$. When the term is non-vanishing, it represents a current flowing in one particular direction, pointing out the arrow of time for either positive or negative frequencies. The current is a 'canonical current' and is clearly related to the fundamental commutator for the scalar field in the limit $+ \to -$.

Using equation (78) it is now possible to express (75) in the form

$$S_{CTP} = \int dV_x \, dV_{x'} \frac{1}{2} \phi^a S_{ab} \phi^b \tag{81}$$

and thus the closed time path field equations may be found by varying this action with respect to the $+$ and $-$ fields:

$$\frac{\delta S_{CTP}}{\delta \phi_+(x)} = (-\Box + m^2)\phi_+(x) + \frac{1}{2}\int dV_{x'} \left\{ (\overline{A} + i\overline{C})\phi_+(x') + (\tilde{B} - i\overline{C})\phi_-(x') + \overline{\gamma}^\mu(\partial_\mu \phi_-(x')) + \overline{\partial^\mu \gamma_\mu}\phi_-(x') \right\} \tag{82}$$

$$\frac{\delta S_{CTP}}{\delta \phi_-(x)} = (\Box - m^2)\phi_-(x) - \frac{1}{2}\int dV_{x'} \left\{ (\overline{A} - i\overline{C})\phi_+(x') + (\tilde{B} + i\overline{C})\phi_-(x') + \overline{\gamma}^\mu(\partial_\mu \phi_+(x')) + \overline{\partial^\mu \gamma_\mu}\phi_+(x') \right\} \tag{83}$$

$$\tag{84}$$

and setting the right hand side to zero, which introduces the notation

$$\begin{aligned} \overline{A}(x, x') &= \frac{1}{2}\left(A(x, x') + A(x', x)\right) \\ \tilde{A}(x, x') &= \frac{1}{2}\left(A(x, x') - A(x', x)\right) \\ \overline{\partial^\mu \gamma_\mu}(x, x') &= \frac{1}{2}\left(\overset{x}{\partial^\mu} \gamma_\mu(x, x') + \overset{x'}{\partial^\mu} \gamma_\mu(x', x) \right). \end{aligned} \tag{85}$$

The canonical commutation relations which derive from the action principle (see equation (9)) are unchanged by these modifications, since they depend only on $\overline{x}$ and therefore cancel in the commutator. This is key feature in any consistent description of non-equilibrium phenomena.

To solve this system of non-local equations, the best strategy will be to look for the Green functions, or the inverse of the operator S_{ab}. This is the method adopted by Lawrie [3]. Although a common strategy will be used here, the method will be somewhat different in spirit. The variational approach used in ref. [1] will not be used here. Owing to the non-locality, it is clear that the inverse operator cannot be a translationally invariant function in the general case. It must be formally dependent on both Cartesian coordinate differences and the average coordinate:

$$\tilde{x} \equiv \frac{1}{2}(x - x') \tag{86}$$

$$\overline{x} \equiv \frac{1}{2}(x + x'). \tag{87}$$

Moreover, since the operator contains off-diagonal terms, which typically signify a non-trivial density matrix, it is natural to look for a solution based on the form of equation (72), generalized to include a dependence on $\overline{x}$. Although

this sounds like an innocent step, it is far from a trivial undertaking since it introduces non-linearities in the spectrum of excitations which must be handled in a self-consistent way. It is useful to work with the quantity $H(x,x')$, from which all the Green functions can be obtained (actually the Wightman function in disguise). Using either the field equations or the matrix equation

$$S_{ab}G^{bc} = \delta_a{}^c \delta(x,x') \tag{88}$$

one obtains equations of motion for the quantity $H(x,x')$ (see equation (53)). Not all of these equations are independent, owing to the symmetry in equation (76). In particular, their consistency requires that $C=0$ which is now chosen explicitly. It is sufficient to consider

$$(-\Box + m^2)H(x,x') + \int dV_{x''} \left(\overline{A}(x,x'') - \tilde{B}(x,x') - \overline{\gamma}^{\mu}(x,x'') \overset{x''}{\partial}_{\mu} - \overline{\partial_{\mu}\gamma^{\mu}}(x,x'') \right) H(x'',x) = 0 \tag{89}$$

on the understanding that $H(x,x') = H(x',x)^*$. This relation is to be supplemented by the canonical commutation relations for the field, which appear in equation (88) in the form

$$\partial_{t'} \left(H(x,x') - H(x',x) \right) \Big|_{t=t'} = i\delta(\mathbf{x},\mathbf{x}') \tag{90}$$

and complete the consistency of the equations of motion.

The next step in obtaining an intuitive formalism is to introduce a (local) momentum space technique by Fourier transforming $\tilde{x}$ and retaining a dependence on the average coordinate $\overline{x}$:

$$H(x,x') = \int \frac{d^n k}{(2\pi)^n} e^{ik(x-x')} H(k,\overline{x}). \tag{91}$$

A suitable ansatz for this function, which generalizes the dispersion relation and the one-particle distribution function $f(k_0)$ is

$$H(k,\overline{x}) = 2\pi\theta(k_0)[1+f(k_0,\overline{x})]\delta(-k_0^2+\omega^2(k,\overline{x})). \tag{92}$$

The spacetime dependent function $f(k,\overline{x})$ is often referred to as the Wigner function and signifies the inhomogeneity in particle occupation numbers. The generalized dispersion relation takes generic form $-k_0^2+\omega^2=0$. In the free particle limit $\omega^2 = \mathbf{k}^2 + m^2$. It is the determination of this dispersion relation which is of specific importance, since this determines the spectrum of excitations for particles in the plasma-field, and forms the basis of all perturbation theory when the sources represent self-interactions.

It can be verified that, since $H(k,\overline{x})$ depends only on the average coordinate, the commutation relations are preserved (see equation (90)). A more useful form of (91) is obtained on performing the integration over k_0. This eliminates the dubious derivative of the delta-function from subsequent relations and leads to a number of helpful insights.

$$H(x,x') = 2\pi \int \frac{d^{n-1}k}{(2\pi)^{n-1}} e^{ik_{\mu}(x-x')^{\mu}} \frac{(1+f(k_0,\overline{x}))}{2|\omega|} \tag{93}$$

where it is understood that $k_0 = |\omega|$. Finally, it is useful to define the derivative with respect to the average coordinate $\overline{\partial} = \frac{1}{2}(\partial_x + \partial_{x'})$ and the quantities

$$F_{\mu} = \frac{\partial_{\mu} f}{1+f} = \frac{1}{2}\overline{F}_{\mu} = \frac{1}{2}\overline{\partial}_{\mu} \ln(1+f) \tag{94}$$

$$\Omega_{\mu} = \frac{\partial_{\mu}\omega}{\omega} = \frac{1}{2}\overline{\Omega}_{\mu} = \frac{1}{2}\overline{\partial}_{\mu} \ln|\omega|. \tag{95}$$

IV. DISPERSION RELATIONS

To determine the dispersion relations for given sources it is useful to distinguish three cases which will be referred to as the local, translationally invariant and inhomogeneous cases respectively. In the local case, the sources are

proportional to a delta function. In the translationally invariant case $A(x,x') = A(x-x')$ and in the inhomogeneous case $A(x,x') = A(\tilde{x},\overline{x})$.

There are two ways in which one can proceed with the determination of the dispersion relations. One is to separate real and imaginary parts and the other is to used complexified momenta. The latter has several advantages and makes straightforward contact with the classical theory of normal modes. It will be used exclusively for determining the spectral relations. Separating real and imaginary parts on the other hand allows one to identify imaginary contributions as a Boltzmann/Vlasov equation, illustrating nicely the intimate relationship between transport and dissipation [14].

In order to extract information from the equations it is necessary to undertake an approximation scheme in which only low order derivatives are kept in x. This is equivalent to an adiabatic (or quasi-static) scheme in which the development of the system if slow in comparison to fluctuations, so that fast and slow moving variables separate in an assumed way. In fact this is already built into the assumed form of the solution for the Green function, since without such an assumption, there are no grounds for assuming that $\tilde{x}$ and $\overline{x}$ would separate in the prescribed manner. For most purposes this approximation is quite sound. For the present, there seems to be no way of eliminating the approximation.

A. Local sources

In the local case, the equation satisfied by $H(x,x')$ is

$$[-\Box + m^2 + \overline{A}(x) - \overline{\partial^\mu \gamma_\mu} - \overline{\gamma}^\mu \partial_\mu] = 0. \tag{96}$$

Note that, since $\tilde{B}$ is an odd quantity, it does not appear in the local limit. Since one is interested in the variables $x-x'$ and $\overline{x}$, it is convenient to Taylor expand x around $\overline{x}$. Under the Fourier transform this takes the appearance

$$\overline{A}(x)H(x,x') \to [\overline{A}(\overline{x}) + \frac{i}{2}(\overline{\partial}_\mu \overline{A})\frac{\partial}{\partial k_\mu} + \ldots]H(k,\overline{x}). \tag{97}$$

It is useful to define a new quantity by

$$T_\mu = \frac{\frac{\partial f}{\partial k}}{(1+f)} \tag{98}$$

(the steepness of the spectral envelope for the Wigner function) so that

$$\frac{1}{H}\frac{\partial H}{\partial k_\mu} = T^\mu - v_g^\mu/\omega \tag{99}$$

where $v_g = \frac{\partial \omega}{\partial k}$ is the group velocity of the dispersing wave-packets. In terms of the quantities (94) and (95) the action of the spacetime derivative operator on $H(x,x')$ gives

$$\overset{x}{\partial}_\mu H(x,x') = 2\pi \int \frac{d^{n-1}k}{(2\pi)^{n-1}} e^{ik(x-x')}\frac{(1+f)}{2|\omega|}[ik_\mu + F_\mu - \Omega_\mu] \tag{100}$$

and subsequent derivatives are obtained in a straightforward way.

Substituting $H(x,x')$ into the equation of motion (96) now leads to a second order differential equation for the frequency $\omega(\overline{x})$:

$$\begin{aligned}\omega^2 - 2(ik^\mu + F^\mu + \frac{1}{2}\overline{\gamma}^\mu)\Omega_\mu + \Omega^2 = \mathbf{k}^2 + m^2 + \overline{A}(\overline{x}) + \frac{i}{2}(\overline{\partial}_\mu \overline{A})(T^\mu - v_g^\mu/\omega) \\ - \overline{\partial^\mu \gamma_\mu} - F^2 - 2ik^\mu F_\mu - \partial^\mu F_\mu - ik^\mu \overline{\gamma}_\mu - \gamma^\mu F_\mu\end{aligned} \tag{101}$$

Clearly this equation presents an insurmountable problem for the purposes of analytic calculation, thus an approximation must be made, based on the adiabatic evolution of the average properties of the system. The lowest order case (which will be sufficient to reveal the features of interest in this paper) is when Ω_μ and derivatives of F_μ may be neglected. This corresponds to a near classical transport of particles, with relatively few of the fluctuations added by the quantum nature of the field. With this approximation the dispersion relation may be written:

$$k^2 + m^2 + \overline{A}(\overline{x}) + \frac{i}{2}(\overline{\partial}_\mu \overline{A})(T^\mu - v_g^\mu/\omega) - \overline{\partial^\mu \gamma_\mu} - F^2 - 2ik^\mu F_\mu - i\gamma^\mu k_\mu - \gamma^\mu F_\mu = 0. \tag{102}$$

This can be separated into a more appealing form as the dispersion relation for a damped oscillator array

$$-\omega^2 - i\Gamma\omega + \omega_0^2 = R \tag{103}$$

where one identifies the natural frequency,

$$\omega_0^2 = \mathbf{k}^2 + m^2 - F^2 \tag{104}$$

the decay constant,

$$\Gamma = -\frac{1}{2\omega}(\overline{\partial}_\mu \overline{A})(T^\mu - v_g^\mu/\omega) + \frac{2}{\omega}k^\mu(F_\mu - \frac{1}{2}\gamma_\mu) \tag{105}$$

and force term

$$R = \gamma^\mu F_\mu + \overline{\partial^\mu \gamma_\mu} - \overline{A}(\overline{x}). \tag{106}$$

One notices how the effective mass of the theory is reduced by the gradient of the Wigner function F_μ, indicating that rapid transport could lead to a second order phase transition. This might also lead to anomalous dispersion.

In a true linear oscillator array R, Γ and ω_0 would all be independent of the frequency ω. In equation (105) only the zeroth component of the last term is independent of ω. This indicates that the decay/amplification of certain modes in time is oscillator-like, but that the spatial modes are multiplied by a factor of $\mathbf{k}/\omega$, the inverse phase velocity, which has a critical value when $m/\mathbf{k}$ is a maximum. This signifies the effect which a gap in the frequency spectrum can have in leading to anomalous dispersion in the 'plasma'. At high frequencies $\Gamma \to k^0(F_0 - \frac{1}{2}\gamma_0)$ and the system is oscillator-like. At low frequencies, damping is dominated by the external potential $\overline{A}$ and by transport as one might expect.

B. Translationally-invariant sources

In the translationally invariant case, all variables are a function of $x - x'$. One may therefore fully Fourier transform the sources:

$$\begin{aligned}
\overline{A}(x - x') &= \int \frac{d^n k}{(2\pi)^n} e^{ik_\mu (x-x')^\mu} \overline{A}(k) \\
\overline{\gamma}^\mu(x - x') &= \int \frac{d^n k}{(2\pi)^n} e^{ik_\mu (x-x')^\mu} \overline{\gamma}^\mu(k) \\
\tilde{B}(x - x') &= \int \frac{d^n k}{(2\pi)^n} e^{ik_\mu (x-x')^\mu} i\tilde{B}(k).
\end{aligned} \tag{107}$$

Note that, since $\tilde{B}$ is an antisymmetric function a factor of i is introduced to keep $\tilde{B}(k)$ real. The equation of motion for $H(x, x')$ is now

$$[-\overset{x}{\Box} + m^2]H(x, x') + \int dV_{x''}[\overline{A}(x - x'') - \tilde{B}(x - x') - \overline{\gamma}^\mu(x - x')\overset{x''}{\partial}_\mu - (\overline{\partial^\mu \gamma_\mu})]H(x'', x') = 0. \tag{108}$$

The translational invariance enables the latter spacetime integral to be performed immediately, yielding the dispersion relation

$$k^2 + m^2 + \overline{A}(k) - 2ik^\mu \gamma_\mu - i\tilde{B}(k) = 0. \tag{109}$$

An apparent consequence of the translational invariance is that $F_\mu = 0$ owing to the steady state nature of the system. Comparing the dispersion relation to equation (103), one identifies

$$\begin{aligned}
\Gamma &= \frac{2}{\omega}k^\mu \gamma_\mu + \tilde{B}(k) \\
\omega_0^2 &= \mathbf{k}^2 + m^2 \\
R &= -\overline{A}(k)
\end{aligned} \tag{110}$$

Although the translationally invariant theory describes only steady state disequilibria, it is nevertheless seen that the field oscillations are concentrated around the usual mass-shell ω_0^2 with an amplitude driven by the external force

$$\frac{\overline{A}(k)}{[(\omega^2-\omega_0^2)^2+(\Gamma\omega)^2]^{\frac{1}{2}}} \tag{111}$$

and a quality factor $Q=\omega_0/\Gamma$. Such a steady-state description would be appropriate for an 'infinite laser' i.e. a device which is not affected by any finite size considerations.

C. Inhomogeneous sources

The main case of interest is when the sources and Green functions have a residual dependence on the average position and time. This includes the local limit as a special case:

$$\begin{aligned}\overline{A}(x,x') &= \alpha(x-x')\beta(x+x')\\ \alpha &\to \delta(x-x')\\ \beta &\to \overline{A}(x)=\overline{A}(x').\end{aligned} \tag{112}$$

As usual, one is looking for the eigenspectrum of the quadratic operator acting on $H(x,x')$. The equation satisfied by $H(x,x')$ is now:

$$[-\overset{x}{\Box}+m^2]H(x,x')+\int dV_{x''}\left[\overline{A}(x,x'')-\tilde{B}(x,x'')-\overline{\gamma}^{\mu}(x,x'')\overset{x''}{\partial}_{\mu}-\overline{\partial^{\mu}\gamma_{\mu}}\right]H(x'',x')=0 \tag{113}$$

In the inhomogeneous case there is no dispersion relation consisting of continuous frequencies in general so the dispersion relation will only exist for a discrete set. It is convenient to divide the discussion into two parts: the determination of the dispersion relation and the nature of the restricted set of values which satisfy the dispersion relation.

The problem to be addressed is contained in following form in momentum space:

$$(-\Box+m^2)H(x,x')+\int dV_{x''}\frac{d^nk}{(2\pi)^n}\frac{d^np}{(2\pi)^n}e^{ik(x-x'')+ip(x''-x')}S(p,x''+x')H(k,x+x')=\lambda H(x,x'). \tag{114}$$

The integral over x'' is no longer a known quantity in general, but it is possible to extract an overall Fourier transform by shifting the momentum $p\to p+k$ and defining the average variable of interest $\overline{x}=\frac{1}{2}(x+x')$:

$$\begin{aligned}&(k^2+ik^{\mu}\overline{\partial}_{\mu}-\frac{1}{4}\overline{\Box}+m^2)H(k,\overline{x})+\\ &\quad+\int dV_{x''}\frac{d^np}{(2\pi)^n}S(k,x+x'')H(k+p,x''+x')=\lambda H(k,\overline{x}).\end{aligned} \tag{115}$$

In order to find eigenvalues, it is necessary to extract the factor of $H(k,\overline{x})$ from this expression. This is not possible for arbitrary values of k. It is possible, however, if the momenta are restricted to a denumerable set expressed by the property

$$H(k+p,x''+x')=H(k,x''+x') \tag{116}$$

which implies that $H(k,\overline{x})$ is a periodic function of the momenta. The absence of eigenvalues or the failure of this property leads to the consideration of an infinite iterative mapping of states, which—in the absence of a stable limit—suggests chaotic excitations of the field. This can also be argued geometrically (see the final section). Given this mitigating condition, one has

$$\int dV_{x''}\frac{d^np}{(2\pi)^n}S(k,x+x'')H(k+p,x''+x')=S(k,\overline{x})H(k,\overline{x}). \tag{117}$$

The dispersion relation is now obtained in a straightforward fashion, adopting the same adiabatic approximation as before, and is given by the implicit relation

$$k^2 + m^2 + \overline{A}(k,\overline{x}) + \frac{i}{2}(\overline{\partial}_\mu \overline{A})(T^\mu - v_g^\mu/\omega) - i\tilde{B}(k)$$
$$- \overline{\partial^\mu \gamma_\mu}(k,\overline{x}) - (F-N)^2 - 2ik^\mu(F-N)_\mu - 2i\overline{\gamma}^\mu k_\mu - \overline{\gamma}^\mu(F-N)_\mu = 0. \tag{118}$$

where it is noted that $\{k\}$ is now discontinuous. Note that the antisymmetry of $\tilde{B}(k)$ makes it independent of $\overline{x}$. Comparing to the oscillator equation, one has

$$\begin{aligned}
\Gamma &= -\frac{1}{2\omega}(\overline{\partial}_\mu \overline{A})(T^\mu - v_g^\mu/\omega) + \frac{2}{\omega}k^\mu(F_\mu - N_\mu + \overline{\gamma}_\mu) + \tilde{B}(k) \\
\omega_0^2 &= \mathbf{k}^2 + m^2 - (F-N)^2 \\
R &= \overline{\gamma}^\mu F_\mu + \overline{\partial^\mu \gamma_\mu}(k,\overline{x}) - \overline{A}(k,\overline{x})
\end{aligned} \tag{119}$$

where N_μ will be defined presently.

We now turn to the consequences of the condition in equation (116). There are various precedents for such a relation: one is Green functions defined on a torus (finite temperature, Matsubara formalism, electron band structure); another is the case of Landau levels on a torus. The periodicity is clearly the important factor here. In most of these cases the periodicity is one in real space and the result is a discrete spectrum of eigenvalues. Here the periodicity lies in the momentum itself. In fact the two notions are closely related and a periodic system in real space has Green functions which are periodic in momentum space owing to an infinite summation over discrete frequencies (which is therefore invariant under shifts by a whole number of periods). The relation (116) must be satisfied for all legal values of the momentum, thus the implication is that the system is degenerate—i.e. there exist bands of energy which leave the Green function invariant under certain shifts. These need not all refer to the same band. It is therefore possible to write

$$H(k) = H\left(\sum_l \frac{2\pi l^\mu k_\mu}{P_\mu}\right) \tag{120}$$

where P_μ is the *momentum* periodicity length (which has dimensions of inverse space-length). This finite length must diverge to infinity when the inhomogeneities vanish. There is only one natural momentum/length scale which has these properties, namely

$$L_\mu = P_\mu^{-1} = \overline{\partial}_\mu H(k,\overline{x}). \tag{121}$$

In deriving (117) we have used the fact that

$$\int \frac{d^n k}{(2\pi)^n} \exp(ikx) = \delta(x) \tag{122}$$

Since k is now restricted to a discrete set, the correctness of this relation could now be an issue. It can easily be verified using the formulae

$$\begin{aligned}
\sum_{k=1}^{n} \sin(kx) &= \sin\frac{(n+1)}{2}\sin\frac{(nx)}{2}\operatorname{cosec}\frac{x}{2} \\
\sum_{k=0}^{n} \cos(kx) &= \sin\frac{(n+1)}{2}\cos\frac{(nx)}{2}\operatorname{cosec}\frac{x}{2}
\end{aligned} \tag{123}$$

that an extra finite *imaginary* contribution can arise from the discrete nature of the spectrum, which vanishes in the continuous limit. It will be assumed that such a contribution can be absorbed by a redefinition of the sources.

Although one is looking at periodic functions, the solution for $H(x,x')$ need not be sinusoidal. In the case of Landau levels on the torus [17,18] periodicity is only achieved at the expense of a flux-quantization condition which, again, involves a degeneracy of solutions. There is, in fact, an analogy to this situation here. The extraordinary properties of Landau levels on a torus can be attributed to the non-translational invariance of the electro-magnetic vector potential. The similarity here is the non-translational invariance of the many-particle state as expressed by the dependence on $\overline{x}$. This point will be discussed at greater length in the final section, to avoid its meaning being lost in the present analysis.

The extra terms containing N_μ can now be explained. They arise from the $\overline{x}$ dependence of the momentum space measure:

$$\int \frac{d^n k}{(2\pi)^n} \to \prod_{\mu} \left(\frac{1}{L_\mu} \sum_{l_\mu} \right) \tag{124}$$

giving a contribution

$$N_\mu = \frac{\partial_\mu (L_0 \dots L_{n-1})}{(L_0 \dots L_{n-1})} \tag{125}$$

which compounds the non-linearity. The above restrictions have no special consequences for the Feynman propagator, since the nature of the momentum is not used to obtain it. This is gratifying since the Feynman propagator must always be the literal inverse of the quadratic part of the + time-ordered action. Only the nature of the singularity is altered in accord with the modified dispersion:

$$G_F(k) = \frac{1}{-k_0^2 + \omega^2 - i\epsilon} + 2\pi i f(k, \overline{x}) \delta(-k_0^2 + \omega^2) \theta(k_0). \tag{126}$$

The appearance of a natural length scale, connected to the inhomogeneities of a non-equilibrium system, is an important feature for two reasons. Firstly, the spontaneous generation of a length scale implies the possibility of domain formation, or a cellular localization in the field. Secondly, the dependence of the Green functions on themselves implies that the stable solutions of the system can be regarded as fixed points of an iterative map. Such maps have been studied in connection with classical chaotic systems [19].

In the present case, the function $H(k, \overline{x})$ depends not merely on itself but on its derivative. For exponential-like solutions one could expect that this would amount to the same thing, up to a constant multiplier. The situation would then be something akin to $H = H(\lambda H)$, for some constant λ. This bears a noteworthy similarity to Feigenbaum's functional equation which can be written

$$g(x) = \alpha g(g(x/\alpha)) \tag{127}$$

subject to a boundary value, or rewriting:

$$g(g(\lambda x)) = \lambda g(x). \tag{128}$$

This equation has an analytic solution as a power series

$$g(z) = 1 + c_1 z^2 + c_2 z^4 + \dots \tag{129}$$

where a limiting value is approached through a geometric progression with Feigenbaum ration $\mathcal{F} = 4.66$ and universal scaling factor $\alpha = -2.5$. Solutions to this equation which fall outside the fixed point behaviour can be expected to lead to chaotic behaviour. This strongly suggests that the non-equilibrium Green functions must exhibit universal behaviour or chaos in their approach to stable behaviour. In other words, the approach to equilibrium need not be of the simple damped or over-damped form of a linear oscillator array—it could easily entail a chaotic attractor.

V. ENTROPY, TEMPERATURE AND THE KMS CONDITION

For systems close enough to a thermal state, it is possible to define an approximate temperature and entropy. The entropy of the system may be defined in various ways, often based only on combinatorial considerations of the micro-canonical picture. Here it is convenient to define an 'oscillator effective entropy' which is easily related to quantities which arise in the analysis. Suppose the Wigner function is given by the approximate equilibrium form

$$f(k_0, \overline{x}) = \left(\exp(\beta(\overline{x}) \omega(\overline{x})) - 1 \right)^{-1} \tag{130}$$

then one has

$$\overline{F}_\mu = -f^2 \left[\left(\frac{\overline{\partial}_\mu \beta}{\beta} + \overline{\Omega}_\mu \right) \beta \omega \right] \tag{131}$$

and, classically, the statistical entropy S is

$$S = k \left(\ln Z + \beta \langle \omega \rangle \right). \tag{132}$$

For a harmonic oscillator, one has (see for example ref. [20])

$$\ln Z = -\mathrm{Tr}(1 - e^{-\beta\omega}) - \frac{1}{2}\beta\langle\omega\rangle, \tag{133}$$

thus the oscillator entropy may be defined as

$$S = \frac{1}{2}\beta\langle\omega\rangle + \mathrm{Tr}\ln(1+f). \tag{134}$$

This motivates the definition of a simple measure of entropy for the oscillator array, given by

$$S_E(\overline{x}) = \int \frac{d^n k}{(2\pi)^n}\theta(k_0)\ln(1+f)\delta(-k_0^2+\omega^2). \tag{135}$$

The rate of change of this entropy is then

$$\begin{aligned}\overline{\partial}_\mu S_E &= \overline{\partial}_\mu \int \frac{d^{n-1}k}{(2\pi)^{n-1}}\frac{\ln(1+f)}{2|\omega|}\Bigg|_{k>0} \\ &= \int \frac{d^{n-1}k}{(2\pi)^{n-1}}\frac{(1+f)}{2|\omega|}[\overline{F}_\mu - \overline{\Omega}_\mu]\end{aligned} \tag{136}$$

This quantity can be compared to (100). It shows that the entropy gradient can be thought of as a 'connection' for the field modes. The generation of entropy is therefore fundamentally connected with the flow of particle occupation numbers and the 'downgrading' of the frequency spectrum—i.e. the rate at which energy becomes unavailable to do work.

As mentioned earlier, the effect of a non-trivial density matrix, either at the initial time or later, is reflected in the off-diagonal sources and Green functions. If one imagines that the sources $A_{\pm\pm}$ arise from a coupling to another oscillator system [10] or that they represent the self-interaction of the field to order ϕ^4, then $A_{\pm\pm}$ is essentially the Green function for the field concerned and one would therefore expect the KMS condition to hold for the sources at equilibrium—now in the form

$$\theta(|\omega|)A_{+-}(\omega) = e^{\beta\omega}\theta(-|\omega|)A_{-+}(\omega). \tag{137}$$

This condition does not hold in general, but for an isoentropic process, in terms of the defined quantities at $\gamma^\mu = 0$, one therefore has

$$\tilde{B}(\omega) = -e^{\beta\omega}\tilde{B}(\omega) \tag{138}$$

It is verified that

$$\frac{\theta(\omega)A_{+-}(\omega)}{\theta(-\omega)A_{-+}(\omega)} = e^{\beta|\omega|} \tag{139}$$

giving $A_{+-} = \sinh(\frac{1}{2}\beta|\omega|)a(\omega)$ for some $a(\omega)$ or

$$B(\omega) = \frac{1-e^{-\beta|\omega|}}{1+e^{\beta|\omega|}} = \tanh(\frac{1}{2}\beta|\omega|) \tag{140}$$

which agrees with Schwinger's result [10]. Note that the initial state $f(\overline{x}_i)$ and its subsequent development enter only as boundary conditions to the Green functions and the Wigner function. The changing form of $f(\overline{x})$ is determined solely by the sources $A_{\pm\pm}$. Thus, if the sources do not evolve, neither does $f(\overline{x})$ and neither does the implicit density matrix. In the perturbation around free field theory [1], $f(\overline{x})$ always represents the state of the system at the initial time.

In the approach to equilibrium one normally expects that dependence on the average coordinate $\overline{x}$ to disappear. This is an expression of what is often called 'loss of memory' of the initial state, since $\overline{x}$ is measured relative to the initial time. An equilibrium state (thermal or otherwise) is, by its nature, either static or periodic, thus the resulting Wigner function $f(k_0, \overline{x})$ must either be independent of $\overline{x}$ or a periodic function of this variable. One of the advantages of the present formulation is that one sees how the sources are responsible for this loss of memory. Since the sources drive the system, $f(k_0, \overline{x})$ can never become $\overline{x}$ independent as long as the sources are $\overline{x}$ dependent. Thus equilibrium

will only be secured by accounting for the back-reaction of the sources to the behaviour of the field. Explicit equations of motion for the sources have not been considered here.

An example of a periodic 'equilibrium' is the case of Rabi oscillations in the laser (see ref. [21] for a review), in which the source and the field enter into a pendulum-like flip-flop behaviour. An example of this will be given in the final section.

The decay of field modes is exponential, per mode and is mediated by the source $\gamma^{\mu}(x,x')$ and the gradient of the potential $A(x,x')$. This does not preclude other behaviour for the Wigner function. For example, in the simplest case close to equilibrium in which the system is quasi-static and $\overline{A}=\tilde{B}=0$, with almost no external force (see equation (119)), one has $\gamma_{\mu}\sim F_{\mu}$ and thus $\partial^{\mu}F_{\mu}+F^2\sim 0$ giving $F_{\mu}\sim x_{\mu}^{-1}$— a 'long tail' power law decay which parallels the decay of harmonic waves in curved spacetime [22].

VI. CALCULATION OF EXPECTATION VALUES

The closed time path formalism codifies the causal relationship between source and response, for the computation of expectation values in a general mixed state. Since it is redundant except as a calculational aid, it's introduction should be justified by an example. The causality of the method is not affected by the introduction of the sources $A_{\pm\pm}$, but the dissipative dynamics are. Normally a fundamental Gaussian theory can never show dissipation, but in the present situation one has sources which can drive the field modes and redistribute energy.

There are two cases of interest. In a self-interacting theory one might identify $A_{\pm\pm}$ with the correlation function for the field itself $\lambda G_{\pm\pm}$, giving rise to dispersion relation of the approximate form

$$k^2+m^2+\lambda \mathrm{Tr}(k^2+m^2)^{-1}=0. \tag{141}$$

This is like the variational method used in ref. [1]. Lawrie [3] takes the view that the sources can effect a renormalization of a self-interacting theory by choosing them in such a way as to 'minimize' the effect of higher order perturbative contributions. In either case, the effective 'resummation' induced by the sources makes it possible to see damping of field modes at the one-loop (Gaussian) level.

Consider the response of the field to the source $J(x)$, in the presence of $A_{\pm\pm}$. One is interested in the causal expectation value of the field at time $\overline{t}$, given the state of the system at the initial time. The time dependence, in the present formalism is now contained entirely within the sources—or equivalently the dispersion relation. That the CTP generator leads to a causal result is easily verified by realizing that the expectation value of the field is always coupled to the sources by the retarded n point functions. For an arbitrary action $S[\phi]$,

$$\begin{aligned}
\langle\phi(x)\rangle &= -i\frac{\delta}{\delta J_+(x)}\bigg|_{+=-}\langle 0|0\rangle_{\pm}\\
&= \frac{1}{2}\int dV_{x'}[2G_{++}(x,x')+G_{+-}(x,x')+G_{-+}(x,x')]J(x')+\ldots\\
&= \frac{1}{2}\int dV_{x'}[2G_{++}(x,x')+G^{(+)}-G^{(-)}]J+\ldots\\
&= \int dV_{x'}[G_{++}(x,x')-G^{(-)}]J(x')+\ldots\\
&= \int dV_{x'}G_{ret}(x,x')J(x')+\ldots
\end{aligned} \tag{142}$$

thus the expectation value depends only on retarded times. Furthermore, the result is real (being a probability) since the retarded Green function is explicitly the real part of the Wightman functions, restricted to retarded times by a step function:

$$G_{ret}(x,x')=-\theta(t-t')[G^{(+)}(x,x')+G^{(+)*}(x,x')]. \tag{143}$$

Making use of the integral representation (58), one has

$$G_{ret}(x,x')=-i\int\frac{d\omega}{(2\pi)}\frac{d^nk}{(2\pi)^n}\frac{\exp(-i\omega(t-t')+i\mathbf{k}(\mathbf{x}-\mathbf{x}'))}{(\omega+i\epsilon)}\left(G^{(+)}(k)+G^{(-)}(k)\right). \tag{144}$$

Relabelling and inserting the momentum-space forms for the Wightman functions from (72), one has

$$G_{ret}(k) = -\int d\omega \frac{1}{k_0 - \omega + i\epsilon}\left(\frac{1}{\omega_+}\delta(\omega - \omega_+) - \frac{1}{\omega_-}\delta(\omega + \omega_-)\right) \tag{145}$$

where ω_+ and $-\omega_-$ are the positive and negative frequency solutions to the appropriate dispersion relation. These are complex numbers in general, owing to the non-vanishing imaginary part labelled as Γ. Now, since unitarity demands that $G^{(+)}(x,x')$ be the complex conjugate of $G^{(-)}(x,x')$, it is clear that

$$\omega_+^* = \omega_-. \tag{146}$$

It is assumed here that the dispersion relation has two complex roots. The quantity appearing in the delta function in equation (92) is then $-k_0^2 + \omega_+\omega_-$ which may also be written $-k_0^2 + \omega^*\omega$. To avoid confusion with previous notation for the absolute value, the complex modulus will not be denoted $|\omega|$. This indicates that, in spite of the complex momenta in the dispersion relation, whose role it is to capture dissipation and transport/kinetic effects, the 'mass shell' constraint is real. The simplest expression for the retarded Green function is therefore

$$G_{ret}(k) = -\left(\frac{1}{2\omega_+(k_0 - \omega_+ + i\epsilon)} - \frac{1}{2\omega_-(k_0 - \omega_- + i\epsilon)}\right). \tag{147}$$

This expression is not manifestly real, since it is a momentum space result. However, if one defines $2i\tilde{\omega} = \omega_- - \omega_+$ and $2\overline{\omega} = \omega_+ + \omega_-$, where $\tilde{\omega}$ and $\overline{\omega}$ are real, then it is possible to write

$$G_{ret}(k) = \frac{1}{\omega^*\omega}\left(\frac{(i\tilde{\omega}k_0 - \omega^*\omega + 2\overline{\omega}^2)(-k_0^2 + \omega^*\omega - 4ik_0\tilde{\omega})}{(-k_0^2 + \omega^*\omega)^2 + 16k_0^2\tilde{\omega}^2}\right). \tag{148}$$

This may be compared to equation (111) and reduces to

$$\frac{1}{-k_0^2 + \omega^2} \tag{149}$$

when $\omega^* = \omega$ and $\epsilon \to 0$. Since the imaginary part of (148) is odd with respect to the momentum variable k, the Fourier transform back to configuration space is real, as expected. The desired expectation value is therefore manifestly real and causal, and the time dependence since the initial time is contained entirely in the $\overline{x}$ dependence of the frequency ω.

VII. REFORMULATION

In the preceding sections, it has been shown how dissipation and amplification of spectral modes can be incorporated into the dispersion of a quadratic theory, for suitably adiabatic processes. It is now practical interest to show that the same results can be presented in another significant form by introducing a 'covariant derivative' D_μ which commutes with the average development of the field state. This description parallels the structure of a gauge theory (in momentum space) with an imaginary charge. Alternatively one may speak of a generalized chemical potential for the 'gauge' field.

Consider the derivative

$$D_\mu = \partial_\mu - a_\mu \tag{150}$$

and its square

$$D^2 = \Box - \partial^\mu a_\mu - 2a^\mu\partial_\mu + a^\mu a_\mu. \tag{151}$$

Without any approximation, it is straightforward to show that, in the general inhomogeneous case,

$$\begin{aligned}(-\Box + m^2)H(x,x') = 2\ \pi \int \frac{d^{n-1}k}{(2\pi)^{n-1}}\frac{(1+f)}{2|\omega|}e^{ik(x-x')} \\ \{-(ik_\mu + F_\mu - \Omega_\mu - N_\mu)^2 - \partial^\mu(ik_\mu + F_\mu - \Omega_\mu - N_\mu)\}.\end{aligned} \tag{152}$$

It is then natural to make the identification

$$\begin{aligned}a_\mu &= F_\mu - \Omega_\mu - N_\mu + \overline{\gamma}_\mu \\ &= \partial_\mu S_E(k) - N_\mu + \overline{\gamma}_\mu\end{aligned} \tag{153}$$

where the meaning of this notation is such that the expression only defined when all objects are under the momentum integration—this is to be understood in all future expressions. The field a_μ is clearly related to the rate of increase of the entropy S_E, the damping factor γ^μ and the rarefaction of the localized cells N_μ. One now has:

$$\begin{aligned}(-D^2+m^2)H(x,x') &= 2\pi\int\frac{d^{n-1}k}{(2\pi)^{n-1}}\frac{(1+f)}{2|\omega|}\{-(ik_\mu-\overline{\gamma}_\mu)^2)-\partial^\mu(ik_\mu-\overline{\gamma}_\mu)\}\\ &= 2\pi\int\frac{d^{n-1}k}{(2\pi)^{n-1}}\frac{(1+f)}{2|\omega|}\{k^2+2ik^\mu\overline{\gamma}_\mu-\overline{\gamma}^2-i(\partial^\mu k_\mu)+(\partial^\mu\overline{\gamma}_\mu)\}.\end{aligned} \tag{154}$$

Adding the appropriate source combinations for the inhomogeneous case one has, without approximation, the differential equation satisfied by $H(x,x')$:

$$\left[-D^2+m^2+\overline{\gamma}^2(k,\overline{x})+\overline{A}(k,\overline{x})-\tilde{B}(k)+\frac{i}{2}(\overline{\partial}_\mu\overline{A})(T^\mu-v_g^\mu/\omega)\right]_k H(x,x')=0 \tag{155}$$

where the appearence of the subscript k to the bracket serves to remind that the equation only exists under the momentum integral. The local limit is simply

$$[-D^2+m^2-\overline{\gamma}^2(x)+\overline{A}(x)]_k H(x,x')=0. \tag{156}$$

The 'gauge' field a_μ couples via an imaginary unit-charge plays the role of a generalized chemical potential on the manifold of positive energy solutions for the real scalar field (the chemical potential has no meaning for the full field, since particle numbers are not conserved). Suppose now that one defines the analogue of the field strength tensor

$$f_{\mu\nu}=\partial_\mu a_\nu-\partial_\nu a_\mu. \tag{157}$$

In many cases one will have $f_{\mu\nu}=0$, thus one can 'gauge transform' the field, which maps

$$\begin{aligned}\phi(k)\rightarrow\phi(k)e^{\int a_\mu dx^\mu} &= \phi(k)e^{-\int(F_\mu-\Omega_\mu-N_\mu-\overline{\gamma}_\mu)dx^\mu}\\ &= \phi(k)e^{-S_E-\int(N_\mu+\overline{\gamma}_\mu)dx^\mu}.\end{aligned} \tag{158}$$

This shows the explicit decay (amplification) of the k-th field mode. The latter relation shows that this process involves an increase in the effective oscillator entropy of the system.

In terms of the above formulation, the spectral content of the bosonic theory reduces to the problem of finding the eigenvalues of the operator D^2. In particular one can use the body of experience gained in the study of gauge theories to attack the problem. With an adiabatic approximation for $f(\overline{x},k)$, a_μ has a series expansion in powers of $\overline{x}$. Thus for quasi static systems

$$a_\mu=(c_0+c_1\overline{x}+\ldots)_\mu. \tag{159}$$

The effective field strength $f_{\mu\nu}$ need not always be zero. Two situations might arise: (i) the Wigner function might contain a logarithmic singularity, as in the case where vortices are present, and (ii) the source $\overline{\gamma}_\mu$ could contain components which specifically drive the macroscopic field in a given way. A simple example of the latter is the analogue of Rabi oscillations in the laser, in which the field oscillates between two states in a regular way. Here, this oscillation is driven by the source $\overline{\gamma}_\mu$ or perhaps by a pulsation of the inhomogeneity scale, and occurs from the linear terms in equation (159). The current $J=\phi_2\overset{\leftrightarrow}{D}^{\gamma}\phi_1$ behaves like a magnetic influence on the system (doing no net work). Simplifying to the case of a $(1+1)$ dimensional system, one may write

$$a_\mu\sim\overline{\gamma}_\mu=|\gamma|\epsilon_{\mu\nu}\overline{x}^\nu \tag{160}$$

for constant $|\gamma|$ and $\mu,\nu=0,1$. This corresponds to a harmonic 'flip-flop' motion between field and source. It is also directly analogous to the well known problem of Landau levels in an effective magnetic field $|\gamma|$.

The localization in spacetime resulting from the inhomogeneity scale suggests that such oscillations may take place locally in cellular regions. A simplified model for this is to impose periodic boundary conditions on the cells, generating a kind of global field coherence (this is admittedly motivated by technical simplicity rather than physical reasoning). One is therefore led to the study of Landau levels on the torus—a system which has been studied at some length [18,17], and will not be re-analyzed here.

A significant feature of the Landau problem on the torus is that the periodicity enforces a flux quantization condition on the field. Here this translates into the following relation:

$$\overline{\partial}_0 H(k,\overline{x})\overline{\partial}_1 H(k,\overline{x})|\gamma| = 2\pi n \tag{161}$$

for integer n. This relation indicates that nearest neighbour cells might engage in cooperative oscillations (i.e. the size of cells is quantized in units of the local inhomogeneity scale). This is clearly a far less stringent condition here than in the case of a true periodic torus, since the inhomogeneity scale varies in space and time thus the meaning of strict quantization is lost. However, it indicates that one can expect a tiling of spacetime by oscillation cells. Since the size of the cells might be highly irregular, the tiling behaviour is most likely to be chaotic unless special geometrical boundary conditions can enforce a regularity on the field. This is an alternative expression of the behaviour deduced from the Green function in equation (116).

VIII. CONCLUSION

Schwinger's closed time path action principle has been applied to the neutral scalar meson, off-equilibrium, in the presence of long-range, inhomogeneous sources. The method of dispersion relations is used to find formal expressions for Green functions which reflect the absorbative and amplifying processes in the normal modes of the field. In the case of self-interacting theories, the sources can be thought of as representing ϕ^{2n} interactions to one-loop order, effecting a resummation of the theory. The effect of rapid transport (large F_μ) is to induce a change in the sign of the mass squared, indicating a second order phase transition and anomalous dispersion.

If significant inhomogeneities or long range interactions exist, the field naturally forms localized cells with (to lowest order) a periodic relationship to the natural inhomogeneity length/time scale. This is shown from the viewpoint of the Green functions and by recourse to an analogy with Landau levels on the torus. Since the length scale is determined by non-linear considerations one can expect chaotic behaviour with islands of order (stable fixed points) along the approach to equilibrium. A simple analogue of Rabi oscillations in the laser is shown to arise as a leading order behaviour in $\overline{x}$.

The method used in the this work has the advantage of combining the fundamental aspects of an operator field theory with the usefulness of the action principle. The use of generating functional ultimately leads to functional integral forms, as used almost exclusively in the literature. However, the introduction of the functional integral is scarcely necessary using the present method and often has the undesirable effect of turning the discussion of causality into one of complicated paths of integration in the complex plane.

Comparing to other works reveals both differences and similarities. Lawrie [3], for example, treats the quantity γ^μ as an explicitly written imaginary part of the spectrum of excitations. He ignores F_μ, but does not ignore Ω_μ. This is a somewhat different approximation which has a more distant relationship to classical transport theory. In fact, since the appearance of F_μ and Ω_μ in a_μ is identical, up to a sign, the form of dynamics might well be independent of the approximation used in this work—understandable as a reparameterization of an equivalent problem. Lawrie further considers ϕ^4 theory and uses a renormalization-like philosophy to determine the sources self-consistently thereby effecting a resummation as noted in equation (141). Calzetta and Hu [1] use a variational principle to determine the effective action for a self-interacting boson theory. This is the same idea as in ref. [3], expressed in extremely aesthetic formalism and containing important insights into the subject; the solution to their method is, in practice, more difficult to attain however and thus results are mainly formal. Neither of these works consider the implications of non-local effects. Another interesting approach is the Schrödinger quantization approach in ref. [11]. This makes a contact with the Schwinger action principle at a more subtle level and, focusing on somewhat different issues, uncovers features absent in other formulations of non-equilibrium physics.

It is important to extend the present analysis to include both fermions and spin-1 bosons (true gauge fields). The latter is probably a difficult task in view of the problems which can arise in gauge fixing. Again, the action principle approach, starting from the operator field theory is likely to be the most informative approach. The appearance of discrete spectra and magnetic like effects makes the present work very interesting to the study of the fractional quantum Hall effect. In particular, the pseudo-gauge field formulation might have interesting connections with the statistical gauge field employed in the Chern-Simons gauge theory picture. These and other outstanding issues will be discussed in future work.

I am grateful to I.D. Lawrie for helpful discussions.

APPENDIX A: RETARDED N-POINT FUNCTIONS

The retarded n-point functions are defined by

$$(n=0) \quad R(x) = = \phi(x)$$
$$(n=1) \quad R(x,x_1) = = -i\theta(x-x_1)[\phi(x),\phi(x_1)]$$
$$(n>1) \quad R(x,x_1\ldots x_n) = (-i)^n \sum_{P_i} \theta(x-x_1)\theta(x_1-x_2)\ldots\theta(x_{n-1}-x_n) \times$$
$$[[[\phi(x),\phi(x_1)],\phi(x_2)],\ldots\phi(x_n)] \tag{A1}$$

where P_i signifies all the permutations of the indices on x_i. This is strictly only defined when all the x_i are different. The coincidence limit is often defined by recourse to the momentum representation.

The retarded functions have the following properties: (i) $R(x,x_1\ldots x_n)$ vanishes if any $x_i > x$ (with respect to the time), (ii) $R(x,x_1\ldots x_n)$ is a symmetric function of $x_1\ldots x_n$ and (iii) the retarded Green functions are always defined with respect to a special point x which is later than all other points.

It may be verified explicitly that the Schwinger-Symanzik generating functional

$$Z_{ret}(x) = \left(T^\dagger e^{-iJ\phi}\right) \frac{\delta}{\delta J(x)} \left(T e^{iJ\phi}\right) \tag{A2}$$

generates the n-point functions according to the rule

$$R(x,x_1\ldots x_n) = (-i)^n \frac{\delta^n}{\delta J^n} Z_{ret}(x)\Big|_{J=0}. \tag{A3}$$

The step functions are enforced by explicit cancellation of field operators for times outside the bounds of the constraints. The above generating functional is clearly related to the closed time path generator, and it is easy to see that one may also write

$$R(x,x_1\ldots x_n)\left(\frac{\delta}{\delta J_-} + \frac{\delta}{\delta J_+}\right)^n (-i)^n \frac{\delta}{\delta J_+}\langle 0|0\rangle_\pm\Big|_{J_\pm=0}. \tag{A4}$$

Finally, it can be observed that that the Hermite polynomials are generated by the generating functional

$$H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2} \tag{A5}$$

and can therefore be expected to play an important role in the computation of the transformation function for a quadratic theory.

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