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On the canonical reduction of spherically symmetric gravity

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Abstract

In a thorough paper Kuchař has examined the canonical reduction of the most general action functional describing the geometrodynamics of the maximally extended Schwarzschild geometry. This reduction yields the true degrees of freedom for spherically symmetric general relativity. The essential technical ingredient in Kuchař's analysis is a canonical transformation to a certain chart on the gravitational phase space which features the Schwarzschild mass parameter M_S (expressed off-shell in terms of ADM-like variables) as a canonical coordinate. In this paper we reveal the geometric interpretation of Kuchař's canonical transformation. We do this by appealing to the theory of quasilocal energy-momentum in general relativity given by Brown and York. We find Kuchař's transformation to be a "sphere-dependent boost to the rest frame" (defined by vanishing quasilocal momentum). Furthermore, our formalism is robust enough to include the pure-dilaton model of Callan, Giddings, Harvey, Strominger, and Witten. Therefore, besides reviewing Kuchař's original work for the Schwarzschild case from the framework of hyperbolic geometry, we present new results concerning the canonical reduction of Witten-black-hole geometrodynamics. Finally, addressing a recent work of Louko and Whiting, we discuss some delicate points concerning the canonical reduction of the "thermodynamical action," which is of central importance in the path-integral formulation of gravitational thermodynamics.

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I. INTRODUCTION

In a thorough paper (Ref. [1], hereafter referred to as KVK) Kuchař has examined the canonical reduction of the most general action functional describing the geometrodynamics of the maximally extended Schwarzschild geometry. This reduction yields the true degrees of freedom associated with general relativity subject to ansatz of spherical symmetry. (The canonical reduction of spherically symmetric gravity has also been considered in detail, but along a somewhat different line, by Kastrup and Thiemann. [2]) The key technical ingredient in Kuchař's analysis is a canonical transformation to a certain chart on the gravitational phase space which features the Schwarzschild mass parameter M (expressed off-shell in terms of ADM-like variables) as a canonical coordinate. Potential applications of the new reduced formalism include examinations, from the canonical viewpoint, of spherically symmetric collapse, the Hawking effect, and gravitational thermodynamics. Indeed, Louko and Whiting have already made such an application. Applying Kuchař's method to a spatially bounded exterior region of the Schwarzschild black hole, they have constructed the Schwarzschild thermodynamical (canonical) partition function completely within the Lorentzian Hamiltonian framework. (Ref. [3], hereafter referred to as LW) Their canonical partition function is in agreement with previous results derived via the Euclidean-path-integral method. [4] The starting point in LW is the "thermodynamical action," which is of central importance in the path-integral formulation of gravitational thermodynamics. A very delicate issue in the analysis of LW concerns the treatment of the thermodynamical action's boundary terms under the canonical reduction via the KVK method.

In this paper we reveal the geometric interpretation of Kuchař's canonical transformation. By appealing to notions of quasilocal energy and momentum in general relativity which have been given by Brown and York [5,6], we interpret Kuchař's canonical transformation as a "sphere-dependent boost to the rest frame" (defined by vanishing quasilocal momentum). The main point is the following. On an arbitrary (spherically symmetric) spatial slice Σ the parameter φ describing the local boost between the slice Eulerian observers at a point and the rest-frame observers at the same point can be constructed from the canonical variables of Σ (if known in a tiny spatial region surrounding the point of interest). Furthermore, we work in a framework which is robust enough to include the pure-dilaton model of Witten and Callan, Giddings, Harvey, and Strominger (Refs. [7,8], hereafter CGHSW). Therefore, besides reviewing some of Kuchař's original work for the Schwarzschild case from the framework of hyperbolic geometry, we present new results concerning the canonical reduction of Witten-black-hole geometrodynamics. We show that the canonical transformation of KVK can be made in the pure-dilaton case. Therefore, the potential applications of the KVK formalism, listed in the first paragraph, are also relevant for two-dimensional pure-dilaton gravity. Finally, with our general framework we address some of the delicate points, first considered in LW, concerning the canonical reduction of the "thermodynamical action." Our conceptual framework supports the difficult technical steps taken in LW. All of our results are given for both the Schwarzschild and pure-dilaton case.

A few technical points demand some comment at the outset. As mentioned, the analysis of KVK concerns the full Kruskal spacetime, the maximally extended Schwarzschild geometry. The canonical variables used in KVK are defined on spatial slices which cut completely across the Kruskal diagram, and therefore have to obey appropriate boundary conditions in the asymptotic regions. In crossing from one spatial infinity to the other, the slices of KVK are allowed to cross the horizons in a completely general way. This introduces some technical difficulties at the horizons, especially when one is considering Kuchař's canonical transformation. However, as demonstrated in KVK, with care these difficulties may be surmounted. We choose to confine our attention entirely to the right static region of the Kruskal diagram. At first, we work with the time history of the static region lying between concentric spheres. Thus we avoid many of the technical difficulties faced by Kuchař at the outset. We could, of course, work in the full Kruskal diagram, but the essential points of this paper do not demand that we do so. However, since we do chose to bypass a technical treatment of the horizons, questions concerning how to handle such horizon difficulties remain for the CGHSW pure-dilaton model. However, notationally we adopt nearly the same conventions as KVK. Therefore, we expect that with the present paper as a stepping stone, the interested reader could -with minimal effort- convert any and all of the horizon arguments given in KVK into corresponding arguments applicable to the pure-dilaton case.

The layout of this paper is as follows. In § 2, the preliminary section, we describe the relevant kinematics of our spacetime geometry. Since the spacetime geometry is spherically symmetric, it proves convenient to work with a toy 1+1 dimensional spacetime $\mathcal{M}$. In reality, the points of $\mathcal{M}$ are round spheres. In § 3 we derive quasilocal[1] energy and momentum expressions for the physical fields defined on generic spatial slices of $\mathcal{M}$. The method used to derive the quasilocal expressions is a Hamilton-Jacobi analysis of an appropriate action principle for $\mathcal{M}$. In § 4 we use the developed notions of quasilocal energy and momentum to underscore the geometric significance of Kuchař's canonical transformation. This section also considers the reduction of the canonical action with the boundary conditions adopted in this work. The last § 5 considers the canonical reduction of the thermodynamical action.

II. PRELIMINARIES

A. Spacetime $\mathcal{M}$

Since we deal exclusively with spherically symmetric spacetimes, we chose to work with a simplified $1+1$ formalism. Therefore, consider a $1+1$ dimensional spacetime region $\mathcal{M}$ which is bounded spatially. The region $\mathcal{M}$ consists of a collection of one dimensional spacelike slices Σ. The letter Σ denotes both a foliation of $\mathcal{M}$ into spacelike slices and a generic leaf of such a foliation. However, for the initial spacelike slice we reserve the special symbol t' (also the value of the coordinate time on the initial slice), and, likewise, for the final spacelike slice we reserve the symbol t'' (also the value of the coordinate time on the final slice). On spacetime $\mathcal{M}$ we have global coordinates (t, r), and a generic spacetime

[1]We use the adjective "quasilocal" because from the four-dimensional viewpoint the energy and momentum are associated round two-spheres. Thus these "quasilocal" quantities are associated with points of $\mathcal{M}$.

point[2] is $B(t,r)$. Every level-time slice Σ has two spatial boundary points B_i (at $r = r_i$) and B_o (at $r = r_o$). Assume that along Σ the coordinate r increases monotonically from B_i to B_o. We represent the timelike history $B_i(t) \equiv B(t,r_i)$ by $\bar{\mathcal{T}}_i$ (unbarred $\mathcal{T}$ is reserved for another meaning) and refer to it as the *inner boundary.* Likewise, we represent the timelike history $B_o(t) \equiv B(t,r_o)$ by $\bar{\mathcal{T}}_o$ and refer to it as the *outer boundary.* Later on, when we deal with black-hole solutions, we will "seal" the inner boundary. In other words, the time development at the inner boundary will be arrested, and the point B_i will correspond to a bifurcation point in a Kruskal-like diagram. We denote the *corner* points of our spacetime as follows: $B'_i \equiv B(t',r_i)$, $B'_o \equiv B(t',r_o)$, $B''_i \equiv B(t'',r_i)$, and $B''_o \equiv B(t'',r_o)$.

B. Foliations and spacetime decompositions

The spacetime metric is g_{ab}. The metric on a generic Σ slice is Λ^2, and the metric on both $\bar{\mathcal{T}}_i$ and $\bar{\mathcal{T}}_o$ is denoted by $-\bar{N}^2$. In terms of the Σ foliation, the metric may be written the ADM form [19]

$$g_{ab}\mathrm{d}x^a\mathrm{d}x^b = -N^2\mathrm{d}t^2 + \Lambda^2\left(\mathrm{d}r + N^r\,\mathrm{d}t\right)^2 \ , \tag{2.1}$$

with N and N^r denoting the familiar *lapse* and *shift.* The vector field

$$u = \frac{1}{N}\left(\frac{\partial}{\partial t} - N^r\frac{\partial}{\partial r}\right) \tag{2.2}$$

is the unit, timelike, future-pointing normal to the Σ foliation.

We can also consider a *radial* foliation of $\mathcal{M}$ by a family of one-dimensional timelike surfaces which extend from $\bar{\mathcal{T}}_i$ outward to $\bar{\mathcal{T}}_o$ (for black-hole solutions $\bar{\mathcal{T}}_i$ may be a degenerate sheet). [9] These are constant-r surfaces. Like before, we loosely use the letter $\bar{\mathcal{T}}$ both to denote the radial foliation and a generic leaf of this foliation. We call $\bar{\mathcal{T}}$ leaves *sheets*, whereas we have called Σ leaves *slices* (this is an informal convention). In terms of the $\bar{\mathcal{T}}$ foliation, the $\mathcal{M}$ metric takes the form

$$g_{ab}\mathrm{d}x^a\mathrm{d}x^b = \bar{\Lambda}^2\mathrm{d}r^2 - \bar{N}^2\left(\mathrm{d}t + \bar{\Lambda}^t\mathrm{d}r\right)^2 \ , \tag{2.3}$$

where $\bar{\Lambda}$ and $\bar{\Lambda}^t$ are the *radial lapse* and *radial shift.* The unit, spacelike, $\bar{\mathcal{T}}$-foliation normal is

$$\bar{\mathsf{n}} = \frac{1}{\bar{\Lambda}}\left(\frac{\partial}{\partial r} - \bar{\Lambda}^t\frac{\partial}{\partial t}\right) \ . \tag{2.4}$$

We also define the $\bar{\mathcal{T}}$ *boundary* normal $\bar{n}$ on $\bar{\mathcal{T}}_i$ and $\bar{\mathcal{T}}_o$ by the requirement that it always be *outward*-pointing on these boundary elements. On the outer boundary $\bar{\mathcal{T}}_o$ the outward normal is $\bar{n} = \bar{\mathsf{n}}$, while on the inner boundary $\bar{\mathcal{T}}_i$ the outward normal is $\bar{n} = -\bar{\mathsf{n}}$.

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[2]One may consider each point of $\mathcal{M}$ to be round two-sphere with radius $R(t,r)$, where R is the radius function.

By equating the coefficients of the above forms of g_{ab}, we obtain the following relations between the "barred" and "unbarred" variables:

$$\bar{N} = N/\gamma \tag{2.5a}$$
$$\bar{\Lambda} = \gamma\Lambda \tag{2.5b}$$
$$\Lambda N^r/N = -\bar{N}\bar{\Lambda}^t/\bar{\Lambda}\,. \tag{2.5c}$$

Here $\gamma \equiv (1-\mathsf{v}^2)^{-1/2}$ is the local relativistic factor associated with the velocity with $\mathsf{v} \equiv \Lambda N^r/N = -\bar{N}\bar{\Lambda}^t/\bar{\Lambda} = -\bar{\mathsf{v}}$.

Consider the collection of spacetime points $B(t,r)$. Fixation of the r coordinate gives a collection of points $B(t)$ which foliates the sheet $\bar{\mathcal{T}}$ (of course, $\bar{\mathcal{T}}$ could be either $\bar{\mathcal{T}}_i$ or $\bar{\mathcal{T}}_o$). The normal associated with this foliation of $\bar{\mathcal{T}}$ is $\bar{u} = \bar{N}^{-1}\partial/\partial t$. Note that on $\bar{\mathcal{T}}$ the vector fields u and $\bar{u}$ need not coincide. Also, fixation of the t coordinate gives a collection of points $B(r)$ which foliates the slice Σ. The normal associated this foliation of Σ is $\mathsf{n} = \Lambda^{-1}\partial/\partial r$. Again, we define a boundary normal n such that at the inner boundary n is $-\mathsf{n}$, the outward-pointing normal of B_i as embedded in Σ, while at the outer boundary n is n, the outward-pointing normal of B_o as embedded in Σ. On the inner and outer boundaries n and $\bar{n}$ need not coincide. It is easy to verify the following point-wise boost relations:

$$\bar{u} = \gamma u + \mathsf{v}\gamma\mathsf{n} \tag{2.6a}$$
$$\bar{\mathsf{n}} = \gamma\mathsf{n} + \mathsf{v}\gamma u\,. \tag{2.6b}$$

We can write the boost relations for the boundary normals as

$$\bar{u} = \gamma u + v\gamma n \tag{2.7a}$$
$$\bar{n} = \gamma n + v\gamma u\,, \tag{2.7b}$$

by simply defining v such that $v = -\mathsf{v}$ on $\bar{\mathcal{T}}_i$ and $v = \mathsf{v}$ on $\bar{\mathcal{T}}_o$. This is a convention that we use throughout the paper. Regular letters represent objects associated with the boundary and have the appropriate sign for each boundary element built in. The same letters but in boldface or sans serif type represent the same objects but with a fixed sign (always the sign appropriate for the outer boundary). This is a very useful convention. Note that our use of sans serif characters has nothing at all to do with the use of sans serif characters used in KVK and LW.

C. Extrinsic curvatures

The extrinsic curvature of a Σ slice as embedded in $\mathcal{M}$ is

$$\mathrm{k} \equiv -\frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}u^a\right)\,. \tag{2.8}$$

It is easy to see that $\mathrm{k} = u_a b^a$, where $b^a \equiv n^b\nabla_b n^a$ is the spacetime "acceleration" of n. We will use $b^\perp \equiv -u_a b^a$ to denote the orthonormal component of the one-dimensional vector field b. One may also consider spacelike slices $\bar{\Sigma}$ which are everywhere orthogonal by assumption to the $\bar{\mathcal{T}}$ sheets. Since the spacetime-filling extension of the $\bar{\mathcal{T}}$ sheets from $\bar{\mathcal{T}}_o$ to $\bar{\mathcal{T}}_i$ is arbitrary, the $\bar{\Sigma}$ slices are almost as general as the Σ slices. However, the $\bar{\Sigma}$ slices are restricted by the requirement that their normal vector field coincides with $\bar{u}$ on the the boundary elements $\bar{\mathcal{T}}_o$ and $\bar{\mathcal{T}}_i$. We describe such slices as *clamped*. When the velocity v defined above is set to zero on the boundary, then the Σ slices are clamped. (In which case, there is no longer a need to make a distinction between barred and unbarred slices.) Also define an extrinsic curvature associated with the $\bar{\Sigma}$ slices,

$$\bar{\mathrm{k}} \equiv -\frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}\bar{u}^a\right) . \tag{2.9}$$

Like before, $\bar{\mathrm{k}} = \bar{u}_a\bar{b}^a$, where $\bar{b}^a = \bar{n}^b\nabla_b\bar{n}^a$ is the spacetime "acceleration" of $\bar{n}$.

The extrinsic curvature associated with the $\bar{\mathcal{T}}$ boundary elements is defined by

$$\bar{\vartheta} \equiv -\frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}\bar{n}^a\right) , \tag{2.10}$$

We have that $\bar{\vartheta} = -\bar{n}_b\bar{a}^b$, where $\bar{a}^b \equiv \bar{u}^a\nabla_a\bar{u}^b$ is the spacetime acceleration of $\bar{u}$. We may also consider a foliation $\mathcal{T}$ generated by the u-Eulerian histories of points in the Σ slices. At the boundary, the $\mathcal{T}$ sheets may be "crashing into" or "emerging from" the actual boundary elements $\bar{\mathcal{T}}_o$ and $\bar{\mathcal{T}}_i$. Nevertheless, one can define an extrinsic curvature

$$\vartheta \equiv -\frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}n^a\right) . \tag{2.11}$$

The value of ϑ at a particular boundary point of $\bar{\mathcal{T}}$ is associated with the $\mathcal{T}$ sheet intersecting that point. As before, $\vartheta = -n_b a^b$, where $a^b \equiv u^a\nabla_a u^b$ is the spacetime acceleration of u. We use the notation $a^{\vdash} \equiv n_b a^b$ for the orthonormal components of the one-dimensional vector field a. Since by assumption the metric-compatible connection associated with $\mathcal{M}$ is torsion-free, $a^{\vdash} = n[\log N]$.

With our transformation equations (2.7), one can derive the following "splitting" formulas for $\bar{\mathrm{k}}$ and $\bar{\vartheta}$:

$$\bar{\mathrm{k}} = \gamma\mathrm{k} + v\gamma\vartheta - \bar{n}[\eta] \tag{2.12a}$$

$$\bar{\vartheta} = \gamma\vartheta + v\gamma\mathrm{k} - \bar{u}[\eta] , \tag{2.12b}$$

where $\eta \equiv \tanh^{-1} v = \frac{1}{2}\log|(1+v)/(1-v)|$. Note that $\eta = \tanh^{-1} -\mathsf{v}$ on $\bar{\mathcal{T}}_i$ and $\eta = \tanh^{-1}\mathsf{v}$ on $\bar{\mathcal{T}}_o$. In accord with our conventions also set $\boldsymbol{\eta} \equiv \tanh^{-1}\mathsf{v}$. In obtaining these formulas, it helps to realize that $\delta\eta = \gamma^2\delta v$.

III. ACTION AND QUASILOCAL ENERGY-MOMENTUM

We begin this section by precisely defining the type of action functional associated with our bounded spacetime region $\mathcal{M}$ which is of interest in this work. We will discuss in detail the action's associated variational principle, paying strict attention to all boundary terms. This is the background work necessary to derive expressions for the quasilocal energy, momentum, and "stress" associated with a bounded slice Σ.

A. Variational principle

Our analysis begins with the following action functional:

$$S^{1} = \tfrac{1}{4}\,\alpha\int_{\mathcal{M}} \mathrm{d}^2x\sqrt{-g}e^{-2\Phi}\left[\mathcal{R} + 2yg^{ab}\partial_a\Phi\partial_b\Phi + 2\lambda^2 y\exp\left[(2-y)2\Phi\right]\right]$$
$$+\tfrac{1}{2}\,\alpha\int_{t''}^{t'} \mathrm{d}r\Lambda e^{-2\Phi}\mathrm{k} - \tfrac{1}{2}\,\alpha\int_{\bar{\mathcal{T}}} \mathrm{d}t\bar{N}e^{-2\Phi}\bar{\vartheta} - \tfrac{1}{2}\,\alpha e^{-2\Phi}\eta\Big|_{B'}^{B''}\,, \tag{3.1}$$

where $\mathcal{R}$ is the scalar curvature of $\mathcal{M}$ built from the metric g_{ab}, and the scalar field Φ is the celebrated *dilaton*. The variable y is an as-yet unspecified number, and λ is a positive constant with dimensions of inverse length. Finally, α is another positive (and possibly dimensionful) constant. The $\mathcal{M}$ integral in our action corresponds to a subclass of models within the larger framework generalized dilaton theories. [10] In (3.1) we have used the short-hand notation

$$\int_{t'}^{t''} = \int_{t''} - \int_{t'}\ , \tag{3.2}$$

where the one should note that here t' and t'' represent the initial and final slices and are not integration parameters. Also, it is understood that $\bar{\mathcal{T}}$ represents the not-simply-connected total timelike boundary $\bar{\mathcal{T}}_i\bigcup\bar{\mathcal{T}}_o$. Therefore, $\bar{\mathcal{T}}$ expressions stand for the sum of an inner-boundary and an outer-boundary expression. Likewise, B' and B'' expressions stand for the sum of an inner-corner and an outer-corner expression. The boundary terms in the above action ensure that its associated variational principle features fixation of induced metric and the dilaton on the boundary. Symbolically, we could collect the boundary terms into one expression

$$\left(S^{1}\right)_{\partial\mathcal{M}} = \tfrac{1}{2}\,\alpha\int_{\partial\mathcal{M}} \mathrm{d}x\sqrt{{}^{1}g}e^{-2\Phi}\mathrm{k}\,, \tag{3.3}$$

where here k is used for the extrinsic curvature over the whole boundary $\bar{\mathcal{T}}\bigcup t'\bigcup t''$. The corner-point contributions in (3.1) are included because, though the corner points are a set of measure zero in the integration of $e^{-2\Phi}\mathrm{k}$ over all of $\partial\mathcal{M}$, the term k becomes infinite at the corners, since the boundary normal changes discontinuously from u to $\bar{n}$ at these points.[3] Finally, we write S^{1} for the action, because we anticipate the need to append to the action a Gibbons-Hawking subtraction term $-S^{0}[\bar{N},\Lambda,\Phi]$ (a functional of the fixed boundary data). [11,5] In this case the full action would be $S \equiv S^{1} - S^{0}$. We briefly consider the more general action later in the appendix. Also we could add to the action a matter contribution S^m. Most of our work in this section, §3 on quasilocal energy-momentum, would be unaffected by an S^m contribution to the action, as long as the matter fields were minimally coupled. However, an S^m contribution would affect the following sections devoted to the canonical reduction. Therefore, for the sake of simplicity we do not consider an S^m further. More comments on the notation will follow below when it is appropriate.

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[3]Such corner terms where first considered for four-dimensional gravity in Refs. [9].

We are interested in spherically symmetric general relativity (SSGR). A suitable action for SSGR is given by (3.1) with the choices $y = 1$, $\alpha = \lambda^{-2}$. Note that for the SSGR case, we set α equal to the *dimensionful* constant λ^{-2}. This gives our action the units of action in four dimensions. The appendix shows that this action does indeed describe general relativity with the ansatz of spherical symmetry (where the *four dimensional* action principle is associated with the time history ${}^4\mathcal{M}$ of a spatial region bounded by concentric spheres). In this correspondence with SSGR, it turns out that the radius of a round sphere is given by

$$R = \sqrt{\frac{A}{4\pi}} = \frac{e^{-\Phi}}{\lambda}\,, \tag{3.4}$$

where A stands for the proper area of the sphere.

In this paper we also consider the CGHSW pure-dilaton model, which corresponds to $y = 2$ and α taken to be a dimensionless positive number. The favorite choice in the literature has been $\alpha = 2/\pi$. But α remains essentially arbitrary. The arbitrariness of α for the CGHSW model corresponds to the freedom to shift the dilaton by a constant $\Phi \rightarrow \Phi - \log\sqrt{\alpha'}$ (a freedom not present for the SSGR action). Under such a shift, $\alpha \rightarrow \alpha'' = \alpha\alpha'$. We would actually prefer to set $\alpha = 2$ for reasons which become clear later. Nevertheless, when dealing with the CGHSW model, this paper leaves α arbitrary, so the reader can pick her or his favorite convention. For the SSGR case the action S^1 has dimensions of length-squared, while for the CGHSW case the action S^1 is dimensionless. This difference in units will propagate throughout all the formulae to follow. However, the freedom of allowing α to be either λ^{-2} or a plain number will automatically keep track of the correct units for both cases. The formulae to follow will depend on y and α, and the reader is free to choose whether they hold for SSGR or the CGHSW theory. For the SSGR case ($y = 1, \alpha = \lambda^{-2}$) all of our conventions have been tailored to match those of KVK and LW.

The first step is to compute the variation of the action. One can compute the variation in a number of ways, but the fastest way is the following. Note that $\mathcal{R}$ is a pure divergence. Then use an integration by parts followed by an appeal to Stokes' theorem on the $\mathcal{R}$ term in the action. This leads to cancelation of most of the boundary terms. This short calculation and resulting form of the action are given later in the discussion in the text preceding (4.3). Vary the resulting form of the action (4.3) to find

$$\begin{aligned}\delta S^1 &= \text{(terms giving the equations of motion)}\\ &\quad + \int_{t'}^{t''} \mathrm{d}r\,(P_\Lambda \delta\Lambda + P_\Phi \delta\Phi) + \int_{\bar{\mathcal{T}}} \mathrm{d}t\,\left(\bar{\Pi}_{\bar{N}}\delta\bar{N} + \bar{\Pi}_\Phi \delta\Phi\right) + \alpha e^{-2\Phi}\eta\delta\Phi\Big|_{B'}^{B''}\,,\end{aligned} \tag{3.5}$$

where we have defined the momenta

$$P_\Lambda \equiv \alpha e^{-2\Phi} u[\Phi] \tag{3.6a}$$

$$P_\Phi \equiv -\alpha e^{-2\Phi}\Lambda\,(\mathrm{k} + y u[\Phi]) \tag{3.6b}$$

$$\bar{\Pi}_{\bar{N}} \equiv -\alpha e^{-2\Phi}\bar{n}[\Phi] \tag{3.6c}$$

$$\bar{\Pi}_\Phi \equiv \alpha e^{-2\Phi}\bar{N}\left(\bar{\vartheta} + y\bar{n}[\Phi]\right)\,, \tag{3.6d}$$

Inspection of the variation of the action shows that P_Λ is the gravitational momentum conjugate to Λ. Likewise, $\bar{\Pi}_{\bar{N}}$ is the gravitational momentum conjugate to $\bar{N}$, where now conjugacy is defined with respect to $\bar{\mathcal{T}}$. The momentum conjugate to the dilaton field is P_Φ. Later in the discussion we will also address the interpretation of the momentum-like quantity $\bar{\Pi}_\Phi$. Note that we could define a momentum $p_\Phi \equiv \alpha e^{-2\Phi}\eta$ in some sense conjugate to Φ at the corners. However, we chose not to do this.[4] If the corner terms in the action (3.1) had not been included, then the variational principle would have featured fixation of η on the corner points. Fixing Φ at the corners seems to be more in harmony with the fact that Φ is fixed on $\bar{\mathcal{T}}$, t', and t''.

Our momenta P_Λ and P_Φ agree with the analysis of KVK. To see this take the SSGR case and use the fact that $P_R = -(1/R)P_\Phi$. One then finds precisely Kuchař's momenta,

$$P_\Lambda = -N^{-1}R\left(\dot{R} - N^r R'\right) \tag{3.7a}$$

$$P_R = -N^{-1}\left[R\left(\dot{\Lambda} - (\Lambda N^r)'\right) + \Lambda\left(\dot{R} - N^r R'\right)\right]. \tag{3.7b}$$

To get the last expression, we have used the definition (2.8) in the preliminary section to find $\mathrm{k} = -(N\Lambda)^{-1}\left[\dot{\Lambda} - (\Lambda N^r)'\right]$. (Note, however, that due to well-entrenched notation for the dilaton, we unfortunately must break with the KVK convention of using Greek letters only to represent spatial densities like Λ. Though represented by a Greek letter, the dilaton Φ is a scalar and its momentum P_Φ is a density.)

B. Quasilocal energy-momentum

As advertised, our variational principle has been rigged so that the induced metric $(\Lambda^2, -\bar{N}^2)$ and the dilaton Φ are fixed as boundary data. In particular, the lapse of proper time between the initial and final slices is fixed in the variational principle, since this information is encoded in the $\bar{\mathcal{T}}$ metric $-\bar{N}^2$. This is the key feature exploited in the Brown-York theory [5,6] of quasilocal stress-energy-momentum in general relativity. Following the basic line of reasoning in this theory, we will "read off" from the variation of the action what geometric expressions play the role of quasilocal energy and momentum in our theory.

We begin by writing the boundary terms in the variation (3.5) of the action in the following suggestive way:

$$\left(\delta S^1\right)_{\partial\mathcal{M}} = -\int_{t'}^{t''} \mathrm{d}r\,(J\delta\Lambda + \Lambda T\delta\Phi) - \int_{\bar{\mathcal{T}}} \mathrm{d}t\left(\bar{E}\delta\bar{N} + \bar{N}\bar{S}\delta\Phi\right) + \alpha e^{-2\Phi}\eta\delta\Phi\Big|_{B'}^{B''}, \tag{3.8}$$

where we have defined the *scalars*

$$\bar{E} \equiv -\bar{\Pi}_{\bar{N}} = \alpha e^{-2\Phi}\bar{n}[\Phi] \tag{3.9a}$$

$$J \equiv -P_\Lambda = -\alpha e^{-2\Phi}u[\Phi] \tag{3.9b}$$

$$\bar{S} \equiv -\bar{\Pi}_\Phi/\bar{N} = -\alpha e^{-2\Phi}(\bar{\vartheta} + y\bar{n}[\Phi]) \tag{3.9c}$$

$$T \equiv -P_\Phi/\Lambda = \alpha e^{-2\Phi}(\mathrm{k} + yu[\Phi]). \tag{3.9d}$$

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[4]So, strictly speaking, when we pass to the canonical form of the theory, the action will be in a mixed Hamiltonian-Lagrangian form.

Do not confuse $\bar{S}$ in the above set with the action functional S^1.

We interpret $\bar{E}$ as the *quasilocal energy* associated with the $\bar{u}$ observers at $\bar{\mathcal{T}}$. In other words, $\bar{E}$ is the total energy of the gravitational and dilaton fields which live on a spatial slice $\bar{\Sigma}$ which is orthogonal to $\bar{\mathcal{T}}$. The slice $\bar{\Sigma}$ has $\bar{u}$ as its timelike normal *at the boundary* $\bar{\mathcal{T}}$.[5] Note that, when evaluated on solutions to the field equations, $\bar{E}$ is minus the rate of change of the classical action (or Hamilton-Jacobi principal function) with respect to a unit stretch in $\bar{N}$, where $\bar{N}$ controls the lapse of proper time between neighboring points on $\bar{\mathcal{T}}$. [5] Our notation is somewhat compact, as there are actually inner-boundary and outer-boundary contributions to consider for $\bar{E}$. Indeed,

$$\bar{E} = \bar{E}_i + \bar{E}_o = \alpha e^{-2\Phi}\, \bar{\mathsf{n}}[\Phi]\Big|_{B_i}^{B_o} \tag{3.10}$$

is the total quasilocal energy. There are also inner and outer contributions to $\bar{S}$. It is easy to obtain the expression for the quasilocal energy associated with the u observers at $\bar{\mathcal{T}}$ (which is the energy of the spanning slice Σ to which u is the timelike normal). From the results of Ref. [6], we know that we merely need to "unbar" the $\bar{E}$ expression to find

$$E \equiv \alpha e^{-2\Phi}\, n[\Phi]\,. \tag{3.11}$$

Note that E depends on the canonical variables of the Σ, specifically Λ and Φ, so E is a functional on the Σ phase space (as $\bar{E}$ is a functional on the $\bar{\Sigma}$ phase space). Though we have not yet given an interpretation for $\bar{S}$, also define its unbarred version

$$S \equiv -\alpha e^{-2\Phi}\left[-(a \cdot n) + yn[\Phi]\right]\,. \tag{3.12}$$

Note that S does not depend solely on Σ Cauchy data, as it depends on the spacetime acceleration a of u. So far the expression E is really the total energy $E_i + E_o$, and also $S = S_i + S_o$. However, it is more convenient to associate an E and S with each separate boundary point of a particular slice. Therefore, *often in the remainder of this work, and depending on the context, the expressions for E (and $\bar{E}$) and S (and $\bar{S}$) are associated with a single spacetime point.* Furthermore, it proves useful in the next section to have an energy expression for *each* point[6] of Σ. However, we have a sign ambiguity, because each Σ point could be viewed as an inner or an outer boundary point. For the sake of definiteness, define

$$\mathsf{E} \equiv \alpha e^{-2\Phi}\mathsf{n}[\Phi] = \alpha e^{-2\Phi}\Phi'/\Lambda\,. \tag{3.13}$$

For some expressions, like E^2, the sign ambiguity cancels, so we can use E^2 or E^2.

---

[5]Actually, $\bar{E}$ is the energy associated with *any* slice in the equivalence class determined by this condition. The slice $\bar{\Sigma}$ can be "wiggled" in the interior as long as its ends remain clamped to the boundary $\bar{\mathcal{T}}$.

[6]Recall that the Σ points are spheres, at least in the SSGR case. So we are not really defining a local energy, and certainly not a local energy *density*.

We consider J to be a *quasilocal momentum*. Notice that on-shell J is minus the rate of change of the Hamilton-Jacobi principle function with respect to a unit stretch in Λ, where Λ controls the lapse of proper distance between neighboring radial leaves B of Σ. Such a variation in the boundary data corresponds to a dilation of Σ. At a glance one sees that J and T are also a functionals on the Σ phase space.

To sharpen our understandings of the quantities E, J, S, and T, let us examine the correspondence of our work thus far with the theory of quasilocal stress-energy-momentum for general relativity as laid out in Refs. [5,6,12]. These deal with the full theory, but we may specialize their results to the case of spherical symmetry. Take our two-dimensional metric g_{ab} from the preliminary section and adjoin to it the metric of a round sphere. The result is the four-dimensional spherically symmetric metric

$$ {}^4g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -N^2\mathrm{d}t^2 + \Lambda^2\left(\mathrm{d}r + N^r\mathrm{d}t\right)^2 + R^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) . \tag{3.14} $$

Now every point $B(r,t)$ of $\mathcal{M}$ is a round sphere of radius $R(t,r)$. Therefore, our $1+1$ spacetime region $\mathcal{M}$ has now been promoted to a four-dimensional time history ${}^4\mathcal{M}$ of a three-dimensional spatial region which lies between concentric spheres. Slices Σ and sheets $\bar{\mathcal{T}}$ now correspond to three-dimensional submanifolds of ${}^4\mathcal{M}$. In particular, the boundary elements $\bar{\mathcal{T}}_i$ and $\bar{\mathcal{T}}_o$ are now 2+1 hypersurfaces in ${}^4\mathcal{M}$. Consider the timelike 2+1-dimensional hypersheets $\mathcal{T}$ which are generated by the integral curves of the Σ normal u. Just like in the $1+1$ scenario, these sheet may "emerge from" or "crash into" the actual boundary $\bar{\mathcal{T}}$. Pick one such sheet which intersects the spatial boundary. By construction the normal n of $\mathcal{T}$ as embedded in ${}^4\mathcal{M}$ is orthogonal to u. Now, in the notation of Ref. [5,6] k represents the trace of the extrinsic curvature of a two surface (in the present case a round sphere) as embedded in a three-dimensional spacelike slice Σ. *This is not our* k (block-Roman script) defined in the preliminary section. Be careful to make the distinction between k and k. Also, as in Ref. [6], we use ℓ to represent the trace of the extrinsic curvature of same round sphere as embedded in a $\mathcal{T}$ sheet. Quick calculations show that

$$ \begin{aligned} k &= -2(R\Lambda)^{-1}R' = 2n[\Phi] \\ \ell &= -2(RN)^{-1}(\dot{R} - N^r R') = 2u[\Phi] , \end{aligned} \tag{3.15} $$

where we have used the correspondence (3.4) and assumed that the round sphere of interest is the outer one. Moreover, for the four-dimensional scenario it is straightforward to define the acceleration components $(a \cdot n)$ and $(b \cdot u)$. With the ansatz of spherical symmetry, the four-dimensional expressions for $(a \cdot n)$ and $(b \cdot u)$ are the obvious generalizations of the formulas for $(a \cdot n)$ and $(b \cdot u)$ given for the two-dimensional scenario in the preliminary section.

Now simply divide E, J, S, and T by $(4\pi\alpha)e^{-2\Phi}$ (this is the spherical area for the SSGR case). This gives the following "surface densities":

$$ \begin{aligned} \varepsilon &\equiv (4\pi\alpha)^{-1}e^{2\Phi}E = (8\pi)^{-1}k \\ j_{\vdash} &\equiv (4\pi\alpha)^{-1}e^{2\Phi}J = -(8\pi)^{-1}\ell \\ s &\equiv (4\pi\alpha)^{-1}e^{2\Phi}S = (8\pi)^{-1}\left(2a\cdot n - yk\right) \\ t &\equiv (4\pi\alpha)^{-1}e^{2\Phi}T = (8\pi)^{-1}\left(2b\cdot u + y\ell\right) . \end{aligned} \tag{3.16} $$

The expressions on the right-hand side may now be interpreted as four-dimensional quantities. Note that for both of the models that we are considering in this paper the ε and $j_\vdash$ expressions[7] match the four-dimensional expressions for ε and $j_\vdash$ given in Refs. [5,6,12]. As expected, for the SSGR case, our s and t match s^a_a, the trace of the *spatial stress* tensor s^{ab}, and t^a_a, the trace of the *temporal stress* tensor t^{ab}, of those references. One should note that the indices on s^{ab} and t^{ab} are associated with the round sphere not our two-dimensional spacetime $\mathcal{M}$. The surface density s^{ab} describes the flux of the a component of momentum in the b direction. [5] Our S may be thought of as the integrated trace of this spatial stress tensor.

C. Boost relations and invariants

Return to the two-dimensional scenario and consider again the clamped spacelike slice $\bar{\Sigma}$ which has a normal vector field which agrees with the normal $\bar{u}$ *on the boundary* $\bar{\mathcal{T}}$ (again, defining an equivalence class of such slices). Clearly the Eulerian observers of $\bar{\Sigma}$ at $\bar{\mathcal{T}}$ coincide with the natural observers in the boundary. We may define a set of quasilocal quantities for these observers,

$$\bar{E} = \alpha e^{-2\Phi}\bar{n}[\Phi] \tag{3.17a}$$
$$\bar{J} = -\alpha e^{-2\Phi}\bar{u}[\Phi] \tag{3.17b}$$
$$\bar{S} = -\alpha e^{-2\Phi}(\bar{\vartheta} + y\bar{n}[\Phi]) \tag{3.17c}$$
$$\bar{T} = \alpha e^{-2\Phi}(\bar{\mathrm{k}} + y\bar{u}[\Phi])\,. \tag{3.17d}$$

Notice that the set $\bar{E}$, $\bar{J}$, $\bar{S}$, and $\bar{T}$ has the same dependence on the Cauchy data of $\bar{\Sigma}$ as the set E, J, S has on the Cauchy data of Σ (but both S and $\bar{S}$ do not depend on Cauchy data alone). Now the slice Σ need not be clamped to the boundary $\bar{\mathcal{T}}$ in our formalism. Hence, in general Σ and $\bar{\Sigma}$ are different slices which intersect at the same boundary point of interest. We will refer to a switch of the spatial slice spanning a particular boundary point as a *generalized boost* or simply a boost. Properly speaking, a generalized boost is a switch of the equivalence class of spanning slices. The behavior of the quasilocal quantities under boosts,

$$\bar{E} = \gamma E - v\gamma J \tag{3.18a}$$
$$\bar{J} = \gamma J - v\gamma E \tag{3.18b}$$
$$\bar{S} = \gamma S - v\gamma T + \alpha e^{-2\Phi}\bar{u}[\eta] \tag{3.18c}$$
$$\bar{T} = \gamma T - v\gamma S - \alpha e^{-2\Phi}\bar{n}[\eta]\,. \tag{3.18d}$$

follows immediately from the boost relations (2.7) and the splittings (2.12).

---

[7] We write $j_\vdash$, because in the four-dimensional case one deals with a momentum surface density j_k which may have components tangential to the generic two-surface B of interest. In this case $j_\vdash = (j \cdot n)$, where n is the normal of B as embedded in Σ.

Clearly the expression $-E^2 + J^2$ is invariant under boosts.[8] We may multiply it by any function of the dilaton field or add to it any function of the dilaton field and retain a boost-invariant expression. Therefore, it is not completely unnatural to introduce the invariant

$$M = (2\alpha\lambda)^{-1} e^{y\Phi} \left[-E^2 + J^2 + (\alpha\lambda)^2 \exp\left[-2y\Phi\right]\right] . \tag{3.19}$$

Note that M has units of length for the SSGR case and units of inverse length for the CGHSW case. It turns out that on-shell (on solutions to the field equations) M is a completely conserved quantity (constant in time and space). Moreover, for the case of SSGR we find that $M = M_S$, where M_S is Kuchař's canonical expression for the Schwarzschild mass parameter,

$$M_S = \frac{P_\Lambda^2}{2R} - \frac{R}{2}\left(\frac{R'}{\Lambda}\right)^2 + \frac{R}{2} . \tag{3.20}$$

From the four-dimensional spacetime perspective, the expression for M_S corresponds to several mass definitions in general relativity, when spherical symmetry is assumed. One is the Hawking mass [13,14]

$$M_H = \frac{1}{8\pi}\sqrt{\frac{A}{16\pi}} \int_B \mathrm{d}^2x \sqrt{\sigma} \left[-\tfrac{1}{2}(k^2 - \ell^2) + {}^2\!R\right] , \tag{3.21}$$

where ${}^2\!R$ is the scalar curvature of the two-surface B and the boost-invariant combination $\frac{1}{2}(k^2 - \ell^2)$ is more often written as a product of spin coefficients. The factor A is the area of B which is included so that the whole expression has units of energy. With the ansatz of spherical symmetry, the Hawking mass coincides[9] with the Ashtekar-Hansen expression, [14,15]

$$M_{AH} = \frac{1}{8\pi}\sqrt{\frac{A}{16\pi}} \int_B \mathrm{d}^2x \sqrt{\sigma} \sigma^{\mu\nu} \sigma^{\lambda\kappa} C_{\mu\lambda\nu\kappa} , \tag{3.22}$$

where $C_{\mu\lambda\nu\kappa}$ is the Weyl tensor of (3.14), and here the two-metric $\sigma_{\mu\nu} = {}^4g_{\mu\nu} - n_\mu n_\nu + u_\mu u_\nu$ serves as the projection operator into the sphere. We stress that, assuming spherical symmetry, M_H and M_{AH} correspond to Kuchař's M_S even off-shell. They can be expressed purely in terms of the canonical data of spacelike slices.[10] For Schwarzschild both M_H

[8] In accord with the comment given before (3.13), here we are working at a single $\mathcal{M}$ point.

[9] See Ref. [12] for the details.

[10] This is not true of, say, the Komar integral [16]. The Komar integral for round spheres in the Schwarzschild geometry yields the mass parameter (the on-shell value of M_S), but is not equivalent to M_S as a canonical expression. Indeed, the Komar integral depends on the lapse function associated with the static-time slices.

and M_{AH} yield the on-shell value of M_S even for finite two-spheres, but in general one must consider the suitable asymptotic limit to get the ADM mass. For general closed two surfaces in general spacetimes, the Hawking and Ashtekar-Hansen expressions can be "built" as a combination of quasilocal boost invariants. [6,12]

For the pure dilaton case set $M_W = 2M/\alpha$. By expressing M_W in covariant form,

$$M_W = \lambda^{-1} e^{-2\Phi} \left(\lambda^2 - g^{ab} \partial_a \Phi \partial_b \Phi \right) , \tag{3.23}$$

we see that it is the "local mass" of Tada and Uehara. [17] Such a quantity was also considered by Frolov in Ref. [18]. With an argument originally given by KVK for the Schwarzschild case, the appendix shows that M_W is the canonical expression for the mass parameter of the Witten black hole. The appendix also shows that the ADM energy [19] at spatial infinity (associated with the preferred static-time slices) of the Witten black hole is the on-shell value of M. This is the reason we would prefer to set $\alpha = 2$.

IV. CANONICAL THEORY

This section is devoted to the canonical form of the theory. We first sketch the Legendre transformation which yields the canonical form of the action. We then vary the canonical action, paying strict attention to all boundary terms. Finally, we consider the canonical transformation of KVK and write down a new-canonical-variable version of the action (3.1) which is particularly amenable to canonical reduction.

A. Form of the canonical action

Passage to the canonical form of the action (3.1) demands that we write the action in $1+1$ form as a preliminary step. This is easily done with three ingredients. The first is the splitting result (2.12b) for the $\bar{\mathcal{T}}$ extrinsic curvature $\bar{\vartheta}$. The second is the identity

$$2y g^{ab} \partial_a \Phi \partial_b \Phi = 2y \left[- \left(\frac{\dot{\Phi}}{N} \right)^2 + 2N^r \left(\frac{\dot{\Phi}\Phi'}{N^2} \right) + \left(\frac{\Phi'}{\Lambda\gamma} \right)^2 \right] , \tag{4.1}$$

where γ is the relativistic factor of the preliminary section. The third and final ingredient is the realization that for our two dimensional spacetime the Ricci scalar is a pure divergence,

$$\mathcal{R} = -2\nabla_b \left[\mathrm{k} u^b + (a \cdot n) n^b \right] . \tag{4.2}$$

Using these three ingredients, one can quickly cast (3.1) into the form

$$S^1 = \tfrac{1}{4}\alpha \int_{\mathcal{M}} \mathrm{d}^2 x N \Lambda e^{-2\Phi} X + \int_{\bar{\mathcal{T}}} \mathrm{d}t \, \alpha e^{-2\Phi} \eta \dot{\Phi} , \tag{4.3}$$

where X is the expression

$$
\begin{aligned}
X = & -\frac{4}{N\Lambda}\left[\Lambda \mathrm{k}\left(\dot{\Phi} - N^r\Phi'\right) + \frac{N'\Phi'}{\Lambda}\right] \\
& + 2y\left[-\left(\frac{\dot{\Phi}}{N}\right)^2 + 2N^r\left(\frac{\dot{\Phi}\Phi'}{N^2}\right) + \left(\frac{\Phi'}{\Lambda\gamma}\right)^2\right] + 2\lambda^2 y \exp\left[(2-y)2\Phi\right] .
\end{aligned} \tag{4.4}
$$

With these formulas it is not extremely difficult to express the canonical action as follows:

$$
S^{1} = \int_{\mathcal{M}} \mathrm{d}^2x\left(P_\Lambda\dot{\Lambda} + P_\Phi\dot{\Phi} - N\mathcal{H} - N^r\mathcal{H}_r\right) + \int_{\bar{\mathcal{T}}} \mathrm{d}t\left(\alpha e^{-2\Phi}\eta\dot{\Phi} - \bar{N}\bar{E}\right) , \tag{4.5}
$$

where here $\bar{E}$ is shorthand for $\gamma E - v\gamma J$. It is important to realize that in the canonical picture the equation $\bar{E} = \alpha e^{-2\Phi}\bar{n}[\Phi]$ is not necessarily valid, for equality implicitly assumes the *canonical equation of motion* $P_\Lambda = \alpha e^{-2\Phi}u[\Phi]$. Respectively, the Hamiltonian constraint and the momentum constraint have the form

$$
\begin{aligned}
\mathcal{H} = & \, \alpha^{-1}e^{2\Phi}\left[P_\Phi P_\Lambda + \tfrac{1}{2}y\Lambda(P_\Lambda)^2\right] \\
& + \alpha e^{-2\Phi}\left[\left(2 - \tfrac{1}{2}y\right)\frac{\Phi'^2}{\Lambda} - \frac{\Phi''}{\Lambda} + \frac{\Phi'\Lambda'}{\Lambda^2} - \tfrac{1}{2}\lambda^2\Lambda y \exp\left[(2-y)2\Phi\right]\right]
\end{aligned} \tag{4.6a}
$$

$$
\mathcal{H}_r = P_\Phi\Phi' - \Lambda P_\Lambda' . \tag{4.6b}
$$

Again, for the case of SSGR our results match those of KVK and LW. Notice that it is P_Λ which appears differentiated in the momentum constraint $\mathcal{H}_r$. This is to be expected, as Λ is a scalar density.

B. Variation of the canonical action

Straightforward but fairly tedious manipulations establish that the variation of the canonical action is

$$
\begin{aligned}
\delta S^{1} = & \,(\text{terms which enforce the constraints and give the} \\
& \quad \textit{Hamiltonian} \text{ equations of motion}) + \int_{t'}^{t''} \mathrm{d}r\left(P_\Lambda\delta\Lambda + P_\Phi\delta\Phi\right) \\
& - \int_{\bar{\mathcal{T}}} \mathrm{d}t\left[\bar{E}\delta\bar{N} + \bar{N}\bar{S}\delta\Phi - \left(\bar{N}\bar{J} + \alpha e^{-2\Phi}\dot{\Phi}\right)\delta\eta\right] + \alpha e^{-2\Phi}\eta\delta\Phi\Big|_{B'}^{B''} ,
\end{aligned} \tag{4.6}
$$

where here $\bar{E}$, $\bar{J}$, and $\bar{S}$ are shorthand for the expressions

$$
\bar{E} = \gamma E - v\gamma J \tag{4.7a}
$$

$$
\bar{J} = \gamma J - v\gamma E \tag{4.7b}
$$

$$
\bar{S} = \gamma S - v\gamma T + \alpha e^{-2\Phi}\bar{u}[\eta] . \tag{4.7c}
$$

As mentioned, one should be careful, for while E, J, S, and T are built from the canonical variables $(\Lambda, P_\Lambda; \Phi, P_\Phi)$ in the same way as before, *now the momenta* P_Λ *and* P_Φ *need not have the forms given in* (3.6) (which are canonical equations of motion). The term which appears multiplied by the variation $\delta\eta$ vanishes when the canonical equation of motion for P_Λ holds. Therefore, η is not a quantity which is held fixed in our variational principle.

C. Canonical Transformation

In this subsection we perform Kuchař's canonical transformation on the phase-space pairs $(\Lambda, P_\Lambda; \Phi, P_\Phi)$. In order to grasp the underlying hyperbolic geometry of this canonical transformation, we first need to collect a few results and observations.

Consider a black-hole solution which extremizes the action (3.1) (either a Schwarzschild black hole or a Witten black hole, depending on whether y is 1 or 2). Associated with this solution, there is a preferred family of static-time slices, the collection of constant-Killing-time level surfaces. For the Schwarzschild-black-hole case let $T(t,r)$ denote the Killing time, and for the Witten-black-hole case let $\tau(t,r)$ denote the Killing time. Now, given a particular $\mathcal{M}$ point B, we may interpret it as a boundary point of the static-time slice $\tilde{\Sigma}$ which contains it. Our construction defines the *rest frame* $(\tilde{u},\tilde{n})$ at B, where $\tilde{u}$ is the normal of $\tilde{\Sigma}$ as embedded in $\mathcal{M}$ and $\tilde{n}$ is the normal of B as embedded in $\tilde{\Sigma}$. If B is also considered to be a point of the boundary $\bar{\mathcal{T}}$, then in general $\tilde{\Sigma}$ does not define the same frame at B as the slice Σ or the slice $\bar{\Sigma}$ considered before. We know how to compute the energy $\tilde{E}$ and momentum $\tilde{J}$ of the bounded static-time slice $\tilde{\Sigma}$,

$$\begin{aligned}\tilde{E} &= \alpha e^{-2\Phi}\tilde{n}[\Phi]\\ \tilde{J} &= -\alpha e^{-2\Phi}\tilde{u}[\Phi]\,.\end{aligned} \tag{4.7}$$

Clearly $\tilde{E}$ and $\tilde{J}$ depend on the Cauchy data of $\tilde{\Sigma}$ in the same way that E and J depend on the Cauchy data of Σ.

Now, it is a fact that $\tilde{J}=0$, which is why we refer to $(\tilde{u},\tilde{n})$ as the rest frame at B. The existence of the rest frame at B leads to a remarkable fact: *at B the parameter φ associated with the boost from the frame (u,n) defined by Σ to the rest frame $(\tilde{u},\tilde{n})$ is determined by the canonical variables of Σ.*[11] Indeed, with $w \equiv J/E$ the boost from the Σ frame to the rest frame is parameterized by

$$\varphi \equiv \tfrac{1}{2}\log\left|\frac{1+w}{1-w}\right| = \tfrac{1}{2}\log\left|\frac{E+J}{E-J}\right|\,. \tag{4.8}$$

Notice that $\tilde{u} = \psi u + w\psi n$ with $\psi = (1-w^2)^{-1/2}$. At this stage we have a sign ambiguity in our expressions, since we did not say whether B is an inner or an outer boundary point. For the sake of definiteness in what follows, we often want to assume that B is taken as an outer boundary point and make this distinction notationally. Therefore, we use E, defined before in (3.13), and also $\mathsf{w} = J/\mathsf{E}$ which defines $\boldsymbol{\varphi} \equiv \frac{1}{2}\log|(1+\mathsf{w})/(1-\mathsf{w})|$. Notice that the sign ambiguities cancel in the expressions $E\varphi = \mathsf{E}\boldsymbol{\varphi}$ and $E^2 = \mathsf{E}^2$.

Since we have a canonical expression for $\boldsymbol{\varphi}$, it is easy to write down a new set of constraints which generate unit displacements with respect to the static-time slices. Moreover, the new constraints will depend only on the canonical variables of Σ, so we can consider them off the constraint surface in phase space (in which case the canonical data of Σ does not obey the Σ constraints). Indeed, define

---

[11]Which need to be known only in a tiny neighborhood of B.

$$\mathsf{H} = \psi\mathcal{H} + \mathsf{w}\psi(\mathcal{H}_r/\Lambda) \tag{4.9a}$$
$$\mathsf{H}_\vdash = \psi(\mathcal{H}_r/\Lambda) + \mathsf{w}\psi\mathcal{H}\,. \tag{4.9b}$$

Note that, for instance, H is not really of the form $-(\tilde{u}\cdot u)\mathcal{H} + (\tilde{u}\cdot n)(\mathcal{H}_r/\Lambda)$ when the canonical variables do not obey the Σ constraints. However, when computing a Poisson bracket $\{G, \mathsf{H}\}$, where G is a functional of the canonical variables, one finds that all the brackets $\{G, \mathsf{w}\}$ which arise come multiplied by either by factor of $\mathcal{H}$ or $\mathcal{H}_r$. Therefore, on-shell H generates unit evolution normal to the static-time slices.

Define the new canonical variables in terms of the old ones as follows:

$$M = \tfrac{1}{2}\,\alpha\lambda e^{-y\Phi}(1-F) \tag{4.10a}$$
$$P_M = (\alpha\lambda)^{-1}e^{y\Phi}F^{-1}\Lambda P_\Lambda \tag{4.10b}$$

$$\Psi = \Phi \tag{4.10c}$$
$$P_\Psi = P_\Phi + \tfrac{1}{2}y(1+F^{-1})\Lambda P_\Lambda + \alpha e^{-2\Phi}\boldsymbol{\varphi}'\,, \tag{4.10d}$$

where M is the boost invariant (3.19) and in terms of the old variables F and $\boldsymbol{\varphi}$ are shorthand for

$$F = (\alpha\lambda)^{-2}\exp\left[(y-2)2\Phi\right]\left[(\alpha\Phi'/\Lambda)^2 - (e^{2\Phi}P_\Lambda)^2\right] \tag{4.11a}$$

$$\boldsymbol{\varphi} = \tfrac{1}{2}\log\left|\frac{\alpha e^{-2\Phi}\Phi' - \Lambda P_\Lambda}{\alpha e^{-2\Phi}\Phi' + \Lambda P_\Lambda}\right|\,. \tag{4.11b}$$

Notice that $\boldsymbol{\varphi} = \frac{1}{2}\log|F_-/F_+|$, where

$$F_\pm = (\alpha\lambda)^{-1}e^{y\Phi}(\mathsf{E} \mp J)\,. \tag{4.12}$$

Evidently then, another expression for F of key importance is

$$F = F_+F_- = (\alpha\lambda)^{-2}e^{y2\Phi}\left(E^2 - J^2\right)\,. \tag{4.13}$$

For the SSGR case $e^{-\Psi} = \lambda\mathsf{R}$ and $P_\Psi = -\mathsf{R}P_\mathsf{R}$, in the notation of KVK and LW. Also for this case, one finds that $P_M = -T'$, because as shown in KVK the canonical expression for (minus) the radial derivative of the Killing time is given by $-T' = (RF)^{-1}\Lambda P_\Lambda$. The situation is the same for the CGHSW model, as in this case $P_M = -\tau'$. We show in the appendix that the canonical expression for (minus) the radial derivative of the Witten-black-hole Killing time is $-\tau' = (\alpha\lambda)^{-1}e^{2\Phi}F^{-1}\Lambda P_\Lambda$. One may prove that the transformation $(\Lambda, P_\Lambda; \Phi, P_\Phi) \to (M, P_M; \Psi, P_\Psi)$ is canonical for our boundary conditions by verifying the identity

$$P_\Lambda\delta\Lambda + P_\Phi\delta\Phi - P_M\delta M - P_\Psi\delta\Psi = \delta\left[\Lambda(-J + E\varphi)\right] - \left(\alpha e^{-2\Phi}\delta\Phi\boldsymbol{\varphi}\right)'\,, \tag{4.14}$$

which upon integration over r shows that the difference between the old Liouville form and the new Liouville is an exact form. Hence, the transformation is canonical.

Recall that for simplicity we wish to restrict our attention to the right-static region of the relevant Kruskal diagram. Now, in fact, for both SSGR and the CGHSW model F is the canonical expression for $\tilde{N}^2$, where $\tilde{N}$ is the lapse function associated with the static-time slices. On-shell, the event horizon of a particular black-hole solution is the locus of points determined by $F = 0$. We may ensure that we are working exclusively in a static region of the Kruskal diagram by choosing our boundary conditions appropriately and by excluding solutions for which $F = 0$ somewhere on $\mathcal{M}$. Where F is nonzero, the above transformation may be inverted,

$$\Lambda = \left[\lambda^{-2} F^{-1} (\Psi')^2 \exp\left[(y-2)2\Psi\right] - F(P_M)^2\right]^{1/2} \tag{4.15a}$$

$$P_\Lambda = \frac{\alpha\lambda e^{-y\Psi} F P_M}{\left[\lambda^{-2} F^{-1} (\Psi')^2 \exp\left[(y-2)2\Psi\right] - F(P_M)^2\right]^{1/2}} \tag{4.15b}$$

$$\Phi = \Psi \tag{4.15c}$$

$$P_\Phi = P_\Psi - \tfrac{1}{2}\alpha\lambda y e^{-y\Psi} P_M (1+F) - \alpha e^{-2\Psi} \boldsymbol{\varphi}', \tag{4.15d}$$

where in terms of the new variables F and $\boldsymbol{\varphi}$ are shorthand for

$$F = 1 - 2(\alpha\lambda)^{-1} e^{y\Psi} M \tag{4.16a}$$

$$\boldsymbol{\varphi} = \tfrac{1}{2}\log\left|\frac{\Psi' - \lambda\exp\left[(2-y)\Psi\right] F P_M}{\Psi' + \lambda\exp\left[(2-y)\Psi\right] F P_M}\right| . \tag{4.16b}$$

As mentioned in the introduction, KVK considers the canonical transformation and its inverse in all regions of the Kruskal diagram for SSGR. Moreover, this reference provides a detailed treatment of the (singular) behavior of the transformation at the horizon. We expect that a similar treatment with essentially the same results can be carried out for the CGHSW case.

As shown in KVK for the SSGR case, the payoff obtained by using the new variables comes when considering the constraints (4.6). Since on solutions to the constraints M is a constant, one knows that M' must be a sum of constraints. Indeed, direct calculation establishes that

$$M' = F^{1/2}\mathsf{H} . \tag{4.17}$$

Since $F^{1/2}$ is the lapse $\tilde{N}$, we see that M' is the generator of Killing-time evolution. It is nice to learn that

$$P_\Psi = \lambda^{-1}\exp\left[(y-2)\Phi\right] F^{-1/2}\mathsf{H}_\vdash \tag{4.18}$$

is also a sum of constraints and so weakly vanishes. The new canonical variables are related to the constraints associated with the static-time slices in a very simply way: $\mathsf{H} = F^{-1/2}M'$ and $\mathsf{H}_\vdash = \lambda\exp\left[(2-y)\Psi\right] F^{1/2} P_\Psi$. Using these relations, we may write the old constraints in terms of the new variables as

$$\mathcal{H} = \psi \mathsf{H} - \mathsf{w}\psi \mathsf{H}_{\vdash}$$
$$\mathcal{H}_r = \Lambda\left(\psi \mathsf{H}_{\vdash} - \mathsf{w}\psi \mathsf{H}\right) , \tag{4.19}$$

where here Λ is given in (4.15a) and w must be expressed in terms of the new variables. With the list (4.15), it is not hard to show that

$$\mathsf{w} = -\lambda(\Psi')^{-1} \exp\left[(2-y)\Psi\right] F P_M \, . \tag{4.20}$$

It is now straightforward, if tedious, to express the old constraints in terms of the new variables,

$$\begin{aligned} \mathcal{H} &= \frac{\lambda^{-1}F^{-1}\Psi' \exp\left[(y-2)\Psi\right] M' + \lambda F \exp\left[(2-y)\Psi\right] P_\Psi P_M}{\left[\lambda^{-2}F^{-1}(\Psi')^2 \exp\left[(y-2)2\Psi\right] - F(P_M)^2\right]^{1/2}} \\ \mathcal{H}_r &= P_M M' + P_\Psi \Psi' \, . \end{aligned} \tag{4.21}$$

As noted in KVK, in terms of the new variables it is relatively simple to show that the Poisson bracket of M with either $\mathcal{H}$ or $\mathcal{H}_r$ vanishes weakly.

D. Canonical reduction

The goal of this subsection is to use the new canonical variables to find a reduced action principle which -in a certain sense- corresponds to the canonical action (4.5). However, we will be adding boundary terms to (4.5) before the reduction is made, so we should clearly state what we have in mind to begin with and why. Several aspects of the path-integral formulation of gravitational thermodynamics motivate fixation of $\bar{N}$ and Φ on the timelike boundary $\bar{\mathcal{T}}$ as the features of central importance which need to be preserved as we modify the original action (4.5). In path-integral expressions for gravitational partition functions, the sum over histories includes only spacetimes for which the initial and final slices are identified. In this scenario the gauge-invariant information of $\bar{N}$ (the lapse of proper time between the identified initial slice t' and final slice t'') is essentially the inverse temperature, which is fixed in the canonical ensemble. [4] Regarding the fixation of the dilaton Φ on the boundary $\bar{\mathcal{T}}$, from a four-dimensional perspective this feature allows the area of the boundary of the system to be fixed as a boundary condition.

In what follows we modify our original canonical action (4.5) in several steps. Each step is -at least heuristically- justified. The result will be an action $S^{\mathit{1}}_{\ddagger}$ (for the SSGR case this action is closely related to one considered by LW) which is particularly amenable to canonical reduction via the new variables. Moreover, as we will explicitly demonstrate, the new action $S^{\mathit{1}}_{\ddagger}$ retains fixation of $\bar{N}$ and Φ on $\bar{\mathcal{T}}$, important for the above mentioned reasons, as features of its associated variational principle. Our analysis provides some conceptual justification for several technical steps taken in LW.

Let us go through the steps of modifying S^{1}. We know from (4.14) that addition of the boundary terms

$$-\omega[\Lambda, P_\Lambda, \Phi]\big|_{t'}^{t''} = \int_{t'}^{t''} \mathrm{d}r \Lambda\left(J - E\varphi\right) \tag{4.22}$$

to the canonical action (4.5) gives the new action

$$S^{1}_{\dagger} \equiv \int_{\mathcal{M}} \mathrm{d}^2x \left(P_M \dot{M} + P_\Psi \dot{\Psi} - N\mathcal{H} - N^r\mathcal{H}_r\right) + \int_{\bar{\mathcal{T}}} \mathrm{d}t \left(\alpha e^{-2\Psi}\eta\dot{\Psi} - \bar{N}\bar{E}\right) , \tag{4.23}$$

where here we consider all the quantities as expressed in terms of the old variables. The vanishing of the original set of constraints is equivalent to the vanishing of M' and P_Ψ. To take advantage of this fact, re-express the constraint terms in the action as $N\mathcal{H} + N^r\mathcal{H}_r = N^M M' + N^\Psi P_\Psi$, where the new Lagrange multipliers are

$$N^M = F^{-1/2}\left(\psi N - \mathsf{w}\psi\Lambda N^r\right) \tag{4.24a}$$

$$N^\Psi = \lambda \exp\left[(2-y)\Psi\right] F^{1/2}\left(\psi\Lambda N^r - \mathsf{w}\psi N\right) . \tag{4.24b}$$

At this stage the new Lagrange multipliers still depend on the old multipliers and the old canonical variables. In particular, note that in terms of the old variables

$$N^\Psi = -\alpha^{-1}e^{2\Phi}\bar{N}\left(\gamma J - v\gamma \mathsf{E}\right) , \tag{4.25}$$

where we have used the expression (4.13) and the fact that $\mathsf{w} = J/\mathsf{E}$. With the canonical equation of motion for P_Λ, one can show that $N^\Psi = -\alpha^{-1}e^{2\Phi}\bar{N}\bar{J} = \dot{\Phi}$.

Had we merely passed to the new canonical variables, without redefining the Lagrange multipliers, the variational principle associated with (4.23) would have featured fixation of $\bar{N}$ and Φ on $\bar{\mathcal{T}}$ automatically. However, with the Lagrange multipliers redefined (and in a way which absorbs some of the canonical variables) all bets are off. We must cleverly choose the appropriate $\bar{\mathcal{T}}$ boundary term for the new-variable version of the action. In passing from the old constraints to M' and P_Ψ, we are effectively performing the Lorentz boost from the frame (u, n) to the rest frame $(\tilde{u}, \tilde{\mathsf{n}})$ at each point on Σ. A point-wise boost[12] has been performed on the old constraints and Lagrange multipliers. However, we have not included the boundary term in the boost. It seems that the correct way to incorporate the effect of the boost into the boundary term is to reference the existing boost parameter η against the parameter φ associated with the boost to the rest frame. This is achieved by adding a $\bar{\mathcal{T}}$ boundary term to the action $S^{1}_{\dagger}$, with the result

$$S^{1}_{\ddagger} \equiv \int_{\mathcal{M}} \mathrm{d}^2x \left(P_M \dot{M} + P_\Psi \dot{\Psi} - N^M M' - N^\Psi P_\Psi\right) + \int_{\bar{\mathcal{T}}} \mathrm{d}t \left[\alpha e^{-2\Psi}(\eta - \varphi)\dot{\Psi} - \bar{N}\bar{E}\right] , \tag{4.26}$$

On-shell, the boost parameter in the new action

$$\rho \equiv \eta - \varphi = -\tfrac{1}{2}\log\left|\frac{\bar{E}+\bar{J}}{\bar{E}-\bar{J}}\right| \tag{4.27}$$

is associated with the local boost between the rest frame $(\tilde{u}, \tilde{n})$ and the boundary frame $(\bar{u}, \bar{n})$. At this stage, the terms $\bar{E}$ and $\bar{J}$ are still shorthand expressions for $\gamma E - v\gamma J$ and $\gamma J - v\gamma E$, respectively.

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[12]Or, depending on the viewpoint, a sphere-wise boost.

We now wish to express all terms in the action $S^1_\ddagger$ solely in terms of the new variables and freely variable N^M and N^Ψ. Using the expressions (4.15), one can easily express the quasilocal energy and momentum in terms of the new variables,

$$E = \frac{\epsilon\alpha e^{-2\Psi}\Psi'}{\left[\lambda^{-2}F^{-1}(\Psi')^2\exp\left[(y-2)2\Psi\right] - F(P_M)^2\right]^{1/2}} \tag{4.28a}$$

$$J = \frac{-\alpha\lambda e^{-y\Psi}FP_M}{\left[\lambda^{-2}F^{-1}(\Psi')^2\exp\left[(y-2)2\Psi\right] - F(P_M)^2\right]^{1/2}}\,, \tag{4.28b}$$

where the factor $\epsilon \equiv (\bar{n}\cdot\bar{\mathsf{n}})$ takes care of the appropriate sign on each of the boundary elements. Moreover, we must now regard N and N^r (which along with Λ are hidden in the v's and γ's which are in turn hidden in ρ) as depending on the new variables. It is not difficult to invert the relations (4.24) to get the needed expressions. Also, a short calculation shows that the boundary lapse $\bar{N} = N/\gamma$ has a fairly nice expression in terms of the new variables,

$$\bar{N}^2 = F\left(N^M\right)^2 - \lambda^{-2}\exp\left[(y-2)2\Psi\right]F^{-1}\left(N^\Psi\right)^2\,. \tag{4.29}$$

However, the situation at hand remains quite problematic. We would like to use the fact that $M' = 0$ to define a radially independent $\mathbf{m}(t) = M(t)$ with conjugate momentum $\mathbf{p}(t) = \int_\Sigma \mathrm{d}r P_M(t,r)$. Since we have seen that P_M is minus the radial derivative of the Killing time, the momentum $\mathbf{p}$ would then be the difference between the Killing-time values for the boundary points B_i and B_o. Indeed, our canonical-reduction goal is to insert the solutions of the constraints into the action to find a reduced action which is expressed in terms of the pair $(\mathbf{m},\mathbf{p})$. However, P_M appears in the $\bar{\mathcal{T}}$ boundary term explicitly. Therefore, even if we perform the r integration in the action to define $\mathbf{p}$, the boundary terms still contain factors of P_M.[13]

A solution to the problem at hand is to make an appeal to the equations of motion. For the moment, let us go back to considering the action $S^1_\ddagger$ as depending on the old variables. Notice that *using the canonical equation of motion for* P_Λ, one may write

$$\bar{J} = -\alpha e^{-2\Phi}(\dot{\Phi}/\bar{N}) \tag{4.30a}$$

$$\bar{E} = -\epsilon\alpha e^{-2\Phi}\sqrt{\lambda^2\exp\left[(2-y)2\Phi\right]F + (\dot{\Phi}/\bar{N})^2}\,, \tag{4.30b}$$

where we have appealed to the form (4.13) for F and again used ϵ to take care of the appropriate sign for each boundary element. Note that by convention we take the positive square root. Since for a classical solution the dilaton is a "bad" radial coordinate in the sense

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[13] The reader might suspect that this problem was introduced when we performed the heuristically-justified reference $\eta \to (\eta - \varphi)$ to get the action $S^1_\ddagger$. But the boundary term in the action $S^1_\dagger$ suffers from the same problem. Indeed, η is built from v which is in turn built from Λ, and hence in term of the new variables η depends on P_M.

that $\Phi' < 0$, our *unreferenced* energy expression (3.13) is negative, and likewise the barred version of this expression is negative. This is why the minus sign has been introduced in (4.30b). The expression for the quasilocal energy with the flat-space reference contribution is considered in the appendix and at length in Ref. [5]. It is trivial to write these new expressions for $\bar{E}$ and $\bar{J}$ in terms of the new variables. Fortunately, these expressions have no dependence on P_M. Using these expressions instead of those in (4.28), we find that in terms of the new variables our action is

$$S^1_{\ddagger} = \int_{\mathcal{M}} \mathrm{d}^2x \left(P_M \dot{M} + P_\Psi \dot{\Psi} - N^M M' - N^\Psi P_\Psi\right) + \int_{\bar{\mathcal{T}}} \mathrm{d}t \left(\alpha e^{-2\Psi}\rho\dot{\Psi} - \bar{N}\bar{E}\right) , \tag{4.31}$$

where $\bar{N}\bar{E} = -\epsilon\alpha e^{-2\Psi}\sqrt{\lambda^2 \exp\left[(2-y)2\Psi\right]\bar{N}^2 F + (\dot{\Psi})^2}$, with $\bar{N}^2$ shorthand for the expression (4.29) and F shorthand for expression (4.16a). Also, now the parameter specifying the boost between the boundary and rest frames takes the form

$$\rho = -\tfrac{1}{2}\log\left|\frac{\sqrt{\lambda^2 \exp\left[(2-y)2\Psi\right]\bar{N}^2 F + (\dot{\Psi})^2} + \epsilon\dot{\Psi}}{\sqrt{\lambda^2 \exp\left[(2-y)2\Psi\right]\bar{N}^2 F + (\dot{\Psi})^2} - \epsilon\dot{\Psi}}\right| . \tag{4.32}$$

Now let us verify that the variational principle associated with the above action possesses the features we demand. The constraints and equations of motion are

$$M' = 0 \tag{4.33a}$$
$$P_\Psi = 0 \tag{4.33b}$$
$$\dot{M} = 0 \tag{4.33c}$$
$$\dot{P}_M = N^M \tag{4.33d}$$
$$\dot{\Psi} = N^\Psi \tag{4.33e}$$
$$\dot{P}_\Psi = 0\,. \tag{4.33f}$$

In terms of the old variables, we have already seen that the equation (4.33e) holds when the canonical equation of motion for P_Λ is assumed. Upon variation of the action, we find the boundary terms

$$\left(\delta S^1_{\ddagger}\right)_{\partial\mathcal{M}} = \int_{t'}^{t''} \mathrm{d}t \left(P_M \delta M + P_\Psi \delta\Psi\right) + \int_{\bar{\mathcal{T}}} \mathrm{d}t \left(\bar{\Pi}_{\bar{N}}\delta\bar{N} + \bar{\Pi}_\Psi \delta\Psi + \bar{\Pi}_M \delta M\right) + \alpha e^{-2\Psi}\rho\delta\Psi\Big|^{B''}_{B'} , \tag{4.34}$$

where now the $\bar{\mathcal{T}}$ momenta are the following:

$$-\bar{E} = \bar{\Pi}_{\bar{N}} = \epsilon\alpha e^{-2\Psi}\sqrt{\lambda^2 \exp\left[(2-y)2\Psi\right]F + (\dot{\Psi}/\bar{N})^2} \tag{4.35a}$$
$$\bar{\Pi}_\Psi = \tfrac{1}{2}\bar{N}\left[y\bar{E}(1+F^{-1}) - 2\alpha e^{-2\Psi}\bar{u}[\rho]\right] \tag{4.35b}$$
$$\bar{\Pi}_M = -\epsilon N^M + (\alpha\lambda)^{-1}e^{y\Psi}F^{-1}\bar{N}\bar{E}\,. \tag{4.35c}$$

Again, $\epsilon = (\bar{n}\cdot\mathsf{n})$ takes care of the correct sign for each boundary element. Note that $\bar{\Pi}_\Psi$ is not the same as the $\bar{\Pi}_\Phi$ in the list (3.6). Although the $\bar{\Pi}_{\bar{N}}$ found here is not the expression (3.6c), it agrees with this expression on-shell. Recalling that $\bar{N}^2$ stands for (4.29), one finds that $\bar{\Pi}_M$ vanishes when the equation of motion (4.33e) holds. Therefore, M need not be held fixed on $\bar{\mathcal{T}}$ in the variational principle associated with $S^{\mathit{1}}_{\ddagger}$, as the equations of motion ensure that the boundary term with $\bar{\Pi}_M$ vanishes for arbitrary variations δM about a classical solution.

The reduced action $I_{\ddagger}$, expressed in terms of $\mathbf{m}$ and $\mathbf{p}$ defined earlier, is obtained by solving the constraints and inserting these solutions back into the action $S_{\ddagger}$. From the result (4.34) for the variation of the action $S^{\mathit{1}}_{\ddagger}$, we know that the reduced action,

$$I^{\mathit{1}}_{\ddagger} = \int_{t'}^{t''} \mathrm{d}t\, \mathbf{p}\dot{\mathbf{m}} + \int_{\bar{\mathcal{T}}} \mathrm{d}t\, \alpha e^{-2\Psi} \Bigg[\epsilon \sqrt{\lambda^2 \exp\left[(2-y)2\Psi\right] \bar{N}^2 F + (\dot{\Psi})^2} \\ -\tfrac{1}{2}\dot{\Psi} \log \left| \frac{\sqrt{\lambda^2 \exp\left[(2-y)2\Psi\right] \bar{N}^2 F + (\dot{\Psi})^2} + \epsilon\dot{\Psi}}{\sqrt{\lambda^2 \exp\left[(2-y)2\Psi\right] \bar{N}^2 F + (\dot{\Psi})^2} - \epsilon\dot{\Psi}} \right| \Bigg] , \tag{4.36}$$

possesses the variational principle we desire. We leave the derivation of $\delta I_{\ddagger}$ as a simple exercise for the interested reader. In the reduced action $\bar{N}$ is positive and independent, and $F = 1 - 2(\alpha\lambda)^{-1} e^{y\Psi}\mathbf{m}$. Also, in the expression for $I_{\ddagger}$ the t' and t'' represent integration parameters rather than spacelike slices as they did before.

V. THE THERMODYNAMICAL ACTION

For both the Schwarzschild and the Witten black hole we are interested in applying the canonical action principle to a static exterior region with spatial boundary (*including* the bifurcation point) of the relevant Kruskal diagram. Such an application of the action principle is the appropriate one when studying the equilibrium thermodynamics of black holes. [4] In such a scenario, with the covariant form of the action there is no inner boundary, since the bifurcation point is a set of measure zero in the integration over $\mathcal{M}$. Nevertheless, in the canonical picture the bifurcation point is a boundary point of every spacelike slice, which implies that the canonical coordinates must obey certain fall-off conditions as the bifurcation point is approached. Moreover, in the thermodynamical paradigm, when the initial and final spacelike slice are identified, one must worry about *regularity conditions* at the bifurcation point which ensure that the geometry is smooth. [11,4,20] To handle these issues, we will use a technique due to Brown and York. [21] The basic idea is to work with an inner boundary but with boundary conditions which effectively seal it. The main ingredient in this technique is a new action functional, which differs from (3.1) by boundary terms. The purpose of this section is to introduce this new action principle and to study its canonical reduction via the new canonical variables.

A. Alternative canonical action principle

Starting with the canonical action (4.5), we define the new action

$$S^{\mathit{1}}_{*} \equiv S^{\mathit{1}} - \tfrac{1}{2}\int_{\bar{\mathcal{T}}_i} \mathrm{d}t \bar{N}\bar{S} + \tfrac{1}{2}\, \alpha e^{-2\Phi}\eta \Big|_{B_i'}^{B_i''} . \tag{5.1}$$

Note that only an inner boundary term has been added to the original action. It is easy to show that the canonical form of S^1_* has the boundary terms

$$\left(S^1_*\right)_{\partial\mathcal{M}} = \int_{\bar{\mathcal{T}}_o} \mathrm{d}t\left[\alpha e^{-2\Phi}\eta\dot{\Phi} - \bar{N}\bar{E}\right] + \int_{\bar{\mathcal{T}}_i} \mathrm{d}t\left[\tfrac{1}{2}\,\alpha e^{-2\Phi}\dot{\eta} - \bar{N}\left(\bar{E} + \tfrac{1}{2}\bar{S}\right)\right] . \tag{5.2}$$

Using the expressions (4.7a) and (4.7c), one finds that the inner boundary term is

$$\left(S^1_*\right)_{\bar{\mathcal{T}}_i} = -\int_{\bar{\mathcal{T}}_i} \mathrm{d}tN\left[E - vJ + \tfrac{1}{2}(S - vT)\right] . \tag{5.3}$$

It is also relatively straightforward to compute the variation of S^1_*. With the result (4.6) it follows that the $\bar{\mathcal{T}}$ and corner contributions to the variation of S^1_* are

$$\begin{aligned}&(\delta S^1_*)_{\bar{\mathcal{T}},B'',B'} = \\ &-\int_{\bar{\mathcal{T}}_o} \mathrm{d}t\left[\bar{E}\delta\bar{N} + \bar{N}\bar{S}\delta\Phi - \left(\bar{N}\bar{J} + \alpha e^{-2\Phi}\dot{\Phi}\right)\delta\eta\right] + \alpha e^{-2\Phi}\eta\delta\Phi\Big|^{B''_o}_{B'_o} \\ &-\int_{\bar{\mathcal{T}}_i} \mathrm{d}t\left[\bar{E}\delta\bar{N} + \tfrac{1}{2}e^{-2\Phi}\delta\left(e^{2\Phi}\bar{N}\bar{S}\right) - \left(\bar{N}\bar{J} + \alpha e^{-2\Phi}\dot{\Phi}\right)\delta\eta\right] + \tfrac{1}{2}\,\alpha e^{-2\Phi}\delta\eta\Big|^{B''_i}_{B'_i} .\end{aligned} \tag{5.4}$$

We describe the boundary conditions at the inner boundary as *completely open* because $\bar{N}$ and $e^{2\Phi}\bar{S}$ are held fixed, as opposed to closed or *microcanonical* boundary conditions characterized by fixation of the energy $\bar{E}$ and fixation of Φ (effectively the surface area). [21] With some work and the formulas in (4.7), one can show that the inner-boundary and inner-corner-point contributions in the above variation may be combined into the expression

$$(\delta S^1_*)_{\bar{\mathcal{T}}_i,B''_i,B'_i} = -\int_{\bar{\mathcal{T}}_i} \mathrm{d}t\left[(E - vJ)\,\delta N + \tfrac{1}{2}e^{-2\Phi}\delta\left(Ne^{2\Phi}S - vNe^{2\Phi}T\right) - (N/\gamma^2)J\delta\eta\right] . \tag{5.5}$$

Now we introduce fall-off conditions on the fields which seal the inner boundary. What we have in mind is a general foliation of our spatially bounded static region $\mathcal{M}$. All of the spatial slices meet at the bifurcation point, but otherwise are essentially arbitrary. Our phase space is the set of fields $(\Lambda, P_\Lambda; \Phi, P_\Phi)$ with the appropriate fall-off conditions near the bifurcation point. The needed fall-off conditions have already been given in LW for the specific case of the Schwarzschild geometry. For convenience and without loss of generality, take the inner boundary, the bifurcation point, to be located at $r_i = 0$ and the outer boundary to be located at $r_o = 1$. The boundary conditions given in LW are the following:

$$\Lambda(t,r) = \Lambda_0(t) + O(r^2) \tag{5.6a}$$
$$\Phi(t,r) = \Phi_0(t) + \Phi_2(t)r^2 + O(r^4) \tag{5.6b}$$
$$P_\Lambda(t,r) = O(r^3) \tag{5.6c}$$
$$P_\Phi(t,r) = O(r) \tag{5.6d}$$
$$N(t,r) = N_1(t)r + O(r^3) \tag{5.6e}$$
$$N^r(t,r) = N^r_1(t)r + O(r^3)\,, \tag{5.6f}$$

where $O(r^n)$ stands for a term whose magnitude as $r \to 0$ is bounded by r^n times a constant. Also, as $r \to 0$, the k'th derivative of such a term is similarly bounded by r^{n-k} times a constant for $1 \leq k \leq n$. Note that the time development at the inner boundary has been arrested as the lapse vanishes there. In effect these boundary conditions seal the inner boundary. One can show that these boundary conditions are consistent with the equations of motion. In other words, the Hamiltonian evolution preserves the above boundary conditions, provided that the initial data obeys both the above set of fall-off conditions and the constraints (4.6) on the initial spacelike slice Σ and provided that the lapse and shift also obey the above fall-off conditions.[14] Moreover, the dynamical equation for Φ and the above fall-off conditions imply that $\dot{\Phi}_0 = 0$. Imposition of the Hamiltonian constraint (4.6a) as $r \to 0$ yields the relation

$$(\Lambda_0)^2 = -4y^{-1}\lambda^{-2}\Phi_2 \exp\left[(y-2)2\Phi_0\right] , \tag{5.7}$$

which shows that Φ_2 is negative for classical solutions. Let us quickly compare this result with the Schwarzschild result found in LW. Setting $R(t,r) = R_0(t) + R_2(t)r^2 + O(r^4)$ near the bifurcation point, one finds that $\lambda R_0 = e^{-\Phi_0}$ and $\lambda R_2 = -\Phi_2\, e^{-\Phi_0}$. Therefore, for the SSGR case the above expression is $(\Lambda_0)^2 = 4R_0R_2$, which is the LW result.

Application of the fall-off conditions (5.6) to the inner boundary term (5.3) shows that now the new action (5.1) has the following form:

$$\begin{aligned} S^{\mathit{1}}_{*} = &\int_{\mathcal{M}} \mathrm{d}^2x \left(P_\Lambda\dot{\Lambda} + P_\Phi\dot{\Phi} - N\mathcal{H} - N^r\mathcal{H}_r\right) \\ &+ \int_{\bar{\mathcal{T}}_o} \mathrm{d}t \left(\alpha e^{-2\Phi}\eta\dot{\Phi} - \bar{N}\bar{E}\right) + \tfrac{1}{2}\int_{t'}^{t''} \mathrm{d}t\ \alpha e^{-2\Phi}(N'/\Lambda)\Big|_{r=0} , \end{aligned} \tag{5.8}$$

where in the last integral t' and t'' now represent integration parameters and not manifolds as they have before. We refer to (5.8) as the *thermodynamical action* because of its importance in the path-integral formulation of gravitational thermodynamics. This is the appropriate action with which to study the canonical ensemble for spherically symmetric black holes. [4] Using (5.5) in tandem with the fall-off conditions (5.6), one finds the following boundary terms in the variation of the thermodynamical action:

$$\begin{aligned} (\delta S^{\mathit{1}}_{*})_{\mathcal{T},B'',B'} = &\tfrac{1}{2}\int_{t'}^{t''} \mathrm{d}t\ \alpha e^{-2\Phi}\delta\left(N'/\Lambda\right)\Big|_{r=0} \\ &- \int_{\bar{\mathcal{T}}_o} \mathrm{d}t \left[\bar{E}\delta\bar{N} + \bar{N}\bar{S}\delta\Phi - \left(\bar{N}\bar{J} + \alpha e^{-2\Phi}\dot{\Phi}\right)\delta\eta\right] + \alpha e^{-2\Phi}\eta\delta\Phi\Big|_{B'_o}^{B''_o} . \end{aligned} \tag{5.9}$$

For the $y = 1$ case this is precisely the action and variational principle considered in LW. Notice that we have the same boundary conditions at the outer boundary as before: $\bar{N}$ and Φ are held fixed on this surface in the variational principle associated with (5.8). As spelled out in LW, the quantity N'/Λ which is fixed at the bifurcation point in the variational principle has a direct physical interpretation. In fact, N'/Λ is the time rate of change a certain boost parameter. Each Σ slices defines a timelike normal u at the bifurcation point. As the Σ slicing develops in time this vector is continuously boosted at the rate $(N'/\Lambda)|_{r=0} = N_1/\Lambda_0$.

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[14]Note that one must appeal to (5.7) when showing that the boundary conditions (5.6) are consistent with the $\dot{P}_\Lambda$ equation of motion.

B. Canonical reduction of the thermodynamical action

We want a new-variable version of the thermodynamical action which is amenable to canonical reduction. From the work in §4 we already know how to modify/handle the outer boundary term when passing to the new canonical variables. Except for a minor difference, we will handle the outer boundary term in the thermodynamical action just like in the previous section. Therefore, the last section has already addressed several of the delicate issues concerning the canonical reduction of the thermodynamical action. However, there is a new feature of the thermodynamical action which we need to worry about. This new feature concerns the quantity N'/Λ which is fixed at the bifurcation point. We have already discussed why fixation of $\bar{N}$ and Φ (now only at the outer boundary) are important features of the variational principle. The boundary integral at the bifurcation point is also of importance. Indeed, in the thermodynamical paradigm black-hole entropy arises from this term. [4,20] For the Schwarzschild case LW has shown that for applications to gravitational thermodynamics is is crucial to retain fixation of N'/Λ at the bifurcation point when passing to a new-variable version of the action. We also regard this feature of the action principle as the feature of central importance which needs to be preserved. Though essentially reviewing the work of LW, this subsection shows how the Louko-Whiting formalism extends to the CGSHW pure-dilaton case. Therefore, our discussion has relevance to the thermodynamics of pure-dilaton gravity.

Let us present the quantities with which we will construct a new-variable version of the thermodynamical action. The new canonical variables are the same as before, since the transformation (4.10) remains canonical with the boundary conditions adopted for the thermodynamical action. As before, the term $-\left(\alpha e^{-2\Phi}\delta\Phi\boldsymbol{\varphi}\right)'$ in the identity (4.14) gives two boundary terms. The one at the outer boundary vanishes as Φ is held fixed on this boundary element. Moreover, $\boldsymbol{\varphi}$ vanishes as $r\rightarrow 0$ so the inner boundary term also vanishes. Hence, upon integration (4.14) still shows that the difference between the old and new Liouville forms is an exact form. The new shift N^{Ψ} is again defined by (4.24b). However, following Louko and Whiting, we define a *different* new lapse

$$\mathsf{N} = -N^{M}\left(\frac{yM}{\alpha}\right)\left(\frac{\alpha\lambda}{2M}\right)^{2/y}. \tag{5.10}$$

Recall that M has units of length for the SSGR case and units of inverse length for the CGHSW case. It is easy to see that N^M is dimensionless for both cases, and, therefore, for both cases N has units of inverse length. It turns out that this choice for the lapse must be made in order to ensure that we retain fixation of the boost rate N'/Λ at the bifurcation point as feature of variational principle. In terms of N the boundary lapse is given by

$$\bar{N}^2 = \left(\frac{\alpha}{yM}\right)^2\left(\frac{2M}{\alpha\lambda}\right)^{4/y} F\mathsf{N}^2 - \lambda^{-2}\exp\left[(y-2)2\Psi\right]F^{-1}\left(N^{\Psi}\right)^2 . \tag{5.11}$$

The fall-off conditions (5.6) imply the following fall-off conditions for the new variables:

$$M(t,r) = \tfrac{1}{2}\alpha\lambda\exp\left[-y\Psi_0(t)\right] + M_2(t)r^2 + O(r^4) \tag{5.12a}$$

$$\Psi(t,r) = \Psi_0(t) + \Psi_2 r^2 + O(r^4) \tag{5.12b}$$
$$P_M(t,r) = O(r) \tag{5.12c}$$
$$P_\Psi(t,r) = O(r) \tag{5.12d}$$
$$\mathsf{N}(t,r) = \mathsf{N}_0(t) + O(r^2) \tag{5.12e}$$
$$N^\Psi(t,r) = N_2^\Psi(t) r^2 + O(r^4)\,, \tag{5.12f}$$

where

$$\Psi_0 = \Phi_0 \tag{5.13a}$$
$$\Psi_2 = \Phi_2 \tag{5.13b}$$
$$M_2 = -\tfrac{1}{2}\alpha\lambda\Phi_2 \exp(-y\Phi_0)\left[y + 4\lambda^{-2}(\Lambda_0)^{-2}\Phi_2 \exp[(y-2)2\Phi_0]\right] \tag{5.13c}$$
$$\mathsf{N}_0 = -\tfrac{1}{4}y\lambda^2 N_1 \Lambda_0(\Phi_2)^{-1} \exp\left[(2-y)2\Phi_0\right] \tag{5.13d}$$
$$N_2^\Psi = 2\Phi_2 N_1^r\,. \tag{5.13e}$$

We also have that

$$F = 4\lambda^{-2}(\Phi_2)^2(\Lambda_0)^{-2}\exp[(y-2)2\Phi_0] r^2 + O(r^4)\,. \tag{5.13}$$

For the SSGR case these fall-off results for the new variables match those given in LW. Equation (5.7) implies that

$$(N'/\Lambda)|_{r=0} = -\tfrac{1}{4}y\lambda^2 N_1\Lambda_0(\Phi_2)^{-1}\exp\left[(2-y)2\Phi_0\right]\,, \tag{5.14}$$

which in turn gives $\mathsf{N}_0 = N_1/\Lambda_0$. Hence, we want to ensure that the Lagrange multiplier N is fixed at the bifurcation point in our new variational principle.

Let us now consider the new-variable version of the thermodynamical action and show that it has the correct variational principle. We could write down a general action which covers both the SSGR and CGSHW cases, but the expression is a bit unseemly. Therefore, let us examine both cases separately. For the SSGR case we have

$$\begin{aligned} S^1_\diamond \equiv & \int_{\mathcal{M}} \mathrm{d}^2x \left(P_M \dot{M} + P_{\mathsf{R}}\dot{\mathsf{R}} + 4\mathsf{N}MM' - N^{\mathsf{R}}P_{\mathsf{R}}\right) \\ & + \int_{t'}^{t''} \mathrm{d}t\, 2M^2\mathsf{N}\Big|_{r=0} + \int_{\bar{\mathcal{T}}_o} \mathrm{d}t \left[-\rho\mathsf{R}\dot{\mathsf{R}} - \bar{N}\bar{E}\right]\,, \end{aligned} \tag{5.15}$$

with the boundary lapse given by

$$\bar{N}^2 = 16M^2F\mathsf{N}^2 - F^{-1}\left(N^{\mathsf{R}}\right)^2\,. \tag{5.16}$$

To get these expressions we have use the facts that $e^{-\Psi} = \lambda\mathsf{R}$, $P_\Psi = -\mathsf{R}P_{\mathsf{R}}$, and $N^\Psi = -N^{\mathsf{R}}/\mathsf{R}$ (all in the notation of KVK and LW). Also, $\bar{E}$ and ρ are still given by (4.30b) and (4.32), respectively, but now one must express them in terms of R. This is precisely the action considered in LW. For the CGHSW pure-dilaton model we find

$$\begin{aligned} S^1_\diamond \equiv & \int_{\mathcal{M}} \mathrm{d}^2x \left[P_M\dot{M} + P_\Psi\dot{\Psi} + \lambda^{-1}\mathsf{N}M' - N^\Psi P_\Psi\right] \\ & \int_{t'}^{t''} \mathrm{d}t\lambda^{-1}M\mathsf{N}\Big|_{r=0} + \int_{\bar{\mathcal{T}}_o} \mathrm{d}t \left[\alpha e^{-2\Psi}\rho\dot{\Psi} - \bar{N}\bar{E}\right]\,, \end{aligned} \tag{5.17}$$

with

$$\bar{N}^2 = \lambda^{-2} F \mathsf{N}^2 - \lambda^{-2} F^{-1} \left(N^{\Psi}\right)^2 . \tag{5.18}$$

In contrast to the SSGR case, for the CGHSW model the Lagrange multipler $N^M|_{r=0}$ does specify, apart only from a dimensionful constant, the boost rate of the Σ normal u at the bifurcation point.

The variation of (5.15) has already been considered in LW, so we will only consider the variation of (5.17). The equations of motion derived from (5.17) are the same as those given in (4.33), except that now $\dot{P}_M = -\lambda^{-1}\mathsf{N}$. Upon variation of the action (5.17), we find the boundary terms

$$\begin{aligned}\left(\delta S^{1}_{\diamond}\right)_{\partial\mathcal{M}} &= \int_{t'}^{t''} \mathrm{d}t \left(P_M \delta M + P_\Psi \delta\Psi\right) + \int_{t'}^{t''} \mathrm{d}t \lambda^{-1} M \delta \mathsf{N}\Big|_{r=0} \\ &+ \int_{\bar{\mathcal{T}}_o} \mathrm{d}t \left(\bar{\Pi}_{\bar{N}} \delta\bar{N} + \bar{\Pi}_\Psi \delta\Psi + \bar{\Pi}_M \delta M\right) + \alpha e^{-2\Psi} \rho \delta\Psi\Big|^{B''_o}_{B'_o} .\end{aligned} \tag{5.19}$$

In the first integral t' and t'' represent spacelike slices, while in the second integral they are integration parameters. The $\bar{\mathcal{T}}_o$ momenta in (5.19) are essentially the same as the outer-boundary ones in (4.35), except that now $\bar{N}$ stands for (5.18) and the momenta $\bar{\Pi}_M = \lambda^{-1}\mathsf{N} + (\alpha\lambda)^{-1} e^{y\Psi} F^{-1} \bar{N}\bar{E}$. Like before, plugging in the explicit formula (5.18) for $\bar{N}^2$, one finds that $\bar{\Pi}_M$ vanishes when the equation of motion $\dot{P}_M = -\lambda^{-1}\mathsf{N}$ holds. Therefore, M need not be held fixed on $\bar{\mathcal{T}}_o$ in the variational principle associated with $S^{1}_{\diamond}$. It is now straightforward to pass to the reduced form of the thermodynamical action.

VI. DISCUSSION

We conclude with some comments concerning the possible extension of the KVK and LW formalisms to other two-dimensional models of gravity. Recently, important progress has been made in the field of two-dimensional gravity with the realization that a huge class of two-dimensional models can be described within the framework of the so-called *Poisson-sigma models* (PSM's) of Schaller and Strobl. [22] For all such models there exists an absolutely conserved quantity C (referred to as a *Casimir function* in the Poisson-sigma model language) which is analogous to our M expression (3.19), and recently Kummer and Widerin have explored the relationship between the PSM C and notions of quasilocal energy for such models. [23] Many of our results, especially those concerning our general treatment of quasilocal energy-momentum, seem to extend to the general PSM formalism. In particular, the absolutely conserved quantity C can be interpreted as a quasilocal boost invariant. [24] Extension of the canonical-reduction method of KVK to PSM theory also seems possible, though several technical difficulties lie in the way. For instance, one encounters an almost limitless variety of singularity structures when considering the set of all PSM's. [25] For SSGR the canonical transformation of KVK is singular at the horizon. Similar technical difficulties are likely to surface when applying the KVK method to any two-dimensional model. Since the collection of all possible Penrose diagrams obtainable from PSM's is so large, it is questionable whether or not a fully unified treatment for the canonical reduction of all PSMs is possible. On the other hand, the richness of singularity structures that PSM gravitation offers provides a promising testing ground for gravitational thermodynamics. The appropriate thermodynamical action, as expressed in the LW formalism, would be a crucial ingredient in any study of PSM thermodynamics via reduced canonical variables.

VII. ACKNOWLEDGMENTS

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APPENDIX A: REDUCTION OF THE HILBERT ACTION WITH BOUNDARY TERMS

Consider again the *four-dimensional* spacetime region ${}^4\mathcal{M}$ described in the last two paragraphs of §3.B. The action functional associated with ${}^4\mathcal{M}$ is the standard Hilbert action, complete with the TrK terms needed to ensure that the four-dimensional action principle features fixation of the induced three-metric on all of the elements of the three-boundary $\partial^4\mathcal{M}$. In this appendix we will insert the metric ansatz (3.14) into the four-dimensional spacetime action and then integrate out the angular variables. This procedure will yield a reduced action principle for the fields g_{ab} and R (or equivalently Φ) which are defined on the toy $1+1$ dimensional spacetime described in the preliminary section. The variational principle for the reduced action still features fixation of the induced metric on the boundary, and the equations of motion derived from the reduced action are the Einstein equations with ansatz of spherical symmetry.[15]

The four-dimensional spacetime action is [27,9]

$$16\pi S^1 = \int_{{}^4\mathcal{M}} \mathrm{d}^4x\sqrt{-{}^4g}\,\Re + 2\int_{t'}^{t''} \mathrm{d}^3x\sqrt{h}K - 2\int_{\bar{\mathcal{T}}} \mathrm{d}^3x\sqrt{-\bar{\gamma}}\bar{\Theta} - 2\int_{B'}^{B''} \mathrm{d}^2x\sqrt{\sigma}\eta\,, \tag{A1}$$

where $\Re$ is the Ricci scalar of ${}^4\mathcal{M}$, h_{ij} is the induced three-metric on the spacelike manifolds t' and t'', and γ_{ij} is the induced three-metric on the timelike boundary elements $\bar{\mathcal{T}}$.[16] Also,

[15] We do not prove this, but see, for instance, Ref. [26].

[16] As in the two-dimensional scenario, it is understood that $\bar{\mathcal{T}}$ represents the not-simply-connected total timelike boundary manifold $\bar{\mathcal{T}}_i \bigcup \bar{\mathcal{T}}_o$. Therefore, $\bar{\mathcal{T}}$ expressions stand for the sum of an inner-boundary and an outer-boundary expression.

$K_{\mu\nu} = -h_{\mu}^{\lambda 4}\nabla_{\lambda} u_{\nu}$ is the extrinsic curvature of t' or t'' as embedded in ${}^4\mathcal{M}$. In this definition u_μ is the future-pointing normal of t' or t'', and we have used spacetime coordinates for convenience, so $h_{\mu\nu} = {}^4g_{\mu\nu} + u_\mu u_\nu$. Likewise, the extrinsic curvature of $\bar{\mathcal{T}}$ as embedded in ${}^4\mathcal{M}$ is $\bar{\Theta}_{\mu\nu} = -\bar{\gamma}_{\mu}^{\lambda 4}\nabla_\lambda \bar{n}_\nu$ where $\bar{n}_\mu$ is the outward-pointing spacelike normal of $\bar{\mathcal{T}}$ and here $\bar{\gamma}_{\mu\nu} = {}^4g_{\mu\nu} - \bar{n}_\mu \bar{n}_\nu$. Finally, on the corners B' and B'' (each is the disjoint union of an inner and outer sphere) the two-metric is σ_{ab} and $\eta = \sinh^{-1}(u \cdot \bar{n})$.

Let us collect a few results needed for the reduction. From the form of the line element (3.14) one can compute the components of K^i_j, the Σ extrinsic curvature tensor. The set of nonzero components is the following:

$$K^r_r = -(N\Lambda)^{-1}\left[\dot{\Lambda} - (\Lambda N^r)'\right] \tag{A2a}$$

$$K^\theta_\theta = K^\phi_\phi = -(NR)^{-1}\left(\dot{R} - N^r R'\right) \tag{A2b}$$

Since we work with one index up and one down and since h_{ij} is diagonal, these are also the orthonormal components of K^i_j with respect to the standard triad. So we have that $(n \cdot K \cdot n) = n_i n_j K^{ij} = K^r_r$, where n_j is the outward-pointing normal of a round sphere B as embedded in a constant-t spacelike slice Σ of ${}^4\mathcal{M}$. Treating the time-radial piece g_{ab} of the full four-metric (3.14) as if it were a true metric in its own right, we can compute its curvature scalar $\mathcal{R}[g]$. The result is

$$\mathcal{R} = -2(N\Lambda)^{-1}\left(\Lambda K^r_r\right)^{\bullet} - 2(N\Lambda)^{-1}\left(\Lambda^{-1}N' - \Lambda N^r K^r_r\right)' . \tag{A3}$$

Now we turn to the reduction. Let us start with the spacetime volume integral in (A1). Use the identity[17] [27]

$$\Re[{}^4g] = {}^3R[h] + K_{\mu\nu}K^{\mu\nu} - (K)^2 - 2\nabla_\mu\left(Ku^\mu + a^\mu\right) , \tag{A4}$$

where a^μ is the spacetime acceleration of u^μ, to split this volume integral into three pieces:

$$(\text{term 1}) = \int_{{}^4\mathcal{M}} \mathrm{d}^4x\sqrt{-{}^4g}\,{}^3R \tag{A5a}$$

$$(\text{term 2}) = \int_{{}^4\mathcal{M}} \mathrm{d}^4x\sqrt{-{}^4g}\left[K_{ij}K^{ij} - (K)^2\right] \tag{A5b}$$

$$(\text{term 3}) = -2\int_{{}^4\mathcal{M}} \mathrm{d}^4x\,\partial_\mu\left[\sqrt{-{}^4g}\left(Ku^\mu + a^\mu\right)\right] . \tag{A5c}$$

Focus attention on

$$(\text{term 3}) = -2\int_{{}^4\mathcal{M}} \mathrm{d}^4x\left\{\left[\sqrt{-{}^4g}\left(Ku^t + a^t\right)\right]^{\bullet} + \left[\sqrt{-{}^4g}\left(Ku^r + a^r\right)\right]'\right\} , \tag{A6}$$

where one has

[17]Here we let ${}^3R$ stand for the Σ curvature scalar to avoid confusing it with the radius function R.

$$
\begin{aligned}
u^t &= 1/N &\text{(A7a)}\\
u^r &= -N^r/N &\text{(A7b)}\\
a^t &= 0 &\text{(A7c)}\\
a^r &= N^{-1}\Lambda^{-2}N'\,. &\text{(A7d)}
\end{aligned}
$$

After some work and with $\sqrt{-{}^4\!g} = N\Lambda R^2 \sin\theta$, this term can be written as

$$
\begin{aligned}
(\text{term } 3) = &\int_{{}^4\mathcal{M}} \mathrm{d}^4x\sqrt{-{}^4\!g}\left[\mathcal{R} + 4K^r_r K^\theta_\theta - 4\Lambda^{-2}(\log N)'(\log R)'\right]\\
&-2\int_{{}^4\mathcal{M}} \mathrm{d}^4x \sin\theta\left[\left(2\Lambda R^2 K^\theta_\theta\right)^\bullet - \left(2\Lambda R^2 N^r K^\theta_\theta\right)'\right]. \qquad \text{(A8)}
\end{aligned}
$$

Straightforward manipulations show that

$$
(\text{term } 2) = \int_{{}^4\mathcal{M}} \mathrm{d}^4x \sin\theta\left[-2K^\theta_\theta K^\theta_\theta - 4K^r_r K^\theta_\theta\right]. \qquad \text{(A9)}
$$

Recombining the three terms, we get

$$
\begin{aligned}
\int_{{}^4\mathcal{M}} \mathrm{d}^4x\sqrt{-{}^4\!g}\,\Re = &\int_{{}^4\mathcal{M}} \mathrm{d}^4x\sqrt{-{}^4\!g}\left[\mathcal{R} + {}^3\!R - 2K^\theta_\theta K^\theta_\theta - 4\Lambda^{-2}\left(\log N\right)'\left(\log R\right)'\right]\\
&-2\int_{{}^4\mathcal{M}} \mathrm{d}^4x \sin\theta\left[\left(2\Lambda R^2 K^\theta_\theta\right)^\bullet - \left(2\Lambda R^2 N^r K^\theta_\theta\right)'\right]. \qquad \text{(A10)}
\end{aligned}
$$

Now the explicit expression for the Σ Ricci scalar is

$$
{}^3\!R = -4\Lambda^{-2}R^{-1}R'' + 4\Lambda^{-3}R^{-1}\Lambda' R' - 2\Lambda^{-2}R^{-2}(R')^2 + 2R^{-2}\,. \qquad \text{(A11)}
$$

With this expression and the result for K^θ_θ, one can show that

$$
\begin{aligned}
{}^3\!R - 2K^\theta_\theta K^\theta_\theta &= 2R^{-2}\left[-u[R]u[R] + n[R]n[R] + 1 - 2\Lambda^{-1}\left(\Lambda^{-1}RR'\right)'\right]\\
&= 2R^{-2}\left[g^{ab}\partial_a R\partial_b R + 1 - 2\Lambda^{-1}\left(\Lambda^{-1}RR'\right)'\right]. \qquad \text{(A12)}
\end{aligned}
$$

After a bit of re-shuffling, one finds the following expression for the spacetime volume term in the action:

$$
\begin{aligned}
\int_{{}^4\mathcal{M}} \mathrm{d}^4x\sqrt{-{}^4\!g}\,\Re = &\int_{{}^4\mathcal{M}} \mathrm{d}^4x\sqrt{-{}^4\!g}\left[\mathcal{R} + 2R^{-2}g^{ab}\partial_a R\partial_b R + 2R^{-2}\right]\\
&-2\int_{{}^4\mathcal{M}} \mathrm{d}^4x \sin\theta\left[\left(2\Lambda R^2 K^\theta_\theta\right)^\bullet - \left(2\Lambda R^2 N^r K^\theta_\theta - 2\Lambda^{-1}NRR'\right)'\right]. \qquad \text{(A13)}
\end{aligned}
$$

Now we turn to the boundary terms in the action. First consider

$$
2\int_{t'}^{t''} \mathrm{d}^3x\sqrt{h}K = 2\int_{t'}^{t''} \mathrm{d}^3x\sqrt{h}K^r_r + 4\int_{t'}^{t''} \mathrm{d}^3x\Lambda R^2 \sin\theta K^\theta_\theta\,. \qquad \text{(A14)}
$$

For the $\bar{\mathcal{T}}$ boundary term we have

$$-2\int_{\bar{\mathcal{T}}} \mathrm{d}^3x\sqrt{-\bar{\gamma}}\bar{\Theta} =$$
$$-2\int_{\bar{\mathcal{T}}} \mathrm{d}^3x\sqrt{-\bar{\gamma}}\bar{\Theta}^t_t - 2\int_{\bar{\mathcal{T}}_o} \mathrm{d}^3x\sqrt{-\bar{\gamma}}\left[\bar{\Theta}^\theta_\theta + \bar{\Theta}^\phi_\phi\right] - 2\int_{\bar{\mathcal{T}}_i} \mathrm{d}^3x\sqrt{-\bar{\gamma}}\left[\bar{\Theta}^\theta_\theta + \bar{\Theta}^\phi_\phi\right] , \tag{A15}$$

where we have simply written explicitly both the inner and outer boundary terms for part of this integral. Now $\sigma_{ij}\bar{\Theta}^{ij} = \bar{\Theta}^\theta_\theta + \bar{\Theta}^\phi_\phi$, so we can apply the the result[18]

$$\sigma_{ij}\bar{\Theta}^{ij} = \gamma k + v\gamma\sigma_{ij}K^{ij} . \tag{A16}$$

For the case at hand, $\sigma_{ij}K^{ij} = 2K^\theta_\theta$ and

$$k = \pm 2\Lambda^{-1}R^{-1}R' \tag{A17}$$

with $-$ at the outer sphere and $+$ at the inner sphere. Therefore, the $\bar{\mathcal{T}}$ integral can be expressed as

$$-2\int_{\bar{\mathcal{T}}} \mathrm{d}^3x\sqrt{-\bar{\gamma}}\bar{\Theta} = -2\int_{\bar{\mathcal{T}}} \mathrm{d}^3x\sqrt{-\bar{\gamma}}\bar{\Theta}^t_t - 2\int_{\bar{\mathcal{T}}_i}^{\bar{\mathcal{T}}_o} \mathrm{d}^3x N\sin\theta\left[-2\Lambda^{-1}RR' + 2\mathsf{v}R^2K^\theta_\theta\right] , \tag{A18}$$

where we have used $\sqrt{-\bar{\gamma}}\gamma = NR^2\sin\theta$ (note that the two γ's on the lefthand side are different, the second is the relativistic factor). Also, in the above expression

$$\int_{\bar{\mathcal{T}}_i}^{\bar{\mathcal{T}}_o} = \int_{\bar{\mathcal{T}}_o} - \int_{\bar{\mathcal{T}}_i} , \tag{A19}$$

and $\mathsf{v} = \Lambda N^r/N$. The v in (A16) is $-\mathsf{v}$ on $\bar{\mathcal{T}}_i$ and v on $\bar{\mathcal{T}}_o$.

Using the results (A13), (A14), and (A18), we see that with the metric ansatz (3.14) the spacetime action (A1) reduces to

$$\begin{aligned} 16\pi S^{\mathit{1}} = &\int_{^4\mathcal{M}} \mathrm{d}^4x\sqrt{-{}^4\!g}\left[\mathcal{R} + 2R^{-2}g^{ab}\partial_a R\partial_b R + 2R^{-2}\right] \\ &+2\int_{t'}^{t''} \mathrm{d}^3x\sqrt{h}K^r_r - 2\int_{\bar{\mathcal{T}}} \mathrm{d}^3x\sqrt{-\bar{\gamma}}\bar{\Theta}^t_t - 2\int_{B'}^{B''} \mathrm{d}^2x\sqrt{\sigma}\eta , \end{aligned} \tag{A20}$$

where $K^r_r = (n\cdot K\cdot n) = (b\cdot u)$ and $\bar{\Theta}^t_t = -(\bar{u}\cdot\bar{\Theta}\cdot\bar{u}) = -(\bar{a}\cdot\bar{n})$. Rename these $\mathrm{k} \equiv K^r_r$ and $\bar{\vartheta} \equiv \bar{\Theta}^t_t$. None of the quantities in the above action has any angular dependence. Therefore, we simple integrate over the angular variables to find

$$\begin{aligned} S^{\mathit{1}} = &\tfrac{1}{4}\int_M \mathrm{d}^2x\sqrt{-g}R^2\left[\mathcal{R} + 2R^{-2}g^{ab}\partial_a R\partial_b R + 2R^{-2}\right] \\ &+\tfrac{1}{2}\int_{t'}^{t''} \mathrm{d}r\Lambda R^2\mathrm{k} - \tfrac{1}{2}\int_{\bar{\mathcal{T}}} \mathrm{d}t\bar{N}R^2\bar{\vartheta} - \tfrac{1}{2}\,R^2\eta\Big|_{B'}^{B''} . \end{aligned} \tag{A21}$$

---

[18]Proving this result is an exercise with projection operators. Note that the extrinsic curvature of B as embedded in Σ is defined in spacetime coordinates by $k_{\mu\nu} = -\sigma^\lambda_\mu D_\lambda n_\nu$, where D_μ is the Σ intrinsic covariant derivative operator compatible with $h_{\mu\nu}$ and $\sigma_{\mu\nu} = {}^4g_{\mu\nu} + u_\mu u_\nu - n_\mu n_\nu$. On a Σ covector like n_ν, the action of D_μ is $D_\mu n_\nu = h^\lambda_\mu h^\kappa_\nu {}^4\nabla_\lambda n_\kappa$. Also remember that $\bar{n} = \gamma n + v\gamma u$.

If one prefers, one can write the action in terms of the dilaton $\Phi \equiv -\frac{1}{2}\log \lambda R$. The result,

$$
\begin{aligned}
S^{1} = & \tfrac{1}{4}\lambda^{-2}\int_{M} \mathrm{d}^2x\sqrt{-g}e^{-2\Phi}\left[\mathcal{R} + 2g^{ab}\partial_a\Phi\partial_b\Phi + 2\lambda^2 e^{2\Phi}\right] \\
& + \tfrac{1}{2}\lambda^{-2}\int_{t'}^{t''} \mathrm{d}r\Lambda e^{-2\Phi}\mathrm{k} - \tfrac{1}{2}\lambda^{-2}\int_{\bar{T}} \mathrm{d}t\bar{N}e^{-2\Phi}\bar{\vartheta} - \tfrac{1}{2}\lambda^{-2}\, e^{-2\Phi}\eta\Big|_{B'}^{B''}\,,
\end{aligned}
\tag{A22}
$$

is precisely the action (3.1) from the first section with the choices $y = 1$ and $\alpha = \lambda^{-2}$.

APPENDIX B: THE WITTEN BLACK HOLE

In this section we have two goals in mind. The first is to compute the energy at spatial infinity (associated with the static-time slices $\tilde{\Sigma}$) for the Witten black hole. The second is to derive expressions, depending on the canonical variables of an arbitrary spacelike slice Σ, for the dilaton black-hole mass parameter M_W and (minus) the radial derivative of the Killing time τ'. Our procedure for obtaining such expressions is nearly identical to that found in KVK. Multiplying the expression we find for M_W by $\alpha/2$, one finds the boost invariant M given in (3.19).

1. Line element and asymptotic energy

In Kruskal-like coordinates the black-hole solution of the CGHSW theory is given by the line element

$$
\mathrm{d}s^2 = -\left(M_W/\lambda - \lambda^2 uv\right)^{-1}\mathrm{d}u\mathrm{d}v\,,
\tag{B1}
$$

along with the following expression for the dilaton:

$$
e^{-2\Phi} = M_W/\lambda - \lambda^2 uv\,.
\tag{B2}
$$

Our form of the line element corresponds to the one given in Ref. [26] with $u = x^-$, $v = x^+$, and the discussion which follows uses the conventions of that reference. We are only interested in the right static region of the Kruskal diagram associated with (B1).

We wish to compute the energy at infinity associated with the preferred static-time slices $\tilde{\Sigma}$, so we first have to find these slices. Consider the new coordinates (τ,σ) defined by [26]

$$
\lambda u = -e^{-\lambda(\tau-\sigma)}
\tag{B3a}
$$

$$
\lambda v = e^{\lambda(\tau+\sigma)}\,.
\tag{B3b}
$$

Note that the coordinate patch (τ,σ) only covers the right static region of the Kruskal diagram. In terms of these coordinates the line element reads

$$
\mathrm{d}s^2 = \left(1 + e^{-2\lambda\sigma}M_W/\lambda\right)^{-1}\left(-\mathrm{d}\tau^2 + \mathrm{d}\sigma^2\right)\,.
\tag{B4}
$$

Notice that as $\sigma \to \infty$ the line element becomes flat. Further, notice that as $\sigma \to \infty$ the dilaton behaves as

$$\Phi = -\lambda\sigma - \frac{M_W}{2\lambda} e^{-2\lambda\sigma} + O\left(e^{-2\lambda\sigma}\right) , \tag{B5}$$

where $O(e^{-2\lambda\sigma})$ stands for higher powers in $e^{-2\lambda\sigma}$. At spatial infinity the black-hole solution approaches the *linear dilaton*, the vacuum solution of the theory. For the linear dilaton $\Phi = -\lambda\sigma$ and the line element is Minkowskian.

The static-time slices $\tilde{\Sigma}$ are level surfaces of constant τ. We shall pick one and evaluate the energy at an outer boundary point B_o. Now the normal of such a point as embedded in $\tilde{\Sigma}$ has been denoted $\tilde{n} = \tilde{\mathbf{n}}$. Since only the static-time slices are of interest now, we shall drop the tilde which appears on $\tilde{\Sigma}$, $\tilde{n}$, and other objects. The history of B_o with respect to the Eulerian observers of the Killing time slices is the timelike sheet $\mathcal{T}_o$ defined by $\sigma = \sigma_o \geq -\infty$, where σ_o is a finite constant. This means that the dilaton Φ is also fixed to a constant value Φ_o on $\mathcal{T}_o$. Now it turns out that the energy expression for B_o as embedded in the selected static slice diverges in the limit that $\sigma_o \to \infty$.

In order to obtain a finite energy at spatial infinity, we must reference the energy against the linear dilaton vacuum *before* taking the limit. The expression for the quasilocal energy *with reference point* is

$$E = \alpha e^{-2\Phi} \left(n[\Phi] - n[\Phi]|^0\right) . \tag{B6}$$

This expression has been obtain by comparison with the known expression for quasilocal energy in general relativity, [5]

$$E = (8\pi)^{-1} \int_B \mathrm{d}^2 x \sqrt{\sigma} \left(k - k^0\right) . \tag{B7}$$

In this expression $\sqrt{\sigma}$ is the square root of the determinant of the B metric (in our case that of a round sphere $R^2 \sin\theta$) and k^0 is the trace of the extrinsic curvature of a two surface isometric to B but which is embedded in three-dimensional flat Euclidean space.[19] The origin of the reference term k^0 can be traced to the freedom to add a subtraction term (a functional of the fixed boundary data) to the four-dimensional spacetime action without affecting the variational principle. [5] Likewise, the reference point contribution in (B6) arises from the freedom to append a subtraction term $-S^0$ to our base action (3.1). By inspecting (B7), we see that the correct way to calculate the referenced energy is to first calculate $n[\Phi] - n[\Phi]|^0$ and then multiply by the nonlinear "determinant" factor $\alpha e^{-2\Phi}$.

We shall compute the quasilocal energy for the black hole with the subtraction term $n[\Phi]|^0$ determined by the linear-dilaton vacuum. For the black-hole solution the outward-pointing normal to points embedded in the constant τ slices is

$$n = \left(1 + e^{-2\lambda\sigma} M_W/\lambda\right)^{1/2} \partial/\partial\sigma \, . \tag{B8}$$

---

[19] Such a construction is not possible for a generic two surface. Of course, such a construction is always possible when B is a round sphere, the relevant case for this work.

With this one finds

$$n[\Phi] = -\lambda\left(1 + e^{-2\lambda\sigma}M_W/\lambda\right)^{-1/2}. \tag{B9}$$

A similar calculation for the case of the linear dilaton gives,

$$n[\Phi]|^0 = -\lambda\,. \tag{B10}$$

Hence, for the point B_o located at a $\sigma = \sigma_o$ as embedded in the constant time slice, the associated referenced quasilocal energy is

$$E = \alpha\lambda\left[e^{2\lambda\sigma} + M_W/\lambda\right]\left[1 - \left(1 + e^{-2\lambda\sigma}M_W/\lambda\right)^{-1/2}\right]\Big|_{\sigma=\sigma_o}. \tag{B11}$$

We then have that $\lim_{\sigma_o\to\infty} E = \frac{1}{2}\alpha M_W$. Hence we obtain the on-shell value of M given in (3.19) as the asymptotic energy associated with the static time slices. Note that the asymptotic energy is M_W if we make the choice $\alpha = 2$ for the CGHSW model.

2. Canonical expressions for M_W and $-\tau'$

Use of the dilaton itself as the radial coordinate casts the line element (B4) in the Schwarzschild-like form

$$\mathrm{d}s^2 = -F\mathrm{d}\tau^2 + F^{-1}\left(\mathrm{d}\Phi/\lambda\right)^2, \tag{B12}$$

where $F \equiv 1 - e^{2\Phi}M_W/\lambda$. The horizon is located at $\Phi = -(1/2)\log(M_W/\lambda)$, or equivalently at $R = \sqrt{M_W/\lambda^3}$, and we know from the Kruskal form of the line element (B1) that the geometry is perfectly regular at the horizon ($uv = 0$). The goal know is to obtain canonical expressions for τ' and M_W.

To get the desired expressions follow the method of KVK and assume that $\tau = \tau(t,r)$ and $\Phi = \Phi(t,r)$. It proves convenient to define a dimensional dilaton $\bar{\Phi} \equiv \Phi/\lambda$. Now expand the differentials $\mathrm{d}\tau = \dot{\tau}\mathrm{d}t + \tau'\mathrm{d}r$ and $\mathrm{d}\bar{\Phi} = \dot{\bar{\Phi}}\mathrm{d}t + \bar{\Phi}'\mathrm{d}r$ and plug these into the line element (B12). Comparison of the result with the ADM form of the metric (2.1) gives the following equations:

$$\Lambda^2 = -F(\tau')^2 + F^{-1}(\bar{\Phi}')^2 \tag{B13a}$$

$$\Lambda^2 N^r = -F\dot{\tau}\tau' + F^{-1}\dot{\bar{\Phi}}\bar{\Phi}' \tag{B13b}$$

$$-N^2 + \left(\Lambda N^r\right)^2 = -F(\dot{\tau})^2 + F^{-1}(\dot{\bar{\Phi}})^2\,. \tag{B13c}$$

From these it is straightforward to obtain the following expressions for the lapse and shift:

$$N^r = \frac{-F\dot{\tau}\tau' + F^{-1}\dot{\bar{\Phi}}\bar{\Phi}'}{-F(\tau')^2 + F^{-1}(\bar{\Phi}')^2} \tag{B14a}$$

$$N = \frac{\dot{\bar{\Phi}}\tau' - \dot{\tau}\bar{\Phi}'}{\sqrt{-F(\tau')^2 + F^{-1}(\bar{\Phi}')^2}}\,. \tag{B14b}$$

In obtaining the formula for N, we have taken a square root. Therefore, we need to verify that we have taken this root in such a way that the lapse is positive in the right static region of the Kruskal diagram, since we want our spacelike slices to advance everywhere into the future. Note that the dilaton is a "bad" radial coordinate in the sense that $\Phi \to -\infty$ as one approaches spatial infinity, whereas the preliminary section has assumed that the radial coordinate r increases in the direction of spatial infinity. Therefore, in the right static region $t = \tau$ and $r = -\bar{\Phi}$ are "good" coordinates, and, using these, we see that the lapse is positive everywhere in the right static region. The next step is to insert the last two expressions into the formula for the momenta

$$P_\Lambda = \alpha e^{-2\Phi} N^{-1} (\dot{\Phi} - N^r \Phi') \,, \tag{B15}$$

and after some algebra this insertion yields the first of our desired expressions

$$-\tau' = (\alpha\lambda)^{-1} e^{2\Phi} F^{-1} \Lambda P_\Lambda \,. \tag{B16}$$

Now, using this expression for $-\tau'$ in the first equation of (B13), we find the canonical expression for F,

$$F = (\alpha\lambda)^{-2} \left[(\alpha\Phi'/\Lambda)^2 - (e^{2\Phi} P_\Lambda)^2 \right] \,. \tag{B17}$$

Solving for M_W, one finds the second desired expression,

$$M_W = (\alpha\lambda)^{-2} \lambda e^{2\Phi} (P_\Lambda)^2 - \lambda^{-1} e^{-2\Phi} (\Phi'/\Lambda)^2 + \lambda e^{-2\Phi} \,. \tag{B18}$$

Notice that $M = \frac{1}{2}\alpha M_W$ is precisely the boost invariant (3.19) with the appropriate choices for the CGHSW model.

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