hep-th/9508056 CERN-TH/95-217
NYU-TH-95/07/02 IFUM/515/FT

Heterotic-Type II String Duality and the H-Monopole Problem

Luciano Girardello[a], Massimo Porrati[b], and Alberto Zaffaroni[c]

a Dipartimento di Fisica, Università di Milano, via Celoria 16, 20133 Milan, Italy[1 2]

b Department of Physics, NYU, 4 Washington Pl., New York, NY 10003, USA[3]

c Centre de Physique Théorique, Ecole Polytechnique, F-91128 Palaiseau CEDEX, France[4 5]

ABSTRACT

Since T-duality has been proved only perturbatively and most of the heterotic states map into solitonic, non-perturbative, type II states, the 6-dimensional string-string duality between the heterotic string and the type II string is not sufficient to prove the S-duality of the former, in terms of the known T-duality of the latter. We nevertheless show in detail that the perturbative T-duality, together with the heterotic-type II duality, does imply the existence of heterotic H-monopoles, with the correct multiplicity and multiplet structure. This construction is valid at a generic point in the moduli space of heterotic toroidal compactifications.

CERN-TH/95-217
August 1995

[1]and Theory Division, CERN, CH-1211 Geneva 23, Switzerland
[2]e-mail girardello@vaxmi.mi.infn.it
[3]e-mail porrati@mafalda.physics.nyu.edu
[4]Laboratoire Propre du CNRS UPR A.0014
[5]e-mail zaffaron@orphee.polytechnique.fr

The standard approach to string theory is intrinsically perturbative: one is given a recipe whereby one computes, say, the g-loop contribution to an S-matrix element in terms of a (super)conformal 2-d field theory on a genus-g Riemann surface. Naturally, any technique shedding light on the non-perturbative dynamics of strings is of the utmost importance. One such technique is based on the conjecture that some strongly interacting string models can be rewritten in terms of other, weakly interacting, "dual," string models.

One of the better understood among string dualities is that between the heterotic string, compactified to 6 dimensions on a 4-torus T_4, and the type IIA superstring, compactified on K_3 [1, 2]. Evidence supporting this conjecture has been given in refs. [1, 2, 3, 4]. If this 6-dimensional heterotic-type II duality holds, it implies various results.

One of the most important is that, upon further compactification of both the heterotic and type II strings to 4 dimensions on a two-torus T_2, there exists a "duality of dualities" [5] (see also [6]) between the two strings. This property consists in the following: both the heterotic string, compactified on $T_4 \times T_2$, and the type II string, compactified on $K_3 \times T_2$, have $N = 4$, $d = 4$ supersymmetry. They are both invariant under a discrete group of target space dualities (see [7] for a review on this matter). This group contains the direct product $SL(2, Z) \otimes SL(2, Z)$. The first $SL(2, Z)$ acts on the complex structure of the torus T_2, and it is called "U-duality." The second, called "T-duality," acts by fractional linear transformations on the complex field $T = B_{56} + i\sqrt{G}$, where $\sqrt{G}$ is the volume of the two-torus and B_{56} comes from the dimensional reduction on T_2 of the universal antisymmetric tensor of strings. Both theories are also conjectured to be invariant under a coupling-constant duality, the "S-duality," which also forms an $SL(2, Z)$ group. As shown in [5, 2], under heterotic-type II duality, the T- and S-dualities are interchanged. Thus, one may be tempted to conclude that S-duality follows automatically from the 6-dimensional heterotic-type II duality, since T-duality is a well-established, perturbative symmetry of strings [6].

This statement is not correct as it stands: *perturbative* T-duality is not sufficient to prove S-duality in the dual string. One obvious reason is that, for instance, the type II *perturbative* spectrum contains no state charged under the vectors coming from the Ramond-Ramond sector. These vectors are mapped by the heterotic-type II duality into gauge fields in the Cartan subalgebra of the heterotic gauge group ($E_8 \otimes E_8$, for instance). Conversely, heterotic states charged under the gauge group must be mapped by heterotic-type II duality into solitonic (non-perturbative) states of the type II string. This means that in order to prove, say, that the heterotic string compactified on $T_4 \times T_2$ is S-dual, one needs to prove that the type II string is T-dual *non-perturbatively.* Thus one has to find the action of T-duality on the non-perturbative, solitonic spectrum of this string etc. This task is obviously as complicated as a direct proof of S-duality for the heterotic string.

On the other hand, perturbative T-duality of the type II string can still be of use in trying to prove S-duality of the heterotic string: one may discover perturbative states of the type II string, transforming among themselves under T-duality, which map under heterotic-type II duality into perturbative, as well as *non-perturbative states* of the heterotic string, transforming among themselves under S-duality.

The purpose of this paper is to study in detail this scenario, where the perturbative T-duality of one string gives non-perturbative information about S-duality on the dual string. In

[6] i.e. a symmetry holding order by order in the string loop expansion.

particular, we will show that the rigorously proved *perturbative* T-duality of the type II string together with the conjectured heterotic-type II duality in 6 dimensions imply the existence of H-monopoles [8], with the right multiplicity and super-multiplet structure.

The paper is organized as follows: first we write the low-energy effective action of the type II superstring compactified on $K_3 \times T_2$, and show that T-duality is not a manifest symmetry of this action: to prove that two models related by a T-duality are in fact equivalent, one needs a Poincaré duality involving the field strengths of the vectors coming from the Ramond-Ramond sector. This fact already shows that perturbative T-duality is not the whole story for type II strings. Indeed, an equivalence between theories involving a Poincaré duality among two-forms is intrinsically non-perturbative, when acting on charged fields [7]. Then, we write the effective action of the heterotic string, compactified on $T_4 \times T_2$, and we show in details how the heterotic-type II duality works. In particular, we show how the S-, T- and U- dualities of this heterotic compactification relate to the corresponding dualities (S', T' and U') of the type II string. After that, we study the perturbative spectrum of the type II string, and identify the states related to "G-poles," i.e. heterotic states charged with respect to the vectors A^i_μ, $i = 4, 5$, which come from the dimensional reduction of the 6-dimensional metric. The G-poles saturate an appropriate Bogomol'nyi bound, thus their mass is non-renormalized [9]. These states must be mapped by S-duality into H-monopoles, i.e. into states magnetically charged under vectors $B_{\mu i}$, coming from the dimensional reduction of the 6-dimensional antisymmetric tensor. We thus come to the last and main result of the paper: we will show that the *perturbative* spectrum of the type II string contains all the states corresponding to the H-monopoles of the heterotic string, with the correct multiplicities. These states are mapped by the perturbative T-duality of the type II string into the type II partners of the G-poles. Thus, at least in this case, we can prove that perturbative T-duality and the 6-dimensional string-string duality do provide non-perturbative information about S-duality.

Let us begin by showing how the T-duality acts on the low-energy effective action of the type II superstring compactified on $K_3 \times T_2$. We start with the bosonic part of 6-dimensional theory obtained by compactifying the type II string on K_3,

$$\begin{aligned} & \int d^6x(\sqrt{-G}\{e^{-\Phi}[R + G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{12}G^{\mu\mu'}G^{\nu\nu'}G^{\sigma\sigma'}H_{\mu\nu\sigma}H_{\mu'\nu'\sigma'} \\ + & \frac{1}{8}G^{\mu\nu}Tr(\partial_\mu\hat{M}\hat{L}\partial_\nu\hat{M}\hat{L})] - G^{\mu\mu'}G^{\nu\nu'}F^a_{\mu\nu}(\hat{L}\hat{M}\hat{L})_{ab}F^b_{\mu'\nu'}\} \\ - & \frac{1}{4}\epsilon^{\mu\nu\sigma\rho\tau\eta}B_{\mu\nu}F^a_{\sigma\rho}\hat{L}_{ab}F^b_{\tau\eta}). \end{aligned} \tag{1}$$

The 24 Abelian gauge fields, with field strength $F^a_{\mu\nu}$, come from the reduction of the vector field and the third-rank antisymmetric field in ten dimensions, both originating in the Ramond-Ramond sector of the superstring. The symmetric matrix-valued scalar field $\hat{M}$ parametrizes

[7]The reason is simple: an "electrically charged" particle, coupling to the Ramond-Ramond field strength $F^{R-R}_{\mu\nu}$ with charge $g_e = \sqrt{4\pi/\mathrm{Im}\,T}$, is mapped into a "magnetically charged" particle, coupling to $\tilde{F}^{R-R}_{\mu\nu} \equiv (1/2)\epsilon^{\mu\nu\rho\sigma}F^{R-R}_{\rho\sigma}$ with charge $g_m = \sqrt{4\pi T\bar{T}/\mathrm{Im}\,T}$. At $\mathrm{Re}\,T = 0$, this equation becomes $g_m = 4\pi/g_e$, that is, a non-perturbative equation relating strong coupling to weak coupling.

the coset $O(4,20)/(O(4)\times 0(20))$ and satisfies $\hat{M}\hat{L}\hat{M}=\hat{L}$, where

$$\hat{L}=\left(\begin{array}{ccc} 0 & I_4 & 0 \\ I_4 & 0 & 0 \\ 0 & 0 & -I_{16} \end{array}\right). \tag{2}$$

Its components come from the reduction of the metric and antisymmetric tensor field. It is crucial that

$$H_{\mu\nu\sigma}=\partial_\mu B_{\nu\sigma}+\text{cyclic permutations}. \tag{3}$$

without Chern-Simons term.

In order to perform the dimensional reduction from 6 to 4 dimensions, it is convenient to introduce tangent indices, and a parametrization [16] in which the 6-dimensional vielbein is

$$\hat{e}_{\hat{\mu}}^{\hat{r}}=\left(\begin{array}{cc} e_\mu^r & A_\mu^i E_i^a \\ 0 & E_j^a \end{array}\right). \tag{4}$$

From now on, we shall use, whenever necessary, hatted fields and indices to denote 6-dimensional quantities ($\hat{\mu}=(\mu,i), i=5,6$ and $\hat{r}=(r,a), a=5,6$). Internal indices are raised and lowered by the metric $h_{ij}=E_i^a\delta_{ab}E_j^b$.

We obtain new scalars h_{ij}, B_{ij}, A_i^a from the internal components of the 6-dimensional metric, antisymmetric tensor and gauge fields. 4 new vectors, $A_\mu^i, B_{\mu i}$, come from the off-diagonal terms of the metric and antisymmetric tensor. It is convenient to perform the reduction by starting from the tangent-index expressions, using the following redefinition

$$\begin{aligned} \hat{e}_r^{\hat{\mu}}\hat{e}_s^{\hat{\nu}}\hat{B}_{\hat{\mu}\hat{\nu}} &= e_r^\mu e_s^\nu B_{\mu\nu} \\ \hat{e}_r^{\hat{\mu}}\hat{B}_{\hat{\mu}i} &= e_r^\mu B_{\mu i} \\ \hat{e}_r^{\hat{\mu}}\hat{A}_{\hat{\mu}}^a &= e_r^\mu A_\mu^a \end{aligned} \tag{5}$$

and then convert back to world indices. With the new definitions,

$$\begin{aligned} B_{\mu\nu} &= \hat{B}_{\mu\nu}+\frac{1}{2}(A_\mu^i B_{\nu i}-A_\nu^i B_{\mu i})-A_\mu^i B_{ij}A_\nu^j \\ B_{\mu i} &= \hat{B}_{\mu i}+B_{ij}A_\mu^j \\ A_\mu^a &= \hat{A}_\mu^a-A_i^a A_\mu^i. \end{aligned} \tag{6}$$

The dimensionally reduced Lagrangian reads:

$$\begin{aligned} & \int d^4x(\sqrt{-g}\{e^{-\phi}[R+g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\frac{1}{12}g^{\mu\mu'}g^{\nu\nu'}g^{\sigma\sigma'}H_{\mu\nu\sigma}H_{\mu'\nu'\sigma'} \\ + & \frac{1}{8}g^{\mu\nu}Tr(\partial_\mu\mathcal{M}J\partial_\nu\mathcal{M}J)+\frac{1}{8}g^{\mu\nu}Tr(\partial_\mu\hat{M}\hat{L}\partial_\nu\hat{M}\hat{L})]+2\sqrt{h}g^{\mu\nu}h^{ij}\partial_\mu A_i^a(\hat{L}\hat{M}\hat{L})_{ab}\partial_\nu A_j^b\} \\ - & \int d^4x\sqrt{-g}\{e^{-\phi}[\frac{1}{4}F_{\mu\nu}^A\mathcal{M}_{AB}F^{\mu\nu B}]+\sqrt{h}(F_{\mu\nu}^a+G_{\mu\nu}^iA_i^a)(\hat{L}\hat{M}\hat{L})_{ab}(F^{\mu\nu b}+G^{\mu\nu j}A_j^b)\} \\ + & \int d^4x[-\frac{1}{2}B_{56}\epsilon^{\mu\nu\sigma\rho}(F_{\mu\nu}^a+G_{\mu\nu}^iA_i^a)\hat{L}_{ab}(F_{\sigma\rho}^b+G_{\sigma\rho}^jA_j^b)+\epsilon^{\mu\nu\sigma\rho}\epsilon^{ij}H_{\mu\nu i}A_j^a\hat{L}_{ab}(F_{\sigma\rho}^b+\frac{1}{2}G_{\sigma\rho}^jA_j^b) \\ - & \frac{1}{3}\epsilon^{\mu\nu\sigma\rho}H_{\mu\nu\sigma}\epsilon^{ij}A_i^a\hat{L}_{ab}\partial_\rho A_j^b]. \end{aligned} \tag{7}$$

Here we defined a shifted dilaton field $\phi = \Phi - \log\sqrt{h}$, and the curvatures

$$F^A_{\mu\nu} = \left(\begin{array}{c} G^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu \\ H_{\mu\nu i} = \partial_\mu B_{i\,\nu} - \partial_\nu B_{i\,\mu} \end{array} \right) \quad A = (i,j). \tag{8}$$

The kinetic term of vectors is given by the matrices

$$\mathcal{M} = \left(\begin{array}{cc} h_{ij} - B_{ik}h^{kl}B_{lj} & B_{ik}h^{kj} \\ -h^{ik}B_{kj} & h^{ij} \end{array} \right), \quad J = \left(\begin{array}{cc} 0 & I_2 \\ I_2 & 0 \end{array} \right). \tag{9}$$

In the reduction, $H_{\mu\nu\sigma}$ acquired a Chern-Simons term with respect to the vectors coming from the metric and antisymmetric tensor,

$$H_{\mu\nu\sigma} = \partial_\mu B_{\nu\sigma} - \frac{1}{2}(A^i_\mu H_{\nu\sigma i} + B_{\mu i} G^i_{\nu\sigma}) + \text{cyclic permutations}. \tag{10}$$

From string theory we know that the compactification on a two-torus induces an $O(2,2;Z)$ symmetry of the perturbative spectrum (see [7] and the discussion of the spectrum below). $O(2,2;Z)$ is split into two copies of $SL(2;Z)$, called T- and U-duality. Let us show how they act on the effective Lagrangian. At first, we must recall that $O(2,2;Z)$ acts on the matrix $\mathcal{M}$ [7] in the following way

$$O(2,2;Z): \quad \mathcal{M} \to \Omega\mathcal{M}\Omega^T. \tag{11}$$

The kinetic term for the scalars is obviously invariant, and so also the kinetic term for the vectors $F^A_{\mu\nu}$, provided we simultaneously trasform the field strengths as follows:

$$O(2,2;Z): \quad F^A_{\mu\nu} \to (\Omega^{-1})_{BA} F^B_{\mu\nu}. \tag{12}$$

The terms containing the gauge fields $F^a_{\mu\nu}$ are more complicated since U and T act in a very different way. To see this, we must look more carefully at the action of T and U separately. A useful parametrization for the two-dimensional matrices h and B is;

$$h + iB = \sqrt{h}\left[\frac{1}{U_2}\left(\begin{array}{cc} U_1^2 + U_2^2 & U_1 \\ U_1 & 1 \end{array} \right)\right] + iB_{56}\left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) = \sqrt{h}\tilde{h} + iB_{56}.\epsilon \tag{13}$$

Introducing the two complex numbers $U = U_1 + iU_2$, $T = B_{56} + i\sqrt{h}$, $O(2,2;Z)$ acts as a copy of the standard $SL(2,Z)$ linear fractional transformations on both variables,

$$\begin{array}{l} X \to \dfrac{aX+b}{cX+d}, \quad P(X) \to \omega P(X)\omega^T \\ X = T, U, \quad \omega = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right), P(X) = \dfrac{1}{X_1}\left(\begin{array}{cc} X_1^2 + X_2^2 & X_1 \\ X_1 & 1 \end{array} \right). \end{array} \tag{14}$$

Given the following expression for the matrix $\mathcal{M}$,

$$\mathcal{M} = \left(\begin{array}{cc} 1 & 0 \\ 0 & -\epsilon \end{array} \right)\left[\frac{1}{T_1}\left(\begin{array}{cc} T_1^2 + T_2^2 & T_1 \\ T_1 & 1 \end{array} \right) \otimes \frac{1}{U_2}\left(\begin{array}{cc} U_1^2 + U_2^2 & U_1 \\ U_1 & 1 \end{array} \right)\right]\left(\begin{array}{cc} 1 & 0 \\ 0 & \epsilon \end{array} \right), \tag{15}$$

it is easy to determine the embedding of the T- and U-dualities in the full group $O(2,2;Z)$. Let us begin with the U-duality. The $O(2,2;Z)$ matrix corresponding to the $SL(2,Z)$ transformation ω, and the corresponding transformation of the gauge fields $F^A_{\mu\nu}$, are

$$\Omega_U = \begin{pmatrix} \omega & 0 \\ 0 & (\omega^{-1})^T \end{pmatrix}, \quad H_{i\,\mu\nu} \to (\omega^{-1})^{ji} H_{j\,\mu\nu},\ G^i_{\mu\nu} \to \omega_{ij} G^j_{\mu\nu}. \tag{16}$$

Imposing the following transformation on the scalars

$$A^a_i \to \omega_{ij} A^a_j \tag{17}$$

we see that the U-duality is a manifest symmetry of the low-energy effective Lagrangian.

T-duality, on the other hand, is more delicate. The corresponding $O(2,2;Z)$ transformation is

$$\Omega_T = \begin{pmatrix} a & b\epsilon \\ -c\epsilon & d \end{pmatrix}, \tag{18}$$

and we see that under the two generators of $SL(2,Z)$ the gauge fields transform as follows

$$T \to T+1: \quad \begin{array}{c} G^i_{\mu\nu} \to G^i_{\mu\nu} \\ H_{i\,\mu\nu} \to \epsilon_{ij} G^j_{\mu\nu} + H_{i\,\mu\nu} \end{array} \qquad T \to -\frac{1}{T}: \quad \begin{array}{c} G^i_{\mu\nu} \to \epsilon^{ij} H_{j\,\mu\nu} \\ H_{i\,\mu\nu} \to \epsilon_{ij} G^j_{\mu\nu} \end{array}. \tag{19}$$

Notice that the generator of $T \to T+1$, usually realized in a trivial manner, here requires a non-trivial transformation of the gauge field $H_{i\,\mu\nu}$, since it multiplies a term that, in this case, is not a topological invariant. It is easy to verify the invariance of the Lagrangian under the combined transformations (18) and (19), corresponding to the generator of $T \to T+1$.

For what regards the generator of $T \to -1/T$, the kinetic term for scalars involves only the matrix h_{ij} so it is trivially invariant if the scalars do not transform. However, the presence of a factor of $\sqrt{h}$ in front of the kinetic term for the 24 gauge fields $F^a_{\mu\nu}$, implies that the T-duality cannot be realized as a symmetry of the Lagrangian, but it must involve a Poincaré duality on $F^a_{\mu\nu}$. Equivalently, T-duality is realized only on the equations of motion. This is most easily seen by adding to the action the Lagrange multiplier

$$\int d^4x \epsilon^{\mu\nu\sigma\rho} C^a_{\mu\nu} F^a_{\sigma\rho}, \tag{20}$$

which enforces the Bianchi identities for $F^a_{\mu\nu}$. Notice that now $F^a_{\mu\nu}$ is an independent variable and appears only polynomially in the Lagrangian. If we perform a duality transformation on the modulus T and the gauge fields $F^A_{\mu\nu}$, the action is obviously not invariant, but when re-expressed in terms of the dual gauge field $C^a_{\mu\nu}$, using the $F^a_{\mu\nu}$ equations of motion,

$$F^a_{\mu\nu} = \epsilon^{ij} H_{\mu\nu i} A^a_j - T_1 (C^a_{\mu\nu} + G^i_{\mu\nu} A^a_i) + T_2 (\hat{M}\hat{L})^{ab} (\tilde{C}^b_{\mu\nu} + \tilde{G}^i_{\mu\nu} A^b_i), \tag{21}$$

the action reacquires the original form. We have used the following convention for the dual gauge field

$$\tilde{F}^{\mu\nu} = \frac{1}{2\sqrt{-g}} \epsilon^{\mu\nu\sigma\rho} F_{\sigma\rho}. \tag{22}$$

We see that an $SL(2,Z)$ transformation on T, combined with the explicit rotation (19) on the gauge fields $F^A_{\mu\nu}$, and the Poincaré duality (21) on the gauge fields $F^a_{\mu\nu}$, is a symmetry of the

Lagrangian. This is how the T-duality is realized on the low-energy effective action. The equations of motion are of course invariant. Let us collect for further reference the transformations of the gauge fields

$$
\begin{aligned}
T \to T+1 \quad &: \quad \begin{array}{c} G^i_{\mu\nu} \to G^i_{\mu\nu} \\ H_{i\,\mu\nu} \to \epsilon_{ij} G^j_{\mu\nu} + H_{i\,\mu\nu} \\ F^a_{\mu\nu} \to F^a_{\mu\nu} \end{array} \\
T \to -\frac{1}{T} \quad &: \quad \begin{array}{c} G^i_{\mu\nu} \to \epsilon^{ij} H_{j\,\mu\nu} \\ H_{i\,\mu\nu} \to \epsilon_{ij} G^j_{\mu\nu} \\ F^a_{\mu\nu} \to \epsilon^{ij} H_{i\,\mu\nu} A^a_j - T_1 (F^a_{\mu\nu} + G^i_{\mu\nu} A^a_i) + T_2 (\hat{M}\hat{L})^{ab} (\tilde{F}^b_{\mu\nu} + \tilde{G}^i_{\mu\nu} A^b_i) \end{array}
\end{aligned} \tag{23}
$$

Next, we must examine the duality between heterotic and type II strings. The equivalence we need is between the heterotic string theory compactified to 6 dimensions on a 4-torus, and the type II superstring compactified on K_3 [1, 2, 3, 4]. At the level of low-energy Lagrangians, the string-string duality is realized by the following redefinitions of the 6-dimensional fields,

$$
\Phi' = -\Phi, \quad G'_{\mu\nu} = e^{-\Phi} G_{\mu\nu}, \quad \sqrt{-G'} e^{-\Phi'} H'^{\mu\nu\sigma} = \frac{1}{6} \epsilon^{\mu\nu\rho\sigma\tau\eta} H_{\sigma\tau\eta}. \tag{24}
$$

This redefinition maps the equations of motion of the type II Lagrangian (1) into the equations of motion of the heterotic Lagrangian

$$
\begin{aligned}
&\int d^6x \sqrt{-G'} e^{-\phi'} [R' + G'^{\mu\nu} \partial_\mu \Phi' \partial_\nu \Phi' - \frac{1}{12} G'^{\mu\mu'} G'^{\nu\nu'} G'^{\sigma\sigma'} H'_{\mu\nu\sigma} H'_{\mu'\nu'\sigma'} \\
&+ \frac{1}{8} G'^{\mu\nu} Tr(\partial_\mu \hat{M}' \hat{L} \partial_\nu \hat{M}' \hat{L}) - G'^{\mu\mu'} G'^{\nu\nu'} F'^a_{\mu\nu} (\hat{L}\hat{M}'\hat{L})_{ab} F'^b_{\mu'\nu'}],
\end{aligned} \tag{25}
$$

where

$$
H'_{\mu\nu\rho} = \partial_\mu B'_{\nu\rho} - 2 A'^a_\mu \hat{L}_{ab} F'^b_{\nu\rho} + \text{cyclic permutations}. \tag{26}
$$

Notice that the crucial ingredient of the redefinition is a Poincaré duality on the third-rank form H in 6 dimensions. The equivalence of the two models can be seen also by adding to the type II action (1) a Lagrange multiplier

$$
\int d^6x \epsilon^{\mu\nu\rho\sigma\tau\eta} H_{\mu\nu\rho} \tilde{H}'_{\sigma\tau\eta}. \tag{27}
$$

Trading H for $\tilde{H}'$, and using the equations of motion, we recover (after a Weyl rescaling, a change of sign of the dilaton, and defining H' as $\tilde{H}'$ plus the Chern-Simons contribution) the heterotic Lagrangian (25).

Compactifying further to 4 dimensions on a two-torus, using the same field redefinitions of the type II case, and paying attention to the Chern-Simons term, we obtain the reduced Lagrangian (where the primes are suppressed for simplicity)

$$
\begin{aligned}
&\int d^4x \sqrt{-g} e^{-\phi} [R + g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{12} g^{\mu\mu'} g^{\nu\nu'} g^{\sigma\sigma'} H_{\mu\nu\sigma} H_{\mu'\nu'\sigma'} \\
&+ \frac{1}{8} g^{\mu\nu} Tr(\partial_\mu \mathcal{N} L \partial_\nu \mathcal{N} L) - \frac{1}{4} F^\alpha_{\mu\nu} \mathcal{N}_{\alpha\beta} F^{\mu\nu\beta}],
\end{aligned} \tag{28}
$$

where we have defined

$$F^{\alpha}_{\mu\nu} = \begin{pmatrix} G^{i}_{\mu\nu} \\ H_{i\,\mu\nu} \\ 2F^{a}_{\mu\nu} \end{pmatrix}, \quad L = \begin{pmatrix} 0 & I_6 & 0 \\ I_6 & 0 & 0 \\ 0 & 0 & -I_{16} \end{pmatrix}. \tag{29}$$

Here, $H_{\mu\nu\tau}$ has acquired a full Chern-Simons term

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} - \frac{1}{2} A^{\alpha}_{\mu} \hat{L}_{\alpha\beta} F^{\beta}_{\nu\rho} + \text{cyclic permutations}. \tag{30}$$

The scalar matrix $\mathcal{N}$ is expressed in terms of the original fields of the theory as follows:

$$\begin{pmatrix} h + (C-B)h^{-1}(C+B) + A(\hat{L}\hat{M}\hat{L})A & -(C-B)h^{-1} & (C-B)h^{-1}(A\hat{L}) + A(\hat{L}\hat{M}\hat{L}) \\ -h^{-1}(C+B) & h^{-1} & -h^{-1}(A\hat{L}) \\ (\hat{L}A)h^{-1}(C+B) + (\hat{L}\hat{M}\hat{L})A & -(\hat{L}A)h^{-1} & (\hat{L}A)h^{-1}(A\hat{L}) + (\hat{L}\hat{M}\hat{L}) \end{pmatrix}, \tag{31}$$

where

$$C_{ij} = \frac{1}{2} A^{a}_{i} \hat{L}_{ab} A^{b}_{j}. \tag{32}$$

The heterotic 4-dimensional string also possesses an S-duality [17], which acts on the complex coupling constant,

$$\lambda = \psi + i e^{-\phi}, \quad H^{\mu\nu\rho} = -(\sqrt{-g})^{-1} e^{\phi} \epsilon^{\mu\nu\rho\sigma} \partial_\sigma \psi, \tag{33}$$

as a linear fractional transformation, and on the gauge fields as the Poincaré duality

$$\lambda \to \frac{a\lambda + b}{c\lambda + d} \quad F^{a}_{\mu\nu} \to (c\lambda_1 + d) F^{a}_{\mu\nu} - c\lambda_2 (L\mathcal{N})_{ab} \tilde{F}^{a}_{\mu\nu}. \tag{34}$$

The type II and heterotic 4-dimensional Lagrangians are obviously, but not manifestly, equivalent. When reduced to 4 dimensions, the simple 6-dimensional redefinition (24) becomes a less obvious redefinition (Poincaré duality) of $H_{\mu\nu\tau}, G_{\mu\nu}, G_{ij}, B_{ij}$. In particular, under string-string duality, the T-duality of the type II superstring is mapped to the S-duality of the heterotic string. In fact, by reducing eq. (24) to 4 dimensions (taking into account the Chern-Simons term in the heterotic side), we learn that

$$B_{56} \to \lambda'_1 \quad \frac{1}{6} \epsilon_{\mu\nu\tau\sigma} H_{\nu\tau\sigma} \to \sqrt{-g'} e^{-\phi'} \partial^{\mu} B'_{56}. \tag{35}$$

Thus, if we exchange the role of $\sqrt{h}$ and $e^{-\phi}$ after the redefinition (35), the T modulus is mapped in the complex coupling constant λ' and vice-versa. The redefinition of the gauge field H_i reads

$$H'_{\mu\nu i} - G^{j}_{\mu\nu}(C_{ij} + B'_{ij}) - A^{a}_{i} \hat{L}_{ab} F^{b}_{\mu\nu} = \frac{e^{-\phi}}{(\sqrt{h})^2} h_{ij} \epsilon^{jk} (\tilde{H}_{\mu\nu k} - \tilde{G}^{t}_{\mu\nu} B_{kt}). \tag{36}$$

It is now a simple, though tedious, exercise to check that the T-duality transformations (23) for the type II gauge fields $(G^{i}_{\mu\nu}, H_{i\,\mu\nu}, F^{a}_{\mu\nu})$, when re-expressed in terms of the primed variables, reproduce exactly the S-duality transformations (34) of the heterotic gauge fields $(G'^{i}_{\mu\nu}, H'_{i\,\mu\nu}, F'^{a}_{\mu\nu})$.

Let us now turn to the study of the type II superstring (perturbative) spectrum. We want to find states that may correspond to massive, short multiplets of the $N = 4$ supersymmetry algebra [10]. These multiplets have the same number of components as a "long" $N = 2$ multiplet, and their common supersymmetric mass saturates a Bogomol'nyi bound. This means that their squared mass is proportional to the sum of squares of some Abelian (central) charges Q_e^I, $I = 1, .., 6$, and a constant matrix M_{IJ}:

$$m^2_{\text{short multiplet}} = Q_e^I M_{IJ} Q_e^J. \tag{37}$$

The charges Q_e^I are "electric," because no perturbative state can have a non-zero "magnetic" charge. To identify these charges, and find which type II massless vector they correspond to, we must recall some elementary facts about superstring theory [11].

The spectrum of the type II superstring compactified on $K_3 \times T_2$ can be easily written in the light-cone gauge, and in the Ramond-Neveu-Schwarz formalism. We will work at a generical point of the separate moduli space of K_3 and T_2, but with a compactification for which $A_i^a = 0$, since otherwise the corresponding conformal field theory is no longer constructed with free fields and involves non-trivial R-R deformations. We need only consider the space-time bosons, since fermions can be obtained from them by space-time supersymmetry transformations. Since one can choose independently the boundary conditions of the left- and right-moving world-sheet fermions, as either antiperiodic (Neveu-Schwarz b.c. or NS) or periodic (Ramond b.c. or R), one obtains 4 sectors in the Hilbert space of string states, denoted as usual by NS-NS, NS-R, R-NS or R-R. The space-time bosons of the type II string arise from both the NS-NS and the R-R sectors. Their mass is given by the standard light-cone formula ($\alpha' = 1/2$):

$$\frac{1}{2}m^2 = N_R^{ST} + L_0^{T_2} + L_0^{K_3} - \frac{1}{2} = N_L^{ST} + \bar{L}_0^{T_2} + \bar{L}_0^{K_3} - \frac{1}{2}. \tag{38}$$

Here N_R^{ST} (N_L^{ST}) is the usual right (left) transverse space-time oscillator number. In detail, we have two free bosons (the transverse space-time coordinates $X^1(\sigma,\tau)$, $X^2(\sigma,\tau)$) together with their world-sheet supersymmetric partners, the Neveu-Schwarz fermions $\psi_R^\mu(\sigma-\tau)$, $\psi_L^\mu(\sigma+\tau)$, $\mu = 1, 2$. Their mode expansion is

$$\begin{aligned}
X^\mu(\sigma,\tau) &= x^\mu + k^\mu\tau + \frac{i}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}[\alpha^\mu_{R\,n}e^{-in(\tau-\sigma)} + \alpha^\mu_{L\,n}e^{-in(\tau+\sigma)}],\\
\psi^\mu_R(\sigma-\tau) &= \sum_{r\in Z+a/2}\psi^\mu_{R\,r}e^{-ir(\tau-\sigma)}, \quad \psi^\mu_L(\sigma+\tau) = \sum_{r\in Z+a/2}\psi^\mu_{L\,r}e^{-ir(\tau+\sigma)},\\
[\alpha^\mu_{R\,n}, \alpha^\nu_{R\,m}] &= [\alpha^\mu_{R\,n}, \alpha^\nu_{R\,m}] = m\delta^{\mu\nu}\delta_{n+m,0}, \quad [\alpha^\mu_{R\,n}, \alpha^\nu_{L\,m}] = 0,\\
\{\psi^\mu_{R,r}, \psi^\nu_{R\,p}\}_+ &= \{\psi^\mu_{L,r}, \psi^\nu_{L\,p}\}_+ = \delta^{\mu\nu}\delta_{q+p,0}, \quad \{\psi^\mu_{R,r}, \psi^\nu_{L\,p}\}_+ = 0.
\end{aligned} \tag{39}$$

Here a is a constant equal to 1 in the NS sector and to 0 in the R sector, while k^μ is the transverse space-time momentum. The oscillator number N_R reads

$$N_R^{ST} = \sum_{n>0}\alpha^\mu_{R\,-n}\alpha^\mu_{R\,n} + \sum_{r\in N+a/2} r\psi^\mu_{R\,-r}\psi^\mu_{R\,r} + \frac{1-a}{8}. \tag{40}$$

The formula for N_L is obtained by replacing R with L throughout this equation.

$L_0^{T_2}$ ($\bar{L}_0^{T_2}$) is the right (left) Virasoro operator of two free bosons $X^i(\sigma,\tau)$, $i = 4, 5$, compactified on a two-dimensional torus ($X^i \approx X^i + 2\pi n$, $n \in Z$), together with their fermionic right-and left-moving superpartners $\psi^i_R(\sigma-\tau)$, $\psi^i_L(\sigma+\tau)$. Their mode expansion is [12, 7]

$$\begin{aligned}
X^i(\sigma,\tau) &= x^i + m^i\sigma + h^{ij}(n_j - B_{jk}m^k)\tau + \frac{i}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}[\alpha^i_{R\,n}e^{-in(\tau-\sigma)} + \alpha^i_{L\,n}e^{-in(\tau+\sigma)}],\\
\psi^i_R(\sigma-\tau) &= \sum_{r\in Z+a/2}\psi^i_{R\,r}e^{-ir(\tau-\sigma)}, \quad \psi^i_L(\sigma+\tau) = \sum_{r\in Z+a/2}\psi^i_{L\,r}e^{-ir(\tau+\sigma)},\\
[\alpha^i_{R\,n},\alpha^j_{R\,m}] &= [\alpha^i_{R\,n},\alpha^j_{R\,m}] = mh^{ij}\delta_{n+m,0}, \quad [\alpha^i_{R\,n},\alpha^j_{L\,m}] = 0,\\
\{\psi^i_{R,\,r},\psi^j_{R\,p}\}_+ &= \{\psi^i_{L,\,r},\psi^j_{L\,p}\}_+ = h^{ij}\delta_{q+p,0}, \quad \{\psi^i_{R,\,r},\psi^j_{L\,p}\}_+ = 0.
\end{aligned} \tag{41}$$

h_{ij} and B_{ij} are, respectively, the metric and antisymmetric tensor of the two-torus. The integers $m^i \in Z$ are the winding numbers, while the $n_i \in Z$ are the momenta. Thanks to eq. (41), the Virasoro operators on the two-torus read

$$\begin{aligned}
L_0^{T_2} + \bar{L}_0^{T_2} &= \frac{1}{2}Z^t\mathcal{M}Z + N_R^B + N_R^F + N_L^B + N_L^F + \frac{1-a}{4},\\
L_0^{T_2} - \bar{L}_0^{T_2} &= m^in_i + N_R^B + N_R^F - N_L^B - N_L^F,\\
N_R^B &= \sum_{n>0} h_{ij}\alpha^i_{R\,-n}\alpha^j_{R\,n} \quad N_R^F = \sum_{r\in N+a/2} r\psi_{R\,-r}\psi_{R\,r},\\
N_L^B &= \sum_{n>0} h_{ij}\alpha^i_{L\,-n}\alpha^j_{L\,n} \quad N_L^F = \sum_{r\in N+a/2} r\psi_{L\,-r}\psi_{L\,r}.
\end{aligned} \tag{42}$$

The matrix $\mathcal{M}$ is the same as the one we encountered in the dimensional reduction of the low-energy effective action, and Z is a column vector:

$$Z = \begin{pmatrix} m^i \\ n_i \end{pmatrix}. \tag{43}$$

From eq. (41) we learn that the form of the internal, two-dimensional momenta is

$$\begin{aligned}
p^i_R &= \frac{1}{\sqrt{2}}(m^i - h^{ij}(n_j - B_{jk}m^k))\\
p^i_L &= \frac{1}{\sqrt{2}}(m^i + h^{ij}(n_j - B_{jk}m^k)).
\end{aligned} \tag{44}$$

The Hamiltonian restricted to zero modes reads:

$$H = \frac{1}{2}(p_R^2 + p_L^2) \equiv \frac{1}{2}(p^i_Rp^i_R + p^i_Lp^j_L)h_{ij}. \tag{45}$$

The compactifications [7] are in one-to-one correspondence with the even self-dual Lorentzian lattice $\Gamma^{(1,1)}$ spanned by the vectors (p_R, p_L), which have indeed even-integer Lorentzian norm:

$$p_L^2 - p_R^2 = 2n_im^i \in 2Z. \tag{46}$$

The moduli space of toroidal compactifications is therefore isomorphic to $O(2,2;R)/(O(2)\times O(2))$. The spectrum is known to be invariant under the T-duality transformation $O(2,2;Z)$, which acts as a linear transformation on Z: $Z\to\omega Z$, $\omega\in SL(2,Z)$.

The superstring coordinates, compactified on K_3, describe an $N=4$ two-dimensional superconformal field theory. For our purposes, we do not need the complete spectrum of the theory, but only the lowest conformal-weight states in the NS and R sectors. All unitary representations of the $N=4$ superconformal algebra have been classified in [13]. They are labelled by the conformal weight, h, and by an internal $SU(2)$ spin l. For central charge $c=6$ (recall that K_3 has real dimension 4), and since we have two superconformal $N=4$ algebras, with Virasoro operators L_0 and $\bar{L}_0$, respectively, the generic K_3 state belongs to an irreducible representation of $SU(2)\otimes SU(2)$, and reads:

$$|h,l,\bar{h},\bar{l}\rangle; \quad L_0|h,l,\bar{h},\bar{l}\rangle = h|h,l,\bar{h},\bar{l}\rangle, \quad \bar{L}_0|h,l,\bar{h},\bar{l}\rangle = \bar{h}|h,l,\bar{h},\bar{l}\rangle. \tag{47}$$

As shown in ref. [14], the lowest-weight states of the NS-NS sector are: a) the (unique) $SL(2,R)$-invariant vacuum $|0,0,0,0\rangle$. b) 20 4-vectors (representations $(1/2,1/2)$ of $SU(2)\otimes SU(2)$) with conformal weights $h=\bar{h}=1/2$: $|1/2,1/2,1/2,1/2\rangle$. The vacuum $|0,0,0,0\rangle$ is even under the standard GSO projection [15] of the type IIA superstring, while the (20) states $|1/2,1/2,1/2,1/2\rangle$ are odd. In the R-R sector, the lowest conformal-weight states are obtained by the spectral flow from the above NS-NS states [14]. In particular, the spectral flow of the vacuum gives rise to a R-R state $|1/4,1/2,1/4,1/2\rangle$ (a 4-vector of $SU(2)\otimes SU(2)$), while the spectral flow of each 4-vector $|1/2,1/2,1/2,1/2\rangle$ gives rise to a singlet $|1/4,0,1/4,0\rangle$. Thus, in total, in the R-R sector, we find 24 states with conformal weights $h=1/4$, $\bar{h}=1/4$.

The Hilbert space of the conformal field theory corresponding to the compactification on $T_2\times K_3$ is just the direct product of the Hilbert spaces of the theories on the transverse space-time coordinates T_2, and K_3. The physical states are odd under the GSO projection. As we are interested only in states saturating a Bogomol'nyi bound, we must find the charges Q_e^I of our perturbative type II string states. This is easily done by noticing that the 4-dimensional Abelian vectors, which come from the dimensional reduction of the 6-dimensional theory on T_2, are, using our previous notations, $\hat{B}_{\mu i}=B_{\mu i}+A^j_\mu B_{ij}$ and $\hat{A}_{\mu i}=A^j_\mu h_{ij}$. The light-cone vertices corresponding to the fields A^i_μ, $B_{\mu i}$ read, at zero 4-dimensional momentum,

$$\begin{aligned} V(B_{\mu i}) &= \int d\tau d\sigma(\partial_\sigma X^\mu \partial_\tau X^i - \partial_\tau X^\mu \partial_\sigma X^i), \\ V(A^i_\mu) &= \int d\tau d\sigma[\partial_\sigma X^\mu(h_{ij}\partial_\sigma X^j + B_{ij}\partial_\tau X^j) - \partial_\tau X^\mu(h_{ij}\partial_\tau X^j + B_{ij}\partial_\sigma X^j)]. \end{aligned} \tag{48}$$

From these equations, one can extract the charges Q_i, associated with A^i_μ, and $\tilde{Q}^i$, associated with $B_{\mu i}$:

$$Q_i = \int d\sigma(h_{ij}\partial_\tau X^j + B_{ij}\partial_\sigma X^j) = n_i, \quad \tilde{Q}^i = \int d\sigma \partial_\sigma X^i = m^i. \tag{49}$$

Let us note that these charges are not those that appear in the asymptotic expression of the gauge fields, because of the non-canonical kinetic terms of the vectors A^A_μ. If we add a source $\int d^4x\sqrt{-g}A^A_\mu J^\mu_A$ to the Lagrangian (7), we learn from the equations of motion that, for large r (and $A^a_i=0$),

$$F^A_{0i} \to \frac{q^A}{r^2}, \quad q^A = \frac{1}{e^{-\phi}}(\mathcal{M}^{-1})^{AB}Q_B, \quad Q_A = \begin{pmatrix} Q_i \\ \tilde{Q}^i \end{pmatrix}. \tag{50}$$

By comparing eq. (37) with the mass formula eq. (38), and using the explicit formulae just derived above, we find that the Bogomol'nyi bound is saturated only when either $N_R^{ST}+N_R^F+$

$N_R^B + h = (1+a)/4$ (no constraint on the left-moving oscillator numbers and the conformal weight on K_3), or $N_L^{ST} + N_L^F + N_L^B + \bar{h} = (1+a)/4$ (no constraint on the right-movers) and explicitely reads:

$$m^2 = \frac{1}{2} Z^t (\mathcal{M} + J) Z = p_R^2 \ \text{ or } p_L^2. \tag{51}$$

Let us now turn to the heterotic string compactified on T^6. The supersymmetric right-movers are two transverse space-time and 6 internal free bosons, together with their fermionic superpartners. The non-supersymmetric left-movers are two space-time and 6 internal free bosons coming from the compactification on $T^4 \times T^2$, and 16 free bosons in the Cartan subalgebra of the gauge group $E_8 \times E_8$. The conventions and the normalizations for oscillators are the same as for the type II string, so let us exhibit only the difference in the zero modes of the bosons

$$\begin{aligned} X_R^i &= x_R^i + \sqrt{2} p_R^i (\sigma - \tau) + \text{oscillators}, \quad i = 5, ..., 10 \\ X_L^{i,I} &= x_L^{i,I} + \sqrt{2} p_L^{i,I} (\sigma - \tau) + \text{oscillators}, \quad i = 5, ..., 10, I = 1, ..., 16. \end{aligned} \tag{52}$$

Here the internal momenta depend on the compactification data so that the vector (p_R, p_L) belongs to the even self-dual Lorentzian lattice $\Gamma^{(6,22)}$. We refer to [7] for the explicit expression of the momenta. Here we limit ourselves to two particular cases. When the background vectors of $E_8 \times E_8$ are set to zero, formula (44) is reproduced, with the only difference that the indices now run from 5 to 10. When, instead, the winding number is set to zero, we obtain [12]:

$$\begin{aligned} p_L^I &= e^I \\ p_L^i &= \frac{1}{\sqrt{2}} h^{ij} (n_j + A_i^I e^I) \\ p_R^i &= p_L^i \end{aligned} \tag{53}$$

where e^I is an element in the root lattice of $E_8 \times E_8$. Formulae for the Virasoro operators analogous to those given for the type II superstring are valid for the heterotic string; here we recall only the main differences. Since we are interested in the bosonic part of the spectrum (the fermionic part follows for space-time supersymmetry), we will consider only the NS sector. The mass-shell condition now reads

$$\frac{1}{2} m^2 = N_R^{ST} + L_0^{T_6} - \frac{1}{2} = N_L^{ST} + \bar{L}_0^{T_6} - 1. \tag{54}$$

The Bogomol'nyi condition is now realized when $N_R^{TOT} = 1/2$, with the left-moving oscillators constrained by

$$N_L^{TOT} - 1 = \frac{1}{2} (p_R^2 - p_L^2). \tag{55}$$

and reads

$$m^2 = p_R^2 = \frac{1}{2} Q^\alpha (N + L)_{\alpha\beta} Q^\beta, \tag{56}$$

where the vector Q^α contains the momentum, the winding number, and the charges under $E_8 \times E_8$, $Q^\alpha = (n_i, m^i, e^I)$. The asymptotic expression of the gauge fields are given by

$$F_{0i}^\alpha \to \frac{q^\alpha}{r^2}, \quad q^\alpha = \frac{1}{e^{-\phi}} (L \mathcal{N} L)_{\alpha\beta} Q^\beta. \tag{57}$$

The point of view of searching evidence for S-duality in the heterotic string strongly reduces the number of perturbative string states that we can consider: this is because our knowledge of the non-perturbative solitonic spectrum of the string is limited to the solutions of the low-energy action for the massless states. Thus, we are led to consider only perturbative string states which can become light in some compactification limit (as, for instance, the large-radius limit). Only these states are mapped by S-duality into light solitonic states, which may be found as solutions of the equations of motion of the low-energy effective action of the string [17]. Therefore, we shall consider only states with zero winding number, for which a field theory limit exists. To satisfy the condition $N_R^{TOT} = 1/2$, we apply to the vacuum $|0\rangle$ the 8 left-moving fermionic oscillators $\psi^{\mu,i}_{R-1/2}$, which give rise by themselves to the bosonic part of a 16-dimensional representation of $N = 4$ supersymmetry (1 vector and 5 scalars). The simplest state satisfying the Bogomol'nyi bound has $N_L = 0$ and $p_L^2 = p_R^2 + 2$. This condition can be realized only with states that are charged under $E_8 \times E_8$ (53); however, they are uncharged with respect to the vectors coming from the metric and antisymmetric tensor, since they have neither internal momentum nor winding number on the compactification torus ($n_i = m^i = 0$). Under an S-duality transformation, the $p_L^2 = p_R^2 + 2$ states are mapped into the well-known BPS monopoles. Since we want to compare the heterotic with the type II string in which these states are solitonic, non-perturbative excitations (as any state charged with respect to $E_8 \times E_8$), we will not consider them. More interesting, from our point of view, are the states with $N_L = 1$ and $p_L^2 = p_R^2$. They are charged only with respect to the vectors coming from the metric; indeed, formula (53) implies $e^I = 0$ and the winding number is zero by assumption. These states are obtained by applying the 24 left-moving bosonic oscillators X^A_{-1} to the vacuum $|0\rangle$. By tensoring the Lorentz indices, we get 21 vector representations and 1 spin-2 representation of $N = 4$ supersymmetry. These states are mapped by S-duality into solitonic solutions, magnetically charged under the vectors coming from the antisymmetric tensor; such solutions are known as H-monopoles, and they are therefore predicted to have multiplicity 21+1 [17]. These are the states we want to find in the type II string. States with $N_L > 1$ are easily seen to need non-zero winding number and are not expected to have a field theory limit.

The direct study of the multiplicity of the H-monopoles has not given a complete answer because of the difficulty in analysing the moduli space of the solution [8]. We want to bypass this problem by looking for these states in the dual type II string, where the counterparts of heterotic H-monopoles are perturbative. Indeed, by using eq. (9) and (50), and setting $A_i^a = 0$, one can check that the fields $H'_{\mu\nu i} - G^j_{\mu\nu}(C_{ij} + B'_{ij}) - A^a_i \hat{L}_{ab} F^b_{\mu\nu}$ and $h_{ij}\epsilon^{jk}(\tilde{H}_{\mu\nu k} - \tilde{G}^t_{\mu\nu} B_{kt})$, which appear in eq. (36) have an asymptotical charge that is simply the winding number of the string state in, respectively, the heterotic and the type II string. With a closer look at the same equation, we learn that the type II string state corresponding to the heterotic state with $p_L^2 = p_R^2$ and $N_L = 1$, is both electrically charged and with zero winding number. We can therefore search for it in the perturbative spectrum of the type II string.

In the NS-NS sector the mass formula reads:

$$\frac{1}{2}m^2 = \frac{1}{2}p_R^2 + h + N_R^{TOT} - \frac{1}{2} = \frac{1}{2}p_L^2 + \bar{h} + N_L^{TOT} - \frac{1}{2} \tag{58}$$

and the Bogomol'nyi bound can be realized by imposing $N_R = 1/2$ or $h = 1/2$ (or the same condition in the left sector). The constraint always reads $p_R^2 = p_L^2$, and implies, using formula (58), that we are indeed considering states with zero winding number and a field theoretical interpretation. We find 16 states of the form $\psi^A_{L-1/2}\psi^B_{R-1/2}|0\rangle$, $A, B = (\mu, i)$, and 80 states $|1/2, 1/2, 1/2, 1/2\rangle$. On the other hand in the R-R sector the mass formula

$$\frac{1}{2}m^2 = \frac{1}{2}p_R^2 + h + N_R^{TOT} - \frac{1}{4} = \frac{1}{2}p_L^2 + \bar{h} + N_L^{TOT} - \frac{1}{4} \tag{59}$$

can be realized only by $h = \bar{h} = 1/4$, with the same constraint $p_R^2 = p_L^2$. The Ramond vacuum of the space-time and T^2 fermions (4 left-movers and 4 right-movers) $|s, \alpha\rangle$ is labelled by the space-time helicity ($s = 1, 0, 0, -1$) and by the T^2 "helicity" ($\alpha = 1, 0, 0, -1$). The physical states are obtained by multiplying this vacuum by the 4 states obtained from the spectral flow of the identity, $|1/4, 1/2, 1/4, 1/2\rangle$, and the 20 states $|1/4, 0, 1/4, 0\rangle$, and projecting over GSO odd states. The GSO projection leaves a total of $384/4 = 96$ physical states. This is exactly the multiplicity of 192 bosonic states found in the perturbative spectrum of the heterotic string. The helicities of all these 192 states, given in the table below, arrange exactly into the bosonic parts of 21 $N = 4$ short vector multiplets and 1 spin-2 short multiplet, as illustrated in table 1.

State	Helicity	Multiplicity
$\psi^\mu_{L-1/2}\psi^\nu_{R-1/2}\|0\rangle$	± 2	1
	0	2
$\psi^\mu_{L-1/2}\psi^i_{R-1/2}\|0\rangle$ $\psi^i_{L-1/2}\psi^\mu_{R-1/2}\|0\rangle$	± 1	4
$\psi^i_{L-1/2}\psi^j_{R-1/2}\|0\rangle$	0	4
$\|1/2, 1/2, 1/2, 1/2\rangle$	0	80
$\|\pm 1, \alpha\rangle \otimes [\|1/4, 1/2, 1/4, 1/2\rangle \oplus \|1/4, 0, 1/4, 0\rangle]$	± 1	24 (after GSO)
$\|0, \alpha\rangle \otimes [\|1/4, 1/2, 1/4, 1/2\rangle \oplus \|1/4, 0, 1/4, 0\rangle]$	0	48 (after GSO)

Table 1

If we now perform a T-duality on such states, we obtain, obviously, a state with non-zero winding number only. Note that in the heterotic string, the H-monopoles exist for $\lambda'_1 = 0$ [8]; translated in the type II vocabulary, this means $B_{56} = 0$. Using once more formula (36) with $A^a_i = 0$, we get a state with the correct quantum numbers to be identified with the H-monopole. Its existence and its correct multiplicity are now guaranteed by the perturbatively proved T-duality of the type II string. Clearly, we have found only states charged with respect to the 5 or 6 components of the gauge fields G and H; the other H-monopoles can be obtained by a T-duality in the heterotic string.

In conclusion, the string-string duality turns out to be useful in studying the non-perturbative dynamics of the heterotic string. Obviously, there is a price to be paid: in the type II string, many states, which are present in the perturbative spectrum of the heterotic string, appear as solitons. The study of these solitons, which lies beyond the scope of this paper, would be a most powerful test of the string-string duality itself.

Acknowledgements

L.G. and A.Z. would like to thank the Department of Physics of NYU for its kind hospitality. M.P would like to thank the Aspen Center for Physics, where part of this work was completed.

L.G. is supported in part by the Ministero dell' Università e della Ricerca Scientifica e Tecnologica, by INFN and by ECC, contracts SCI-CT92-0789 and CHRX-CT92-0035. M.P. is supported in part by NSF under grant no. PHY-9318171. A.Z. is supported in part by ECC, Projects ERBCHGCTT93073, SCI-CT93-0340, CHRX-CT93-0340.

References

[1] C.M. Hull and P.K. Townsend, Nucl. Phys. B438 (1995) 109.

[2] E. Witten, Princeton preprint IASSNS-HEP-95-18, hep-th/9503124.

[3] A. Sen, Tata Inst. preprint TIFR-TH-95-16, hep-th/9504027.

[4] J.A. Harvey and A. Strominger, Chicago preprint EFI-95-16, hep-th/9504047.

[5] M.J. Duff, Nucl. Phys. B442 (1995) 47.

[6] J.H. Schwarz and A. Sen, Phys. Lett. B312 (1993) 105.

[7] A. Giveon, M. Porrati and E. Rabinovici, Phys. Rep. 244 (1994) 77.

[8] J.P. Gauntlett and J.A. Harvey, Chicago preprints EFI-94-11, hep-th/9403072 and EFI-94-36, hep-th/9407111.

[9] E. Witten and D. Olive, Phys. Lett. B78 (1978) 97.

[10] S. Ferrara, C.A. Savoy and B. Zumino, Phys. Lett. B100 (1981) 393.

[11] J.H. Schwarz, Phys. Rep. 89 (1982) 223.

[12] K.S. Narain, M.H. Sarmadi and E. Witten, Nucl. Phys. B279 (1987) 369.

[13] T. Eguchi, H. Ooguri and A. Taormina, Nucl. Phys. B315 (1989) 193.

[14] N. Seiberg, Nucl. Phys. B303 (1988) 286.

[15] F. Gliozzi, J. Scherk and D. Olive, Phys. Lett. B65 (1976) 282; Nucl. Phys. B122 (1977) 253.

[16] J. Scherk and J. Schwarz, Nucl. Phys. B153 (1979) 61; J. Maharana and J. Schwarz, Nucl. Phys. B390 (1993) 3.

[17] A. Sen, Int. J. Mod. Phys. A9 (1994) 3707.