Discussing the U(1)-Problem of QED$_2$ without Instantons[1]

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Abstract

We construct QED$_2$ with mass and flavor and an extra Thirring term. The vacuum expectation values are carefully decomposed into clustering states using the U(1)-axial symmetry of the considered operators and a limiting procedure. The properties of the emerging expectation functional are compared to the proposed θ-vacuum of QCD. The massive theory is bosonized to a generalized Sine-Gordon model (GSG). The structure of the vacuum of QED$_2$ manifests itself in symmetry properties of the GSG. We study the U(1)-problem and derive a Witten-Veneziano-type formula for the masses of the pseudoscalars determined from a semiclassical approximation.

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1 Introduction

There is a long history of attempts to study problems of four dimensional field theories in low dimensional models. Maybe the most prominent example is the U(1)-gauge theory in two dimensions first analyzed by Schwinger [1] and therefore christened Schwinger model. It has many features in common with QCD. It shows confinement, mass generation of the would-be Goldstone particle via the axial anomaly and allows for topologically nontrivial gauge field configurations. The model can be made more realistic by introducing several flavors and mass terms. The resulting model which we refer to as QED_2 is less simple to analyze. In this paper we study QED_2 using the Euclidean functional integral approach.

Of course this project is inspired by some '4d mysteries', as should be any investigation of toy models. Namely the topics that will be attacked are the construction of the θ-vacuum, the U(1)-problem and Witten-Veneziano type formulas. Those problems are closely related to each other.

According to the common wisdom, the θ-vacuum [2, 3] is supposed to be the formal superposition of topological sectors

$$| \, \theta \, \rangle \; = \; \sum_{l=-\infty}^{l=+\infty} e^{-i\theta l} | \, l \, \rangle \; . \tag{1}$$

$|l\rangle$ denotes the states formally corresponding to the sector of classical pure gauges that wind l-times around compactified space. The mathematical status of (1) is of course rather formal, since it is e.g. unclear how to normalize $|\theta\rangle$. For QED_2 a similar, but also formal double vacuum structure was derived in [4]. It has to be remarked, that the explicit form (1) of the θ-vacuum has dissappeared in later work, in favor of an implicit characterization of the θ-vacuum through its symmetry properties under large gauge transformations (see e.g. [5]). However for QED_2 we will construct the θ-vacuum functional explicitely.

Another rather formal manipulation often is used when one studies the contribution of instanton sectors to Euclidean functional integrals. Usually expressions like $\mathcal{D}[A]_n$ show up, which are then meant to denote a measure over gauge field configurations with fixed winding number n. It is indeed a rather challenging project to marry the idea of a winding number which heavily relies on the continuity of the classical gauge fields with the concept of functional measures living on distributions [6]. For QED_2 on a torus where the infrared problems are absent, an interesting construction of the measure with winding number was obtained in [7]-[14].

Despite their mathematical problems, the topological ideas have played a fruitful role in gauge theories and sometimes are a good semiclassical guideline. In two dimensions where the mathematical analysis of the models is much simpler there is of course a more elegant strategy. In this paper we show that for QED_2 it is possible to construct explicitely a proper clustering vacuum functional $\langle .. \rangle^\theta$ without relying on instantons or a formal θ-vacuum like (1). Afterwards it is certainly legitimate to compare the properties of $\langle .. \rangle^\theta$ with the formal properties of (1).

The second topic that motivies this 2d study is the U(1)-problem [15]-[20]. In QCD (as well as in QED_2) the axial U(1) current j_5 acquires an anomaly when quantizing the theory. In both cases it is possible [21] to rewrite the right hand side of the anomaly equation as a total divergence. Thus one can define a new current $\tilde{j}_5$ which is conserved. Ignoring the fact that $\tilde{j}_5$ is not gauge invariant and thus rather unphysical, one can formally implement the U(1)-axial symmetry related to $\tilde{j}_5$. It is known that nature does not respect this symmetry and thus one could expect a Goldstone particle that corresponds to this broken symmetry. Since the quarks are massive, one only can hope to find an approximate Goldstone particle, i.e. a light pseudoscalar. Based on the paper by Weinberg [15] for the case of two flavors it is believed that the η-meson

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which has the right quantum numbers is too heavy to play the role of an approximate Goldstone particle. For three flavors the corresponding particle is the η'. The U(1)-problem now is the absence of a fourth (ninth) light pseudoscalar. As will be discussed below, up to some restrictions due to the Coleman theorem, the U(1)-problem of QED$_2$ is formulated equivalently.

Also for the analysis of the U(1)-problem of QED$_2$ the strategy will be to study the problem using only mathematically rigorous methods, which is of course much easier in two dimensions. The lessons on the U(1)-problem will be drawn afterwards and we will discuss what could be learned for QCD.

Finally we will also analyze Witten-Veneziano-type formulas. They connect the masses of the pseudoscalar mesons to the topological susceptibility. It was pointed out in [22] (see also [23, 24]) that there is a problem with Witten's original derivation [25, 26] and an alternative proof was given. It was argued that the topological susceptibility on the right hand side of the original Witten-Veneziano formula has to be replaced by its contact term. In a lattice approach [27] the formula was generalized to QCD with three flavors of massive quarks. We show that in QED$_2$ a Witten-Veneziano-type formula for the masses of the pseudoscalars determined from a semiclassical approximation holds.

The paper is organized as follows. In the next section we discuss the model and its symmetries. In order to treat the mass term perturbatively we include a Thirring term as an UV-regulator and also a space-time cutoff. This is followed by a section where we outline the construction using mass perturbation series.

In Section 4 we show that the vacuum state constructed so far does not cluster and thus the vacuum fails to be unique. We cure this problem by defining a new vacuum functional $\langle .. \rangle^\theta$ in a mathematically rigorous way, which involves the symmetry properties of the operators and a limiting procedure. The properties of $\langle .. \rangle^\theta$ give rise to chiral selection rules which we discuss.

The massive model will be constructed using the bosonization prescriptions which we establish in Section 5. The bosonic model turns out to be a generalized Sine-Gordon model (GSG). Properties of the vacuum functional $\langle .. \rangle^\theta$ manisfest themselves as symmetry properties of the GSG which we discuss. In particular it will turn out that the symmetry that would correspond to the axial U(1) symmetry of QED$_2$ can not be implemented in the GSG. This leads to the conclusion that the U(1)-problem of QED$_2$ does not exist.

Using a semiclassical approximation we finally establish that a Witten-Veneziano-type formula holds for QED$_2$.

In Section 6 the relevance and also the limitations of the lessons on QED$_2$ for the physics of QCD will be discussed.

2 The model

The Euclidean action of the model that will be constructed is given by

$$S[\overline{\psi}, \psi, A, h] \;=\; S_G[A] + S_h[h] + S_F[\overline{\psi}, \psi, A, h] + S_M[\overline{\psi}, \psi] \,. \tag{2}$$

The action for the gauge field reads

$$S_G[A] \;=\; \int d^2x \left(\frac{1}{4} F_{\mu\nu}(x) F_{\mu\nu}(x) + \frac{1}{2} \lambda \Big(\partial_\mu A_\mu(x) \Big)^2 \right) \,. \tag{3}$$

A gauge fixing term is included that will be considered in the limit $\lambda \to \infty$ which ensures $\partial_\mu A_\mu = 0$ (transverse gauge). As usual $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, denotes the field strength tensor.

The fermion action is a sum over N flavor degrees of freedom

$$S_F[\overline{\psi}, \psi, A, h] = \sum_{b=1}^{N} \int d^2x\, \overline{\psi}^{(b)}(x)\gamma_\mu\Big(\partial_\mu - ieA_\mu(x) - ig^{1/2}h_\mu(x)\Big)\psi^{(b)}(x) . \tag{4}$$

For the Euclidean γ-algebra we choose $\gamma_1 = \sigma_1, \gamma_2 = \sigma_2, \gamma_5 = \sigma_3$, where σ_i are the Pauli-matrices. In addition to the gauge field an auxiliary field h_μ couples to the fermions in exactly the same way as A_μ. Its action is given by

$$S_h[h] = \frac{1}{2} \int d^2x\, h_\mu(x)\Big(\delta_{\mu\nu} - \lambda'\partial_\mu\partial_\nu\Big)h_\nu(x) . \tag{5}$$

$S_h[h]$ is simply a white noise term plus a term that makes h_μ transverse in the limit $\lambda' \to \infty$.

In order to understand the role of the auxiliary field one can formally integrate over h_μ. This leads to the Thirring term

$$S_T[\overline{\psi}, \psi] = \frac{1}{2}\, g\, \int d^2x\, \sum_{b=1}^{N} j^{(b)}{}^T_\mu(x) \sum_{c=1}^{N} j^{(c)}{}^T_\mu(x) , \tag{6}$$

for the transverse part of the U(1)-current $\sum_{b=1}^{N} j^{(b)}{}^T_\mu$. The superscript T denotes projection on the transverse direction

$$j^{(b)}{}^T_\mu := T_{\mu\nu}\overline{\psi}^{(b)}\gamma_\nu\psi^{(b)} \quad , \quad T_{\mu\nu} := \delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\triangle} , \tag{7}$$

and $T_{\mu\nu}$ is the corresponding projector. The purpose of this Thirring term is to make the short distance singularity of

$$\langle\, \overline{\psi}^{(b)}(x)\psi^{(b)}(x)\, \overline{\psi}^{(b)}(y)\psi^{(b)}(y)\, \rangle , \tag{8}$$

integrable. The quoted expression is a typical term showing up in a power series expansion of the mass term (9) (see below). It has to be integrated over $d^2x d^2y$ which is possible only if an ultraviolet regulator such as the Thirring term is included.

Since the mass term will be treated perturbatively, we denote it separately

$$S_M[\overline{\psi}, \psi] = -\sum_{b=1}^{N} m^{(b)} \int d^2x\, \chi_\Lambda(x)\, \overline{\psi}^{(b)}(x)\psi^{(b)}(x) . \tag{9}$$

$m^{(b)}$ are the fermion masses for the various flavors. For the perturbation expansion it is necessary to introduce an infrared cutoff. Here we use a space-time cutoff, namely a finite rectangle Λ in space-time, and χ_Λ denotes its characteristic function.

For vanishing fermion masses $m^{(b)}$, the Lagrangian of the model has the symmetry $\mathrm{SU(N)}_L \times \mathrm{SU(N)}_R \times \mathrm{U(1)}_V \times \mathrm{U(1)}_A$ as is the case for QCD. When quantizing the massless theory the axial U(1)-current

$$j_{5\,\mu}(x) := \sum_{b=1}^{N} \overline{\psi}^{(b)}(x)\gamma_\mu\gamma_5\psi^{(b)}(x) , \tag{10}$$

acquires the anomaly

$$\partial_\mu j_{5\,\mu}(x) = 2N \frac{e}{2(\pi + gN)}\, \varepsilon_{\mu\nu}\partial_\mu A_\nu(x) + \text{contact terms} . \tag{11}$$

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It has to be remarked that the coupling constant g for the Thirring term shows up in the anomaly equation. This is due to the fact that it is the U(1) vector current which enters the Thirring term, leading to an extra contribution to the anomaly.

As in QCD, the anomaly breaks the symmetry down to $\text{SU(N)}_L \times \text{SU(N)}_R \times \text{U(1)}_V$. Since the right hand side of the anomaly equation (11) is a divergence, the formal arguments [21] that were applied in QCD to define a conserved current $\tilde{j}_5$ can be repeated. Thus when considering the symmetry properties, the toy model is adequate for studying the problematic aspects in the formulation of the U(1)-problem.

3 Outline of the construction

In [23] it was shown that for a perturbative treatment of the determinant of the massive Schwinger model one would have to evaluate infinitely many Feynman diagrams all of the same order in the fermion mass. Thus the strategy will be to expand the mass term $\exp(-S_M[\overline{\psi},\psi])$

$$\langle P[\overline{\psi}, \psi, A, h]\rangle = \frac{1}{Z}\langle P[\overline{\psi}, \psi, A, h] e^{-S_M[\overline{\psi},\psi]}\rangle_0 = \frac{1}{Z}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\langle P[\overline{\psi}, \psi, A, h]\, S_M[\overline{\psi}, \psi]^n\rangle_0 . \quad (12)$$

The normalization constant $Z := \langle \exp(-S_M)\rangle_0$ also has to be expanded with respect to the fermion masses. It has to be remarked that the Thirring term (6), as well as the cutoff Λ are essential for this expansion.

The expectation values with subscript 0 showing up in (12) are the expectation values of the massless model which are formally given by the path integral expression

$$\langle P[\overline{\psi}, \psi, A, h]\rangle_0 := \frac{1}{Z_0}\int \mathcal{D}h\mathcal{D}A\mathcal{D}\overline{\psi}\mathcal{D}\psi \; P[\overline{\psi}, \psi, A, h]\, e^{-S_G[A]-S_h[h]-S_F[\overline{\psi},\psi,A,h]} . \quad (13)$$

When integrating out the fermions (N flavors) one obtains $\det[\partial\!\!\!/ - i(e\,A\!\!\!/ + g^{1/2}\,h\!\!\!/)]^N$. The fermion determinant is only defined when an ultraviolet and infrared cutoff (for instance a finite space-time lattice, [28, 29]) is introduced. The determinant can then be normalized to 1 for $e, g = 0$, by replacing it with $\det[1 - K(A, h)]$ where $K(A, h) = i(e\,A\!\!\!/ + g^{1/2}\,h\!\!\!/)\partial\!\!\!/^{-1}$. In two dimensions this determinant can be computed explicitey, using the idea of regularized fermion determinants (see e.g. [30]). If we assume that the vector potentials A_μ and h_μ satisfy some mild regularity and falloff conditions at infinity to make them square integrable [30], the answer is

$$\det[1 - K(A, h)] = \exp\left(-\frac{1}{2\pi}\left\|eA^T + g^{1/2}h^T\right\|_2^2\right), \quad (14)$$

where A_μ^T and h_μ^T are the transverse parts of the vector fields

$$A_\mu^T = \left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\triangle}\right)A_\nu \quad , \quad h_\mu^T = \left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\triangle}\right)h_\nu . \quad (15)$$

The action for A_μ as well as for h_μ are quadratic forms. Thus together with the logarithm of the fermion determinant (14), the measure including the effective action for the gauge field and the auxiliary field can be given a precise mathematical meaning in terms of Gaussian functional integrals.

Since Formula (14) for the fermion determinant mixes h_μ and A_μ we perform a shift of the auxiliary field

$$h' = h + \frac{eg^{1/2}N}{\pi + gN}TA , \quad (16)$$

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in order to decouple the two fields. T is the transverse projector defined in (7). Thus we have

$$\frac{1}{Z^0}\mathcal{D}h\mathcal{D}Ae^{-S_G[A]-S_h[h]}\det[\partial\!\!\!/ - i(e\,A\!\!\!/ + g^{1/2}\,h\!\!\!/)]^N \;\rightsquigarrow\; d\mu_Q[A]d\mu_C[h'] \,, \qquad (17)$$

where $d\mu_Q[A]$ and $d\mu_C[h']$ denote Gaussian measures for the gauge field A_μ and the shifted auxiliary field h'_μ, with covariances given by

$$Q_{\mu\nu} \;=\; \left(-\triangle+\frac{e^2 N}{\pi+gN}\right)^{-1}T_{\mu\nu} \quad,\quad C_{\mu\nu} \;=\; \frac{\pi}{\pi+gN}T_{\mu\nu}\,. \qquad (18)$$

Both expressions are quoted already in the transverse limit i.e. $\lambda\to\infty$ and $\lambda'\to\infty$.

The final ingredient for the solution of massless QED_2 is the fermion propagator $G(x,y;B)$ in an external field $B_\mu = eA_\mu + g^{1/2}h_\mu$. The corresponding Green's function equation reads

$$\gamma_\mu\big(\partial_\mu - iB_\mu(x)\big)G(x,y;B) \;=\; \delta(x-y)\,. \qquad (19)$$

The solution was already found by Schwinger [1] and is given by

$$G(x,y;B) \;=\; G^0(x-y)e^{i[\Phi(x)-\Phi(y)]}\,, \qquad (20)$$

where

$$\Phi(x) \;=\; -\int d^2z D(x-z)\Big(\partial_\mu B_\mu(z) + i\gamma_5\varepsilon_{\mu\nu}\partial_\mu B_\nu(z)\Big)\,. \qquad (21)$$

$G^0(x)$ denotes the propagator for free massless fermions given by $G^0(x) = \frac{1}{2\pi}\frac{\gamma_\mu x_\mu}{x^2}$, and $\varepsilon_{\mu\nu}$ is the antisymmetric tensor. $D(x) = -\triangle^{-1}(x)$ denotes the propagator for massless bosons. Expressing h_μ in terms of h'_μ and using the fact that A_μ and h'_μ are transverse fields one ends up with

$$G(x,y;A,h') \;=\; \frac{1}{2\pi}\frac{1}{(x-y)^2}\begin{pmatrix} 0 & e^{-[\chi(x)-\chi(y)]}\,\overline{(\tilde{x}-\tilde{y})} \\ e^{+[\chi(x)-\chi(y)]}\,(\tilde{x}-\tilde{y}) & 0 \end{pmatrix}\,, \qquad (22)$$

where

$$\chi(x) \;:=\; \frac{\varepsilon_{\mu\nu}\partial_\mu}{\triangle}\left(\frac{e\pi}{\pi+gN}A_\nu + g^{1/2}h'_\nu\right) \quad,\quad \tilde{x} := x_1 + ix_2\,. \qquad (23)$$

As was discussed above the construction of the determinant requires a falloff condition for the external fields A_μ, h_μ, which rules out nonvanishing winding number. They also require a UV-cutoff which we impose as follows. The scalar field $\chi(x)$ at the single space-time point x will be replaced by the convolute

$$\chi(x) = \longrightarrow \int d^2\xi\,\chi(\xi)\,\delta_n(\xi-x) \;=:\; \Big(\chi,\delta_n(x)\Big)\,, \qquad (24)$$

where $\delta_n(x)$ denotes a δ-sequence peaked at x

$$\delta_n(\xi-x) := \int \frac{d^2p}{(2\pi)^2}\,e^{-\frac{|p|}{n}}\,e^{ip(\xi-x)}\,. \qquad (25)$$

Thus the propagator takes the form

$$G(x,y;A,h) = \frac{1}{2\pi}\frac{1}{(x-y)^2}\begin{pmatrix} 0 & e^{-\left(\chi,\delta_n(x)-\delta_n(y)\right)}\,\overline{(\tilde{x}-\tilde{y})} \\ e^{+\left(\chi,\delta_n(x)-\delta_n(y)\right)}\,(\tilde{x}-\tilde{y}) & 0 \end{pmatrix}\,. \qquad (26)$$

When one considers the limit $n\to\infty$ in the end, some of the operators will have to be multiplied with a wave function renormalization constant (see Eq. (42) below) diverging as $n\to\infty$.

Since the dependence of the propagator on the external fields A_μ and h'_μ is exponential, only Gaussian functional integrals are needed to solve the model.

4 Construction of a unique vacuum

It turns out that the expectation functional for the massless model constructed so far violates clustering, and thus the vacuum state is not unique. In [31] an explicit, mathematically rigorous definition of a new expectation functional $\langle..\rangle_0^\theta$ which clusters, was given for $g = 0$. We adapt this construction to $g > 0$ and compare the properties of $\langle..\rangle_0^\theta$ with the expected properties of the θ-vacuum of QCD.

4.1 Violation of clustering

To identify the operators that violate clustering, it is useful (see [31] for the $g = 0$ case) to start with an ansatz containing only the chiral densities $\overline{\psi}^{(b)} P_\pm \psi^{(b)}$, $(P_\pm := (1 \pm \gamma_5)/2)$ and discuss the effect of adding vector currents and other modifications later. Define

$$C(\tau) := C_1(\tau) - C_2 \tag{27}$$

where

$$C_1(\tau) := \Big\langle \prod_{b=1}^{N} \prod_{i=1}^{n_b} \overline{\psi}^{(b)}(x_i^{(b)} + \hat{\tau}) P_+ \psi^{(b)}(x_i^{(b)} + \hat{\tau}) \prod_{i=1}^{m_b} \overline{\psi}^{(b)}(y_i^{(b)} + \hat{\tau}) P_- \psi^{(b)}(y_i^{(b)} + \hat{\tau})$$

$$\times \prod_{i=1}^{n_b'} \overline{\psi}^{(b)}(x'^{(b)}_i) P_+ \psi^{(b)}(x'^{(b)}_i) \prod_{i=1}^{m_b'} \overline{\psi}^{(b)}(y'^{(b)}_i) P_- \psi^{(b)}(y'^{(b)}_i) \Big\rangle_0 \tag{28}$$

and

$$C_2 := \Big\langle \prod_{b=1}^{N} \prod_{i=1}^{n_b} \overline{\psi}^{(b)}(x_i^{(b)}) P_+ \psi^{(b)}(x_i^{(b)}) \prod_{i=1}^{m_b} \overline{\psi}^{(b)}(y_i^{(b)}) P_- \psi^{(b)}(y_i^{(b)}) \Big\rangle_0$$

$$\times \Big\langle \prod_{b=1}^{N} \prod_{i=1}^{n_b'} \overline{\psi}^{(b)}(x'^{(b)}_i) P_+ \psi^{(b)}(x'^{(b)}_i) \prod_{i=1}^{m_b'} \overline{\psi}^{(b)}(y'^{(b)}_i) P_- \psi^{(b)}(y'^{(b)}_i) \Big\rangle_0 . \tag{29}$$

$\hat{\tau}$ denotes the vector of length τ in 2-direction. Violation of the cluster property now manifests itself in a nonvanishing limit

$$\lim_{\tau \to \infty} C(\tau) =: C \neq 0 . \tag{30}$$

It will be obtained for certain n_b, m_b, n_b', m_b'.

$C_1(\tau)$ can be evaluated easily, since due to the exponential dependence of the fermion propagator on the external fields it factorizes into the expectation value for free massless fermions and an integral over the external fields

$$C_1(\tau) = C_1(\tau)_{free} \times I(\tau) . \tag{31}$$

Due to trace identities for the γ algebra $C_1(\tau)_{free}$ does not vanish only for (compare [31])

$$n_b + n_b' = m_b + m_b' , \quad b = 1, ...N . \tag{32}$$

Using Cauchy's identity (see e.g. [32]) the result was computed in [31]

$$C_1(\tau)_{free} = s \left(\frac{1}{2\pi}\right)^{2\sum_b (n_n + n_b)} \prod_{b=1}^{N} \prod_{i,j=1}^{n_b + n_b'} \left(w_i^{(b)} - z_j^{(b)}\right)^{-2} \prod_{1 \le i < j \le n_b + n_b'} \left(w_i^{(b)} - w_j^{(b)}\right)^2 \left(z_i^{(b)} - z_j^{(b)}\right)^2, \tag{33}$$

where s denotes a sign depending on the $n^{(b)}$, $m^{(b)}$. It is irrelevant for the following discussion. The sets $\{w_j^{(b)}\}, \{z_j^{(b)}\}$ for fixed flavor b are given by

$$\{w_j^{(b)}\}_{j=1}^{n_b+n_b'} := \{x_l^{(b)} + \hat{\tau}, x'^{(b)}_k \mid l = 1, ...n_b; \ k = 1, ...n_b'\},$$

$$\{z_j^{(b)}\}_{j=1}^{m_b+m_b'} := \{y_l^{(b)} + \hat{\tau}, y'^{(b)}_k \mid l = 1, ...m_b; \ k = 1, ...m_b'\}. \tag{34}$$

The integral over the external fields A_μ, h_μ can be read off from (26)

$$I(\tau) = \int d\mu_Q[A]d\mu_C[h'] \ \exp\left(-2\sum_{b=1}^N \left[\sum_{i=1}^{m_b}(\chi, \delta_n(x_i^{(b)} + \hat{\tau})) + \sum_{i=1}^{m_b'}(\chi, \delta_n(x'^{(b)}_i))\right]\right)$$

$$\times \ \exp\left(+2\sum_{b=1}^N \left[\sum_{i=1}^{n_b}(\chi, \delta_n(y_i^{(b)} + \hat{\tau})) + \sum_{i=1}^{n_b'}(\chi, \delta_n(y'^{(b)}_i))\right]\right). \tag{35}$$

The result of inserting the δ-sequence (25) and solving the Gaussian integrals can be written as

$$I(\tau) = \exp\left(\sum_{i,j=1}^M V_n(w_i - z_j) - \frac{1}{2}\sum_{i\neq j}^M V_n(w_i - w_j) - \frac{1}{2}\sum_{i\neq j}^M V_n(z_i - z_j)\right). \tag{36}$$

Again we introduced abbreviations for the involved space-time arguments given by

$$\{w_j\}_{j=1}^M := \{x_l^{(b)} + \hat{\tau}, x'^{(b)}_k \mid l = 1, ...n_b; \ k = 1, ...n_b'; \ b = 1, ...N\},$$

$$\{z_j\}_{j=1}^M := \{y_l^{(b)} + \hat{\tau}, y'^{(b)}_k \mid l = 1, ...m_b; \ k = 1, ...m_b'; \ b = 1, ...N\}. \tag{37}$$

Due to (32) both sets contain the same number $M := \sum(n_b + n_b') = \sum(m_b + m_b')$ of elements. The potential V_n showing up in (36) can easily be obtained from the covariances (18) and (25). It reads

$$V_n(x) = 4\int \frac{d^2p}{(2\pi)^2} e^{-2\frac{|p|}{n}} \frac{e^2\pi^2}{(\pi + gN)^2} \frac{1}{p^2 + e^2N/(\pi + gN)} \frac{1}{p^2}\left(1 - \cos(px)\right)$$

$$+ 4\int \frac{d^2p}{(2\pi)^2} e^{-2\frac{|p|}{n}} g \frac{\pi}{\pi + gN} \frac{1}{p^2}\left(1 - \cos(px)\right). \tag{38}$$

In both integrals the infrared problem is cured by the $\left(1 - \cos(px)\right)$ term. The first one even has no ultraviolet problem, and it can be solved after the limit $n \to \infty$ was taken. The other one has to be evaluated for finite n. One obtains (see [23] for the explicit computation of the integrals)

$$V_n(x) = \frac{2}{\pi} \frac{e^2\pi^2}{(\pi+gN)^2} \frac{\pi+gN}{e^2N}\left(\ln|x| + K_0\left(\sqrt{\frac{e^2N}{\pi+gN}}|x|\right) + \ln\left(\frac{1}{2}\sqrt{\frac{e^2N}{\pi+gN}}\right) + \gamma\right)$$

$$+ \frac{2}{\pi} g \frac{\pi}{\pi + gN}\left(\ln|x| + \ln\left(\frac{n}{4}\right) + O\left(\frac{1}{n}\right)\right) = \frac{1}{N}\ln|x| + \tilde{V}(x) + \frac{2\pi g}{\pi+gN}\ln\left(\frac{n}{4}\right) + O\left(\frac{1}{n}\right), \tag{39}$$

where we defined

$$\tilde{V}(x) := \frac{2\pi}{N(\pi + gN)}\left(K_0\left(\sqrt{\frac{e^2N}{\pi+gN}}|x|\right) + \ln\left(\frac{1}{2}\sqrt{\frac{e^2N}{\pi+gN}}\right) + \gamma\right). \tag{40}$$

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K_0 denotes the modified Bessel function and γ is the Euler constant. Thus one ends up with

$$I(\tau) = \left(\frac{n}{4}\right)^{\frac{\pi g}{\pi + gN} 2M} e^{O(\frac{1}{n})} \times \exp\left(\sum_{i,j=1}^{M} \tilde{V}(w_i - z_j) - \frac{1}{2}\sum_{i\neq j}^{M} \tilde{V}(w_i - w_j) - \frac{1}{2}\sum_{i\neq j}^{M} \tilde{V}(z_i - z_j)\right)$$

$$\times \prod_{i,j=1}^{M} \left(w_i - z_j\right)^2 \prod_{i<j}^{M} \left(w_i - w_j\right)^{-2} \left(z_i - z_j\right)^{-2} . \tag{41}$$

As announced, the factor diverging with n, the index of the δ-sequence (25) can be absorbed in a wave function renormalization constant Z for the chiral densities $\overline{\psi}^{(b)} P_\pm \psi^{(b)}$

$$Z := \left(\frac{4}{n}\right)^{\frac{\pi g}{\pi + gN}} . \tag{42}$$

Putting together (33) and (41) one now can discuss the large-τ behaviour of $C_1(\tau)$. $\tilde{V}(x)$ showing up in the expression (41) for $I(\tau)$ depends on x only via the modified Bessel function K_0. Since K_0 approaches zero exponentially, $\exp(\tilde{V}(x))$ goes to a constant for large τ, and the only remaining τ dependence of $I(\tau)$ for $\tau \to \infty$ must come from the rational function of the space-time arguments. Combining this with $C_1(\tau)_{free}$ given in (33) one obtains

$$C_1(\tau) \propto \left(\frac{1}{\tau^2}\right)^E \left(1 + O\left(\frac{1}{\tau}\right)\right) , \tag{43}$$

where the exponent E is given by

$$E = \sum_{a=1}^{N} (n_a - m_a)(n_a - m_a) - \frac{1}{N}\sum_{a,b=1}^{N} (n_a - m_a)(n_b - m_b) = \frac{1}{N}\sum_{a,b=1}^{N} (n_a - m_a)R_{ab}(n_b - m_b) . \tag{44}$$

The matrix R is defined as

$$R_{ab} =: \delta_{ab} N - 1 . \tag{45}$$

The corresponding eigenvalue problem can be solved easily. One finds one eigenvalue 0, and $N-1$ eigenvalues N. The eigenvector x^0 with the eigenvalue 0 is given by $x^0 = 1/\sqrt{N}(1,1,...1)^T$. Hence the quadratic form $x^T R x$ is positive semidefinite, and vanishes only if x is a multiple of x^0. This implies that the exponent E is nonnegative and vanishes only for

$$n_b - m_b = m_b' - n_b' = n \qquad \forall\, b = 1, 2, ... N , \qquad n \in \mathbb{Z} . \tag{46}$$

All those possibilities lead to a nonvanishing limit $C_1(\infty) := \lim_{\tau\to\infty} C_1(\tau)$. In some of the cases $C_1(\infty)$ will be cancelled by C_2 which is given by (29). Using the trace identities for the γ-algebra again (compare [31]), one finds that C_2 does not vanish only for

$$n_b = m_b \quad \text{and} \quad n_b' = m_b' , \quad b = 1, ...N . \tag{47}$$

In these cases C_2 then cancels $C_1(\infty)$ and the operators cluster. Thus violation of clustering in $C(\tau)$ is expressed in the condition

$$n_b - m_b = m_b' - n_b' = n , \quad \forall\, b = 1, 2, ..N , \quad n \in \mathbb{Z} \setminus \{0\} . \tag{48}$$

As was discussed in [31] for the model without Thirring term, the picture does not change when one inserts vector currents as well. The only ingredient used for this result was the exponential dependence of the fermion propagator on the external fields, and the fact that the matrix R defined in (45) is positive semidefinite. The same properties also hold for $g > 0$ and the result of [31] can be taken over to the Schwinger-Thirring model.

In [31] also the symmetry properties of the operators that violate clustering were discussed. Again the result can be taken over. Operators that violate clustering are singlets under $U(1)_V \times SU(N)_L \times SU(N)_R$, but transform nontrivially under $U(1)_A$.

4.2 Definition of the clustering expectation functional $\langle ..\rangle_0^\theta$

The symmetry properties discussed in the last section can be used to decompose the vacuum functional into clustering states. We adapt the prescription given in [31], involving a limiting process which mimics the cluster procedure (30) to the case $g > 0$.

Under a $U(1)_A$ transformation

$$\psi^{(b)} \longrightarrow e^{i\omega\gamma_5}\psi^{(b)} , \tag{49}$$

an arbitrary monomial $\mathcal{B}$ of the fields transforms as

$$\mathcal{B}(\{x\}) \longrightarrow e^{iQ_5(\mathcal{B})\omega} \; \mathcal{B}(\{x\}) \quad , \quad Q_5(\mathcal{B}) \in \mathbb{Z} . \tag{50}$$

The new states $\langle ..\rangle_0^\theta$ labeled by a parameter $\theta \in [-\pi, \pi]$ are defined as follows

$$\langle \; \mathcal{B}(\{x\}) \; \rangle_0^\theta \;\; := \;\; e^{i\theta \frac{Q_5(\mathcal{B})}{2N}} \lim_{\tau\to\infty} \langle \mathcal{U}_\tau(\mathcal{B}) \; \mathcal{B}(\{x\})\rangle_0 . \tag{51}$$

The new expectation value of an operator $\mathcal{B}$ is obtained by correlating $\mathcal{B}$ with a test operator $\mathcal{U}_\tau$ using the old expectation functional and shifting $\mathcal{U}_\tau$ to timelike infinity. The result is multiplied with a phase which depends on the chiral charge $Q_5(\mathcal{B})$. The test operators $\mathcal{U}_\tau(\mathcal{B})$ which also depend on $Q_5(\mathcal{B})$ are defined as

$$\mathcal{U}_\tau(\mathcal{B}) := \begin{cases} \mathcal{N}^{(n)}(\{y\}) \prod_{i=1}^n \prod_{b=1}^N \overline{\psi}^{(b)}(y_i^{(b)}+\hat{\tau})P_\mp\psi(y_i^{(b)}+\hat{\tau}) & \text{for} \quad Q_5(\mathcal{B}) = \pm 2nN, \; n \geq 1 , \\[2mm] 1 \;\; \text{otherwise} . \end{cases} \tag{52}$$

Up to the requirement of being nondegenerate, the arguments $\{y_i^{(b)}\}$ are arbitrary. The normalizing factor $\mathcal{N}^{(n)}(\{y\})$ is defined such that

$$\lim_{\tau'\to\infty} \langle \mathcal{U}_{\tau'}(\mathcal{B}^\dagger)\,\mathcal{U}_\tau(\mathcal{B})\rangle_0 \;=\; 1 . \tag{53}$$

It can be read off from (33), (41)

$$\mathcal{N}^{(n)}(\{y\}) = \left(\frac{1}{2\pi}\right)^{-Nn}\left[\frac{e^2 N}{4(\pi+gN)}e^{2\gamma}\right]^{-\frac{\pi}{\pi+gN}\frac{Nn^2}{2}}$$

$$\times \prod_{b=1}^N \prod_{1\leq i<j\leq n}\left(y_i^{(b)}-y_j^{(b)}\right)^{-2}\exp\left(\frac{1}{2}\sum_{c,d=1}^N\sum_{k,l=1}^n(1-\delta_{cd}\delta_{kl})\tilde{\tilde{V}}\left(y_k^{(c)}-y_l^{(d)}\right)\right), \tag{54}$$

where

$$\tilde{\tilde{V}}(x) \;\; := \;\; \frac{1}{N}\ln(x^2) + \tilde{V}(x) . \tag{55}$$

In [31] the following theorem was shown to hold

Theorem 1 :
i) The cluster decomposition property holds for $\langle \; .. \; \rangle_0^\theta$.
ii) The state $\langle ..\rangle_0$ constructed initially is recovered by averaging over θ

$$\langle \; .. \; \rangle = \frac{1}{2\pi}\int_{-\pi}^\pi \langle \; .. \; \rangle_0^\theta \, d\theta . \tag{56}$$

The theorem establishes that our prescription (51) indeed ensures clustering and thus leads to a unique vacuum. Thus $\langle ..\rangle_0^\theta$ is exactly what is hoped to have been obtained by the expression (1) for the θ-vacuum of QCD. The prescription (51) is mathematically rigorous, whereas (1) has a more formal character. Nevertheless it is interesting to compare the properties of the two constructs. This opens a series of lessons for the topics discussed in the introduction that we will draw from the model.

Lesson 1 : (On the vacuum structure of QED_2.)

The newly defined vacuum functional $\langle ..\rangle_0^\theta$ gives rise to the chirality selection rule

$$\langle \mathcal{B}\rangle_0^\theta \;=\; 0 \quad \text{unless} \quad Q_5(\mathcal{B}) \;=\; 2N\,n \quad, n \in \mathbb{Z}\,. \tag{57}$$

In particular operators $\mathcal{B}$ with nonvanishing $\langle \mathcal{B}\rangle_0^\theta$ have to be singlets under $U(1)_V \times SU(N)_L \times SU(N)_R$. The property (57) can be seen to hold from the discussion of the behaviour of $C_1(\tau)$ in Section 4.1. Eq. (46) immediately leads to (57).

Thus the chiral condensate $\langle \overline{\psi}\psi \rangle$ for N=1 generalizes in the case of several flavors to e.g.

$$\langle \prod_{b=1}^{N} \overline{\psi}^{(b)} P_\pm \psi^{(b)}\rangle_0^\theta \;\neq\; 0\,. \tag{58}$$

This can be understood easily using Coleman's theorem [33], allowing no spontaneous breaking of $SU(N)_A$ in two dimensions (only $U(1)_A$ is explicitly broken by the anomaly). Thus the expectation values of all operators that do not transform trivially under $SU(N)_A$ (such as $\langle \overline{\psi}^{(b)}\psi^{(b)}\rangle$ for some flavor b) have to vanish. The operator in (58) does not fall into this class and acquires a nonvanishing expectation value. Chiral condensates of the type (58) were also discussed in [34], [35], where they were related to classical gauge field configurations with winding number. This is an approach which we avoid here. It has been demonstrated (see [7]-[14], [36] for some recent work) that putting the model on a finite torus leads to breaking of the chiral symmetry and to a nonvanishing condensate $\langle \overline{\psi}^{(b)}\psi^{(b)}\rangle$.

A selection rule equivalent to (57) can formally be obtained for QCD (see e.g. the review article [37]) from the naive construction [2, 3] of the θ-vacuum of QCD. Anyway this result is too naive, since the chiral symmetry is believed to be broken spontaneously in QCD (There is only one instance of a model relevant for QCD, where chiral symmetry breaking is proven [38].). In fact $\langle \theta|\overline{u}u|\theta\rangle \neq 0$ which is the manifestation of chiral symmetry breaking, is one of the main assumptions of current algebra (see [39] for a review).

5   Bosonization and the GSG

In this section it will be shown that expectation values of chiral densities and certain currents within the $\langle .. \rangle_0^\theta$ vacuum functional can be bosonized. For the massive model this gives rise to a generalized Sine-Gordon model (GSG).

5.1 Evaluation of a generating functional

In order to establish the bosonization we evaluate a generating functional where the operators $\overline{\psi}^{(b)} P_\pm \psi^{(b)}$, $j_\mu^{(b)}$ which we want to bosonize enter. It reads

$$E(n_b, m_b; a^{(b)}) := \Big\langle \prod_{b=1}^{N} \prod_{i=1}^{n_b} \overline{\psi}^{(b)}(x_i^{(b)}) P_+ \psi^{(b)}(x_i^{(b)}) \prod_{j=1}^{m_b} \overline{\psi}^{(b)}(y_j^{(b)}) P_- \psi^{(b)}(y_j^{(b)}) e^{ie \sum_{c=1}^{N} (a_\mu^{(c)}, j_\mu^{(c)})} \Big\rangle_0^\theta .$$

(59)

Obviously this is a simple modification of expectation values already considered in the last section. Only a generating exponential has been added where the vector currents $j_\mu^{(b)}$ for the various flavors b, couple to external sources $a_\mu^{(b)}$. Since those sources couple in the same way as the gauge and the auxiliary field, they can be included into the fermion determinant giving rise to (see (14))

$$\prod_{b=1}^{N} \exp\left(-\frac{1}{2\pi} \big\| eA^T + g^{1/2} h^T + ea^{(b)} \big\|_2^2 \right) .$$

(60)

Thus we simply obtain an extra factor in the functional integral over A_μ and h_μ (respectively h'_μ).

To work out the dependence on the sources $a_\mu^{(b)}$ we first consider the case

$$n_b - m_b = 0 \quad , \quad b = 1, 2, \dots, N ,$$

(61)

where the evaluation of $\langle .. \rangle_0^\theta$ is remarkably simple, i.e. it coincides with the naive expectation value (compare (51)). Again the expression factorizes

$$E(n_b, n_b; a^{(b)}) := I(n_b, n_b; a^{(b)}) \times E_{free}(n_b, n_b) ,$$

(62)

and the factor from the functional integral reads

$$I(n_b, n_b; a^{(b)}) = \int d\mu_Q[A] d\mu_C[h'] \ \exp\left(-2 \sum_{b=1}^{N} \sum_{j=1}^{n_b} \Big(\chi^{(b)}, \delta_n(x_j^{(b)}) - \delta_n(y_j^{(b)}) \Big) \right)$$

$$\times \exp\left(-\frac{e^2}{\pi + gN} \Big(A, T \sum_{b=1}^{N} a^{(b)} \Big) - \frac{e\sqrt{g}}{\pi} \Big(h', T \sum_{b=1}^{N} a^{(b)} \Big) - \frac{e^2}{2\pi} \sum_{b=1}^{N} \Big(a^{(b)}, T a^{(b)} \Big) \right) .$$

(63)

The first exponent consists of a sum over the inner products of the δ-sequences centered at the various space-time arguments with the gauge and the auxiliary fields which enter $\chi^{(b)}$. The extra flavor superscript in $\chi^{(b)}$ is due to the fact, that the external sources $a_\mu^{(b)}$ were included into the fermion action and thus show up also in the propagator. Hence the definition (23) for χ has to be generalized to

$$\chi^{(b)} := \frac{\varepsilon_{\mu\nu}\partial_\mu}{\triangle} \left(\frac{e\pi}{\pi + gN} A_\nu + g^{1/2} h'_\nu \right) + e \frac{\varepsilon_{\mu\nu}\partial_\mu}{\triangle} a_\nu^{(b)} .$$

(64)

In the propagator (26), χ has to be replaced by $\chi^{(b)}$ which in turn immediately leads to the first exponent in (63). The exponent in the second line of (63) stems from the determinant (60) and contains only a term linear in A_μ and h'_μ, since the quadratic term is already included in the Gaussian measure.

The functional integral (63) can now be evaluated. The result is an exponential which mixes the sources and the δ-sequences. If the terms quadratic in the sources $a^{(b)}$ are collected, one finds that the corresponding term can be written as

$$-\frac{e^2}{2\pi} \sum_{b,c=1}^{N} \left(\varepsilon_{\mu\nu}\partial_\mu a_\nu^{(b)}, M_{bc}\, \varepsilon_{\rho\sigma}\partial_\rho a_\sigma^{(c')} \right), \tag{65}$$

where the covariance M is given by

$$M = \frac{1}{-\triangle + e^2 N/(\pi + gN)} \left[\frac{\pi}{\pi+gN}\mathbb{1} + \frac{g}{\pi+gN}R \right] + \frac{e^2}{\pi+gN}\frac{-1}{\triangle}\frac{1}{-\triangle + e^2 N/(\pi+gN)}R\,. \tag{66}$$

The numerical matrix R was introduced in (45). As already discussed the eigenvalues of R are 0, and N, where the latter is $N-1$-fold degenerate. Diagonalization of R thus leads to

$$URU^T = \text{diag}(0, N, N, \dots N)\,, \tag{67}$$

where the orthogonal matrix U^T is given by

$$U^T := \left(c^{(1)} \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ . \\ . \\ 1 \end{bmatrix}, c^{(2)} \begin{bmatrix} 1 \\ . \\ . \\ . \\ . \\ 1 \\ -(N-1) \end{bmatrix}, c^{(3)} \begin{bmatrix} 1 \\ . \\ . \\ . \\ 1 \\ -(N-2) \\ 0 \end{bmatrix} \dots c^{(N)} \begin{bmatrix} 1 \\ -1 \\ 0 \\ . \\ . \\ 0 \\ 0 \end{bmatrix} \right). \tag{68}$$

The normalization constants $c^{(I)}$ are given by

$$c^{(1)} := \frac{1}{\sqrt{N}}\,, \quad c^{(2)} := \frac{1}{\sqrt{N-1+(N-1)^2}}\,,$$

$$c^{(3)} := \frac{1}{\sqrt{N-2+(N-2)^2}}\,, \quad \dots, \dots, \quad c^{(N)} := \frac{1}{\sqrt{2}}\,. \tag{69}$$

(67) allows to express the quadratic form (66) in terms of a covariance K diagonal in flavor

$$K := U\, M\, U^T \iff M = U^T\, K\, U\,. \tag{70}$$

Explicitly K is given by

$$K = \begin{pmatrix} \frac{\pi}{\pi+gN}\frac{1}{-\triangle + e^2 N/(\pi+gN)} & & & \\ & \frac{1}{-\triangle} & & \\ & & . & \\ & & & . \\ & & & \frac{1}{-\triangle} \end{pmatrix}. \tag{71}$$

Obviously the covariance K describes one massive and $N-1$ massless particles. Finally the term quadratic in the sources reads

$$-\frac{e^2}{2\pi}\left(\varepsilon_{\mu\nu}\partial_\mu U a_\nu, K\, \varepsilon_{\rho\sigma}\partial_\rho U a_\sigma \right), \tag{72}$$

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where matrix notation in flavor space was used. Eq. (72) suggests to define new sources $A_\mu^{(I)}$ that are linear combinations of the $a_\mu^{(b)}$

$$A_\mu^{(I)} := \sum_{b=1}^{N} U_{Ib} a_\mu^{(b)} \quad \xrightarrow{\underline{U^T = U^{-1}}} \quad a_\mu^{(b)} = \sum_{I=1}^{N} U_{Ib} A_\mu^{(I)} . \qquad (73)$$

Rewriting the coupling term in $E(n_b, n_b; a^{(b)})$ allows the identification of the currents $J_\mu^{(I)}$ that couple to the new sources $A_\mu^{(I)}$

$$\sum_{b=1}^{N} \left(a_\mu^{(b)}, j_\mu^{(b)} \right) = \sum_{I=1}^{N} \sum_{b=1}^{N} \left(U_{Ib} A_\mu^{(I)}, j_\mu^{(b)} \right) := \sum_{I=1}^{N} \left(A_\mu^{(I)}, J_\mu^{(I)} \right) , \qquad (74)$$

where we defined

$$J_\mu^{(I)} := \sum_{b=1}^{N} U_{Ib} j_\mu^{(b)} \qquad I = 1, 2, ... N . \qquad (75)$$

Inspecting the explicit form of the matrix U defined in (68) one can express the new currents also as

$$J_\mu^{(I)} := \sum_{b,c=1}^{N} \overline{\psi}^{(b)} \gamma_\mu H_{bc}^{(I)} \psi^{(c)} \qquad (76)$$

where the N $\times$ N matrices $H^{(I)}$ are generators of a Cartan subalgebra of $U(N)_{flavor}$, and thus the currents (75) will be referred to as Cartan currents in the following. Using (74) one can define a new generating functional $E(n_b, m_b; A^{(I)})$ which now contains the currents $J_\mu^{(I)}$ coupled to the sources $A_\mu^{(I)}$

$$E(n_b, m_b; A^{(I)}) := \Big\langle \prod_{b=1}^{N} \prod_{i=1}^{n_b} \overline{\psi}^{(b)}(x_i^{(b)}) P_+ \psi^{(b)}(x_i^{(b)}) \prod_{j=1}^{m_b} \overline{\psi}^{(b)}(y_j^{(b)}) P_- \psi^{(b)}(y_j^{(b)}) e^{ie \sum_{I=1}^{N} (A_\mu^{(I)}, J_\mu^{(I)})} \Big\rangle_0^\theta . \qquad (77)$$

Using (65) and (33) for $E_{free}(n_b, n_b)$ showing up in (62) one obtains (still $n_b = m_b$)

$$E(n_b, n_b; A^{(I)}) = \left(\frac{1}{2\pi} \right)^{2 \sum_b n_b} \times \exp \left(-\frac{e^2}{2\pi} \sum_{I=1}^{N} \left(\varepsilon_{\mu\nu} \partial_\mu A_\nu^{(I)}, K_{II} \varepsilon_{\rho\sigma} \partial_\rho A_\sigma^{(I)} \right) \right)$$

$$\times \prod_{b=1}^{N} \prod_{j=1}^{n_b} \exp \left(2e \sum_{I=1}^{N} \left(\varepsilon_{\mu\nu} \partial_\mu A_\nu^{(I)}, K_{II} U_{Ib} \left[\delta_n(x_j^{(b)}) - \delta_n(y_j^{(b)}) \right] \right) \right) \times \rho_n(\{x_j^{(b)}\}, \{y_j^{(b)}\}) . \qquad (78)$$

$\rho_n(\{x_j^{(b)}\}, \{y_j^{(b)}\})$ denotes the factor that depends on the space-time arguments. Furthermore it still depends on n, the index of the δ-sequences (25). The wave function renormalization (42) for the chiral densities has to be applied before the limit $n \to \infty$ is taken. We will quote the explicit form of ρ_∞ later.

The result (78) now can be generalized to

$$n_b - m_b = l \qquad l \in \mathbb{Z} \qquad b = 1, 2, ... , N , \qquad (79)$$

which covers all cases where the expectation functional $\langle .. \rangle_0^\theta$ gives nonvanishing results for $E(n_b, m_b; A^{(I)})$. The result can be obtained easily by following the argumentation given in the last section. In fact the term quadratic in the sources is not affected by the θ-prescription,

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and $\rho_n(\{x_j^{(b)}\}, \{y_j^{(b)}\})$ can be read off from (33), (41) immediately. Only the term that mixes the sources with the space time arguments of the chiral densities has to be generalized, but this is straightforward. One ends up with

$$
E(n_b, m_b; A^{(b)}) \;=\; \exp\left(-\frac{e^2}{2\pi} \sum_{I=1}^{N} \left(\varepsilon_{\mu\nu}\partial_\mu A_\nu^{(I)}, K_{II}\, \varepsilon_{\rho\sigma}\partial_\rho A_\sigma^{(I)}\right)\right)
$$

$$
\times \prod_{b=1}^{N} \prod_{j=1}^{n_b} \exp\left(2e \sum_{I=1}^{N} \left(\varepsilon_{\mu\nu}\partial_\mu A_\nu^{(I)}, K_{II}\, U_{Ib}\, \delta(x_j^{(b)})\right)\right)
$$

$$
\times \prod_{b=1}^{N} \prod_{j=1}^{m_b} \exp\left(-2e \sum_{I=1}^{N} \left(\varepsilon_{\mu\nu}\partial_\mu A_\nu^{(I)}, K_{II}\, U_{Ib}\, \delta(y_j^{(b)})\right)\right) \times \rho_\infty(\{x_j^{(b)}\}, \{y_j^{(b)}\})\,, \qquad (80)
$$

where $\rho_\infty(\{x_j^{(b)}\}, \{y_j^{(b)}\})$ is given by (the limit $n \to \infty$ and the wave function renormalization (42) have already been performed)

$$
\rho_\infty(\{x_j^{(b)}\}, \{y_j^{(b)}\}) \;=\;
$$

$$
h(n_b, m_b) \left(\frac{1}{2\pi}\right)^{\sum_b(n_b+m_b)} e^{i\frac{\theta}{N}\sum_b(n_b-m_b)} \times \left(e\sqrt{\frac{N}{\pi+gN}}\,\frac{e^\gamma}{2}\right)^{\frac{\pi}{\pi+gN}\frac{1}{N}\left(\sum_b(n_b-m_b)\right)^2}
$$

$$
\times \exp\left(\sum_{b,c=1}^{N} \sum_{j=1}^{n_b} \sum_{l=1}^{m_{b'}} \tilde{\tilde{V}}(x_j^{(b)} - y_l^{(c)})\right)
$$

$$
\times \exp\left(-\frac{1}{2}\sum_{b,c=1}^{N}\left(\sum_{j=1}^{n_b}\sum_{l}^{n_c}\left(1-\delta_{bc}\delta_{jl}\right)\tilde{\tilde{V}}(x_j^{(b)} - x_l^{(c)}) + \sum_{j=1}^{m_b}\sum_{l=1}^{m_c}\left(1-\delta_{bc}\delta_{jl}\right)\tilde{\tilde{V}}(y_j^{(b)} - y_l^{(c)})\right)\right)
$$

$$
\times \exp\left(-\sum_{b=1}^{N}\sum_{j=1}^{n_b}\sum_{l=1}^{m_b}\ln(x_j^{(b)} - y_l^{(b)})^2\right)
$$

$$
\times \exp\left(\frac{1}{2}\sum_{b=1}^{N}\left(\sum_{j=1}^{n_b}\sum_{l=1}^{n_b}(1-\delta_{jl})\ln(x_j^{(b)} - x_l^{(b)})^2 + \sum_{j=1}^{m_b}\sum_{l=1}^{m_b}(1-\delta_{jl})\ln(y_j^{(b)} - y_l^{(b)})^2\right)\right)\,. \qquad (81)
$$

The factor $h(n_b, m_b)$ is defined as

$$
h(n_b, m_b) \;:=\; \sum_{l=-\infty}^{+\infty}\prod_{b=1}^{N}\delta_{n_b-m_b,l}\,. \qquad (82)
$$

It is equal to one whenever the prescription (51) allows a nonvanishing result for $E(n_b, m_b; A^{(b)})$, otherwise it is zero. $\tilde{\tilde{V}}$ is given by (55). Expression (80) can now be used to identify the correct bosonization.

5.2 Bosonization prescription

Bosonization means that the generating functional $E(n_b, m_b; A^{(b)})$ can also be obtained by computing the vacuum expectation value of a certain functional $\mathcal{F}(n_b, m_b; A^{(b)}; \Phi^{(I)})$ depending on fields $\Phi^{(I)}$ in a bosonic theory. Every operator that was used to define $E(n_b, m_b; A^{(b)})$ will have a transcription in terms of the $\Phi^{(I)}$ which then enters $\mathcal{F}(n_b, m_b; A^{(b)}; \Phi^{(I)})$. Inspecting (80) makes it plausible to try it with Gaussian fields $\Phi^{(I)}$ with some covariances $Q^{(I)}$ which are related to the K_{II} (see (71)). Thus one can express the idea of bosonization in the following formula

$$E(n_b, m_b; A^{(b)}) = \left\langle \mathcal{F}(n_b, m_b; A^{(b)}; \Phi^{(I)}) \right\rangle_{\{Q^{(I)}\}}, \tag{83}$$

where $\langle .. \rangle_{\{Q^{(I)}\}}$ denotes expectation values for the fields $\Phi^{(I)}$ with respect to the covariances $Q^{(I)}$. Two steps have to be done. First define an appropriate covariance $Q^{(I)}$ and then establish the correct transcription of the fermionic operators into bosonic ones.

The definition of the $Q^{(I)}$ is rather simple. We define

$$Q^{(1)} := \frac{1}{-\triangle + e^2 N/(\pi + gN)} = \frac{\pi + gN}{\pi} K_{11}, \tag{84}$$

and

$$Q^{(I)} := \frac{1}{-\triangle} = K_{II} \quad I = 2, ...N. \tag{85}$$

Thus the $Q^{(I)}$ are just the canonically normalized K_{II}. The term in (80) which is quadratic in the sources $A^{(b)}$ then implies the following prescription for the bosonization of the Cartan currents (see also [35] for the $g = 0$ case)

$$J_\nu^{(I)}(x) \longleftrightarrow \begin{cases} -\frac{1}{\sqrt{\pi + gN}} \, \varepsilon_{\mu\nu} \partial_\mu \Phi^{(1)}(x) & I = 1 \\ \\ -\frac{1}{\sqrt{\pi}} \, \varepsilon_{\mu\nu} \partial_\mu \Phi^{(I)}(x) & I = 2, ... N. \end{cases} \tag{86}$$

With this choice the term linear in $A_\mu^{(b)}$ already fixes the structure of the transcription of the chiral densities to (see also [47, 48])

$$\overline{\psi}^{(b)}(x) P_\pm \psi^{(b)}(x) \longleftrightarrow$$

$$\frac{1}{2\pi} c^{(b)} : e^{\mp i 2\sqrt{\pi} \sqrt{\frac{\pi}{\pi + gN}} U_{1b} \Phi^{(1)}(x)} :_{M^{(1)}} \prod_{I=2}^{N} : e^{\mp i 2\sqrt{\pi} U_{Ib} \Phi^{(I)}(x)} :_{M^{(I)}} e^{\pm i \frac{\theta}{N}}, \tag{87}$$

as can be seen from the exponents in (80) linear in the sources. Here $: .. :_{M^{(I)}}$ denotes normal ordering with respect to mass $M^{(I)}$ (see e.g. [40]). Those normal ordering masses as well as the real numbers $c^{(b)}$ are free parameters that will be fixed later. Inserting the prescriptions (86), (87) into the definition of $E(n_b, m_b; A^{(b)})$ one obtains

$$E(n_b, m_b; A^{(b)}) \longleftrightarrow \left(\frac{1}{2\pi}\right)^{\sum_b (n_b + m_b)} e^{i \frac{\theta}{N} \sum_b (n_b - m_b)} \prod_{b=1}^{N} \left(c(b)\right)^{n_b + m_b}$$

$$\left\langle \prod_{b=1}^{N} \prod_{j=1}^{n_b} \left[: e^{-i 2\sqrt{\pi} \sqrt{\frac{\pi}{\pi + gN}} U_{1b} \Phi^{(1)}(x_j^{(b)})} :_{M^{(1)}} \prod_{I=2}^{N} : e^{-i 2\sqrt{\pi} U_{Ib} \Phi^{(I)}(x_j^{(b)})} :_{M^{(I)}} \right] \right.$$

$$\times \prod_{b=1}^{N}\prod_{j=1}^{m_b}\left[: e^{+i2\sqrt{\pi}\sqrt{\frac{\pi}{\pi+gN}}U_{1b}\Phi^{(1)}(y_j^{(b)})} :_{M^{(1)}} \prod_{I=2}^{N} : e^{+i2\sqrt{\pi}U_{Ib}\Phi^{(I)}(y_j^{(b)})} :_{M^{(I)}}\right]$$

$$\times \exp\left(-\frac{ie}{\sqrt{\pi+gN}}\left(A_\mu^{(1)}, \varepsilon_{\nu\mu}\partial_\nu\Phi^{(1)}\right) - \frac{ie}{\sqrt{\pi}}\sum_{I=2}^{N}\left(A_\mu^{(I)}, \varepsilon_{\nu\mu}\partial_\nu\Phi^{(I)}\right)\right)\Bigg\rangle_{\{Q^{(I)}\}}. \tag{88}$$

The Gaussian integrals can be solved easily since they factorize with respect to the $\Phi^{(I)}$. One then obtains for the right hand side of the last equation

$$\exp\left(-\frac{e^2}{2\pi}\left(\varepsilon_{\mu\nu}\partial_\mu A_\nu^{(1)}, \frac{\pi}{\pi+gN}Q^{(1)}\varepsilon_{\rho\sigma}\partial_\rho A_\sigma^{(1)}\right) - \frac{e^2}{2\pi}\sum_{I=2}^{N}\left(\varepsilon_{\mu\nu}\partial_\mu A_\nu^{(I)}, Q^{(I)}\varepsilon_{\rho\sigma}\partial_\rho A_\sigma^{(I)}\right)\right)$$

$$\times \prod_{b=1}^{N}\prod_{j=1}^{n_b}\exp\left(+2eU_{1b}\left(\varepsilon_{\mu\nu}\partial_\mu A_\nu^{(1)}, \frac{\pi}{\pi+gN}Q^{(1)}\delta(x_j^{(b)})\right)\right)$$

$$\times \prod_{b=1}^{N}\prod_{j=1}^{n_b}\exp\left(+2e\sum_{I=2}^{N}U_{Ib}\left(\varepsilon_{\mu\nu}\partial_\mu A_\nu^{(I)}, Q^{(I)}\delta(x_j^{(b)})\right)\right)$$

$$\times \prod_{b=1}^{N}\prod_{j=1}^{m_b}\exp\left(-2eU_{1b}\left(\varepsilon_{\mu\nu}\partial_\mu A_\nu^{(1)}, \frac{\pi}{\pi+gN}Q^{(1)}\delta(y_j^{(b)})\right)\right)$$

$$\times \prod_{b=1}^{N}\prod_{j=1}^{m_b}\exp\left(-2e\sum_{I=2}^{N}U_{Ib}\left(\varepsilon_{\mu\nu}\partial_\mu A_\nu^{(I)}, Q^{(I)}\delta(y_j^{(b)})\right)\right) \times \rho_B(\{x_j^{(b)}\}, \{y_j^{(b)}\}). \tag{89}$$

Comparing (80) and (89) shows immediately that the terms quadratic and linear in the sources $A^{(I)}$ come out correctly. Thus there is left to show

$$\rho_B(\{x_j^{(b)}\}, \{y_j^{(b)}\}) = \rho_\infty(\{x_j^{(b)}\}, \{y_j^{(b)}\}), \tag{90}$$

where $\rho_\infty(\{x_j^{(b)}\}, \{y_j^{(b)}\})$ is given by (81). As mentioned before, the integral over the $\Phi^{(I)}$ factorizes such that $\rho_B(\{x_j^{(b)}\}, \{y_j^{(b)}\})$ reads

$$\rho_B(\{x_j^{(b)}\}, \{y_j^{(b)}\}) = \left(\frac{1}{2\pi}\right)^{\sum_b(n_b+m_b)} e^{i\frac{\theta}{N}\sum_b(n_b-m_b)} \prod_{b=1}^{N}\left(c^{(b)}\right)^{n_b+m_b}$$

$$\times\left\langle \prod_{b=1}^{N}\prod_{j=1}^{n_b} : e^{-i2\sqrt{\pi}\sqrt{\frac{\pi}{\pi+gN}}U_{1b}\Phi^{(1)}(x_j^{(b)})} :_{M^{(1)}} \prod_{l=1}^{m_b} : e^{+i2\sqrt{\pi}\sqrt{\frac{\pi}{\pi+gN}}U_{1b}\Phi^{(1)}(y_l^{(b)})} :_{M^{(1)}} \right\rangle_{Q^{(1)}}$$

$$\times \prod_{I=2}^{N}\left\langle \prod_{b=1}^{N}\prod_{j=1}^{n_b} : e^{-i2\sqrt{\pi}U_{Ib}\Phi^{(I)}(x_j^{(b)})} :_{M^{(I)}} \prod_{l=1}^{m_b} : e^{+i2\sqrt{\pi}U_{Ib}\Phi^{(I)}(y_l^{(b)})} :_{M^{(I)}} \right\rangle_{Q^{(I)}} \tag{91}$$

The expectation values of the normal ordered exponentials are rather simple to evaluate. One only has to take care of the neutrality condition (see e.g. [41]) which has to be fulfilled for normal ordered exponentials of a massless scalar field Φ

$$\lim_{\mu\to0}\langle\prod_{j=1}^{n} : e^{i\Phi(t_j)} :_M\rangle_{C^\mu} = \begin{cases} e^{+\frac{1}{2}\sum_{i=1}^{n}\left(t_i, C^M t_i\right)}e^{-\frac{1}{2}\sum_{i\neq j}\left(t_i, C^0 t_j\right)} & \text{for } \sum_{j=1}^{n} q_j = 0 \\ \\ 0 & \text{for } \sum_{j=1}^{n} q_j \neq 0 \end{cases}, \tag{92}$$

where

$$C^m := \left(-\triangle + m^2\right)^{-1}, \ m = \mu, M \quad ; \quad C^0(x) := \left(-\triangle\right)^{-1}(x) = -\frac{1}{4\pi}\left(\ln(x^2) + 2\gamma - \ln 4\right),$$
$$(93)$$

and $q_j := \int d^2x \, t_j(x)$. In order to obtain a nonvanishing ρ_B the neutrality condition implies

$$\sum_{b=1}^{N} U_{Ib}(n_b - m_b) \overset{!}{=} 0 \quad \forall \ I = 2, 3, ...N \ .$$
$$(94)$$

Interpreting the lines of U_{Ib} as vectors $\vec{r}^{\,(I)}$ (see (68) for the definition of U), the condition reads

$$(\vec{n} - \vec{m}) \cdot \vec{r}^{\,(I)} \overset{!}{=} 0 \quad \forall \ I = 2, 3, ...N \ .$$
$$(95)$$

One finds that the only solution is

$$\vec{n} - \vec{m} \ \propto \ (1, 1, \, ... \, 1) \ .$$
$$(96)$$

Since n_b and m_b are integers this solution is equivalent to multiplication with $h(n_b, m_b)$. Thus the neutrality condition is the mechanism on the bosonic side which reproduces the selection rule steming from the definition (51) of the clustering expectation functional $\langle ..\rangle_0^\theta$.

A straightforward but lengthy computation shows that the equality (90) can be fulfilled by setting the constants $c^{(b)}$ in (87) to

$$c^{(b)} = \left(\frac{M^{(1)}e^\gamma}{2}\right)^{\frac{\pi}{\pi+gN}\frac{1}{N}} \prod_{I=2}^{N} \left(\frac{M^{(I)}e^\gamma}{2}\right)^{(U_{Ib})^2} .$$
$$(97)$$

Thus the bosonization is given by (86) and (87) together with (97). The bosonic model describes the physical sector of the currents and the chiral densities.

Having at hand the bosonization, one can immediately draw a second lesson on the structure of the vacuum functional $\langle ..\rangle_0^\theta$. The vacuum structure manifests itself in symmetry properties of the bosonized model.

Lesson 2 : (On the U(1)-Problem of QED$_2$.)

The axial U(1)-symmetry is not a symmetry on the physical Hilbert space, and there is no U(1)-problem for QED$_2$.

This can be seen rather easily in the N $=$ 2 flavor case. The Lagrangian for the scalar fields $\Phi^{(1)}$, $\Phi^{(2)}$ that bosonize the currents and the chiral densities then, is given by

$$\frac{1}{2}\left(\partial_\mu \Phi^{(1)}\right)^2 + \frac{1}{2}\left(\partial_\mu \Phi^{(2)}\right)^2 + \frac{1}{2}\left(\Phi^{(1)}\right)^2 \frac{2e^2}{\pi + 2g} .$$
$$(98)$$

The bosonization prescription (87) for the left-handed densities gives

$$\overline{\psi}^{(1)}(x)P_+\psi^{(1)}(x) \longleftrightarrow \frac{1}{2\pi}c^{(1)} : e^{-ia\Phi^{(1)}(x)} :_{M^{(1)}} : e^{-ib\Phi^{(2)}(x)} :_{M^{(2)}} e^{i\frac{\theta}{2}} ,$$

$$\overline{\psi}^{(2)}(x)P_+\psi^{(2)}(x) \longleftrightarrow \frac{1}{2\pi}c^{(2)} : e^{-ia\Phi^{(1)}(x)} :_{M^{(1)}} : e^{+ib\Phi^{(2)}(x)} :_{M^{(2)}} e^{i\frac{\theta}{2}} ,$$
$$(99)$$

with $a = \sqrt{2}\pi/\sqrt{\pi + 2g}$ and $b = \sqrt{2\pi}$. The axial transformation (49) acts on the densities via

$$\overline{\psi}^{(b)}(x)P_+\psi^{(b)}(x) \longrightarrow \overline{\psi}^{(b)}(x)P_+\psi^{(b)}(x)\, e^{i2\omega} \; . \tag{100}$$

In the bosonized theory this transformation corresponds to

$$a\, \Phi^{(1)}(x)\, +\, b\Phi^{(2)}(x)\ \longrightarrow\ a\, \Phi^{(1)}(x)\, +\, b\Phi^{(2)}(x)\, -\, 2\omega\; , \tag{101}$$

and

$$a\, \Phi^{(1)}(x)\, -\, b\Phi^{(2)}(x)\ \longrightarrow\ a\, \Phi^{(1)}(x)\, -\, b\Phi^{(2)}(x)\, -\, 2\omega\; . \tag{102}$$

Obviously this is not a symmetry, since ω on the right hand sides of (101), (102) cannot be transformed away, by shifting one of the fields $\Phi^{(1)}, \Phi^{(2)}$ by a constant. $\Phi^{(1)}$ cannot be shifted since it is massive (see (98)). $\Phi^{(2)}$ would have to be shifted by $+2\omega/b$ in order to remove ω in (101) and by $-2\omega/b$ to remove it in (102). Thus $U(1)_A$ is not a symmetry. The generalization of the arguments to $N > 2$ flavors is straightforward.

What this lesson could tell us for QCD, is the simple statement that the U(1)-problem does not exist. The charge $\tilde{Q}_5$ which formally [21] is supposed to generate $U(1)_A$, is not gauge invariant. Thus it has to be doubted [42], that $U(1)_A$ is a symmetry on the physical Hilbert space (maybe it is a symmetry on a 'larger space'). Hence the Goldstone theorem does not apply to the physical sector and there is no reason to expect a physical Goldstone particle. A proof that the unphysical sector decouples from physical amplitudes is of course much more difficult in QCD, than the simple arguments given for QED$_2$ above.

5.3   The massive model and the GSG

In the last subsection it was shown that it is possible to find a common bosonization of the Cartan currents together with the chiral densities. In the mass perturbation series (12) the expansion of the mass term of the action leads to insertions of fermion mass term and thus to insertions of the chiral densities. Thus one can formally identify an interaction term S_{int} for the scalar fields which corresponds to the mass term (9). By inserting the bosonization prescription (87) into (9) one finds

$$S_{int}[\Phi^{(I)}]\ :=\ -\frac{1}{\pi}\sum_{b=1}^{N} m^{(b)}c^{(b)}\int d^2x\, \chi_\Lambda(x)$$

$$:\cos\left(2\sqrt{\pi}\sqrt{\frac{\pi}{\pi+gN}}U_{1b}\Phi^{(1)}(x) + 2\sqrt{\pi}\sum_{I=2}^{N}U_{Ib}\Phi^{(I)}(x) - \frac{\theta}{N}\right): \; . \tag{103}$$

The Wick ordering of the cosine is understood in terms of the perturbation expansion and thus reduces to the Wick ordering of exponentials (compare (87)). Note that the infrared cutoff Λ is taken over. Also the role of the Thirring term which manifests itself in a nonvanishing g shows up in a new light. It leads to the extra factor $\sqrt{\pi/(\pi+gN)}$ attached to the field $\Phi^{(1)}$ in the cosine, which keeps the model below the first collapse point (see e.g. [43]).

In terms of the bosonized model the perturbation series (12) reads

$$\langle P[\{\Phi^{(I)}\}]\rangle\ =\ \frac{1}{Z}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\,\langle\, P[\{\Phi^{(I)}\}]\, S_{int}[\{\Phi^{(I)}\}]^n\,\rangle_Q \; . \tag{104}$$

Using the bosonized model in [23], [44] it is proven that the perturbation series converges for small fermion masses $m^{(b)}$ and space-time cutoff Λ. Using the known techniques for $N > 1$ it

is not possible to remove the cutoff termwise due to the massless fields $\Phi^{(I)}$, $I > 1$ which show up for more than one flavor. For $N = 1$ no such problem is faced [45]. Nevertheless for finite Λ the bosonization is rigorous. Only the perturbative treatment (104) breaks down for $\Lambda \to \infty$, a fact that is related to logarithmic contributions in the small-mass behaviour of the fermion determinant in infinite volume [23]. For further analysis of this non-perturbative behaviour from a different point of view see also [46], [47, 48].

The bosonization to the model with the interaction (103) is a simple generalization of the Coleman isomorphism which maps the one-flavor Schwinger model to the Sine-Gordon model [49, 50]. Models of the type (103) but without the UV-regulator were already discussed in [35]. The classical Lagrangian $\mathcal{L}_{GSG}$ which corresponds to the generalized Sine-Gordon model can be read off from the $Q^{(I)}$ (see (84), (85)) and S_{int}

$$
\mathcal{L}_{GSG} = \frac{1}{2} \sum_{I=1}^{N} \partial_\mu \Phi^{(I)} \partial_\mu \Phi^{(I)} + \frac{1}{2} \left(\Phi^{(1)} \right)^2 \frac{e^2 N}{\pi + gN}
$$

$$
- \frac{1}{\pi} \sum_{b=1}^{N} m^{(b)} c^{(b)} \cos \left(2\sqrt{\pi} \sqrt{\frac{\pi}{\pi+gN}} U_{1b} \Phi^{(1)} + 2\sqrt{\pi} \sum_{I=2}^{N} U_{Ib} \Phi^{(I)} - \frac{\theta}{N} \right) . \tag{105}
$$

It has to be remarked that the GSG defined through (105) bosonizes only the N Cartan currents (75) together with the chiral densities, although there are all together N^2 vector currents in the N-flavor model. In [31] it was shown that there is no common abelian bosonization of all N^2 vector currents. However only the U(1) current $J_\mu^{(1)}$ plays a special role. It gives rise to a heavy state, while the other $N^2 - 1$ states remain light. $N - 1$ of the currents corresponding to light states are now bosonized together with the U(1) current. This is sufficient for the discussion below.

The explicit form (105) of the Lagrangian of the GSG allows to draw another lesson that recovers a property of the θ-vacuum in QCD.

Lesson 3 : (On the vacuum structure of QED$_2$.)

Physics does not depend on θ if at least one of the fermion masses vanishes.

This property can be seen to hold in the bosonized version by the following arguments

$$
U_{N1} = \frac{1}{\sqrt{2}} \quad , \quad U_{N2} = \frac{-1}{\sqrt{2}} \quad , \quad U_{Nb} = 0 \quad \text{for } 3 \leq b \leq N , \tag{106}
$$

for $N \geq 2$ (compare (68)), one obtains for the interaction term in (105)

$$
\sum_{b=1}^{N} m^{(b)} c^{(b)} : \cos \left(2\sqrt{\pi} \sqrt{\frac{\pi}{\pi+gN}} U_{1b} \Phi^{(1)}(x) + 2\sqrt{\pi} \sum_{I=2}^{N} U_{Ib} \Phi^{(I)}(x) - \frac{\theta}{N} \right) :
$$

$$
= m^{(1)} c^{(1)} : \cos \left(2\sqrt{\pi} \sqrt{\frac{\pi}{\pi+gN}} U_{1b} \Phi^{(1)}(x) + 2\sqrt{\pi} \sum_{I=2}^{N-1} U_{Ib} \Phi^{(I)}(x) \right.
$$

$$
\left. + 2\sqrt{\pi} \frac{1}{\sqrt{2}} \Phi^{(N)}(x) - \frac{\theta}{N} \right) :
$$

$$
+ \sum_{b=3}^{N} m^{(b)} c^{(b)} : \cos \left(2\sqrt{\pi} \sqrt{\frac{\pi}{\pi+gN}} U_{1b} \Phi^{(1)}(x) + 2\sqrt{\pi} \sum_{I=2}^{N-1} U_{Ib} \Phi^{(I)}(x) - \frac{\theta}{N} \right) : . \tag{107}
$$

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Since $\Phi^{(N)}$ is a massless field and shows up only in the first term on the right hand side of (107) it can be shifted by a constant in order to change θ. If none of the masses vanishes, $\Phi^{(N)}$ enters the interaction term twice but with different signs, as can be seen from (68). The value of θ cannot be changed by a symmetry transformation then, and physics depends on it.

The above property is believed to hold also for the formal θ-vacuum of QCD [2, 3]. The independence of θ can be seen also from an alternative introduction of the vacuum angle [51, 52]. It is the starting point for the derivation of the Witten-Veneziano type formulas [25]-[27], [22].

5.4 Semiclassical approximation and Witten-Veneziano type formulas

Since in the perturbative treatment the space-time cutoff Λ spoils Lorentz invariance and thus the extraction of physical quantities, one is reduced to a semiclassical approximation of the Lagrangian.

In order to simplify the involved structure of the interaction (105) we consider the special case of all fermion masses being equal $m^{(b)} := m$ for $b = 1, 2 \ .. \ N$. Using the fact that in $\mathcal{L}_{GSG}$ only $\Phi^{(1)}$ plays an extra role one can furthermore restrict the masses $M^{(I)}$ used for normal ordering to $M^{(I)} = M$ for $I = 2, 3 \ .. \ N$. Inserting this restriction in formula (97) for the coefficients $c^{(b)}$ showing up in $\mathcal{L}_{GSG}$ one finds $c^{(b)} := c$ for $b = 1, 2 \ .. \ N$. Together with the restriction for the fermion masses this reduces the semiclassical problem to linear algebra and to the solution of only one transcendental equation. Without this restriction one would have to solve a system of N coupled transcendental equations.

The minima $\Phi_0^{(I)}$ of the potential $V(\Phi^{(I)})$ have to be computed. The potential $V(\Phi^{(I)})$ is given by

$$
V(\Phi^{(I)}) \ := \ \frac{1}{2}\left(\Phi^{(1)}\right)^2 \frac{e^2 N}{\pi + gN} \ - \ \frac{1}{\pi}mc \sum_{b=1}^{N} \cos\left(2\sqrt{\frac{\pi}{N}}\sqrt{\frac{\pi}{\pi+gN}}\Phi^{(1)} + 2\sqrt{\pi}\sum_{I=2}^{N} U_{Ib}\Phi^{(I)} - \frac{\theta}{N}\right) .
\tag{108}
$$

Setting $\frac{\partial}{\partial\Phi^{(J)}}V(\Phi^{(I)})|_{\Phi^{(I)}=\Phi_0^{(I)}} = 0$ gives

$$
\frac{e^2 N}{\pi+gN}\Phi_0^{(1)} + \frac{2mc}{\sqrt{\pi+gN}}\sum_{b=1}^{N}\frac{1}{\sqrt{N}}\sin\left(2\sqrt{\frac{\pi}{N}}\sqrt{\frac{\pi}{\pi+gN}}\Phi_0^{(1)} + 2\sqrt{\pi}\sum_{I=2}^{N} U_{Ib}\Phi_0^{(I)} - \frac{\theta}{N}\right) \ = \ 0 \ , \tag{109}
$$

for $J = 1$ and

$$
\frac{2mc}{\sqrt{\pi}}\sum_{b=1}^{N} U_{Jb}\sin\left(2\sqrt{\frac{\pi}{N}}\sqrt{\frac{\pi}{\pi+gN}}\Phi_0^{(1)} + 2\sqrt{\pi}\sum_{I=2}^{N} U_{Ib}\Phi_0^{(I)} - \frac{\theta}{N}\right) = 0 \ , \tag{110}
$$

for $J = 2, 3, \ ... \ N$. Again one can interpret the lines of U (fixed J in U_{Jb}) as vectors $\vec{r}^{(J)}$ (compare (68)) and denote Eq. (110) as products of two vectors

$$
\vec{r}^{(J)} \cdot \vec{s} \ \overset{!}{=} \ 0 \qquad \forall \ J \ = \ 2, 3 \ ... \ N \ , \tag{111}
$$

where the entries of the vector $\vec{s}$ are given by

$$
s_b := \frac{2mc}{\sqrt{\pi}}\sin\left(2\sqrt{\pi}\sqrt{\frac{\pi}{\pi+gN}}\frac{1}{\sqrt{N}}\Phi_0^{(1)} + 2\sqrt{\pi}\sum_{I=2}^{N} U_{Ib}\Phi_0^{(I)} - \frac{\theta}{N}\right) . \tag{112}
$$

We already found (compare (95),(96)) that the only solution is $\vec{s} \ = \ \lambda \ (1, 1, \ ... \ 1) \ , \ \lambda \ \in \ I\!\!R$. Thus the set of Eqs. (110) is equivalent to

$$
2\sqrt{\pi}\sqrt{\frac{\pi}{\pi+gN}}\frac{1}{\sqrt{N}}\Phi_0^{(1)} + 2\sqrt{\pi}\sum_{I=2}^{N} U_{Ib}\Phi_0^{(I)} - \frac{\theta}{N} \ = \ \arcsin\left(\frac{\lambda\sqrt{\pi}}{2mc}\right) \ , \tag{113}
$$

for all $b = 1, 2, ...N$. Eq. (109) can now be used to express λ in terms of $\Phi_0^{(1)}$ giving $\lambda = -\Phi_0^{(1)}\sqrt{N}e^2/\sqrt{\pi(\pi + gN)}\Phi_0^{(1)}$. Inserting this in (113), multiplying with U_{Jb} and summing over b gives

$$\sum_{I=2}^{N} \delta_{JI}\Phi_0^{(I)} = \delta_{J1}\left\{ \frac{\sqrt{N}}{2\sqrt{\pi}}\left[\frac{\theta}{N} - \arcsin\left(\frac{e^2}{2mc}\sqrt{\frac{N}{\pi+gN}}\Phi_0^{(1)} \right) \right] - \sqrt{\frac{\pi}{\pi+gN}}\Phi_0^{(1)} \right\} , \qquad (114)$$

where we used the orthogonality of U and $\sum_b U_{Jb} = \delta_{J1}\sqrt{N}$ (see(68)). In the last expression the equations for the determination of the minima are decoupled and can be solved easily. For the case $2 \le J \le N$ one obtains the naive solution $\Phi_0^{(J)} = 0$ for $J = 2, 3 .. N$. Of course there exists an infinite countable set of solutions since one can always shift the argument of the cosine in (108) by integer multiples of 2π giving rise to

$$2\sqrt{\pi}\sum_{J=2}^{N} U_{Jb}\Phi_0^{(J)} = n_b 2\pi , \quad n_b \in \mathbb{Z} , \quad \forall \; b = 1, 2, ... N . \qquad (115)$$

Using the orthogonality of U, the last expression can be inverted and one ends up with

$$\Phi_0^{(I)} = \sqrt{\pi}\sum_{b=1}^{N} U_{Ib}\, n_b \quad I = 2, 3, ... N . \qquad (116)$$

The $\Phi_0^{(1)}$ coordinate of the minimum has to fulfill the equation that emerges from (114) setting $J = 1$

$$\frac{1}{\sqrt{N}}\sqrt{\frac{\pi}{\pi+gN}}\Phi_0^{(1)} = \frac{1}{2\sqrt{\pi}}\left[\frac{\theta}{N} - \arcsin\left(\frac{e^2}{2mc}\sqrt{\frac{N}{\pi+gN}}\Phi_0^{(1)} \right) \right] . \qquad (117)$$

Obviously this is a trivial modification of the transcendental equation that determines the minimum in the one flavor case. It has to be solved numerically.

To evaluate the mass matrix of the effective theory around the minima, the Hesse matrix $H_{II'} := \frac{\partial^2 V(\Phi^{(J)})}{\partial\Phi^{(I)}\partial\Phi^{(I')}}\Big|_{\Phi^{(J)}=\Phi_0^{(J)}}$ has to be computed. It can be evaluated easily

$$H = \text{diag}\left(\frac{e^2 N}{\pi+gN}, 0, ... 0 \right) +$$

$$4mc\,\tilde{\lambda}\sum_{b=1}^{N}\begin{bmatrix} \frac{\pi}{\pi+gN}\frac{1}{N} & \sqrt{\frac{\pi}{\pi+gN}}\sqrt{\frac{1}{N}}U_{2b} & \cdot & \cdot & \cdot & \sqrt{\frac{\pi}{\pi+gN}}\sqrt{\frac{1}{N}}U_{Nb} \\ \sqrt{\frac{\pi}{\pi+gN}}\sqrt{\frac{1}{N}}U_{2b} & U_{2b}U_{2b} & \cdot & \cdot & \cdot & U_{2b}U_{Nb} \\ \sqrt{\frac{\pi}{\pi+gN}}\sqrt{\frac{1}{N}}U_{3b} & U_{3b}U_{2b} & & & & \\ \cdot & \cdot & \cdot & & \cdot & \\ \cdot & \cdot & & \cdot & \cdot & \\ \cdot & \cdot & & & \cdot & \\ \sqrt{\frac{\pi}{\pi+gN}}\sqrt{\frac{1}{N}}U_{Nb} & U_{Nb}U_{2b} & \cdot & \cdot & \cdot & U_{Nb}U_{Nb} \end{bmatrix} =$$

$$\text{diag}\left(\frac{e^2 N}{\pi+gN} + \frac{\pi}{\pi+gN}4mc\tilde{\lambda} , \; 4mc\tilde{\lambda} , \; ...... , \; 4mc\tilde{\lambda} \right) . \qquad (118)$$

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The orthogonality of U was used again. $\tilde{\lambda}$ is defined as

$$\tilde{\lambda} := \cos\arcsin\left(\frac{\lambda\sqrt{\pi}}{2mc}\right) = \sqrt{1 - \left(\frac{\lambda\sqrt{\pi}}{2mc}\right)^2} . \tag{119}$$

The Hesse matrix comes out as a positive definite diagonal matrix. The entries have to be interpreted as the squared masses of the fields $\Phi^{(I)}$ in an effective theory around the semiclassical vacua. The masses m_I for the fields $\Phi^{(I)}$ are given by

$$m_1 := \sqrt{\frac{e^2 N}{\pi} + 4mc\tilde{\lambda}}\,\sqrt{\frac{\pi}{\pi + gN}} \qquad \text{for} \quad \Phi^{(1)} , \tag{120}$$

and

$$m_I := \sqrt{4mc\tilde{\lambda}} \qquad \text{for} \quad \Phi^{(I)} , \quad I = 2, 3, \dots N . \tag{121}$$

It is interesting to notice that the masses m_I do not depend on θ. It also has to be emphasized that the semiclassical approximation of the GSG has two regimes where the approximation is good. Firstly this is the case for large fermion masses m which is the usual domain of a semiclassical approximation. Secondly for $m \to 0$ this approximation becomes exact, since the massless model is bosonized by free fields (compare (71)) where the semiclassical approximation already gives the spectrum of the quantized theory. We remark that the linear behaviour (120), (121) in m, is modified in the small m regime for $N > 1$. However for the case of $\theta = 0$, the semiclassical approximation can be computed in closed form without solving a transcendental equation. For the $N = 1$ case one then finds that the semiclassical result then coincides with the first order result [56, 57], of the perturbation expansion in m, which has a sound basis for $N = 1$ [45].

The masses obtained in the semiclassical approximation will now be used to test Witten-Veneziano formulas. Since the semiclassical arguments do not rely on finite g, we set g to zero in the following. Of course one could modify the Witten-Veneziano formula (122) (see below) to include a finite g.

For $g = 0$ the following generalization of the Witten-Veneziano formula quoted in [31] will be shown to hold:

$$m_1^2 - \frac{1}{N-1}\sum_{I=2}^{N} m_I^2 = \frac{4N}{(f_1^0)^2} P^0(0) . \tag{122}$$

f_1^0 denotes the decay constant of the U(1)-pseudoscalar, and $P^0(0)$ is the contact term of the topological susceptibility [22] defined through the spectral representation (see also [23, 31])

$$\frac{e^2}{(2\pi)^2}\int \langle F_{12}(x)F_{12}(0)\rangle e^{-ipx}dx = P^0(0) - \int_0^\infty \frac{d\rho(\mu^2)}{p^2 + \mu^2} . \tag{123}$$

Inserting the mass values (120) and (121) at $g = 0$, one finds that the left hand side of (122) reduces to

$$m_1^2 - \frac{1}{N-1}\sum_{I=2}^{N} m_I^2 = \frac{e^2 N}{\pi} . \tag{124}$$

In [31] $P^0(0)$ and f_1^0 were computed explicitly, and it was shown that the right hand side of (124) can be rewritten as the right hand side of (122) and thus (122) is proven. This result is our Lesson 4.

Lesson 4 : (A Witten-Veneziano-type formula for QED$_2$.)

The masses (determined from a semiclassical approximation) of the particles that correspond to the Cartan currents obey the Witten-Veneziano formula (122).

It has to be remarked that (122) is also a verification of the original form of the Witten-Veneziano formula, because the topological susceptibility of the quenched theory reduces to the contact term [22, 31]. It is not true, however, that the topological susceptibility appearing in the formula expresses a property of the long distance fluctuations of the topological density. In fact it is entirely given by the contact term expressing short distance fluctuations.

6 Discussion

When trying to take over part of the Lessons 1-4 for QED$_2$ to the case of QCD, one of course never should forget the limitations of a two dimensional toy model.

As was discussed, the Coleman theorem [33] determines the form of the chiral selection rule (57) quoted in Lesson 1 for QED$_2$. On the other hand Coleman's theorem does not allow for

$$\langle\, \theta \mid \overline{q}\, q \mid \theta\, \rangle \;\neq 0 \;\;, \tag{125}$$

in QED$_2$ with more than one flavor. For QCD Eq. (125) is one of the main assumptions supporting the belief that QCD is the correct theory for strong interactions. Thus Coleman's theorem limits the relevance of Lesson 1 for QCD.

What Lesson 2 tries to tell us is of more direct relevance for QCD. The charge [21] that is used to formally implement the U(1)-axial symmetry is not gauge invariant. Thus it has to be questioned if the U(1)$_A$ symmetry can be implemented on the physical Hilbert space. If not, the Goldstone theorem does not apply, and there is no reason to expect a light pseudoscalar. Thus QED$_2$ indeed suggests the most simple solution to the U(1)-problem: The U(1)-problem does not exist. As already discussed, the proof that unphysical particles decouple from the physical spectrum is much more involved for QCD.

Lesson 3 recovers a property that is commonly accepted to hold for the θ-vacuum of QCD.

The main limitation of taking over Lesson 4 to QCD, is that the topological susceptibility of QED$_2$ is too simple to model the problems that show up in QCD. In particular χ_{top} of QCD is a composite operator that requires some subtraction procedure determining its properties. As a further limitation it has been pointed out [53] that the structure of the SU(N)-currents that play the role of the pseudoscalar mesons is too simple. In particular they obey the current algebra of free fermions [54, 55] and thus their interaction is reduced to flavor exchange.

Despite the limitations of the lessons for QCD, the study of the corresponding problems in QED$_2$ is interesting, in particular because techniques independent from the concepts used for QCD were developed. For QED$_2$ it has been demonstrated that it is possible to define a clustering vacuum functional without making use of formal instanton arguments. We believe that this is the conceptually clearer way for a two dimensional model where the mathematical structure is rather simple.

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