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Universality Class of $O(N)$ Models

Adrian Patrascioiu

Physics Department, University of Arizona
Tucson, AZ 85721, U.S.A.

and

Erhard Seiler

Max-Planck-Institut für Physik
– Werner-Heisenberg-Institut –
Föhringer Ring 6, 80805 Munich, Germany

Abstract

We point out that existing numerical data on the correlation length and magnetic susceptibility suggest that the two dimensional $O(3)$ model with standard action has critical exponent $\eta = 1/4$, which is inconsistent with asymptotic freedom. This value of η is also different from the one of the Wess-Zumino-Novikov-Witten model that is supposed to correspond to the $O(3)$ model at $\theta = \pi$.

Ever since the seminal paper of Kosterlitz and Thouless [1] it has been accepted that in two dimensions ($2D$) there is a fundamental difference between the Abelian $O(2)$ and the non-Abelian $O(3)$ models: while the first one undergoes a transition (KT) from a phase with exponential decay at high temperature $T(\ T = 1/\beta)$ to a massless one at low temperature, the latter is in the high temperature phase for any $\beta < \infty$. While the first conjecture was proven rigorously by Fröhlich and Spencer [2], the second one remains unproven.

We have advanced several arguments [3, 4, 5] against this accepted belief and in our opinion *all* $O(N)$ models undergo a KT-like transition at some finite β. Since however our arguments do not constitute a rigorous demonstration, it may be useful to get a glimpse at what the truth may be via existing numerical data. Our observation pertains to the value of the critical exponent η, defined as follows: let ξ be the correlation length and χ the magnetic susceptibility. Then η can be defined by the following asymptotic statement:

$$\chi \propto \xi^{2-\eta} \tag{1}$$

i.e. $\ln\chi - (2-\eta)\ln\xi$ should go to a constant as $\xi \to \infty$.

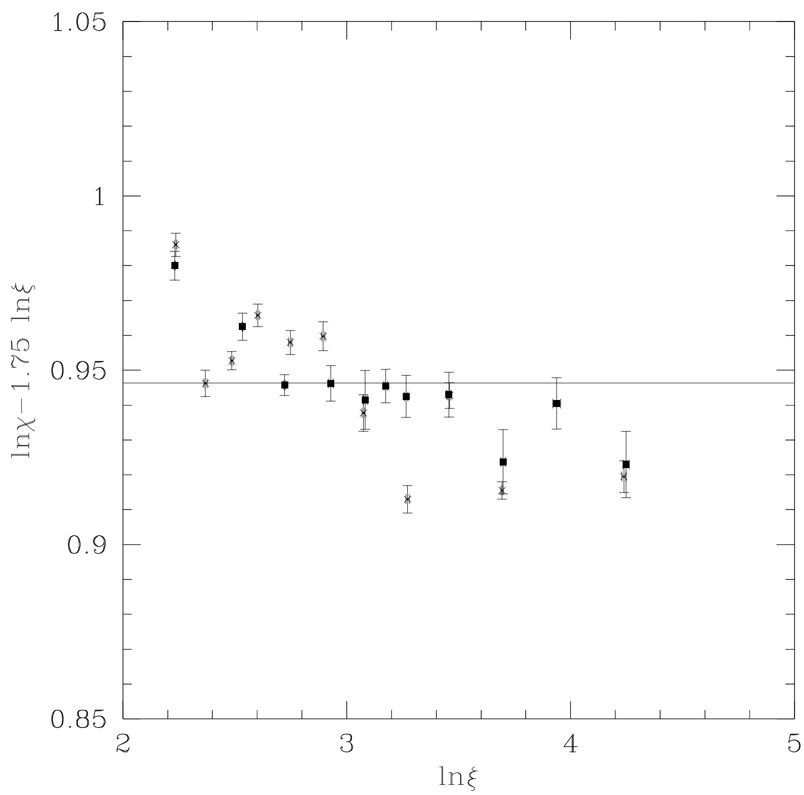

Figure 1: $O(2)$ model, test of $\eta = 1/4$

For the $O(2)$ model certain non-rigorous renormalization group arguments advanced by Kosterlitz [6] predict that $\eta = 1/4$. In Tab.1 we collect the data from [7] and [8]. We could plot $\ln\chi$ vs. $\ln\xi$ and would see a nice straight line with a slope close to 1.75. In fact a fit of $\ln\chi$ to a linear function of $\ln\xi$ yields a slope of $2-\eta = 1.721$, but with a poor fit quality ($\chi^2 = 9.1$ per degree of freedom), probably reflecting underestimation of the errors. It is, however, much more significant to look at the deviations from the predicted slope by plotting $\ln\chi - 1.75\ln\xi$ vs. $\ln\xi$, together with the least square fit to a constant. This is done in Fig.1; the data of [7] are represented by crosses, the ones of [8] by squares. It can be seen that the data still show a tendency to decrease, corresponding to the fact that the aymptotic region has not yet been reached and the fits published in the literature always produced an $\eta > 1/4$, as did ours. But from Fig.1 one can also see a tendency of the data to flatten with increasing ξ, in accordance with the theoretical prediction of $\eta = 1/4$. Fig.1 also shows that obviously the errors were underestimated by a considerable amount.

According to the accepted scenario, the behavior of the $O(3)$ model should be vastly different; indeed perturbative renormalization group arguments predict the following asymptotic behavior ("asymptotic scaling"):

$$\xi \propto \frac{\exp(2\pi\beta)}{\beta} \tag{2}$$

$$\chi \propto \frac{\exp(4\pi\beta)}{\beta^4} \tag{3}$$

From this it follows that for $\xi \to \infty$ one should observe $\ln\chi - 2\ln\xi + 2\ln\beta$ tending to a constant. To test this prediction, we use the published data of [9] as well as the unpublished ones due to [10], which are collected in Tab.2. In Fig.2 we plot $\ln\chi - 2\ln\xi + 2\ln\beta$, which should be constant according to the asymptotic scaling prediction. In Fig.3 we plot instead $\ln\chi - 1.75\ln\xi$ vs. $\ln\xi$, as we did for $O(2)$. Fig.3 also contains a least square fit to a constant. The data from [9] are represented by squares, the ones of [10] by crosses. Note that in both figures we are using the same scale as in Fig.1. Fitting $\ln\chi$ to a linear function of $\ln\xi$, as we did for $O(2)$, gives a slope of $2-\eta = 1.741$, but again with a poor fit quality ($\chi^2 = 11.4$ per degree of freedom). Again this reflects probably mainly that the errors were underestimated.

The following facts are visible from Figs 3 and 2:

(1) The data are not consistent with asymptotic scaling inasmuch the points shown in Fig.2 are not consistent with a constant but show a strong decrease.

(2) The data are consistent with a critical behavior with the same $\eta = 1/4$ that was predicted by the KT theory for the $O(2)$ model.

(3) The errors are almost certainly underestimated.

So we have to conclude that in fact the numerical data disagree with the predictions of asymptotic freedom and instead suggest that both $O(2)$ and

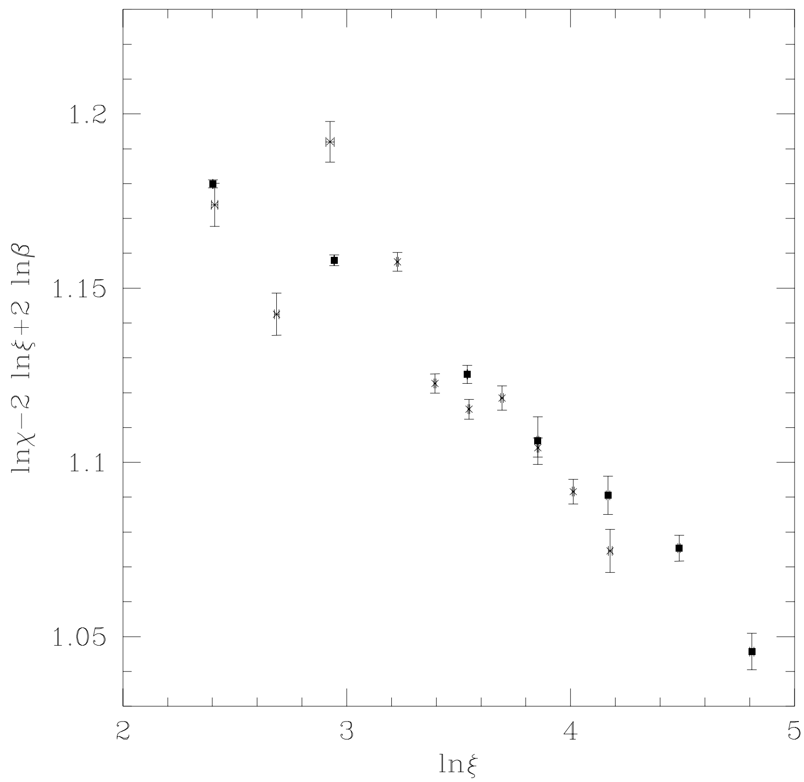

Figure 2: $O(3)$ model, test of asymptotic scaling

$O(3)$ have the same value of η. The data also indicate that the $O(3)$ model with standard action $\sum s(i) \cdot s(j)$ is not in the same universality class as the $\theta = \pi$ $O(3)$ model, which is supposed to have the Wess-Zumino-Novikov-Witten model as its scaling limit [11]; conformal field theory arguments predict η=1, a predicition which a recent paper [12] claims to have verified numerically.

A note of caution: in principle η can also be determined by studying the finite size scaling of the susceptibility $\chi(L)$. Ideally this study should be performed at β_{crt}; in practice though, it suffices to place oneself in a regime where $\xi >> L$. One could ask then what value of η would come out of such a determination. The answer is that one must be careful and use a large enough L. Indeed, in this type of problems, besides the correlation length ξ, there is a second important length ξ_{PT}, the distance over which the system is well ordered. From perturbation theory (PT) one knows that for the $O(3)$ model this second length is $O(\exp(\pi\beta))$. Consequently, if one uses periodic boundary conditions (b.c.) and an L and β such that $\xi_{PT} >> L$, then the system will behave according to PT and one will find the perturbative value of η which is $1/\pi\beta + O(\beta^{-2})$. As we explained in a recent paper [14], a clear indication that such a determination cannot be trusted to reflect the true thermodynamic

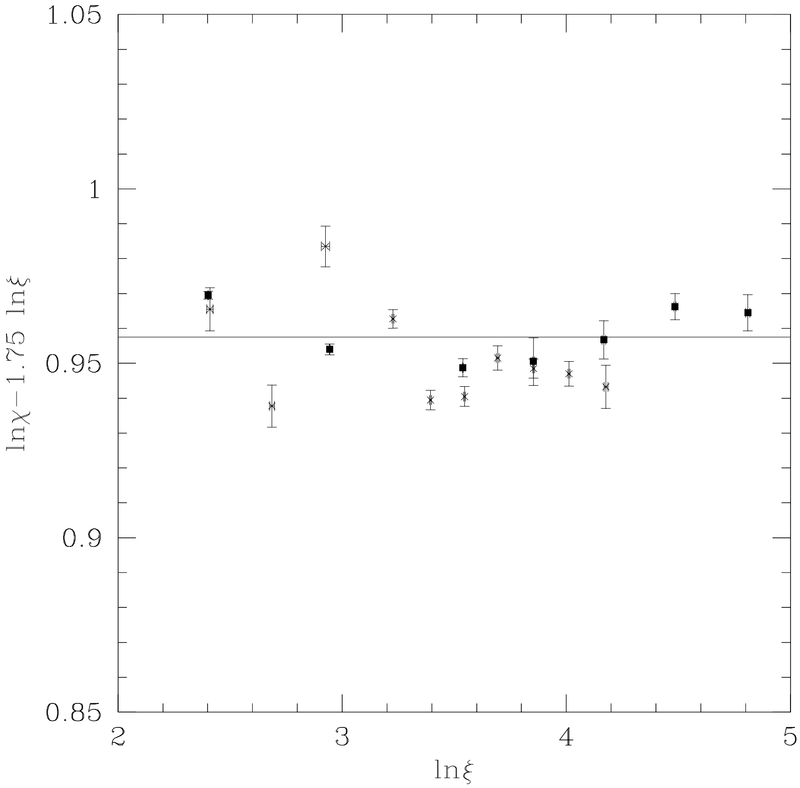

Figure 3: $O(3)$ model, test of $\eta = 1/4$

value of η is the dependence of η upon the b.c. employed. Indeed, as we discussed in ref.[14], in the limit $L \to \infty$, the value of η must be the same if we use periodic b.c. and freeze two spins situated a distance $L/2$ apart in any arbitrary positions; this then gives a criterion for deciding whether one has used a sufficiently large L and whether the determination of η is trustworthy.

Finally we should like to make a remark about the large N-limit of the $O(N)$ model, which is often cited as evidence for the standard picture of $\beta_{crt} = \infty$ for all $N > 2$. As we pointed out in various places (see in particular [15]), this reasoning is faulty. It is true that at fixed $\tilde{\beta} \equiv \beta/N$ both ξ and χ converge as $N \to \infty$ to the values of the spherical model, hence in that limit $\eta = 0$ up to log corrections. But, as we pointed out in [15], the approach to the limit is nonuniform, i.e. it becomes slower and slower as $\tilde{\beta} \to \infty$ or $\xi \to \infty$. With increasing N, $\ln\chi$ as a function of $\ln\xi$ will approach the spherical limit ($N = \infty$) form, but at any fixed N, for large enough β, we expect that it will fall away from that form and that $\ln\chi - 1.75\ln\xi$ will aproach a constant for $\beta \to \infty$.

We are indebted to S.Caracciolo, R.G.Edwards, A.Pelissetto and A.D.Sokal for allowing us to use their data prior to publication; these data form the basis

of the papers [13] and will apear in full in subsequent papers by [16].

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Tab.1a: *χ and ξ for the $O(2)$ model on thermodynamic lattices ($L/\xi \geq 7$).* Data are taken from [7].

β	ξ	χ
.91	9.36(5)	134.17(45)
.92	10.69(8)	162.68(61)
.93	12.03(6)	201.29(53)
.94	13.50(9)	249.90(80)
.95	15.61(10)	319.6(1.1)
.96	18.08(13)	414.1(1.7)
.97	21.66(13)	554.9(2.9)
.98	26.37(19)	764.6(3.0)
.99	31.78(21)	1092.(4.)
1.00	40.20(20)	1604.(4.)
1.02	69.27(59)	4170.(19.)

Tab.1b: *χ and ξ for the $O(2)$ model on thermodynamic lattices ($L/\xi \geq 7$).* Data are taken from [8].

β	ξ	χ
.9090909	9.32(2)	132.41(55)
.9345794	12.61(2)	220.84(86)
.9478673	15.23(5)	302.19(89)
.9615385	18.70(20)	433.7(2.2)
.9708738	21.80(20)	564.1(2.7)
.9756098	23.90(20)	665.0(4.0)
.9803922	26.20(20)	779.0(5.0)
.9900990	31.70(30)	1087.(10.)
1.000000	40.40(40)	1631.(12.)
1.010101	51.30(90)	2520.(24.)
1.020408	70.(1.)	4258.(20.)

Tab.2a: *χ and ξ for the $O(3)$ model on thermodynamic lattices ($L/\xi \geq 7$).* Data are taken from [9].

β	ξ	χ
1.5	11.05(1)	176.4(2)
1.6	19.00(2)	448.4(7)
1.7	34.44(6)	1263.7(3.3)
1.75	47.2(2)	2197.(15.)
1.8	64.5(5)	3823.(21.)
1.85	88.7(5)	6732.(25.)
1.9	122.7(1.1)	11867.(62.)

Tab.2a: *χ and ξ for the $O(3)$ model on thermodynamic lattices ($L/\xi \geq 7$).* Data are taken from [10].

β	ξ	χ
1.5	11.13(16)	178.0(1.1)
1.55	14.69(15)	281.5(1.7)
1.6	18.66(35)	447.6(2.6)
1.65	25.21(13)	742.8(2.0)
1.675	29.77(13)	971.9(2.7)
1.7	34.66(13)	1269.4(3.6)
1.725	40.21(25)	1661.8(5.8)
1.75	47.20(19)	2193.9(6.0)
1.775	55.23(25)	2886.4(10.2)
1.8	65.17(57)	3839.7(23.8)