Bunch motion in the presence of the self-induced voltage due to a reactive impedance. Part I: RF off

E.Shaposhnikova

CERN, Geneva, Switzerland

Abstract

Analytic self-consistent solutions have been found for the nonlinear Vlasov equation describing different types of behaviour with time of an intense bunch under the influence of voltage induced due to a reactive part of broad band impedance. The problem is solved for the particular type of the initial distribution function in longitudinal phase space which is elliptic and corresponds to parabolic line density.

The first part of the paper is devoted to the consideration of the effects in the machine with RF off. In this case induced voltage is changing with time and, as in the case with RF on, can have a significant effect on bunch motion.

Numerical estimations for the SPS show that this effect can be important for manipulations with beam at 26GeV. Measurements of the change in the rate of debunching with intensity can also be used to estimate the value of the impedance.

The same method is applied in the second part of the paper to analyse time dependent effects of potential well distortion when RF is on.

Contents

1 Introduction

It is well known that voltage induced due to the interaction of high intensity long bunches with the low frequency reactive part of the broad band impedance can produce significant potential well distortion when RF is on. Measurements of the resulting bunch-lengthening (or shortening) allow the value of the low frequency part of the impedance to be estimated.

The purpose of the present work is to evaluate the possible effect of the induced voltage on the motion of the intense single bunch when RF is off. Without the focusing effect of the external RF system this bunch normally starts to spread out or debunch. Then the induced voltage which affects the bunch motion is also changing with time.

Debunching is often one of the manipulations with the beam in the machine. Smooth changing of bunch parameters during debunching is also used to measure momentum spread in the bunch and as a method to investigate beam instabilities. However it was noticed already in [1] that measurements of microwave instability threshold during debunching don't give accurate results due to the influence of induced voltage on the variation of beam parameters. Debunching was recently used during studies of the microwave instability threshold for the proton beam in the SPS, [2]. The rate of debunching measured from the decay of the peak line density signal was found to be significantly different from the expected value. It was suggested, [3], that this fast debunching can also be explained by the defocusing effect of induced voltage.

Below the problem is examined in the following way. To introduce convenient definitions we start with the trivial case of the debunching of a low intensity bunch. In the next chapter we consider first the main equations describing the motion of a dense bunch with RF off but in the presence of induced voltage which is changing with time during debunching. With a special choice of the initial distribution function (elliptic in phase space with parabolic line density) this nonlinear problem has exact self-consistent solutions. These solutions, depending on the parameters of the system, describe different kinds of bunch behaviour which are analysed. A defocusing type of induced voltage makes debunching faster in comparison with the zero intensity case. With a focusing induced voltage, increase of intensity first slows down debunching, and then starting from some critical intensity leads to oscillations of the line density. From the exact solutions some simplified expressions are obtained for the variation of beam parameters during the initial part of debunching and these are used later for preliminary numerical estimations of the effect in the SPS. We show the possibility of estimating the low frequency part of the impedance from the measured decay of peak line density during debunching. Finally the variation of microwave instability threshold during debunching is also discussed.

The same method will be used in the second part of this paper to analyse the effect of induced voltage on the evolution of the bunch injected into machine with RF on.

2 Bunch motion with RF off in the low intensity case

Let us start first by considering the debunching of the single bunch in the machine with RF off and when any intensity effects are ignored.

In general at the beginning of debunching the initial distribution function of the bunch is the function of the Hamiltonian H_0 of the system with RF on in the same machine or in the injector:

$$F = F(H_0). \tag{1}$$

If bunches are sufficiently short compared with the RF period, the Hamiltonian of the particle in the single RF system can be written in the form

$$H_0 = \dot{\theta}_0^2 + \Omega^2 \theta_0^2, \tag{2}$$

where Ω is the frequency of linear synchrotron oscillations in the RF system where the bunch was created. Here θ_0 and $\dot{\theta}_0$ are initial values of

$$\theta \quad \text{and} \quad \dot{\theta} = \frac{d\theta}{dt},$$

a pair of conjugate coordinates we shall use to define the position of the single particle in the longitudinal phase space: θ is an azimuthal coordinate measured from the position of the synchronous particle $\theta = 0$ (when RF was on) and $\dot{\theta}$ is connected with momentum deviation $\Delta p = p - p_s$ from the synchronous value p_s by the expression

$$\dot{\theta} = \omega_0 \eta \, \frac{\Delta p}{p_s}. \tag{3}$$

Here $f_0 = \omega_0/(2\pi)$ is the revolution frequency and $\eta = 1/\gamma_t^2 - 1/\gamma^2$.

Distribution function (1) after integration over $\dot{\theta}_0$ gives the initial line density (at $t = 0$):

$$\lambda_0 = \lambda(\theta_0) = \int_{-\dot{\theta}_l}^{\dot{\theta}_l} F(\dot{\theta}_0^2 + \Omega^2 \theta_0^2) d\dot{\theta}_0, \tag{4}$$

where the limits of integration are functions of θ_0

$$\dot{\theta}_l = (H_m - \Omega^2 \theta_0^2)^{1/2} \tag{5}$$

with

$$H_m = \Omega^2 \theta_m^2 = \dot{\theta}_m^2. \tag{6}$$

Above θ_m and $\dot{\theta}_m$ are the maximum values of θ_0 and $\dot{\theta}_0$ in the bunch.

From the normalisation condition we also have

$$N = \int_{-\theta_m}^{\theta_m} \lambda(\theta) d\theta, \tag{7}$$

where N is the number of particles in the bunch.

Single particle motion when RF is off and any intensity effects are ignored is governed by the equation:

$$\frac{d^2\theta}{dt^2} = 0. \tag{8}$$

The solution of this equation for the particle with initial coordinates $(\theta_0, \dot{\theta}_0)$ has the form:

$$\theta = \theta_0 + \dot{\theta}_0 t, \tag{9}$$
$$\dot{\theta} = \dot{\theta}_0. \tag{10}$$

According to Liouville's theorem phase space density doesn't change with time along the particle trajectories. To calculate line density during the debunching process we can then substitute the solutions (9)-(10), rewritten in the form

$$\theta_0 = \theta - \dot{\theta}\, t, \tag{11}$$
$$\dot{\theta}_0 = \dot{\theta}, \tag{12}$$

into the initial distribution function (1), and integrate it over $\dot{\theta}$. For the Hamiltonian (2) we have

$$H_0 = \dot{\theta}_0^2 + \Omega^2\theta_0^2 = q^2\left[\dot{\theta} - \frac{\Omega^2\theta t}{q^2}\right]^2 + \Omega^2\frac{\theta^2}{q^2}, \tag{13}$$

where

$$q = q(t) = (1 + \Omega^2 t^2)^{1/2}. \tag{14}$$

The line density can be written as

$$\lambda(\theta, t) = \frac{1}{q}\int_{-\dot{\Theta}_l}^{\dot{\Theta}_l} F_0\left(\dot{\Theta}^2 + \Omega^2\frac{\theta^2}{q^2}\right)\, d\dot{\Theta}, \tag{15}$$

where the integration variable $\dot{\Theta}$ is defined by

$$\dot{\Theta} = q\left(\dot{\theta} - \frac{\Omega^2\theta t}{q^2}\right), \tag{16}$$

and the integration limits are:

$$\dot{\Theta}_l = \left(H_m - \Omega^2\frac{\theta^2}{q^2}\right)^{1/2} = \dot{\theta}_m\left(1 - \frac{\theta^2}{\theta_m^2 q^2}\right)^{1/2}. \tag{17}$$

The expression for line density (15) should be compared with the expression for the initial line density (4). As a result the line density during debunching can be written in the form

$$\lambda(\theta, t) = \frac{1}{q}\,\lambda_0\left(\frac{\theta}{q}\right). \tag{18}$$

As can be checked easily it satisfies the normalisation condition (7).

Momentum spread along the bunch during debunching can be found from expressions (3) and (13).

A useful characteristic of the debunching rate is the time constant

$$t_d = 1/\Omega. \tag{19}$$

This is the time at which the peak line density is reduced by $\sqrt{2}$ from the initial value. If $t \ll t_d$ the line density doesn't change significantly, and for $t \gg t_d$ the decrease in line density is inversely proportional to time.

As we can see, for low intensity bunches the decay of peak line density during debunching

$$\lambda_p = \frac{\lambda_{p0}}{(1+\Omega^2 t^2)^{1/2}}. \tag{20}$$

is independent of the form of the initial distribution function assuming that it is the function of the Hamiltonian for the short bunch in the single RF system with peak value at $\theta = 0$.

3 Bunch motion with RF off in the high intensity case

3.1 Main equations

Here we consider the situation when an intense bunch created in the single RF system is injected at the moment $t = 0$ into the machine with RF off.

The equations of motion for the particles under these conditions become

$$\frac{d\theta}{dt} = \omega_0 \eta \frac{\Delta p}{p_s}, \tag{21}$$

$$\frac{d(\Delta p)}{dt} = \frac{e}{2\pi R} V_e(\theta, t), \tag{22}$$

where R is the average radius of the machine. The voltage V_e induced by the interaction of the bunch with the low frequency reactive part of the broad band impedance can be presented in the form:

$$V_e(\theta, t) = -L\frac{dI(\theta, t)}{dt} = -Le\omega_0^2 \frac{\partial \lambda(\theta, t)}{\partial \theta}, \tag{23}$$

where $I(\theta, t)$ is the bunch current and L is the effective inductance of the machine connected with the reactive part of the longitudinal coupling impedance by the relation $\omega_0 L = \mathrm{ImZ/n}$. In expression (23) we neglect derivatives $\frac{\partial \lambda(\theta,t)}{\partial t}$ describing the slow dependence of $\lambda(\theta, t)$ on time during debunching.

Equations (21)-(22) are nonlinear since the induced voltage V_e is defined by the derivative of the line density the variation of which with time depends upon the induced voltage. To find a self consistent solution to the system of equations (21)-(22) is equivalent to solving the nonlinear Vlasov equation for distribution function $F = F(\theta, \dot{\theta}, t)$

$$\frac{\partial F}{\partial t} + \dot{\theta}\frac{\partial F}{\partial \theta} + \frac{\omega_0 \eta e V_e(\theta, t)}{2\pi R p_s}\frac{\partial F}{\partial \dot{\theta}} = 0. \tag{24}$$

By analogy with the low intensity case, we can try to find the distribution function at the moment t from the initial distribution function

$$F(\theta, \dot{\theta}, t) = F_0\left(\theta_0(\theta, \dot{\theta}, t), \dot{\theta}_0(\theta, \dot{\theta}, t), 0\right), \tag{25}$$

if initial coordinates θ_0 and $\dot{\theta}_0$ are defined as functions of coordinates θ, $\dot{\theta}$ and t.

In general, there are no regular methods which would allow us to find solutions to nonlinear problem of this type. However, in this particular case it turns out that analytic solutions can be obtained with a special choice for the initial distribution function. Let us consider the case where, at the moment $t = 0$, the bunch has an elliptical distribution function in longitudinal phase space

$$F = \mathcal{F}_0\left(1 - \frac{H_0}{H_m}\right)^{1/2} = \mathcal{F}_0\left(1 - \frac{\theta_0^2}{\theta_m^2} - \frac{\dot{\theta}_0^2}{\dot{\theta}_m^2}\right)^{1/2}, \quad H_0 < H_m. \tag{26}$$

This distribution function corresponds to a bunch with parabolic line density

$$\lambda(\theta_0) = \lambda_{p0}\left(1 - \frac{\theta_0^2}{\theta_m^2}\right), \tag{27}$$

where $\lambda_{p0} = \mathcal{F}_0\dot{\theta}_m\pi/2$ and from the normalisation condition we have also $\lambda_{p0} = 3N/(4\theta_m)$.

According to (18), the line density of low intensity bunches with this type of particle distribution would change with time during debunching as

$$\lambda(\theta, t) = \frac{\lambda_{p0}}{q(t)}\left[1 - \frac{\theta^2}{q^2(t)\theta_m^2}\right], \tag{28}$$

where function $q(t)$ is defined by expression (14).

For the chosen distribution function the induced voltage can be written at the beginning of debunching ($t = 0$) as

$$V_e(\theta, 0) = V_0\,\theta, \tag{29}$$

where we define

$$V_0 = \frac{3Ne\omega_0}{2\theta_m^3}\frac{\mathrm{ImZ}}{n}. \tag{30}$$

Before describing the nonlinear solution let us first introduce some preliminary considerations which may suggest the form in which we can look for the self-consistent solution in the following section.

If we would try to find an approximate solution for this problem by iteration we can use first the zero intensity solution (28) to calculate the induced voltage during debunching. Then the equation of particle motion giving the next approximation is

$$\frac{d^2\theta}{dt^2} - \frac{\epsilon\,\theta}{q^3(t)} = 0, \tag{31}$$

where the parameter

$$\epsilon = \frac{\omega_0^2 \eta e V_0}{2\pi\beta^2 E_s} \tag{32}$$

has the dimensions of frequency squared and can be written also as

$$\epsilon = \mathrm{sgn}(\eta \mathrm{Im Z})\,\Omega_\epsilon^2. \tag{33}$$

Differential equation (31) is linear in θ with a time dependent coefficient. The general solution for the equations of this type can be written in the form

$$\theta = \theta_0\, f_1(t) + \dot{\theta}_0\, f_2(t), \tag{34}$$

where $f_1(t)$ and $f_2(t)$ are the fundamental solutions [1] with Wronskian $W = f_1\dot{f}_2 - \dot{f}_1 f_2 = const.$

The next step is to express the initial coordinates as functions of θ, $\dot{\theta}$ and t and substitute them in the initial distribution function (26). As a result for the next iteration one obtains the equation

$$\frac{d^2\theta}{dt^2} - \frac{\epsilon\,\theta}{q_1^3(t)} = 0, \tag{35}$$

where $q_1(t) = (f_1^2 + \Omega^2 f_2^2)^{1/2}$. As we can see this equation repeats the form of equation (31) but with a different time dependent coefficient. This suggests that if our iteration proccess converges we can search for a closed form solution of the same type as presented by expressions (34) and (35). This is done in the next section.

Note that equation (31) shows the well known fact that if the low frequency part of the coupling impedance of the machine is inductive (ImZ > 0) then the induced voltage has a defocusing effect above transition ($\eta > 0$) and focusing below transition.

3.2 Nonlinear solution

Using the definitions introduced above the nonlinear self-consistent system of equations governing the particle motion during debunching finally can be presented in the form

$$\frac{d^2\theta}{dt^2} + \frac{\epsilon\theta_m^2}{2\lambda_{p0}}\frac{\partial\lambda}{\partial\theta} = 0, \tag{36}$$

[1] We were not able to find analytical expressions for functions f_1, f_2.

$$\lambda(\theta,t) = \mathcal{F}_0 \int_{\dot\theta_1}^{\dot\theta_2} \left[1 - \frac{\theta_0^2(\theta,\dot\theta,t)}{\theta_m^2} - \frac{\dot\theta_0^2(\theta,\dot\theta,t)}{\dot\theta_m^2}\right]^{1/2} d\dot\theta, \tag{37}$$

where $\dot\theta_1 = \dot\theta_1(\theta,t)$ and $\dot\theta_2 = \dot\theta_2(\theta,t)$ are solutions of equation

$$1 - \frac{\theta_0^2(\theta,\dot\theta,t)}{\theta_m^2} - \frac{\dot\theta_0^2(\theta,\dot\theta,t)}{\dot\theta_m^2} = 0. \tag{38}$$

Suppose that for the particle with initial coordinates $(\theta_0,\dot\theta_0)$ the system of equations (36)-(37) has a solution which can be written in the following form

$$\begin{aligned} \theta(t) &= \theta_0\, y_1(t) + \dot\theta_0\, y_2(t), &(39)\\ \dot\theta(t) &= \theta_0\, \dot y_1(t) + \dot\theta_0\, \dot y_2(t), &(40) \end{aligned}$$

where y_1 and y_2 are unknown functions of time with initial conditions:

$$y_1(0) = 1, \quad y_2(0) = 0, \tag{41}$$

$$\dot y_1(0) = 0, \quad \dot y_2(0) = 1. \tag{42}$$

The Wronskian of this system is

$$W = y_1\dot y_2 - \dot y_1 y_2. \tag{43}$$

Suppose that

$$W = const \tag{44}$$

then from initial conditions (41)-(42) it follows that $W = 1$.

This assumption allows us to use the same method as used above in the low intensity case, and express the initial coordinates as functions of coordinates θ and $\dot\theta$ at the moment t

$$\begin{aligned} \theta_0 &= \theta\,\dot y_2 - \dot\theta\, y_2, &(45)\\ \dot\theta_0 &= -\theta\,\dot y_1 + \dot\theta\, y_1. &(46) \end{aligned}$$

Substitution of the expressions (45)-(46) into the initial distribution function (26) makes it possible to find the distribution function at the moment t

$$F = \mathcal{F}_0 \left[1 - \frac{(\theta\dot y_2 - \dot\theta y_2)^2}{\theta_m^2} - \frac{(\dot\theta y_1 - \theta\dot y_1)^2}{\dot\theta_m^2}\right]^{1/2}. \tag{47}$$

Then the line density (37), after integration over $\dot\theta$ of the above expression and using condition (44) becomes

$$\lambda(\theta,t) = \int_{\dot\theta_1}^{\dot\theta_2} F(\theta,\dot\theta,t)d\dot\theta = \frac{\lambda_{p0}}{r}\left[1 - \frac{\theta^2}{r^2\theta_m^2}\right], \tag{48}$$

where we define

$$r = r(t) = (y_1^2 + \Omega^2 y_2^2)^{1/2}. \tag{49}$$

The derivative of the line density can also be find

$$\frac{\partial \lambda}{\partial \theta} = -\frac{\lambda_{p0} 2\theta}{r^3 \theta_m^2}. \tag{50}$$

Now our main differential equation (36) can be rewritten either in the form

$$\frac{d^2\theta}{dt^2} - \frac{\epsilon\theta}{r^3} = 0 \tag{51}$$

or as a system of equations for y_1 and y_2

$$\frac{d^2 y_1}{dt^2} - \frac{\epsilon y_1}{(y_1^2 + \Omega^2 y_2^2)^{3/2}} = 0, \tag{52}$$

$$\frac{d^2 y_2}{dt^2} - \frac{\epsilon y_2}{(y_1^2 + \Omega^2 y_2^2)^{3/2}} = 0. \tag{53}$$

It is interesting to note that this last system of equations is also known to describe the motion of a body in the $(y_1, \Omega y_2)$ plane under the influence of gravitation with attractive force for $\epsilon < 0$ and repulsive for $\epsilon > 0$.

Let us introduce new variables $r(t)$ and $\xi(t)$ according to the following formulae

$$y_1 = r\cos\xi, \quad y_2 = \frac{r}{\Omega}\sin\xi. \tag{54}$$

Then equations (52) - (53), using these new variables, can be transformed into the form:

$$r\ddot{\xi} + 2\dot{r}\dot{\xi} = 0, \tag{55}$$

$$\ddot{r} - r\dot{\xi}^2 - \frac{\epsilon}{r^2} = 0. \tag{56}$$

This system of nonlinear equations has as first integrals of motion

$$r^2\dot{\xi} = C_1, \tag{57}$$

$$\dot{r}^2 + \frac{C_1^2}{r^2} + \frac{2\epsilon}{r} = C_2, \tag{58}$$

Using again gravitation terminology one can say that the first expression corresponds to the second law of Kepler (law of areas) and the second equation describes conservation of energy in the system. Constants C_1 and C_2 are defined by the initial conditions (41)-(42) which now have the form

$$r(0) = 1, \quad \xi(0) = 0, \tag{59}$$

$$\dot{r}(0) = 0, \quad \dot{\xi}(0) = \Omega. \tag{60}$$

Thus for C_1 and C_2 we get

$$C_1 = \Omega, \tag{61}$$

$$C_2 = \Omega^2 + 2\epsilon. \tag{62}$$

We must check now that the Wronskian (43) of the system satisfies our initial assumption (44) that it is constant and equal to one. Indeed using variables $r(t)$ and $\xi(t)$ and the expressions(57), (61) we obtain

$$W = \frac{r^2\dot{\xi}}{\Omega} = \frac{C_1}{\Omega} = 1. \tag{63}$$

Note, that the solutions found above for $y_1(t)$ and $y_2((t)$ define, in fact, from expression (47) a distribution function which is a time dependent solution of the nonlinear Vlasov equation (24). However this method only works due to the special choice of initial distribution function, with which the induced voltage is proportional to θ.

3.3 Analysis of solutions

3.3.1 Exact solutions

The integrals of motion found in the previous section allow us to rewrite expression (58) in the form

$$\frac{\dot{r}^2}{2} + U(r) = 0, \tag{64}$$

where

$$U(r) = -\frac{\Omega^2(r-1)(ar+1)}{2r^2} \tag{65}$$

and constant

$$a = \frac{C_2}{C_1^2} = 1 + 2\frac{\epsilon}{\Omega^2} = 1 + 2\mathrm{sgn}(\eta \mathrm{ImZ})\frac{\Omega_\epsilon^2}{\Omega^2}. \tag{66}$$

This equation can be interpreted as the equation of motion of some particle with the coordinate r in the potential $U(r)$. In reality, according to expression (48), $r(t)$ is a positive defined function which describes the variation with time of bunch length τ

$$\frac{\tau(t)}{\tau(0)} = r(t) \tag{67}$$

or of peak line density

$$\frac{\lambda_p(t)}{\lambda_{p0}} = \frac{1}{r(t)} \tag{68}$$

with the initial condition

$$r(0) = 1. \tag{69}$$

Increasing of r with time means decreasing of peak line intensity - or debunching.

The solution of equation (64) can be written in the following form:

$$\Omega t = \pm \int_1^r \frac{r dr}{\sqrt{\rho(r)}}, \tag{70}$$

where we use the definition

$$\rho(r) = (r-1)(ar+1). \tag{71}$$

If the solution $r(t)$ is known, then the function $\xi(t)$ can be found from (57)-(58) and is defined by the expression:

$$\xi = \pm \int_1^r \frac{dr}{r\sqrt{\rho(r)}}. \tag{72}$$

After integration, we have:

$$r[(\Omega^2 + \epsilon)\cos\xi - \epsilon] = \Omega^2. \tag{73}$$

Depending on the shape of the effective potential $U(r)$ (or value of parameter a) solutions of equation (64) have different character. Let us consider them.

***a*= 1**. If any intensity effects are absent then parameters $\epsilon = 0$ and $a = 1$, and as follows from (65) the potential has the shape

$$U(r) = \frac{\Omega^2}{2}\left(\frac{1}{r^2} - 1\right). \tag{74}$$

This potential is shown in Fig.1. Expression (70) gives in this case the solution (14) already found in Chapter 1 for the low intensity case:

$$r(t) \equiv q(t) = (1 + \Omega^2 t^2)^{1/2}. \tag{75}$$

The behaviour of the normalised peak line density $\lambda_p(t)/\lambda_{p0} = 1/r$ for $\epsilon = 0$ (and $a = 1$) is shown in Fig.2.

For cases where the intensity effects are considered and therefore $\epsilon \neq 0$ there are two main types of solutions of equation (64) which correspond to infinite and finite (periodic) motion. We analyse them below.

***a*> 0**. The integral (70) gives the solution

$$\Omega t = \frac{\sqrt{\rho(r)}}{a} + \frac{a-1}{2a^{3/2}} \ln \frac{|2\sqrt{a\rho(r)} + 2ar + 1 - a|}{1+a}, \tag{76}$$

where $\rho(r)$ is defined by expression (71). Motion in this case is only infinite, which means continuous debunching ($r \to \infty$ with $t \to \infty$). However the character of the debunching is different depending on the value of a. If $a > 1$ the induced voltage has a defocusing effect and debunching is faster compared to the low intensity case. For $a < 1$ debunching is slowed down by the focusing effect of the induced voltage. These two situations are shown in Figs.1-2 together with the low intensity case $a = 1$.

***a*= 0**. This value of parameter a corresponds to the point of bifurcation where the character of the solution is changing. From (70) we can find

$$\Omega t = \frac{2}{3}(r-1)^{3/2} + 2(r-1)^{1/2}. \tag{77}$$

This solution is presented in Figs.1-4.

***a*< 0**. This case can only occur for a focusing type of induced voltage ($\eta \mathrm{ImZ} < 0$) and $\Omega_\epsilon^2 > \Omega^2/2$. The potential $U(r)$ has the shape of a potential well and the solutions describe oscillations of peak line density with time.

If $\mathbf{-1 < \boldsymbol{a} < 0}$ then the solution found from (70) has the form

$$\Omega t = \frac{\sqrt{\rho(r)}}{a} + \frac{1-a}{2a|a|^{1/2}}\left[\arcsin\frac{2ar+1-a}{|1+a|} - \frac{\pi}{2}\right], \tag{78}$$

where $a \neq -1$. In this case oscillations begin with the line density decreasing (which appears for some time as if it is debunching). Oscillation amplitude is defined by the inequality

$$|a| \leq \frac{1}{r} \leq 1. \tag{79}$$

The effective potential well is shown in Fig.3 for $a = -0.3$ together with the $a = 0$ case for comparison. The corresponding behaviour of bunch line density is presented in Fig.4.

The period of the oscillations of the line density is

$$T = \frac{\pi}{\Omega}\frac{(1-a)}{|a|^{3/2}} = \frac{2\pi\Omega_\epsilon^2}{(2\Omega_\epsilon^2 - \Omega^2)^{3/2}}. \tag{80}$$

As can be seen from this expression a bunch with an intensity such that $\Omega_\epsilon^2/\Omega^2 = 1/2$ ($a = 0$) has an infinitely large oscillation period and continuously debunches.

The period and amplitude of the line density oscillations decrease (compare Fig.5 with Fig.4) with growing absolute value of parameter a (which can be for example due to increasing intensity or impedance). As a result at $a \sim -1$, the period $T \sim (2\pi)/\Omega$ but with oscillation amplitude close to zero.

***a*= −1**. This case corresponds to the equilibrium situation when the bunch at the moment $t = 0$ is in the minimum of the potential well $U(r)$ with solution $r = 1$ not changing with time (see Fig.6). With this value of a, $\Omega_\epsilon^2 = \Omega^2$ and the initial bunch is matched to the waveform created by the induced voltage.

For $\boldsymbol{a} < \mathbf{-1}$ the amplitude of peak line density oscillations starts to grow with increasing absolute value of parameter a:

$$1 \leq \frac{1}{r} \leq |a| \tag{81}$$

Oscillations now start at $t = 0$ with the peak line density increasing and have a period also defined by expression (80). The solution found from (70) has the form

$$\Omega t = \frac{\sqrt{\rho(r)}}{|a|} + \frac{1-a}{2|a|^{3/2}}\left[\arcsin\frac{2ar+1-a}{|1+a|} + \frac{\pi}{2}\right], \tag{82}$$

where also $a \neq -1$.

The effective potential well is shown in Fig.6 for $a = -1.3$. Corresponding behaviour of bunch line density is presented in Fig.7.

3.3.2 Approximate solutions

To estimate these effects it is useful to have simplified expressions to describe the variation of beam parameters during debunching ($a > 0$).

As follows from (65), at the beginning of debunching, when $r \sim 1$, we can obtain an approximate solution by replacing Ω^2 by $(\Omega^2 + \epsilon)$ in the formula (14) and then

$$r(t) \simeq [1 + (\Omega^2 + \epsilon)t^2]^{1/2}. \tag{83}$$

According to the assumption made to obtain this expression it is valid only at the start of debunching for times $t \ll 1/(\Omega^2 + \epsilon)^{1/2}$.

For $r \gg 1$ i.e. when the initial distribution is strongly debunched, the asymptotic solution can be again obtained from (65) by using the same formula (14) with $\Omega^2 \rightarrow (\Omega^2 + 2\epsilon)$:

$$r(t) \simeq [1 + (\Omega^2 + 2\epsilon)t^2]^{1/2}. \tag{84}$$

Both these approximations are shown together with the exact solution in Fig.8 for $a = 3$. As can be seen the exact solution lies between these two limits.

For a rough estimation of the time constant for the debunching of the intense beam at the beginning of the process ($t \ll 1/\Omega_\epsilon$ and $t \ll 1/\Omega$) we can use the following approximate formula

$$t_{de} \simeq 1/\sqrt{\Omega^2 + \epsilon}. \tag{85}$$

As can be seen a change in debunching rate due to intensity effects can be used to estimate the inductive part of the broad band impedance if the parameters of initial bunch are known.

However it is interesting to note that if the initial bunch was created in the same machine and later allowed to debunch, then the measured time t_{de} in first approximation is defined only by the external voltage and doesn't depend on intensity. Indeed due to potential well distortion the matched bunch has dimensions defined by

$$\dot{\theta}_m/\theta_m = \Omega = \sqrt{\omega_s^2 - \epsilon},$$

where ω_s is the frequency of synchrotron oscillations with RF on. In this situation the measured debunching time will be

$$t_{de} \simeq 1/\omega_s.$$

3.4 Numerical estimations for the SPS

3.4.1 Defocusing effect

Let us start first with the analysis of the case which we had during the MD study, [2]. The injected bunches of 5-10ns were created in the low frequency RF system (10MHz) of the PS and can be considered there as "short" bunches. For the SPS, the spectrum of these 5-10ns long bunches is situated in the inductive part of the broad band impedance with resonant frequency of 1.3GHz. Previous measurements and estimations show that the space charge impedance of the SPS at 26GeV is much less than this inductive impedance.

Machine parameters were: $E_s = 26\text{GeV}, \gamma_t = 23.4$. For this situation we can expect that the induced voltage has a defocusing character which makes debunching faster. For the 5ns long bunch with emittance $\varepsilon_L = 0.2$eVs we have

$$\Omega = \frac{\dot{\theta}_m}{\theta_m} = \frac{2\eta}{\tau}\frac{\Delta p_m}{p_s} = 0.21 \times 10^3 \text{s}^{-1},$$

which should give a debunching time constant in the low intensity case

$$t_d = 1/\Omega = 5\text{ms}.$$

For an intensity $N = 5 \times 10^{10}$ and the inductive part of the broad band impedance ImZ/n = 20Ω, we obtain from expression (32)

$$\Omega_\epsilon = 0.22 \times 10^3 \text{s}^{-1}.$$

This gives a value $a \simeq 3$. Supposing that the initial bunch was close to parabolic then we can calculate the decay of peak line density. This is shown in Fig.8 as a solid line together with the low intensity case, $a = 1$, for comparison. The debunching time constant, which measurement can provide, is

$$t_{de} \simeq 1/\sqrt{\Omega^2 + \Omega_\epsilon^2} = 3.3\text{ms}.$$

For 10ns long bunches with the same emittance $\varepsilon_L = 0.2eVs$ and intensity $N = 5 \times 10^{10}$ we can compare the values of

$$\Omega = 51.5\text{s}^{-1} \quad \text{and} \quad \Omega_\epsilon = 56.6\text{s}^{-1}.$$

Related to these values are the debunching time constants in the low and high intensity cases,

$$t_d = 19.5\text{ms} \qquad \text{t}_\text{de} \simeq 13.1\text{ms}.$$

In this case parameter $a = 3.4$.

For both sets of measurements, with "short" and "long" bunches, we can observe a significant effect of the induced voltage on the debunching process. Moreover, accurate measurements of the decay of peak line density during debunching in this situation can give important information about the low frequency part of the coupling impedance of the machine.

3.4.2 Focusing effect

The interaction of the bunch with the inductive part of the impedance below transition, or with the capacitive above leads to a focusing effect which we would expect to slow down the debunching of the intense beam. The debunching time constant (at the beginning of the process) becomes

$$t_{de} = 1/\sqrt{\Omega^2 - \Omega_\epsilon^2}. \tag{86}$$

Let us estimate the possible effect in the SPS in the present fixed target mode of operation with machine parameters $E_s = 14\mathrm{GeV}, \gamma_t = 23.4$. Then for high intensity bunches with parameters $\tau = 5\mathrm{ns}, \varepsilon_L = 0.1\mathrm{eVs}, \mathrm{N} = 5 \times 10^{10}$, available from the PS for recent MD studies at 14 GeV, and using again the value $\mathrm{ImZ/n} = 20\Omega$ we obtain

$$\Omega = 0.95 \times 10^3 s^{-1} \quad \text{and} \quad \Omega_\epsilon = 0.68 \times 10^3 s^{-1}.$$

In this case $a = 0.025$. For bunches with a line density close to parabolic the corresponding decay of peak line density is described by the curve shown in Fig.2 for $a = 0$. From the previous analysis of different types of possible solutions in this system, at slightly higher intensities we could expect to observe rebunching.

4 Microwave instability threshold during debunching

The threshold of the microwave instability can be defined from the Boussard criterion. For a parabolic line density it has a form, [4]:

$$\frac{|Z_L|}{n} \leq F^* \frac{E_s|\eta|\beta^2}{eI(\theta)} \left[\frac{\Delta\hat{p}(\theta)}{p}\right]^2, \tag{87}$$

where formfactor $F^* = 0.7\pi 3/2 = 1.05\pi$. Here $\pm\Delta\hat{p}$ is the maximum momentum spread in the bunch at the position θ and I is the local current. During debunching both $\Delta\hat{p}$ and I become not only functions of coordinate θ but also of time. By analogy with the case when RF is on, [5], for the type of particle distribution in phase space chosen, the ratio

$$P = \frac{(\Delta\hat{p})^2}{I}$$

does not depend on coordinate θ and is constant along the bunch even during debunching.

The variation of line density is described by formula (48). The change of maximum momentum spread during debunching can be also calculated

$$\Delta\hat{p}(\theta, t) = \frac{\Delta p_m}{r(t)} \left[1 - \frac{\theta^2}{r^2(t)\theta_m^2}\right]^{1/2}, \tag{88}$$

where $\pm\Delta p_m$ is the maximum momentum spread in the bunch at $t = 0$. As a result we have

$$\frac{P(t)}{P(0)} = \frac{1}{r(t)}. \tag{89}$$

If the effect of induced voltage on the changing of bunch parameters during debunching is ignored then the function $r(t)$ is simply replaced by $q(t) = (1 + \Omega^2 t^2)^{1/2}$.

Using the expressions for Ω and Ω_ϵ, criterion (87) can be rewritten in our definitions as

$$\Omega_\epsilon^2 < \frac{\Omega^2}{r(t)} \frac{|\mathrm{ImZ}|}{|Z_L|}. \tag{90}$$

From the analysis of this expression we can make a few conclusions:

(1) If debunching is used as a method to determine the microwave instability threshold from criterion (87) it is necessary to take into account the effect of induced voltage on the bunch parameters variation. Otherwise (as was noticed first in [1], see also [2]) measurements do not give consistent results.

(2) Measurements of the peak line density variation during debunching give simultaneously information about function $P(t)$, which can be used to define the threshold intensity for microwave instability.

(3) Possible deviations of the value of the debunching time constant of an intense beam t_{de} from the low intensity value t_d should lie within some limits defined by the microwave instability.

5 Conclusions

Analytic solutions of the nonlinear Vlasov equation describing different types of behaviour with time have been found for an intense bunch, with an initial elliptic distribution function, under the effect of of self induced voltage in the machine with RF off.

Voltage induced due to the interaction of the beam with the low frequency part of the coupling impedance of the machine, as in the case with RF on, can have a significant effect on processes when RF is off. Measurements of the change in the rate of debunching with intensity can be used to estimate the value of the impedance supposing that the initial particle distribution in phase space is close to elliptic.

If debunching is used as a method to determine instability thresholds it is necessary to take into account the effect of induced voltage on the variation of the bunch parameters.

Acknowledgements

The author would like to thank T.Linnecar for many usefull discussions of this work, and D.Boussard, A.Hofmann and B.Zotter for reading the manuscript and valuable comments.

References

[1] P.Bramham et al., Longitudinal instabilities of bunched beams in the ISR, IEEE Trans. Nucl. Sci. NS-24, N3, 1977, p.1490.

[2] T.Linnecar, E.Shaposhnikova, Measurements of the microwave instability in the SPS, SL MD Note 164, SPS, January 1995.

[3] D.Boussard, private communication.

[4] D.Boussard, Observation of microwave longitudinal instabilities in the CPS, LABII/RF/Int./75-2, 1975.

[5] A.Hofmann, F.Pedersen, Bunches with local elliptic energy distributions, IEEE Trans. Nucl. Sci., Vol. NS-26, No.3, p.3526, 1979.

Figure 1: Effective potential $U(r)$ describing debunching in low intensity case, $a = 1$, and high intensity case with different types of induced voltage: defocusing, $a = 2$, and focusing, $a = 0$.

Figure 2: Variation of normalised peak line density $1/r$ during debunching for low intensity case, $a = 1$, and high intensity case with defocusing, $a = 2$, and focusing, $a = 0$, type of induced voltage.

Figure 3: Effective potential well $U(r)$ describing oscillations of line density for focusing type of induced voltage.

Figure 4: Variation of normalised peak line density $1/r$ for focusing type of induced voltage, $a = -0.3$ and $a = 0$.

Figure 5: Variation of normalised peak line density $1/r$ for focusing type of induced voltage, $a = -0.6$.

Figure 6: Effective potential well for equilibrium solution, $a = -1$, and oscillation of line density, $a = -1.3$.

Figure 7: Variation of normalised peak line density $1/r$ for focusing type of induced voltage, $a = -1.3$.

Figure 8: Exact (solid line) and approximate (dashed lines) solutions for peak line density variation during debunching of intense beam for defocusing type of induced voltage, $a = 3$, together with exact solution for low intensity case, $a = 1$, (dotted line).