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Consistent string backgrounds and completely integrable 2D field theories

ADEL BILAL

CNRS - Laboratoire de Physique Théorique de l'Ecole Normale Supérieure★
24 rue Lhomond, 75231 Paris Cedex 05, France
bilal@physique.ens.fr

ABSTRACT

After reviewing the β-function equations for consistent string backgrounds in the σ-model approach, including metric and antisymmetric tensor, dilaton and tachyon potential, we apply this formalism to WZW models. We particularly emphasize the case where the WZW model is perturbed by an integrable marginal tachyon potential operator leading to the non-abelian Toda theories. Already in the simplest such theory, there is a large non-linear and non-local chiral algebra that extends the Virasoro algebra. This theory is shown to have two formulations, one being a classical reduction of the other. Only the non-reduced theory is shown to satisfy the β-function equations.

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★ unité propre du CNRS, associé à l'Ecole Normale Supérieure et l'Université Paris-Sud

1. Introduction

Consistent string backgrounds are given by exact conformal field theories. If the string propagating on a curved background is described by the corresponding two-dimensional (non-linear) σ-model action, conformal invariance is equivalent to the vanishing of the associated β-functionals [1]. Part of the string background ("internal dimensions") can of course be replaced by a more general exact conformal field theory. Particular examples are actions that involve exactly marginal operators added to a free bosonic action, as is the case for the Liouville [2] and Toda [3] actions. The latter theories are highly non-trivial, but nevertheless are known to be completely integrable since long [4,5]. They are intimately related to the Gelfand-Dikii hierarchies of integrable partial differential equations generalizing the KdV equation [6].

In this note, I will discuss certain integrable theories that again lead to consistent string backgrounds, i.e. are exactly conformal. From the algebraic point of view, they are associated with non-canonical gradations of simple Lie algebras g where the gradation-zero part g_0 is a *non-abelian* subalgebra. The prototype is the non-abelian Toda theory studied in [7,8]. An important point to be made is that in the formulation of [7,8] the non-abelian Toda theory does not satisfy the β-function equations of [1]. However, as will be shown below, there exists a classically equivalent, and more natural formulation that does.

This note is organized as follows: In section 2, after briefly reviewing the setting of [1] for the β-function equations in the σ-model, we discuss the role of the tachyon potential. In section 3, we show how the WZW-models naturally fit into this setting. In sect. 4, we introduce the non-abelian Toda theory in both classically equivalent formulations and show how the β-function equations are satisfied in one of these formulations. We point out that this should actually be no surprise, at least as far as the equations for the metric, antisymmetric tensor and dilation are concerned, since they are given by those of a WZW-model for g_0.

2. Consistent string backgrounds in the σ-model

Consider string propagation on an arbitrary background described by the (non-linear) σ-model

$$
\begin{aligned}
S_\sigma = &-\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}\left[G_{\mu\nu}(X)g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu + B_{\mu\nu}(X)\epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\right] \\
&+\frac{1}{4\pi}\int d^2\sigma\sqrt{g}R^{(2)}\Phi(X)
\end{aligned}
\tag{2.1}
$$

where $G_{\mu\nu}, B_{\mu\nu}$ and Φ are the background metric, antisymmetric tensor and dilaton field. $\epsilon_{\alpha\beta}$ and $g_{\alpha\beta}$ are the world-sheet antisymmetric ϵ-tensor and metric (with signature $(-,+)$), and $R^{(2)}$ is the corresponding world-sheet curvature scalar. The string tension α' plays the role of a loop-counting parameter ($\sim \hbar$) for the first part of the action. The dilaton-term, being a sort of anomaly cancellation term, is $\mathcal{O}((\alpha')^0)$.

Callan et al have shown [1] that, to lowest non-trivial order in α', the so-defined string theory is conformally invariant, i.e $G_{\mu\nu}, B_{\mu\nu}$ and Φ are consistent string backgrounds, if the following β-function equations are satisfied:

$$
\begin{aligned}
\beta^{G_{\mu\nu}} &\sim \mathcal{R}_{\mu\nu} - \frac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} + 2D_\mu D_\nu\Phi = 0\ , \\
\beta^{B_{\mu\nu}} &\sim D_\rho H^\rho{}_{\mu\nu} - 2(D_\rho\Phi)H^\rho{}_{\mu\nu} = 0\ , \\
\beta^{\Phi} &\sim \frac{N-26}{3\alpha'} + 4D_\mu\Phi D^\mu\Phi - 4D_\mu D^\mu\Phi - \mathcal{R} + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} = 0\ .
\end{aligned}
\tag{2.2}
$$

Here $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$, and $\mathcal{R}_{\mu\nu}$ and $\mathcal{R}$ are the Ricci curvature tensor and scalar computed* from the space-time metric $G_{\mu\nu}$, while D_μ is the corresponding covariant derivative. N is the number of fields X^μ.

In many cases one wishes to study the effect of adding a perturbation to the conformal theory. In particular one can ask when this perturbation is exactly marginal so that the resulting theory remains an exact conformal field theory. If the perturbation simply modifies $G_{\mu\nu}, B_{\mu\nu}$ and Φ one just needs to check whether the new background fields still satisfy eqs. (2.2). But there are many other operators one can add, the simplest of which is the so-called

$\star$ Our sign convention is $\mathcal{R}_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$, $R^\mu{}_{\nu\rho\sigma} = (\partial_\rho\Gamma_\sigma - \partial_\sigma\Gamma_\rho + [\Gamma_\rho,\Gamma_\sigma])^\mu{}_\nu$, $(\Gamma_\rho)^\mu{}_\nu = \Gamma^\mu_{\rho\nu} = \frac{1}{2}G^{\mu\lambda}(\partial_\rho G_{\lambda\nu} + \partial_\nu G_{\rho\lambda} - \partial_\lambda G_{\rho\nu})$.

tachyon potential, i.e. a non-derivative function of the fields X^μ. We could choose to add to the action (2.1) either a term $-\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\sqrt{g}\,\tilde{V}(X)$ or, working in conformal gauge right away, a term

$$S_V = -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\, V(X)\ . \tag{2.3}$$

While $\tilde{V}(X)$ should be a scalar, conformal field theory tells us that, to be marginal, $V(X)$ should be a conformal primary of weight $(1,1)$.†

In the σ-model approach one can again compute the β-functionals now including the effect of V. To lowest order in α' one finds that eqs (2.2) are not modified by the presence of the tachyon potential $V(X)$, while β^V is

$$\beta^V \sim -\frac{1}{2}D^\mu D_\mu V + D^\mu\Phi D_\mu V - \frac{1}{\alpha'}V\ . \tag{2.4}$$

We insisted that these results are to lowest non-trivial order in α'. Indeed, it is known [10] that including higher-order contributions in α' will lead to corrections to eqs (2.2) that involve V, as well as to non-linear terms in V in eq (2.4). As it stands, eq. (2.4) is simply the dimension $(1,1)$-condition in the σ-model language. Typical solutions of $\beta^V=0$ are sums of products of exponentials of the fields.

As an example consider the Liouville theory with $G_{\varphi\varphi}=\alpha'$, $\Phi=Q\varphi$ and $V\sim e^{\beta\varphi}$, coupled to a matter theory with central charge c. Substituting $N=1+c$ in the third eq. (2.2) yields $Q^2=\frac{25-c}{12}$. Equation (2.4) gives $\beta^2-2Q\beta+2=0$, hence the well-known formula $\beta = Q-\sqrt{Q^2-2}=\sqrt{\frac{25-c}{12}}-\sqrt{\frac{1-c}{12}}$. In the classical limit $c\to-\infty$ one correctly gets $\beta\to\frac{1}{Q}$. Upon rescaling $\tilde{\varphi}=\frac{\varphi}{Q}+const$, the Liouville action correctly reads [2] $\frac{25-c}{48\pi}\int \mathrm{d}^2\sigma\left[-(\partial\tilde{\varphi})^2+R\tilde{\varphi}-e^{(\beta Q)\tilde{\varphi}}\right]$ with $\beta Q\sim 1$ in this limit.

Of course, even in the absence of the potential V, eqs (2.2) are valid only to leading order in α' and higher order corrections e.g. to the first equation involve terms like $R_{\mu\lambda\rho\sigma}R_\nu{}^{\lambda\rho\sigma}$. In some cases, however, as for the WZW-models to which we now turn, one knows that the theory is an exact conformal theory, and hence the β-functionals vanish to all orders in α'.

† We also know that an integrable marginal operator moreover needs to have vanishing operator product coefficients c_{VVj} with all primary fields labelled by j that have weight $(1,1)$ [9].

3. WZW-models

It is very easy to compute the background fields $G_{\mu\nu}$ and $H_{\mu\nu\rho}$ for an arbitrary WZW-model for a group $\mathcal{G}$ [11] with action

$$
\begin{aligned}
S = S_1 + S_2 = & \frac{1}{16\pi\alpha'} \int \mathrm{d}^2\sigma \, \mathrm{tr}(g^{-1}\partial_\alpha g)(g^{-1}\partial^\alpha g) \\
& + \frac{1}{24\pi\alpha'} \int \mathrm{d}^2\sigma \mathrm{d}t \, \epsilon^{\alpha\beta\gamma} \, \mathrm{tr}(g^{-1}\partial_\alpha g)(g^{-1}\partial_\beta g)(g^{-1}\partial_\gamma g) \ .
\end{aligned}
\tag{3.1}
$$

Let ξ^μ be coordinates on the group manifold of $\mathcal{G}$ so that $g \equiv g(\xi^\mu(\sigma^\alpha))$. Let $\partial_\mu \equiv \frac{\partial}{\partial \xi^\mu}$. Then $(\partial_\mu g) g^{-1}$ is in the Lie algebra of $\mathcal{G}$ and hence some linear combination of the generators I_a:

$$
\partial_\mu g = U^a_\mu(\xi) I_a g \tag{3.2}
$$

and from equating $\partial_\mu \partial_\nu g$ with $\partial_\nu \partial_\mu g$ one obtains the Maurer-Cartan equations

$$
\partial_\mu U^a_\nu - \partial_\nu U^a_\mu = C^a{}_{bc} U^b_\mu U^c_\nu \tag{3.3}
$$

where $[I_b, I_c] = C^a{}_{bc} I_a$. We normalize the generators as $\mathrm{tr} I_a I_b = -\delta_{ab}$. Furthermore, $C_{acd} C_b{}^{cd} = c_2 \delta_{ab}$ where c_2 is the quadratic Casimir of the adjoint representation.

Using eq. (3.2) it is straightforward to see that the first part of the action reduces to

$$
S_1 = -\frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \, \partial_\alpha \xi^\mu \partial^\alpha \xi^\nu G_{\mu\nu} \tag{3.4}
$$

with

$$
G_{\mu\nu} = \frac{1}{4} U^a_\mu U^a_\nu \ . \tag{3.5}
$$

Thus we interpret the $e^a_\mu = \frac{1}{2} U^a_\mu$ as vielbeins on the group manifold. Then eq. (3.3) is nothing but the zero-torsion equation $\mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0$, provided we identify the spin-connection as

$$
\omega_\mu{}^a{}_b = \frac{1}{2} C^a{}_{bc} U^c_\mu \ . \tag{3.6}
$$

It is an easy excercise to compute the curvature two-form $R^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$ and the

Riemann tensor $R_{\mu\nu\rho\sigma} = e_{a\rho} e^b_\sigma (R^a{}_b)_{\mu\nu}$ from which the Ricci tensor and scalar are obtained as

$$\mathcal{R}_{\mu\nu} = c_2 G_{\mu\nu} \ , \quad \mathcal{R} = c_2 \, \mathrm{dim}\mathcal{G} \ . \tag{3.7}$$

The Ricci tensor is proportional to the metric and the scalar curvature is constant.‡

While the first part S_1 of the WZW-action gives the metric $G_{\mu\nu}$, the WZ-term S_2 will give the antisymmetric tensor field. It is not possible in general to give $B_{\mu\nu}$ directly (otherwise we would have obtained a two-dimensional form of the WZ-term), but it is easy to obtain its field strength $H_{\mu\nu\rho}$ which is all that is needed anyway. Using eq. (3.2) we get

$$S_2 = -\frac{1}{48\pi\alpha'} \int \mathrm{d}^2\sigma \mathrm{d}t \, \epsilon^{\alpha\beta\gamma} \partial_\alpha \xi^\mu \partial_\beta \xi^\nu \partial_\gamma \xi^\rho C_{abc} U^a_\mu U^b_\nu U^c_\rho \tag{3.8}$$

which we want to identify with

$$\begin{aligned} &-\frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \, \epsilon^{\alpha\beta} \partial_\alpha \xi^\mu \partial_\beta \xi^\nu B_{\mu\nu} = -\frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \mathrm{d}t \, \epsilon^{\alpha\beta\gamma} \partial_\gamma (\partial_\alpha \xi^\mu \partial_\beta \xi^\nu B_{\mu\nu}) \\ &= -\frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \mathrm{d}t \, \epsilon^{\alpha\beta\gamma} \partial_\alpha \xi^\mu \partial_\beta \xi^\nu \partial_\gamma \xi^\rho \partial_\rho B_{\mu\nu} \end{aligned} \tag{3.9}$$

(where, as usual, the boundary of the three-dimensional integration region is the two-dimensional manifold of the σ-model) so that

$$H_{\mu\nu\rho} = \frac{1}{4} C_{abc} U^a_\mu U^b_\nu U^c_\rho = 2 C_{abc} e^a_\mu e^b_\nu e^c_\rho \ . \tag{3.10}$$

Since the vielbeins e^a_μ, as well as the structure constants C_{abc} are covariantly constant, it follows that $D_\sigma H_{\mu\nu\rho} = 0$. Furthermore $H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} = 4 c_2 G_{\mu\nu} = 4\mathcal{R}_{\mu\nu}$, so that the three eqs. (2.2) reduce to

$$\begin{aligned} G^{\mu\nu} D_\mu \partial_\nu \Phi &= 0 \ , \\ \partial_\rho \Phi H^\rho{}_{\mu\nu} &= 0 \ , \\ G^{\mu\nu} D_\mu \Phi D_\nu \Phi + \frac{\tilde{N} - 26}{12\alpha'} &= -\frac{\mathrm{dim}\mathcal{G}}{12\alpha'} (1 - 2\alpha' c_2) \ , \end{aligned} \tag{3.11}$$

which determines Φ. Here $\tilde{N}$ is the number of fields other than the ξ^μ, if any.§ For semi-simple

‡ This is reminiscent of a maximally symmetric space. However, only for $SU(2)$ where $\mathcal{G} \simeq S^3$ one has the extra relation $R_{\mu\nu\rho\sigma} \sim G_{\mu\rho} G_{\nu\sigma} - G_{\mu\sigma} G_{\nu\rho}$.

§ Note that the r.h.s. of the last equation is, to order α' and up to an overall factor of $-\frac{1}{12\alpha'}$, the contribution of the WZW-model to the central charge: $c = \frac{x \mathrm{dim}\mathcal{G}}{x + \tilde{h}} = \mathrm{dim}\mathcal{G}(1 - 2\alpha' c_2) + \mathcal{O}(\alpha'^2)$ where $\tilde{h} = c_2/\psi^2$ (ψ^2 being the length squared of the highest root) is the dual Coxeter number, and one identifies the Kac-Moody level x with $\frac{1}{2\alpha'\psi^2}$.

$\mathcal{G}$, the C_{abc} cannot vanish for all a, b for fixed c, hence $H^{\rho}{}_{\mu\nu}$ cannot vanish for all μ, ν for fixed ρ, and one concludes that $\partial_\rho \Phi = 0$. Then, for $\tilde{N} = 0$, unless one fine-tunes α', i.e. unless the WZW Kac-Moody level can be adjusted such that $c_{\rm WZW} = 26$, the third eq. (3.11) is not satisfied.

If, however, we consider the slightly more general situation where in addition to the WZW fields ξ^μ one has $\tilde{N}$ other fields ξ^s, $s = \dim\mathcal{G} + 1, \ldots \dim\mathcal{G} + \tilde{N}$ with flat metric and vanishing $B_{\mu\nu}$ ($U(1)$-factors) then, in these directions $H^s{}_{\mu\nu} = 0$ and $\partial_s \Phi$ can be non-vanishing. The first eq. (3.11) then shows that the dilaton field must be linear in the ξ^s and one has

$$
\begin{aligned}
\Phi &= \sum_{s=\dim\mathcal{G}+1}^{\dim\mathcal{G}+\tilde{N}} a_s \xi^s \\
\sum_{s=\dim\mathcal{G}+1}^{\dim\mathcal{G}+\tilde{N}} a_s^2 &= \frac{26 - \tilde{N} - \dim\mathcal{G}}{12\alpha'} + \frac{c_2 \dim\mathcal{G}}{6}
\end{aligned}
\tag{3.12}
$$

so that all β-function equations (2.2) are satisfied.

4. The non-abelian Toda theory

The (ordinary) conformally invariant Toda field theories based on a Lie algebra g [3] when viewed as a σ-model (2.1) have constant metric $G_{\mu\nu}$ and vanishing antisymmetric tensor, and thus the corresponding β-function equations (2.2) are trivially satisfied, provided one chooses a linear dilaton appropriately. The latter is invisible in the conformal gauge action but controls the improvement term in the stress-energy tensor. These theories correspond to a canonical gradation of g and the gradation zero part g_0 is abelian. Generalizing to non-canonical gradations leads to non-abelian g_0 [5] and non-trivial background fields [7]. The simplest example is the non-abelian Toda theory for $g = B_2$. In the formulation given in [7,8] it corresponds to a σ-model with the metric of the two-dimensional black hole and one additional flat dimension:

$$
G_{\mu\nu} {\rm d}x^\mu {\rm d}x^\nu = {\rm d}r^2 + {\rm th}^2 r \, {\rm d}t^2 + {\rm d}\phi^2 \ , \quad B_{\mu\nu} = 0 \tag{4.1}
$$

and a potential term $V = 4{\rm ch}2r e^{2\phi}$. This model is classically integrable and although its equations of motion are highly non-trivial, their general solution could be explicitly given [7,8] due to the underlying Lie algebraic structure. At the classical (Poisson bracket) level this model has three left-moving conserved quantities T, V^+, V^- of dimension two, as well as three right-moving ones. The chiral equal-time Poisson bracket algebra of these quantities is not only non-linear (like for W-algebras) but also *non-local* due to the appearance of $\epsilon(\sigma-\sigma') = \theta(\sigma-\sigma') - \theta(\sigma'-\sigma)$:

$$\begin{aligned}
\gamma^{-2}\{T(\sigma)\,,\,T(\sigma')\} &= (\partial_\sigma - \partial_{\sigma'})\left[T(\sigma')\delta(\sigma-\sigma')\right] - \frac{1}{2}\delta'''(\sigma-\sigma') \\
\gamma^{-2}\{T(\sigma)\,,\,V^\pm(\sigma')\} &= (\partial_\sigma - \partial_{\sigma'})\left[V^\pm(\sigma')\delta(\sigma-\sigma')\right] \\
\gamma^{-2}\{V^\pm(\sigma)\,,\,V^\pm(\sigma')\} &= \epsilon(\sigma-\sigma')V^\pm(\sigma)V^\pm(\sigma') \\
\gamma^{-2}\{V^\pm(\sigma)\,,\,V^\mp(\sigma')\} &= -\epsilon(\sigma-\sigma')V^\pm(\sigma)V^\mp(\sigma') \\
&\quad + (\partial_\sigma - \partial_{\sigma'})\left[T(\sigma')\delta(\sigma-\sigma')\right] - \frac{1}{2}\delta'''(\sigma-\sigma')\ .
\end{aligned} \tag{4.2}$$

The scale factor γ^2 which controls the central charge plays the role of $\hbar$ or α' already introduced in the classical equations. The classical solution of the non-abelian Toda theory provides a free-field realisation of these chiral generators [8]. It turned out that this algebra can be obtained as the second Gelfand-Dikii symplectic structure associated with a second-order matrix differential operator [12]

$$L = \partial^2 - U\ , \quad U = \begin{pmatrix} T & -\sqrt{2}V^+ \\ -\sqrt{2}V^- & T \end{pmatrix} \tag{4.3}$$

in the same way as the Virasoro algebra is obtained from (4.3) with scalar $U \sim T$. This algebraic structure was generalized to $n\times n$-matrix differential operators of order m and non-linear, non-local algebras like (4.2) that are matrix generalizations of (classical) W_m-algebras [13].

All this was at the level of classical symplectic structures, and it seemed surprisingly difficult to quantize even the simplest algebra (4.2) maintaining the conformal dimensions of the generators equal to two. It is probable that the solution of the quantization problem is linked to the following simple remark.

It is easy to see that the black hole metric (4.1) cannot solve the β-function equations (2.2), i.e. one cannot find a dilation Φ so that all three equations are satisfied: the non-abelian Toda model so defined is classically conformally invariant, but not at the quantum level. The point is that the σ-model action corresponding to (4.1) is *not* the correct starting point.

In the original Lax pair formulation of the integrable model there are four fields μ, ν, r, ϕ whose equations of motion can be written as [14]

$$
\begin{aligned}
&\partial_-(\partial_+\nu - \mathrm{ch}2r\partial_+\mu) = 0 \ , \\
&\partial_+(\partial_-\mu - \mathrm{ch}2r\partial_-\nu) = 0 \ , \\
&2\partial_+\partial_- r = \mathrm{sh}2r(\partial_-\nu\partial_+\mu + 2e^{2\phi}) \ , \\
&\partial_+\partial_-\phi = \mathrm{ch}2re^{2\phi} \ ,
\end{aligned}
\tag{4.4}
$$

which can be obtained from the action

$$
\begin{aligned}
S \sim \int \mathrm{d}^2\sigma \Big[&-\frac{1}{2}\partial_+\nu\partial_-\nu - \frac{1}{2}\partial_+\mu\partial_-\mu + 2\partial_+ r\partial_- r + 2\partial_+\phi\partial_-\phi \\
&+ \mathrm{ch}2r\partial_+\mu\partial_-\nu + 2\mathrm{ch}2r\, e^{2\phi}\Big] \ .
\end{aligned}
\tag{4.5}
$$

We can solve the first two equations (4.4) as

$$
\partial_+\nu - \mathrm{ch}2r\, \partial_+\mu = 0 \ , \quad \partial_-\mu - \mathrm{ch}2r\, \partial_-\nu = 0 \ . \tag{4.6}
$$

Introduce a field t as

$$
\partial_+ t = \mathrm{ch}^2 r\, \partial_+\mu \ , \quad \partial_- t = (1 + \mathrm{th}^2 r)^{-1}\partial_-\mu \ . \tag{4.7}
$$

Then eq. (4.6) implies that $\partial_+\nu = (1 + \mathrm{th}^2 r)\partial_+ t$, $\partial_-\nu = \mathrm{ch}^{-2} r\, \partial_- t$. The compatibility condition of the two equations (4.7) can be written as

$$
\partial_+\partial_- t = -\frac{1}{\mathrm{sh}r\, \mathrm{ch}r}(\partial_+ t\partial_- r + \partial_- t\partial_+ r) \tag{4.8}
$$

while the last two equations (4.4) become

$$
\begin{aligned}
&\partial_+\partial_- r = \frac{\mathrm{sh}r}{\mathrm{ch}^3 r}\partial_+ t\partial_- t + \mathrm{sh}2r\, e^{2\phi} \ , \\
&\partial_+\partial_-\phi = \mathrm{ch}2r\, e^{2\phi} \ .
\end{aligned}
\tag{4.9}
$$

Equations (4.8) and (4.9) are three equations of motion for the three fields r, t, ϕ and can be obtained from the action

$$
S \sim \int \mathrm{d}^2\sigma \left[\partial_+ r\partial_- r + \mathrm{th}^2 r\, \partial_+ t\partial_- t + \partial_+\phi\partial_-\phi + \mathrm{ch}2r\, e^{2\phi}\right] \tag{4.10}
$$

which is the σ-model with the background fields given by (4.1) and a tachyon potential $V = 4\mathrm{ch}2r\, e^{2\phi}$, i.e. the non-abelian Toda action as formulated in [7,8].

While, as already mentioned, the σ-model (4.10) does not satisfy the β-function equations, the model given by (4.5) does, as we now poceed to show. Arranging the fields as $X^1 = \mu, X^2 = \nu, X^3 = r, X^4 = \phi$, the action (4.5) corresponds to

$$G_{\mu\nu} = \frac{1}{4}\begin{pmatrix} -1 & \mathrm{ch}2r & 0 & 0 \\ \mathrm{ch}2r & -1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad , \quad B_{\mu\nu} = \frac{1}{4}\begin{pmatrix} 0 & -\mathrm{ch}2r & 0 & 0 \\ \mathrm{ch}2r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \tag{4.11}$$

and $V = 4\mathrm{ch}2r\, e^{2\phi}$ as before. As far as $G_{\mu\nu}$ and $B_{\mu\nu}$ are concerned, ϕ plays a trivial role, and it is more convenient to consider seperately X^1, X^2 and X^3 with metric $G^{(3)}_{\mu\nu}$. It is then straightforward to see that

$$\mathcal{R}^{(3)}_{\mu\nu} = -2G^{(3)}_{\mu\nu} \tag{4.12}$$

while the full four-dimensional Ricci tensor $\mathcal{R}_{\mu\nu}$ is obtained by adding an extra lign and column of zeros. Relation (4.12) which closely ressembles eq. (3.7) for $SU(2)$ should be no surprise. Indeed, the σ-model part of the action (4.5) can be written as a WZW-model for the zero-gradation subgroup $\mathcal{G}_0$ which is a non-compact version of $SU(2)$, hence the minus sign. Similarly one finds $H_{123} = -\frac{1}{2}\mathrm{sh}2r$ and $H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} = 4\mathcal{R}_{\mu\nu}$, as well as $D_\rho H^\rho{}_{\mu\nu} = 0$ and the β-function equations (2.2) again reduce to eqs. (3.11) with $\dim\mathcal{G} = 3$ and $c_2 = -2$. As in section 3 one determines

$$\Phi = Q\phi\ , \quad Q^2 = \frac{22-d}{12\alpha'} - 1 \tag{4.13}$$

where d is the number of free (flat) extra fields one might want to add. Note that because $\mathcal{R} = -6 = const$, the action (4.5) certainly no longer describes a black hole background.

We have seen that the "kinetic" part of the "correct" non-abelian Toda action (4.5) satisfies the β-function equations (2.2). What about the potential $V = 4\mathrm{ch}2r\, e^{2\phi}$? First, we remark that the normalization factor 4 in front of $e^{2\phi}$ has no meaning since it can be adjusted at will by shifting $\phi \to \phi + const$. Also the action (4.5) was obtained from purely classical considerations and we could just as well have multiplied it by some scale factor γ^{-2}. Redefining all fields $X^\mu \to \gamma X^\mu$ would absorb this factor while changing $\gamma^{-2}V \to 4\gamma^{-2}\mathrm{ch}2\gamma r\, e^{2\gamma\phi}$, so one might just as well consider

$$V = \mathrm{ch} 2\gamma r \, e^{2\gamma\phi} \ . \tag{4.14}$$

This is analogous to the rescaling of the Liouville field discussed in section 2. Of course, in the quantum theory the value of γ is no longer arbitrary. Inserting V into eq. (2.4) one finds that it indeed is a solution of $\beta^V = 0$ provided γ is chosen to satisfy the resulting algebraic equation with solution

$$4\gamma = Q \pm \sqrt{Q^2 - \frac{4}{\alpha'}} \ . \tag{4.15}$$

For all non-negative values of d this leads to a complex value of γ, and it seems that in order to quantize this theory one faces a situation similar to the Liouville theory for $c > 1$.

Whether this is really so is not clear at present. Indeed, the world-sheet symmetry algebra is not just the Virasoro algebra which gives a ghost contribution -26 in β^V and hence in $-Q^2 = \frac{d-4-26}{12\alpha'} + 1$, but the larger algebra (4.2). From the study of superstrings or W-gravity and W-strings [15] one knows that the ghost contributions to the central charge gets modified, and for a larger bosonic chiral algebra it becomes larger (e.g. -100 instead of -26 for the W_3-case). For the algebra (4.2) one would then expect three ghost pairs, all of weight 2 and -1, contributing $3 \times (-26) = -78$ to the central charge, so that the correct value of Q should read

$$Q^2 = \frac{78 - 4 - d}{12\alpha'} - 1 \ , \tag{4.16}$$

and Q and $\sqrt{Q^2 - \frac{4}{\alpha'}}$ remain real as long as $d \leq 26 - 12\alpha'$. Curiously enough, for $\alpha' \to 0$ the upper limit is 26. However, the whole issue of quantization of the theory needs to be studied in detail before one can draw any definit conclusion.

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