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Identical Particles and Quantum Symmetries

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Abstract

We propose a solution to the problem of compatibility of Bose-Fermi statistics with symmetry transformations implemented by quantum groups of Drinfel'd type. We use unitary transformations to conjugate multiparticle symmetry postulates.

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1 Introduction

Quantum groups [1] have deserved much attention in recent years as candidates for generalized symmetry transformations in physics. Among other applications, they look promising in relation to generalized spacetime[2] and/or internal symmetries in QFT (Quantum Field Theory). One way to approach QFT consists first in finding a consistent procedure to implement quantum group transformations in Quantum Mechanics with a finite number of particles, then to pass to QFT through second quantization. Various models describing systems of one particle (see e.g. ref. [4, 3, 6, 5]) or a finite number of *distinct* particles consistently transforming under the action of a quantum group have been constructed so far; as known, the quantum group coproduct plays a specific role in extending the quantum group transformations from the one-particle to the multiparticle system. In this article we would like to study whether the notions of Identical Particles and quantum group transformations are compatible in quantum mechanics (in first quantization).

The setting that we have in mind is a quantum mechanical system transforming under a generalized (symmetry) transformation realizing some Hopf algebra H[3].

In the case that the Hopf algebra under consideration is not co-commutative one might expect that it generates symmetry transformations that are incompatible with the notion of identical particles. In fact, if H is a $*$-quantum group and a representation ρ of H on a Hilbert space $\mathcal{H}$ is known, the action of H on $\mathcal{H}\otimes\mathcal{H}$ defined through the coproduct Δ does not preserve but rather mixes the symmetrical and antisymmetrical subspaces $(\mathcal{H}\otimes\mathcal{H})_{\pm}$ defined by $P_{12}(\mathcal{H}\otimes\mathcal{H})_{\pm}=\pm(\mathcal{H}\otimes\mathcal{H})_{\pm}$ (P_{12} denotes the permutation operator), so that fermions and bosons in the ordinary sense are impossible. Actually, the coproduct does not treat the first and the second tensor factor symmetrically, except when the deformation parameter ($q-1=$, in the $H=U_q\mathbf{g}$ case) vanishes. Since in the ordinary formulation of Q. M. one associates to each *separate* tensor factor one of the two particles, this might lead to the conclusion that the coproduct cannot treat two particles as identical, but,

[2]Such symmetries [2] would be connected to a noncommutative-geometric fundamental structure of spacetime itself.

[3]The transformations may correspond to a symmetry either in the sense that they leave invariant the *dynamics* of the particular system under consideration (e.g. rotation symmetry of its hamiltonian), and therefore are associated to conservation laws for the latter; or in the sense that they leave invariant the *form* of the physical description of *any* system (covariance of the physical description), as it happens e.g. with the Poincaré transformations in Special Relativity.

at most, as "almost identical" (*i.e.* different, in the very end) if the deformation parameter is very small but different from zero. This would result into a drastic and unacceptable discontinuity of the number of allowed states of the two-particle system in the limit of vanishing deformation parameter.

In this work we would like to show that, however, there does exist a way out when we modify our notions of symmetry and anti-symmetry (w.r.t. permutations) associated to bosons and fermions. This is at least possible either in the case where $H = U_q\mathbf{g}$ [1, 7] is one of the standard quantum groups associated to the simple Lie algebras $\mathbf{g}$ of the classical series – the case of $U_q\mathrm{su}(2)$ will be studied in some detail –, or if H is a triangular Hopf algebra arising from the quantization of a solution of the CYBE [8, 9]; in both cases we also need the existence of a complex conjugation $*$. The precise criterium is that H must be the twist of a cocommutative (quasi-) Hopf algebra [10].

In the case where H is a quasitriangular Hopf algebra one might have expected to see anyons arise as a consequence of the braidgroup character of $\mathcal{R}$; however, in our formulation anyons do not seem possible without further modifications.

In section 2 we introduce nonstandard formulations of the (anti-)symmetrization postulates characterizing bosons and fermions, which are obtained by conjugating the standard postulates by some twists F's. Section 3 contains a digression answering the question: When can identical particles be treated as distinct In the twisted approach. In Section 4 we fix the choice of the F's by requiring that the new (anti-)symmetrization postulates are compatible with the quantum group transformations. In section 5 we get insight into the whole subject by looking in more detail at the example of a generic system transforming under the quantum group $U_q(su(2))$.

2 Twisted multiparticle description

Let us forget the coproduct and the issue of quantum symmetry for the moment, and just consider pure quantum mechanics. We will consider a one-particle system, and denote by $\mathcal{H}$ the Hilbert space of its states, by$\mathcal{A}$ the $*$-algebra of observables acting on $\mathcal{H}$. n-particle states and n-particle operators will live in as yet to be determined subspaces of $\mathcal{H}^{\otimes n}$ and $\mathcal{A}^{\otimes n}$ respectively.

Let us consider states of two identical particles. The corresponding state vector

$|\psi^{(2)}\rangle$ will be some element of the tensor product of the one-particle Hilbert space $\mathcal{H}$. Let P_{12} be the permutation map on $\mathcal{H}\otimes\mathcal{H}$: $P_{12}(|a\rangle\otimes|b\rangle)\equiv|b\rangle\otimes|a\rangle$. In the sequel we will also use the symbol τ to denote the abstract permutator of two tensor factors, $\tau(a\otimes b)\equiv b\otimes a$. The fact that we are dealing with identical particles manifests itself in the properties of the state vector under permutation:

$$P_{12}|\psi^{(2)}\rangle = e^{i\nu}|\psi^{(2)}\rangle, \tag{2.1}$$

where $\nu=0$ for Bose-statistics and $\nu=1$ for Fermi-statistics. For the corresponding expectation value of an arbitrary operator $\mathcal{O}\in\mathcal{A}\otimes\mathcal{A}$ we then find

$$\langle\psi^{(2)}|\mathcal{O}|\psi^{(2)}\rangle = \langle\psi^{(2)}|P_{12}\ \mathcal{O}\ P_{12}|\psi^{(2)}\rangle \tag{2.2}$$

because the phases $e^{-i\nu}$ and $e^{i\nu}$ from the bra and the ket cancel. This means that the operators $\mathcal{O}$ and $P_{12}(\mathcal{O})\equiv P_{12}\circ\mathcal{O}\circ P_{12}$ are members of the same equivalence class as far as expectation values go. One particlular representative of each such equivalence class is the symmetrized operator

$$\frac{1}{2}(\mathcal{O}+P_{12}(\mathcal{O}))\in(\mathcal{A}\otimes\mathcal{A})_+. \tag{2.3}$$

It plys a special role because it preserves the two-particle Hilbert spaces for any statistic (2.1), as we remind of below. We can hence avoid redundant operators by restricting $\mathcal{A}\otimes\mathcal{A}$ to the subalgebra

$$(\mathcal{A}\otimes\mathcal{A})_+ := \{a\in\mathcal{A}\otimes\mathcal{A} : [P_{12},a]=0\} \tag{2.4}$$

(note that $[P_{12},a]=0\Leftrightarrow\tau(a)=a$). In this article we will show how to find an analog of $(\mathcal{A}\otimes\mathcal{A})_+$ compatible with quantum group transformations.

We summarize the relevant eqs. characterizing a system of two bosons or fermions:

$$P_{12}|u\rangle_\pm = \pm|u\rangle_\pm \qquad \text{for} \quad |u\rangle_\pm\in(\mathcal{H}\otimes\mathcal{H})_\pm \tag{2.5}$$

$$a:(\mathcal{H}\otimes\mathcal{H})_\pm\to(\mathcal{H}\otimes\mathcal{H})_\pm \qquad \text{for} \quad a\in(\mathcal{A}\otimes\mathcal{A})_+ \tag{2.6}$$

$$*_2:(\mathcal{A}\otimes\mathcal{A})_+\to(\mathcal{A}\otimes\mathcal{A})_+. \tag{2.7}$$

Equation (2.5) defines bosonic $(+)$ and fermionic $(-)$ states as in (2.1). Equation (2.6) follows from $[P_{12},(\mathcal{A}\otimes\mathcal{A})_+]=0$ and shows that symmetrized operators transform boson states into bosons states and fermion states into fermion states.

Similar statements as given here for two particles obviously apply also to states of 3 and more identical particles and to other statistics (anyons).

Can one describe in a non-standard way the system of n identical particles, using what we know for one particle, so that the description is perfectly consistent from the physical viewpoint? Let us concentrate on two-particle systems for the moment:

For a unitary and in general not symmetric operator $F_{12} \in \mathcal{A} \otimes \mathcal{A}$, $F_{12}^{*_2} = F_{12}^{-1}$ where $*_2 = * \otimes *$, we define

$$
\begin{aligned}
(\mathcal{H} \otimes \mathcal{H})_{\pm}^{F_{12}} &= F_{12}(\mathcal{H} \otimes \mathcal{H})_{\pm} && (2.8)\\
P_{12}^{F_{12}} &= F_{12} P_{12} F_{12}^{-1} \qquad (P_{12}|u\rangle \otimes |v\rangle := |v\rangle \otimes |u\rangle) && (2.9)\\
(\mathcal{A} \otimes \mathcal{A})_{+}^{F_{12}} &= F_{12}(\mathcal{A} \otimes \mathcal{A})_{+} F_{12}^{-1} && (2.10)
\end{aligned}
$$

where

$$(\mathcal{A} \otimes \mathcal{A})_{+} = \{a \in \mathcal{A} \otimes \mathcal{A} : [P_{12}, a] = 0 \Leftrightarrow \tau(a) = a\}. \qquad (2.11)$$

We then find in complete analogy to equations (2.5 – 2.7)

$$
\begin{aligned}
& P_{12}^{F_{12}} |u\rangle_{\pm} = \pm |u\rangle_{\pm} \qquad \text{for} \quad |u\rangle_{\pm} \in (\mathcal{H} \otimes \mathcal{H})_{\pm}^{F_{12}} && (2.12)\\
& a : (\mathcal{H} \otimes \mathcal{H})_{\pm}^{F_{12}} \to (\mathcal{H} \otimes \mathcal{H})_{\pm}^{F_{12}} \qquad \text{for} \quad a \in (\mathcal{A} \otimes \mathcal{A})_{+}^{F_{12}} && (2.13)\\
& *_2 : (\mathcal{A} \otimes \mathcal{A})_{+}^{F_{12}} \to (\mathcal{A} \otimes \mathcal{A})_{+}^{F_{12}} && (2.14)
\end{aligned}
$$

and $a^{F_{12}} := F_{12} a F_{12}^{-1}$ is hermitean iff a is. Equation (2.13) follows from

$$[P_{12}^{F_{12}}, (\mathcal{A} \otimes \mathcal{A})_{+}^{F_{12}}] = 0. \qquad (2.15)$$

In general, $(\mathcal{H} \otimes \mathcal{H})_{\pm}^{F_{12}}$ will not be (anti)symmetric, nor will $(\mathcal{A} \otimes \mathcal{A})_{+}^{F_{12}}$ be symmetric. Can we still interpret $(\mathcal{H} \otimes \mathcal{H})_{\pm}$ as the Hilbert space of states of the system of two bosons or fermions of equal type and $(\mathcal{A} \otimes \mathcal{A})_{+}^{F_{12}}$ as the corresponding $*$-algebra of observables? We can. In fact, we have just conjugated the standard description of the 2-particle system through F_{12} into a unitary equivalent one (see also next section).

The idea of conjugation can obviously be generalized to a system of n identical particles: Let $F_{12\ldots n} \in \underbrace{\mathcal{A} \otimes \ldots \otimes \mathcal{A}}_{n\text{-times}}$ be unitary, *i.e.* $(F_{12\ldots n})^{*_n} = (F_{12\ldots n})^{-1}$ where $*_n := \underbrace{* \otimes \ldots \otimes *}_{n\text{-times}}$ and define

$$(\mathcal{H} \otimes \ldots \otimes \mathcal{H})_{\pm}^{F_{12\ldots n}} = F_{12\ldots n}(\mathcal{H} \otimes \ldots \otimes \mathcal{H})_{\pm} \qquad (2.16)$$

$$
\begin{aligned}
P_{12}^{F_{12\ldots n}} &= F_{12\ldots n} P_{12} (F_{12\ldots n})^{-1} && (2.17)\\
&\vdots \\
P_{n-1,n}^{F_{12\ldots n}} &= F_{12\ldots n} P_{n-1,n} (F_{12\ldots n})^{-1} && (2.18)\\
(\mathcal{A}\otimes\ldots\otimes\mathcal{A})_{+}^{F_{12\ldots n}} &= F_{12\ldots n} (\mathcal{A}\otimes\ldots\otimes\mathcal{A})_{+} (F_{12\ldots n})^{-1} && (2.19)
\end{aligned}
$$

where

$$
(\mathcal{A}\otimes\ldots\otimes\mathcal{A})_{+} = \{a\in\mathcal{A}\otimes\ldots\otimes\mathcal{A} : [P_{i,i+1}, a] = 0, i = 1,\ldots n-1\},
$$

and $P_{i,i+1}$ is the permutator of the $i^{th}, (i+1)^{th}$ tensor factors. Then

$$
\begin{aligned}
& P_{i,i+1}^{F_{12\ldots n}} |u\rangle_{\pm} = \pm |u\rangle_{\pm} \qquad \text{for} \quad |u\rangle_{\pm} \in (\mathcal{H}\otimes\ldots\otimes\mathcal{H})_{\pm}^{F_{12\ldots n}} && (2.20)\\
& a : (\mathcal{H}\otimes\ldots\otimes\mathcal{H})_{\pm}^{F_{12\ldots n}} \rightarrow (\mathcal{H}\otimes\ldots\otimes\mathcal{H})_{\pm}^{F_{12\ldots n}} && (2.21)\\
& \qquad\qquad\qquad\qquad \text{for} \quad a \in (\mathcal{A}\otimes\ldots\otimes\mathcal{A})_{+}^{F_{12\ldots n}} && (2.22)\\
& *_{n} : (\mathcal{A}\otimes\ldots\otimes\mathcal{A})_{+}^{F_{12\ldots n}} \rightarrow (\mathcal{A}\otimes\ldots\otimes\mathcal{A})_{+}^{F_{12\ldots n}}. && (2.23)
\end{aligned}
$$

Equation (2.21) follows from

$$
[P_{i,i+1}^{F_{12\ldots n}}, (\mathcal{A}\otimes\ldots\otimes\mathcal{A})_{+}^{F_{12\ldots n}}] = 0. \tag{2.24}
$$

Note that in eqs. (2.20) to (2.24) the twist $F_{12\ldots n}$ does not explicitly appear any more; these equations give an *inthrinsic* characterization of the twisted multiparticle description, involving only the operators $P_{i,i+1}^{F_{12\ldots n}}$. In the next sections it will turn out that, even though the twists which are relevant for the quantum symmetry issue are very hard to compute, the $P_{i,i+1}^{F_{12\ldots n}}$ are much less, see section 5.
Remark: If we replace the nilpotent P_{12} by some braid group generator one could also conjugacy transform anyons.

In next section we will discuss the relation between k-particle states and $(m+k)$-particle states in this non-standard description.

3 Identical versus distinct particles

It is crucial that in some conditions identical particles can be treated as though they were *distinct*. Let us recall why.

One reason can be illustrated by the following simplest example. Assume for instance that we only have two particles of the same type in our laboratory. We prepare their initial states (at some time $t = 0$) independently and in such a way that they are "far apart" from each other. At $t = 0$ we can treat them in three equivalent ways.

1. We can treat them as *distinct* particles and describe them *separatly* (and we are free to describe only one): particle i ($i = 1, 2$) is in a one-particle normalized state $|\psi_i\rangle$ (with $\langle\psi_2|\psi_1\rangle = 0$). A measurement process on the first particle is described by acting on $|\psi_1\rangle$ through a one-particle observable $\mathcal{O}_1 \in \mathcal{A}$, the probability amplitude to find particle 1 in a state $|\psi_1'\rangle$ is $\langle\psi_1'|\psi_1\rangle$. Similarly for the second particle. We allow $|\psi_1\rangle, |\psi_2\rangle$ to range on some orthogonal subspaces $\mathcal{H}_1, \mathcal{H}_2$ of $\mathcal{H}$, $|\psi_1\rangle \in \mathcal{H}_1$, $|\psi_2\rangle \in \mathcal{H}_2$ ($\mathcal{H}_1, \mathcal{H}_2$ could for instance consist of the states describing one particle confined respectively to the regions $\mathcal{R}_1, \mathcal{R}_2$ of the space, where the latter do not overlap ($\mathcal{R}_1 \bigcap \mathcal{R}_2 = \phi$)), and $\mathcal{O}_i : \mathcal{H}_I \to \mathcal{H}_i$, see fig. 1[4].

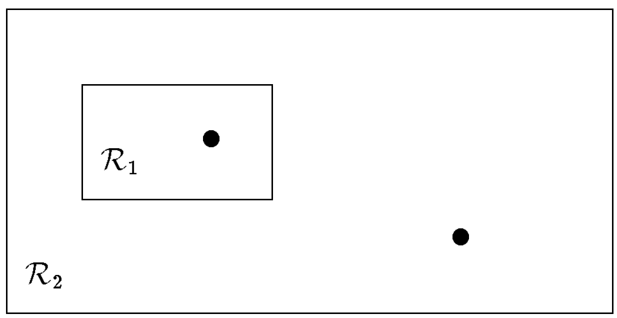

Figure 1: identical particles confined to disjoint regions

2. We can treat them as *distinct* particles forming a *two-particle* system and describe the latter by the state

$$|\psi_d\rangle := |\psi_1\rangle \otimes |\psi_2\rangle \tag{3.1}$$

A measurement process is described by acting on $|\psi_d\rangle$ through a two-particle observable $\mathcal{O}_1 \otimes \mathcal{O}_2$, the probability amplitude to find the two-particle system

[4]More generally, if the preparation were uncomplete, we would assume that particle 1,2 is in a *mixture* of states of $\mathcal{H}_1, \mathcal{H}_2$ respectively.

in a state $|\psi'_d\rangle := |\psi'_1\rangle \otimes |\psi'_2\rangle$ is given by $\langle\psi_d|\psi'_d\rangle = \langle\psi_2|\psi'_2\rangle\langle\psi_1|\psi'_1\rangle$. In the description 1), this amounts respectively to measuring $\mathcal{O}_1$ on the first *and* $\mathcal{O}_2$ on the second, and to the probability amplitude to find particle 1 in state $|\psi'_1\rangle$ *and* particle 2 in state $|\psi'_2\rangle$. If we are interested in measuring $\mathcal{O}_1$ on the first particle as in 1), or in the probability amplitude to find the latter in the same state $|\psi'_1\rangle$ as in 1), we just have to set $\mathcal{O}_2 = id$, $|\psi'_2\rangle = |\psi_2\rangle$ respectively, to find the same results there: in fact, the spectrum of $\mathcal{O}_1 \otimes id$ is that of $\mathcal{O}_1$, and the probability amplitude $\langle\psi_d|\psi'_d\rangle$ reduces in this case to the probability amplitude $\langle\psi_1|\psi'_1\rangle$; in other words, the above settings amount to ignoring the existence of the second particle. This explains in which sense this second description is perfectly equivalent to the first one.

3. We can treat them as *identical* particles forming a two-particle system and describe the latter by the (anti)symmetrized state

$$|\psi\rangle = P^{F_{12}}_{S/A}|\psi_d\rangle, \tag{3.2}$$

where in the *standard* approach to identical particles

$$P^{F_{12}}_{S/A}|\psi_d\rangle := \frac{1}{\sqrt{2}}\left(|\psi_1\rangle \otimes |\psi_2\rangle \pm |\psi_2\rangle \otimes |\psi_1\rangle\right) \in (\mathcal{H} \otimes \mathcal{H})_\pm \tag{3.3}$$

for bosons and fermions respectively (here $P^{F_{12}}_{S/A}$ means either $P^{F_{12}}_{S}$ or $P^{F_{12}}_{A}$. Correspondingly, it is straightforward to check that the same measurement process as in 2) is described now by acting on $|\psi\rangle$ through the symmetrized two-particle observable $\mathcal{O}_1 \otimes \mathcal{O}_2 + \mathcal{O}_2 \otimes \mathcal{O}_1 \in (\mathcal{A} \otimes A)_+$, whereas the probability amplitude to find particle 1 in state $|\psi'_1\rangle$ *and* particle 2 in state $|\psi'_2\rangle$ is equal to the probability amplitude $\langle\psi|\psi'\rangle$ to find the two-particle system in the state $|\psi'\rangle := P^{F_{12}}_{S/A}|\psi'_d\rangle$:

$$\langle\psi|\psi'\rangle = \langle\psi_d|\psi'_d\rangle = \langle\psi_2|\psi'_2\rangle\langle\psi_1|\psi'_1\rangle, \tag{3.4}$$

Again, if we are interested in one particle only, say the first, we will just set $\mathcal{O}_2 = id$, $|\psi'_2\rangle = |\psi_2\rangle$ respectively, as before: in fact, the spectrum of $\mathcal{O}_1 \otimes id + id \otimes \mathcal{O}_1$ on $(\mathcal{H} \otimes \mathcal{H})_\pm$ will be the same as that of $\mathcal{O}_1$, and the probability amplitude $\langle\psi|\psi'\rangle$ will be the same as the probability amplitude $\langle\psi_1|\psi'_1\rangle$. This explains in which sense this third description is equivalent to the previous two.

If we now look at the dynamical evolution of the two particles, it will be no more immaterial which of the three descriptions we use. If there existst some interaction between the two particles, then of course one cannot describe the evolution of the state of one of the two (say the first) only, forgetting the existence of the latter, i.e. description 1) is no more viable for describing the dynamics, and we have to consider the two particles as forming a unique system and an hamiltonian which depends on the observables of both particles. Description 2) (eq. (3.1)) will be still viable (now $|\psi_1\rangle, |\psi_2\rangle$ will depend on time), only as long as $\langle\psi_2|\psi_1\rangle$ remains zero (i.e. the two wavefunctions don't overlap). If the time evolution predicts that at some time $\langle\psi_2|\psi_1\rangle \neq 0$, then also description 2) becomes impossible, and we need to use description 3), which involves in an essential way the quantum statistics; the latter is what happens for instance in a scattering between the two particles. Nevertheless, if for later times $t = t'$ the state $|\psi\rangle$ becomes again a combination of states of the form (3.3) , from that moment description 2) can be implemented again (and description 1) as well, if the interaction becomes negligible).

The example of the scattering perhaps best illustrates one reason why it is important that we have the three equivalent descriptions 1), 2), 3) at $t = 0, t'$, and we know how to go from one to the other: the preparation of the initial states and the measurement on the final states are essentially two independent one-particle preparations/measurements respectively, whereas in the scattering it is crucial to consider the particles as identical.

If instead of two we have $k_1 + k_2$ particles of the same type, and we prepare their states in such a way that k_1 particles are all "far apart" from the other k_2 (e.g. they are confined respectively in non-overlapping regions $\mathcal{R}_1, \mathcal{R}_2$ of the space), then it is easy to realize that we can describe them in three equivalent ways as before:

1) we describe the particles as forming two independent subsystems, such that within each of them the wavefunction is correctly (anti)symmetrized, but particles belonging to different subsystems are considered as distinct, and we describe each of the two subsystems separatly (we are therefore free to describe only one); the wave function of subsystem i will belong to $(\bigotimes^{k_i} \mathcal{H}_i)_\pm$, $i = 1, 2$ where $\mathcal{H}_1$, $\mathcal{H}_2$ are orthogonal subspaces of $\mathcal{H}$.

2) we describe the particles as forming a unique system described by a wavefunction which is the tensor product of the ones describing the two subsystems in case 1), i.e. such that within the first k_1 and the last k_2 tensor factors the wavefunction

is correctly (anti)symmetrized, but particles belonging to different subsystems are still treated as distinct;

3) we describe the particles as forming a unique system described by a wave-function which is completely (anti)symmetric, more precisely is obtained from the one in case 2) by (anti)symmetrizing on all indices, which amounts to treating all particles as equal. If we look at the dynamical evolution of the $k_1 + k_2$ particles, then the same considerations as in the case $k_1 = k_2 = 1$ will apply.

Another deeper reason, why it is crucial that in some conditions identical particles can be treated at least partially as distinct, has a somewhat more philosophical flavor. The considerations done above hold also when k_2 is very large (virtually infinite), i. e. we can apply them to the case in which k_1 particles of the given type form the system that we are really interested to describe "in our laboratory" (as well as its evolution), and the other k_2 are all the other particles of the same typein the universe. Then, what usually happens is:

a) Either we can neglect the interaction between the first and the second subsystem, and simply *forget* the existence of the other k_2 particles; then either of the three descriptions is possible.

b) Or the two subsystems interact, but during their evolution the particles of one subsystem remain "far apart" from the particles of the other; then either description 2) or description 3) is possible (sometimes, when the evolution of the second subsystem is uninfluenced by that of the first one, we can also describe the first subsystem alone as in 1), by introducing an explicitly known time-dependent interaction term in the hamiltonian which represents the interaction of the second subsystem on the first). Note that in case 1), 2) the form of the interaction hamiltonian between subsystem 2 and 1 is such that the time evolution preserves the (anti)symmetry of the wave function in each subsystem.

The fact that both description 3) and either description 1) or description 2) is always possible means the following. To compute any concrete prediction we *don't need* to consider all particles of the given type present in the universe at the same time [description 3)] , but rather use one of the other two; however, in principle *we could*, i.e. we can really apply the postulates of identical particles, through description 3), to *all* particles of the same type in the universe, without finding unconsistent predictions. In other words, the postulates of Quantum Mechanics for identical particles are completely general and self-consistent.

Coming back to our twisted approach to identical particles, if we want it to be physicallly sensible we should check that within its context it is possible to describe two particles prepared in orthogonal states (and similarly more complex systems of k_1+k_2 particles) in three equivalent ways as before.

One can easily verify that this is really possible. In the simplest example of the two particles, for instance, this goes as in the standard approach, except that we have to modify in description 3) the definition (3.3) of $\mathcal{P}^{F_{12}}_{S/A}$

$$\mathcal{P}^{F_{12}}_{S/A}|\psi_d\rangle := F_{12}\frac{1}{\sqrt{2}}\left(|\psi_1\rangle\otimes|\psi_2\rangle \pm |\psi_2\rangle\otimes|\psi_1\rangle\right) \in (\mathcal{H}\otimes\mathcal{H})^{F_{12}}_{\pm}, \tag{3.5}$$

and the symmetrized operators $\mathcal{O}_1\otimes\mathcal{O}_2+\mathcal{O}_2\otimes\mathcal{O}_1 \in (\mathcal{A}\otimes A)_+$ by their twisted versions $F_{12}\left(\mathcal{O}_1\otimes\mathcal{O}_2+\mathcal{O}_2\otimes\mathcal{O}_1\right)F_{12}^{-1} \in (\mathcal{A}\otimes A)^{F_{12}}_{+}$; in other words, we modify the correspondence between states/observables in description 2) and in description 3). These modifications do not invalidate eq. (3.4) and do not change the spectra of the symmetrized operators, therefore the new version of description 3) will be equivalent to descriptions 1), 2) again. It is now easy to understand how in our twisted approach descriptions 1), 2), 3) have to be modified in the general case of k_1+k_2 particles.

4 Quantum Symmetries

So far there was no need for the $F_{12\ldots n}$. Now we take the issue of quantum group symmetries into consideration.

The picture we have in mind is that of a multiparticle quantum mechanical model (consisting of identity particles), on which we would like to implement generalized symmetry transformations through the action of a generic Hopf algebra H [later we will concentrate on the case of a twisted image H of a co-commutative (quasi-) Hopf algebra, like $U_q(g)$]. As given data we take the constituent one-particle system, governed by a $*$-algebra $\mathcal{A}$ of operators that act on a Hilbert space $\mathcal{H}$, a $*$-Hopf algebra H with $\Delta,\varepsilon,S,*$ as coproduct, counit, antipode and complex conjugation, and a unitary realization of H in $\mathcal{A}$.

To construct multiparticle systems that also correctly transform under the Hopf algebra action (and that, in particular, may be symmetric w.r.t. the latter), the key idea will be that properties of the coproduct will have to do with twisted (anti-) symmetry of states. We will find that the coproduct of any element should be considered as being twisted symmetric – even when we are dealing with non-co-commutative Hopf algebras as symmetries.

Let us start by recalling what it means that a one-particle system transforms under the action of H.

4.1 One-particle transformations

To begin, we need a representation ρ of H on $\mathcal{H}$ which realizes H in $\mathcal{A}$:[5]

$$\rho \,:\, H \,\rightarrow\, \mathcal{A}; \tag{4.1}$$

the map ρ is linear and an algebra homomorphism $\rho(xy) = \rho(x)\rho(y)$. It is called a unitary representation if in addition

$$\rho(x)^* = \rho(x^*). \tag{4.2}$$

(For a representation that is not unitary we would find in contrast $\rho(x)^* = \overline{\rho^\vee}(x^*)$, where $\overline{\rho^\vee}$ is the complex conjugate of the contragredient representation. For a matrix representation: $(T^\vee)^i{}_j = S(T^j{}_i)$.)

Let $x \in H$, $\mathcal{O} \in \mathcal{A}$ and $|\psi\rangle \in \mathcal{H}$. The actions of x on the one-particle states $|\psi\rangle$ and and $\mathcal{O}|\psi\rangle$ are given via ρ

$$x \triangleright |\psi\rangle \;=\; \rho(x)|\psi\rangle, \tag{4.3}$$
$$x \triangleright (\mathcal{O}|\psi\rangle) \;=\; \rho(x)\mathcal{O}|\psi\rangle, \tag{4.4}$$

while on the other hand the action of x on a product (that is, on an element of the bigger H-module containing both $\mathcal{A}$ and $\mathcal{H}$) should be computed with the coproduct Δ, *i.e.*

$$x \triangleright (\mathcal{O}|\psi\rangle) = (x_{(1)} \overset{s}{\triangleright} \mathcal{O})(x_{(2)} \triangleright |\psi\rangle). \tag{4.5}$$

Here and in the sequel we will use Sweedler's notation $\Delta(x) \equiv x_{(1)} \otimes x_{(2)}$ for the coproduct (in the RHS a sum of many terms is implicitly understood); similarly, $\Delta^{(n-1)}(x) \equiv x_{(1)} \otimes \ldots \otimes x_{(n)}$ for the $(n-1)$-fold coproduct in Sweedler's notation. As known, it follows that the action of H on the one-particle operator $\mathcal{O}$ is given by

$$x \overset{s}{\triangleright} \mathcal{O} = \rho(x_{(1)})\, \mathcal{O}\, \rho(Sx_{(2)}), \qquad x \in H,\; \mathcal{O} \in \mathcal{A}. \tag{4.6}$$

[5]A given algebra of operators might first have to be extended for this scope.

As a concrete example, the reader may think of the case of quantum mechanics in ordinary three-dimensional space with transformations consisting in ordinary rotations; in that case H is the (undeformed) universal enveloping algebra $U(su(2))$ of the (covering of the) Lie group $SO(3)$. ρ maps elements of $U(su(2))$ into operators acting on $\mathcal{H}$, out of which we can single out unitary operators U realizing finite rotations (i.e. elements of $SO(3)$), as well as hermitean x ones realizing infinitesimal rotations (i.e. elements of $su(2)$) and generating the whole algebra; in these two cases the action (4.6) reduces respectively to conjugating by U, $U\mathcal{O}U^{-1}$, and to taking the commutator $[ix, \mathcal{O}]$. A rotation symmetry of the hamiltonian usually makes elements of $\rho(U(su(2))$ (*e.g.* angular momentum components) as useful observables for studying the dynamics of the system.

4.1.1 Unitary transformations

Under hermitean conjugation an element of $\mathcal{H}$, a "ket", becomes a "bra" which lives in $\mathcal{H}^*$ and transforms under the contragredient representation. This picture should be preserved under transformations. As we know, in the classical case only unitary and—in the infinitesimal case—anti-hermitean transformation operators have the required property. In the general Hopf algebra case the required property is $S(x) = x^*$; we will call such elements of H *quantum unitary.* We stress the point that there are two notions of unitarity which should not be confused: that of a representation, and that of a transformation. Quantum unitary elements also leave the $*$-structure of $\mathcal{A}$ invariant. The condition for an element $u \in H$ to satisfy

$$(u \overset{s}{\triangleright} \mathcal{O})^* = u \overset{s}{\triangleright} \mathcal{O}^* \qquad \forall \mathcal{O} \in A \tag{4.7}$$

is again

$$u^* = S(u) \qquad \text{(quantum unitary operator).} \tag{4.8}$$

This is seen as follows [11]: $*$-conjugating both sides of equation (4.6) we find a condition

$$\rho(Su_{(2)})^* \otimes \rho(u_{(1)})^* \stackrel{!}{=} \rho(u_{(1)}) \otimes \rho(Su_{(2)}), \tag{4.9}$$

or, using that ρ is a unitary representation

$$(Su_{(2)})^* \otimes (u_{(1)})^* \stackrel{!}{=} u_{(1)} \otimes Su_{(2)}. \tag{4.10}$$

Taking the counit $(\varepsilon \otimes id)$ of this equation gives condition (4.8). We want to show that this condition is sufficient:

$$
\begin{aligned}
(u \overset{s}{\triangleright} \mathcal{O})^* &= \Big(\rho(u_{(1)})\, \mathcal{O}\, \rho(Su_{(2)})\Big)^* \\
&= \rho(Su_{(2)})^*\, \mathcal{O}^*\, \rho(u_{(1)})^* \\
&\overset{(4.2)}{=} \rho((Su_{(2)})^*)\, \mathcal{O}^*\, \rho((u_{(1)})^*) \\
&= \rho(S^{-1}(u_{(2)})^*)\, \mathcal{O}^*\, \rho((u_{(1)})^*) \\
&= \rho(S^{-1}(u^*)_{(2)})\, \mathcal{O}^*\, \rho((u^*)_{(1)}) \qquad (4.11)\\
&\overset{(4.8)}{=} \rho(S^{-1}(Su)_{(2)})\, \mathcal{O}^*\, \rho((Su)_{(1)}) \\
&= \rho(S^{-1}Su_{(1)})\, \mathcal{O}^*\, \rho(Su_{(2)}) \\
&= \rho(u_{(1)})\, \mathcal{O}^*\, \rho(Su_{(2)}) \\
&= u \overset{s}{\triangleright} \mathcal{O}^*.
\end{aligned}
$$

In this proof we have used unitarity of the representation ρ and standard facts about $*$-Hopf algebras, like $*\circ S = S^{-1} \circ *$.

Remark: There exist pathological Hopf algebras (*e.g.* with $\tau \circ \Delta = (id \otimes S^2)\Delta$) that are not $*$-Hopf algebras but still allow unitary transformations in a non-standard way.

4.2 Multiparticle transformations

To implement the symmetry transformations (the action of H) on multiparticle systems one makes essential use of the coproduct of H, which enters the game in essentially two different ways.

First, the coproduct is needed to extend the action of H from one-particle *states* to n-particle states; but if the particles are identical, the latter action should preserve the twisted (anti)-symmetry of identical particle states. This will constrain the choice of the F's of section (2), and consequantly also the twisted symmetry of operators, according to formula (2.19). On the other hand, the coproduct enters (in the multiparticle as in the one-particle case) also the way the action of H is defined on operators $\mathcal{O}^{(n)}$ [see formula (4.6) for the one-particle case]; but if the particles are identical this action should preserve the twisted symmetry of the operators. It turns out that both requirements can be satisfied through an appropriate choice of the F's.

4.2.1 Transformation of States

We have so far required that $\mathcal{H}$ be a $*$ H-module, *i.e.* that it carries a $*$ representation of H. The main task in constructing Hilbert spaces for identical particles is then to find an operation of twisted (anti-) symmetrization that is compatible with the action of H, *i.e.* compatible with the twisted symmetry transformations. The action of H on a multiparticle Hilbert space is given once $\rho^{(n)}$ is known. A representation ρ on the 1-particle Hilbert space extends to a unitary representation on the n-particle Hilbert space via the $(n-1)$-fold coproduct of H:

$$\rho^{(n)} = \rho^{\otimes n} \circ \Delta^{(n-1)} : H \to \mathcal{A}^{\otimes n} : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}. \tag{4.12}$$

If ρ is unitary, so is $\rho^{(n)}$, $\rho^{(n)}(x)^{*_n} = \rho^{(n)}(x^*)$, because by hypothesis $(* \otimes *) \circ \Delta = \Delta \circ *$. Here we have used the short-hand notation $*_n := *^{\otimes^n}$

Let $x \in H$ and $|\psi^{(n)}\rangle \in \mathcal{H}^{\otimes n}$, then

$$x \triangleright |\psi^{(n)}\rangle = \rho^{(n)}(x)|\psi^{(n)}\rangle = \rho(x_{(1)}) \otimes \ldots \otimes \rho(x_{(n)})|\psi^{(n)}\rangle. \tag{4.13}$$

where $\Delta^{(n-1)}(x) \equiv x_{(1)} \otimes \ldots \otimes x_{(n)}$ is the $(n-1)$-fold coproduct in Sweedlers notation. As allways we will first consider the case of two particles. Let P_{12} be the permutation operator on $\mathcal{H} \otimes \mathcal{H}$. In the case of a *co-commutative* (*i.e.* symmetric under permutation) coproduct we have

$$P_{12}\left((\rho \otimes \rho)\Delta_c(x)\right) = \left((\rho \otimes \rho)\Delta_c(x)\right) P_{12}$$

and hence

$$P_{12}(x \triangleright |\psi^{(2)}\rangle) = x \triangleright (P_{12}|\psi^{(2)}\rangle).$$

This fact allows us to define symmetrizers $P_S = \frac{1}{2}(I + P_{12})$ and anti-symmetrizers $P_A = \frac{1}{2}(I - P_{12})$ that commute with the action of x, and (anti-) symmetrized Hilbert spaces

$$P_S(\mathcal{H} \otimes \mathcal{H}) \equiv (\mathcal{H} \otimes \mathcal{H})_+, \tag{4.14}$$

$$P_A(\mathcal{H} \otimes \mathcal{H}) \equiv (\mathcal{H} \otimes \mathcal{H})_-, \tag{4.15}$$

that are invariant under the action of x. Similar considerations apply in this case for $n \geq 3$ particles. This happens for instance if $H = U(\mathbf{g})$, $\mathbf{g} = Lie(G)$. Then $U(\mathbf{g})$ is generated by primitive elements X_i with coproduct

$$\Delta^{(n)}(X_i) = \Delta_c^{(n)}(X_i) = X_i \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes X_i; \tag{4.16}$$

$\Delta_c^{(n)}(X_i)$ is invariant under permutations and we can set $F_{12\ldots n} = \mathbf{1} \otimes \ldots \otimes \mathbf{1}$.

But if the coproduct is *not co-commutative*, as it happens for a generic Hopf algebra, then the problem arises that the action of H on $(\mathcal{H} \otimes \mathcal{H})$ will no more preserve the subspaces $(\mathcal{H} \otimes \mathcal{H})_{\pm}$.

While we should not change the form of the coproduct (it is at the very heart of quantum groups and tells us how to act on tensor products) we may however modify our notion of symmetric operators and (anti-) symmetrized Hilbert spaces. We can require

$$\rho^{(n)}(H) \subset (\underbrace{\mathcal{A} \otimes \ldots \otimes \mathcal{A}}_{n\text{-times}})_{+}^{F_{12\ldots n}}, \tag{4.17}$$

so that the system of n identical particles carries a $*$-representation of H as well. This is certainly satisfied if

$$\rho^{(n)}(X) = F_{12..n}\rho_c^{(n)}(X)F_{12..n}^{-1}, \tag{4.18}$$

where $\rho_c^{(n)} := \rho^{\otimes n} \circ \Delta^{(n-1)}$ and Δ_c is a co-commutative coproduct. This has to be read as a condition on both Δ_c and $F_{12..n}$.

If $H = U_q\mathbf{g}$ [1, 7], where $\mathbf{g}$ is the Lie algebra of one of the simple Lie groups of the calssical series, the following theorem due to Drinfel'd will be our guidance to the correct choice of the F's we need to satisfy equations (4.17) and (4.18):

Drinfel'd–Kohno Theorem (Thm. 3.16 in Ref. [10])

1. *There exists an algebra isomorphism* $\phi : U_q\mathbf{g} \leftrightarrow (U\mathbf{g})([[h]])$, *where* $h = \ln q$ *is the deformation parameter.*

2. *If we identify the isomorphic elements of* $U_q\mathbf{g}$ *and* $(U\mathbf{g})([[h]])$ *then there exists an* $\mathcal{F} \in U_q\mathbf{g} \otimes U_q\mathbf{g}$ *such that:*

$$\Delta(a) = \mathcal{F}\Delta_c(a)\mathcal{F}^{-1}, \qquad \forall a \in U_q\mathbf{g} = (U\mathbf{g})([[h]]) \tag{4.19}$$

   where Δ *is the coproduct of* $U_q\mathbf{g}$ *and* Δ_c *is the (co-commutative) coproduct of* $U(\mathbf{g})$.

3. $(U\mathbf{g})([[h]])$ *is a quasi-triangular quasi-Hopf algebra (QTQHA) with universal* $\mathcal{R}_\Phi = q^{t/2}$ *and a quasi-coassociative structure given by an element* $\Phi \in ((U\mathbf{g})^{\otimes 3}([[h]]))$ *that is expressible in terms of* $\mathcal{F}$. $(U\mathbf{g})([[h]])$ *as QTQHA can be transformed via the twist by* $\mathcal{F}$ *into the quasitriangular Hopf algebra* $U_q\mathbf{g}$*; in particular, the universal* $\mathcal{R}$ *of* $U_q\mathbf{g}$ *is given by* $\mathcal{R} = \mathcal{F}_{21}\mathcal{R}_\Phi\mathcal{F}^{-1}$.

Here $(U\mathbf{g})([[h]])$ denotes the algebra of formal power series in the elements of a basis of $\mathbf{g}$, with coefficients being entire functions of h; $(U\mathbf{g})([[h]])|_{h=0} = U\mathbf{g}$. Point 1) essentially says that it is possible to find h-dependent functions of the generators of $U\mathbf{g}$ which satisfy the algebra relations of the Drinfel'd-Jimbo generators of $U_q\mathbf{g}$ and viceversa.

Note: from (4.19) it follows $(\tau \circ \Delta)(a) = \mathcal{M}\Delta(a)\mathcal{M}^{-1}$ with $\mathcal{M} := \mathcal{F}_{21}\mathcal{F}^{-1}$. This is not the usual relation $(\tau \circ \Delta)(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1}$ of a quasitriangular Hopf algebra; the latter is rather obtained by rewriting equation (4.19) in the form $\Delta(a) = \mathcal{F}q^{t/2}\Delta_c(a)q^{-t/2}\mathcal{F}^{-1}$ where $t = \Delta_c(C_c) - 1 \otimes C_c - C_c \otimes 1$ is the invariant tensor $([t, \Delta_c(a)] = 0 \ \ \forall \ \ a \in U\mathbf{g})$ corresponding to the Killing metric, and C_c is the quadratic casimir of $U\mathbf{g}$. $\mathcal{M}$, unlike $\mathcal{R}$, has not nice properties under the coproducts $\Delta \otimes id$, $id \otimes \Delta$. We recall here that the quasitriangular Hopf algebras $U_q\mathbf{g}$ can be obtained as quantizations of Poisson-Lie groups associated with solutions of the MCYBE (modified classical Yang-Baxter equations) corresponding to $\mathbf{g}$.

If the Hopf algebra H can be obtained as the quantization of a Poisson-Lie group associated with a solution of the CYBE classical Yang-Baxter equations corresponding to some $\mathbf{g}$[6], then another (and chronologically preceding) theorem by Drinfel'd [8] states the existence of a different $\mathcal{F}$ with similar properties as in the previous theorem, except that now it is enough to twist $(U\mathbf{g})([[h]])$ equipped with the ordinary *coassociative* structure, in order to obtain H. This means that the quasi-coassociative structure Φ and the quasi-triangular structure $\mathcal{R}_\Phi$ of point 3) inthe theorem reduce to $\Phi = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$, $\mathcal{R}_\Phi = \mathbf{1} \otimes \mathbf{1}$, and that the universal $\mathcal{R}$ is given by $\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1}$. Physically relevant examples of this category of Hopf algebras are, among others, the socalled soft deformations of inhomogeneous groups like the Poincaré [12, 13].

A simple introduction to these topics can be found for instance in Ref. [9].

As shown in Ref. [14], it is very reasonable that one can always choose a unitary $\mathcal{F}$, if H is a compact section of $U_q(g)$ (*i.e.* when $q \in \mathbf{R}$); this is suggested by the fact that on the tensor product of any two representations one can find an orthogonal matrix F intertwining Δ and Δ_c.

These theorems suggest that one can use the twist $\mathcal{F}$ to build F_{12} for a 2-particle sytem. For example:

[6]In this case H is is triangular, *i.e.* $\mathcal{R}_{21}\mathcal{R}_{12} = \mathbf{1}$

1. If $\mathcal{A} = \rho(U_q\mathbf{g})$, then we can define

$$F = \rho^{\otimes^2}(\mathcal{F}).$$

2. If $\mathcal{A} =$ classical Heisenberg algebra $\otimes U^{\text{spin}}(su(2)) \otimes \rho(U_q\mathbf{g})$, were $U_q\mathbf{g}$ plays the role of an internal symmetry, then we can define

$$F_{12} = \text{id}^{(2)}_{\text{Heisenberg}} \otimes \text{id}^{(2)}_{\text{spin}} \otimes F$$

3. If $\mathcal{A}$ is the q-deformed Poincare' algebra of ref. [4, 15], and H is the corresponding q-deformed Lorentz Hopf algebra, realized through ρ in $\mathcal{A}$, then we can again define

$$F_{12} = \rho^{\otimes^2}(\mathcal{F}).$$

   The same applies for other inhomogenous algebras, like the q-Euclidean ones, constructed from the braided semidirect product [15] of a quantum space and of the corresponding homogeneous quantum group. For both of these examples the one-particle representation theory is known [4, 6].

For n-particle systems one can set $F_{12...n} = \rho^{\otimes^n}(\mathcal{F}_{12...n})$, where now we have chosen one particular element $\mathcal{F}_{12...n}$ of $H^{\otimes^n}$ satisfying the condition

$$\Delta(x) = \mathcal{F}_{12...n}\Delta_c(x)(\mathcal{F}_{12...n})^{-1}. \tag{4.20}$$

To obtain one such $\mathcal{F}_{12...n}$ it is enough to act on eq. (4.19) $(n-2)$ times with the coproduct in some arbitrary order. When $n = 3$, for instance, one can use either $\mathcal{F}'_{123} := [(\Delta \otimes id)(\mathcal{F})]\mathcal{F}_{12}$ or $\mathcal{F}''_{123} := [(id \otimes \Delta)(\mathcal{F})]\mathcal{F}_{23}$. These two elements coincide in the case previously mentioned of Hopf algebras associated to solutions of the CYBE, as proved by Drinfeld [8]. In the the case of $U_q\mathbf{g}$, they do not coincide, but nevertheless $\Phi := \mathcal{F}''_{123}(\mathcal{F}'_{123})^{-1} \neq \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$ commutes with $\Delta^{(2)}(H)$. In section (5) we will show (in the $U_q(su(2))$ case) how to find a continuos family of $\mathcal{F}_{123}$ interpolating between $\mathcal{F}'_{123}$ and $\mathcal{F}''_{123}$.

Note: The reader might wonder whether we could use equation $[P_{12}R, (\mathcal{A}\otimes\mathcal{A})_{F_{12}}] = 0$ (where $R = \rho^{\otimes^2}(\mathcal{R})$), instead of eq. (2.15), to single a modified symmetric algebra $(\mathcal{A}\otimes\mathcal{A})'_+$ out of $\mathcal{A}\otimes\mathcal{A}$; in fact, the former is also an equation fulfilled by $\rho^{\otimes^2}(\Delta(H))$ which reduces to the classical eq. (2.4) in the limit $q \to 1$. The reason is that the latter condition is fulfilled *only* by the subalgebra $\rho^{\otimes^2}(\Delta(H)) \subset (\mathcal{A} \otimes \mathcal{A})$ itself, because $q^{t/2}$ does not commute with *all* symmetric operators, but only with the ones corresponding to coproducts. Therefore, the elements of such a $(\mathcal{A}\otimes\mathcal{A})'_+$ would not be enough to be in one-to-one correspondence with the elements of $(\mathcal{A}\otimes\mathcal{A})_+$, *i.e.* would not be enough for our purposes.

Explicit universal $\mathcal{F}$'s for $U_q\mathbf{g}$ are not given in the literature, up to our knowledge; an explicit universal $\mathcal{F}$ for a family of deformations (which include quantizations of solutions of both of a CYBE and of a MCBYE) of the Heisenberg group in one dimension was given in Ref. [16].

However, for most practical purposes one has to deal with representations F of $\mathcal{F}$. A general method for constructing the matrices F acting on tensor products of two arbitrary irreducible representations of compact sections of $U_q\mathbf{g}$ is presented in Ref. [14].

Moreover, in the inthrinsic formulation of the twisted (anti-)symmetrization postulates [eqs. (2.20) – (2.24)] one only needs only the twisted permutators $P_{12\ldots n}^{F_{12\ldots n}}$ (not the $F_{12\ldots n}$ themselves); explicit universal expressions for the latter can be found much more easily, as we show in section 5 for $P_{12}^{\mathcal{F}_{12}}$ in the case $H=u_q(su(2))$.

4.2.2 Transformation of operators

Now we want to see if a consistent transformation of the twisted-symmetric operators can be defined.

As we have seen in section 4.1, the action on one-particle operators which makes eq. (4.5) consistent with eq. (4.4) looks formally like the quantum adjoint action. A subtle but important change in the definition of the action on multiparticle operators is needed in order to get the same goal for multiparticle systems. Therefore, our task in this section is twofold: first we have to find the right action of the Hopf algebra H on tensor products of $\mathcal{A}$, then we have to show that the definition of "twisted symmetric" operators (associated to identical particles) is stable under this action. As before, we assume that ρ is a unitary representation that realizes the Hopf algebra H of transformations in $\mathcal{A}$.

Let $\mathcal{O}^{(n)}\in\mathcal{A}^{\otimes n}$ (or a properly symmetrized subspace), $|\psi_n\rangle\in\mathcal{H}^{\otimes^n}$ (or a properly (anti)symmetrized subspace), then we want, as in the one-particle case,

$$(x_{(1)}\overset{s}{\triangleright}\mathcal{O}^{(n)})(x_{(2)}\triangleright|\psi_n\rangle)=x\triangleright(\mathcal{O}^{(n)}|\psi_n\rangle)=\rho^{(n)}(x)\mathcal{O}^{(n)}|\psi_n\rangle. \tag{4.21}$$

Recalling eq. (4.13) it is easy to see that to satisfy this goal the action (4.6) has to generalize to multiparticle operators in the following way:

$$\begin{aligned} x \overset{s}{\triangleright} \mathcal{O}^{(n)} &= \rho^{(n)}(x_{(1)})\, \mathcal{O}^{(n)}\, \rho^{(n)}(Sx_{(2)}) \\ &= \rho^{\otimes n}(x_{(1)} \otimes \ldots \otimes x_{(n)})\, \mathcal{O}^{(n)}\, \rho^{\otimes n}(Sx_{(2n)} \otimes \ldots \otimes Sx_{(n+1)}). \end{aligned} \tag{4.22}$$

Remark: In the case that $\mathcal{O} = \rho(y)$ with $y \in H$ the action on one-partcle operators is nothing but the adjoint action $x \overset{\mathrm{ad}}{\triangleright} y = x_{(1)} y S(x_{(2)})$. The action on multiparticle operators is however different: For instance in the case that $\mathcal{O}^{(2)} = (\rho \otimes \rho)(y_i \otimes y^i)$ with $y_i \otimes y^i \in H \otimes H$ we get

$$x \overset{s}{\triangleright} (y_i \otimes y^i) = x_{(1)} y_i S x_{(4)} \otimes x_{(2)} y^i S x_{(3)}$$

and *not*

$$x \overset{\mathrm{ad}}{\triangleright} (y_i \otimes y^i) = x_{(1)} \overset{\mathrm{ad}}{\triangleright} y_i \otimes x_{(2)} \overset{\mathrm{ad}}{\triangleright} y^i = x_{(1)} y_i S x_{(2)} \otimes x_{(3)} y^i S x_{(4)}$$

as one might have expected. Both actions "$\overset{\mathrm{ad}}{\triangleright}$" and "$\overset{s}{\triangleright}$" coincide for co-commutative coproducts. The former action treats multiparticle operators as tensor products of H-modules, the latter action is related to the natural Hopf algebra structure on $\Delta(H)$ that is given in Sweedler's book [17]. Briefly, Sweedler's argument is the following. For any given number n $\Delta^{(n-1)}(H)$ can be viewed as a Hopf algebra, with a natural coproduct. Now formula (4.6) is applicable for any n – we just have to take care to use the natural Hopf algebra structure for each of the $\Delta^{(n-1)}(H)$.[7]

The notion of unitary multiparticle transformations generalizes to n particles in an obvious way,

$$(u \overset{s}{\triangleright} \mathcal{O}^{(n)})^* = u \overset{s}{\triangleright} (\mathcal{O}^{(n)})^* \qquad \forall \mathcal{O}^{(n)} \in A \tag{4.23}$$

and again is satisfied if $u^* = S(u)$.

We now want to show that the transformation we have found is compatible with the symmetrization of operators in the twisted multiparticle description. First consider the co-commutative case. Let

$$(\mathcal{A} \otimes \ldots \otimes \mathcal{A})_+ = \{a \in \mathcal{A} \otimes \ldots \otimes \mathcal{A} : [P_{i,i+1}, a] = 0, i = 1, \ldots n-1\}$$

be the completely symmetrized space of n-particle operators. In the case of a co-commutative *i.e. symmetric* coproduct any of the permutation operators $P_{i,i+1}$ will

[7]The action "$\overset{s}{\triangleright}$" was also used in Ref. [18] to define covariance properties of tensors in $H^{\otimes^n}$

commute with the action (4.22):

$$\begin{aligned}\left[P_{i,i+1}\,,\,(x\overset{s}{\triangleright}\mathcal{O}^{(n)})\right] &= \left[P_{i,i+1}\,,\,\rho^{\otimes n}(\Delta_c^{(n-1)}(x_{(1)}))\,\mathcal{O}^{(n)}\,\rho^{\otimes n}(\Delta_c^{(n-1)}(Sx_{(2)}))\right]\\ &= x\overset{s}{\triangleright}[P_{i,i+1}\,,\,\mathcal{O}^{(n)}], \qquad \text{for } x\in H_{\text{co-commutative}}\,.\end{aligned} \tag{4.24}$$

Let $F_{12...n}\in H^{\otimes n}$ such that $\Delta^{(n-1)}(x)=F_{12...n}\Delta_c^{(n-1)}(x){F_{12...n}}^{-1}$ for all $x\in H$. As in the previous section we will use its representation $F_{12...n}\equiv\rho^{\otimes n}(F_{12...n})$ for the similarity transformation of section 2. If we conjugate equation (4.24) with $F_{12...n}$ we find its analog for the non-co-commutative case

$$\left[P_{i,i+1}^{F_{12...n}}\,,\,(x\overset{s}{\triangleright}\mathcal{O}^{(n)})\right]=x\overset{s}{\triangleright}[P_{i,i+1}^{F_{12...n}}\,,\,\mathcal{O}^{(n)}] \qquad \forall x\in H \tag{4.25}$$

and consequently:

$$H\;:\;(\mathcal{A}\otimes\ldots\otimes\mathcal{A})_+^{F_{12...n}}\;\rightarrow\;(\mathcal{A}\otimes\ldots\otimes\mathcal{A})_+^{F_{12...n}}. \tag{4.26}$$

The quantum symmetry is hence compatible with identical particle operators in the twisted multiparticle description.

Remark: The transformation (4.22) is not the only one compatible with the twisted symmetrization. The ordinary commutator $[\rho^{(n)}(x)\,,\,\mathcal{O}^{(n)}]$ also leaves $(\mathcal{A}^{\otimes n})_+^{F_{12...n}}$ invariant, simply because $\rho^{(n)}(x)\in(\mathcal{A}^{\otimes n})_+^{F_{12...n}}$. These two transformations usually coincide in ordinary quantum mechanics. Here they have different interpretations: Let $h\subset H$ be a subalgebra of H. The operator $\mathcal{O}^{(n)}$, $n\geq 1$, is symmetric (*i.e.* invariant) under the transformations generated by $x\in h$ if

$$x\overset{s}{\triangleright}\mathcal{O}^{(n)}=\mathcal{O}^{(n)}\epsilon(x); \tag{4.27}$$

it may be simultaneously diagonalizable with elements in h if

$$[\rho^{(n)}(x)\,,\,\mathcal{O}^{(n)}]=0. \tag{4.28}$$

The two properties coincide if $\Delta(h)\subset h\otimes H$. This can be seen as follows:

$$\rho^{(n)}(x)\mathcal{O}^{(n)}|\psi_n\rangle \overset{(4.13)}{=} x\overset{s}{\triangleright}(\mathcal{O}^{(n)}|\psi_n\rangle) \tag{4.29}$$

$$\overset{(4.21)}{=}(x_{(1)}\overset{s}{\triangleright}\mathcal{O}^{(n)})(x_{(2)}\overset{s}{\triangleright}|\psi_n\rangle) \overset{(4.27)}{=} \varepsilon(x_{(1)})\mathcal{O}^{(n)}(x_{(2)}\overset{s}{\triangleright}|\psi_n\rangle) \tag{4.30}$$

$$=\mathcal{O}^{(n)}(x\overset{s}{\triangleright}|\psi_n\rangle) = \mathcal{O}^{(n)}\rho^{(n)}(x)|\psi_n\rangle \tag{4.31}$$

on any $|\psi_n\rangle\in\mathcal{H}^{\otimes^n}$, so that eq. (4.27) implies eq (4.28); in the same way one proves the converse. The physical relevance of this case is self-evident: if both $\mathcal{O}^{(n)}$ and $\rho^{(\otimes^n)}(x)$ are hermitean, then they can be diagonalized simultaneously; if one of the two, say $\rho^{(\otimes^n)}(x)$, is not hermitean, given an eigenvector $|\psi_n\rangle$ of $\mathcal{O}^{(n)}$, $\rho^{(\otimes^n)}(x)|\psi_n\rangle$ will be another belonging to the same eigenvalue.

5 Explicit example: $H = U_q(su(2))$

We consider as a simple example of a one-particle quantum mechanical system transforming under a quantum group action the case of a q-deformed rotator, $\mathcal{A} \equiv \rho(H) := \rho[U_q(su(2))]$, with $q \in \mathbf{R}^+$. We determine the twisted symmetry of the systems consisting of $n \geq 2$ particles of the same kind.

5.1 $n = 2$ particles

Let us first assume that the states of the system belong to an irreducible $*$-representation of H, namely $\mathcal{H} \equiv V_j$, where V_j denotes the highest weight representation of $U_q(su(2))$ with highest weight $j = 0, \frac{1}{2}, 1, \ldots$. It is very instructive to find out what $(\mathcal{H} \otimes \mathcal{H})_{\pm}^{F_{12}}$ and $(\mathcal{A} \otimes \mathcal{A})_{+}^{F_{12}}$ in this example are.

According to point 1) of the Drinfel'd-Kohno theorem, we can identify $U_q(su(2))$ and $U(su(2))$ as algebras; therefore, V_j can be thought as the representation space of either one. Similarly, $V_j \otimes V_j$ can be considered as the carrier space of a (reducible) representation space of either $U_q(su(2)) \otimes U_q(su(2))$ or $U(su(2)) \otimes U(su(2))$; moreover, $F_{12}(V_j \otimes V_j) = V_j \otimes V_j$. Thus, we can decompose it into irreducible components either of $U_q(su(2))$ or $U(su(2))$, the operators on it being defined as $\rho^{(2)}(X) = \rho^{\otimes^2}[\Delta(X)]$ or $\rho_c^{(2)}(X) = \rho^{\otimes^2}[\Delta_c(X)]$ respectively:

$$V_j \otimes V_j = \begin{cases} \bigoplus_{0 \leq l \leq j} \mathcal{V}^q_{2(j-l)} \oplus \bigoplus_{0 \leq l \leq j - \frac{1}{2}} \mathcal{V}^q_{2(j-l)-1} \\ \bigoplus_{0 \leq l \leq j} \mathcal{V}_{2(j-l)} \oplus \bigoplus_{0 \leq l \leq j - \frac{1}{2}} \mathcal{V}_{2(j-l)-1}; \end{cases} \tag{5.1}$$

here $\mathcal{V}^q_J$ (resp. $\mathcal{V}_J$) denotes the irreducible component of $U_q(su(2))$ (resp. $U(su(2))$) with highest weight J. Moreover, from point 2) of the theorem it follows

$$F_{12}\mathcal{V}_J = \mathcal{V}^q_J, \tag{5.2}$$

Let us recall now that the $\mathcal{V}_J$'s have well-defined symmetry w.r.t the permutation, namely $\mathcal{V}_{2j}, \mathcal{V}_{2(j-1)}, \ldots$ are symmetric, $\mathcal{V}_{2j-1}, \mathcal{V}_{2j-3}, \ldots$ are antisymmetric. This follows from the fact that $\rho_c^{(2)}(X)$ and P_{12} commute. Hence

$$\begin{aligned} (V_j \otimes V_j)_+ &= \bigoplus_{0 \leq l \leq j} \mathcal{V}_{2(j-l)} \\ (V_j \otimes V_j)_- &= \bigoplus_{0 \leq l \leq j - \frac{1}{2}} \mathcal{V}_{2(j-l)-1}. \end{aligned} \tag{5.3}$$

From eq.'s (5.2), (5.4) we finally find

$$
\begin{aligned}
(V_j \otimes V_j)_+^{F_{12}} &:= F_{12}(V_j \otimes V_j)_- = \bigoplus_{0 \le l \le j} \mathcal{V}^q_{2(j-l)} \\
(V_j \otimes V_j)_-^{F_{12}} &:= F_{12}(V_j \otimes V_j)_- \bigoplus_{0 \le l \le j-\frac{1}{2}} \mathcal{V}^q_{2(j-l)-1}.
\end{aligned}
\tag{5.4}
$$

This equation says that the subspaces $\mathcal{V}^q_J \subset V_j \otimes V_j$ have well-defined "twisted symmetry". We can use it to build $(V_j \otimes V_j)_\pm^{F_{12}}$ recalling how the representations $\mathcal{V}^q_J$ are obtained. For this scope, we just have to recall the explicit algebra relations and coproduct:

$$
[h, X^\pm] \;=\; \pm 2X^\pm \qquad\qquad [X^+, X^-] = \frac{q^h - q^{-h}}{q - q^{-1}} \tag{5.5}
$$

$$
\Delta(h) \;=\; \mathbf{1} \otimes h + h \otimes \mathbf{1} \qquad\qquad \Delta(X^\pm) = X^\pm \otimes q^{-\frac{h}{2}} + q^{\frac{h}{2}} \otimes X^\pm. \tag{5.6}
$$

Let $\{|j,m\rangle\}_{m=-j,1-j,\ldots j}$ be an orthonormal basis of V_j consisting of eigenvectors of $\rho(\frac{h}{2})$ with eigenvalues m. As well known, the highest weight vector $\|J,J\rangle \in \mathcal{V}^q_J$ (from which the whole representation $\mathcal{V}^q_J$can be generated by repeated applications of $\rho^{(2)}(X^-)$) is obtained by solving the equation $\rho^{(2)}(X^+)\|J,J\rangle = 0$ for the coefficients a_h of the general ansatz

$$
\|J,J\rangle = \sum_{h=max\{-j,J-j\}}^{min\{j,J+j\}} a_h |j,h\rangle \otimes |j,J-h\rangle. \tag{5.7}
$$

Now we can also understand the difference between $(H \otimes H)_+^{F_{12}}$ and its subalgebra $\rho^{(2)}(H)$:

$$
\rho^{(2)}(H) \ni a : \mathcal{V}^q_J \to \mathcal{V}^q_J, \qquad\qquad (H \otimes H)_+^{F_{12}} \ni b : (V_j \otimes V_j)_\pm^{F_{12}} \to (V_j \otimes V_j)_\pm^{F_{12}}. \tag{5.8}
$$

The elements of $\rho^{(2)}[(H \otimes H)_+ \setminus (H)]$ will in general map $\mathcal{V}^q_J$ out of itself, into some $\mathcal{V}^q_{J'}$ with $J' \neq J$.

If $\mathcal{H}$ carries a reducible $*$-representation of H, it will be possible to decompose it into irreducible representations V_j,

$$
\mathcal{H} = \bigoplus_{j \in \mathcal{J}} V_j \qquad\qquad \mathcal{J} \subset \mathbf{N}_0/2 := \{0, \frac{1}{2}, 1 \ldots\}; \tag{5.9}
$$

then

$$
\mathcal{H} \otimes \mathcal{H} = \bigoplus_{j_1, j_2 \in \mathcal{J}} V_{j_1} \otimes V_{j_2}, \tag{5.10}
$$

and each $V_{j_1} \otimes V_{j_2}$ itself will be a representation. If $j_1 = j_2$, the considerations above apply. If $j_1 \neq j_2$, the irreducible components $\mathcal{V}_J^q$ $(J = |j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2)$ contained in $V_{j_1} \otimes V_{j_2}$ of course *will not* have well-defined symmetry (neither classical nor twisted) under permutations. However, the irreducible components $\tilde{\mathcal{V}}_J^q$ contained in $V_{j_2} \otimes V_{j_1}$ will be characterized by the same set of highest weights J. One can split $\mathcal{V}_J^q \oplus \tilde{\mathcal{V}}_J^q$, and therefore $V_{j_1} \otimes V_{j_2} \oplus V_{j_2} \otimes V_{j_1}$, into the direct sum of one (twisted) symmetric and one (twisted) antisymmetric components

$$[V_{j_1} \otimes V_{j_2} \oplus V_{j_2} \otimes V_{j_1}]_{\pm}^{(F_{12})} = (F_{12})\frac{1}{2}[\mathbf{1} \pm P_{12}]\,[V_{j_1} \otimes V_{j_2} \oplus V_{j_2} \otimes V_{j_1}]_{\pm} \tag{5.11}$$

Let $\{\|J, M\rangle_{12}^q\}_{M=-J,\ldots,J}$ be an orthonormal basis of $\mathcal{V}_J^q$ consisting of eigenvectors of $\rho^{(2)}(h)$ and of $\rho^{(2)}(C_q)$ (C_q denotes the casimir), and let

$$\|J, M\rangle_{12}^q := \sum_{m_1, m_2} \mathcal{C}_{m_1, m_2}^{j_1, j_2}(J, M, q)|j_1, m_1\rangle|j_2, m_2\rangle \tag{5.12}$$

be the explicit decomposition of $\|J, M\rangle_{12}^q$ in the tensor product basis of $V_{j_1} \otimes V_{j_2}$. Then the set $\{\|J, M\rangle_{23}^q\}_{M=-J,\ldots,J}$ with

$$\|J, M\rangle_{23}^q := \sum_{m_1, m_2} \mathcal{C}_{m_2, m_1}^{j_2, j_1}(J, M, q)|j_2, m_2\rangle|j_1, m_1\rangle, \tag{5.13}$$

will be an orthonormal basis of $\tilde{\mathcal{V}}_J^q$ consisting of eigenvectors of $\rho^{(2)}(h)$ and of the casimir $\rho^{(2)}(C_q)$ with the same eigenvalues. Defining

$$\|J, M\rangle_{\pm}^q := N\left(\|J, M\rangle_{12}^q \pm \|J, M\rangle_{23}^q\right), \qquad N^{-1} := \sqrt{2} \tag{5.14}$$

we can easily realize that $\{\|J, M\rangle_{\pm}^q\}_{J,M}$ is an orthonormal basis of $(V_{j_1} \otimes V_{j_2} \oplus V_{j_2} \otimes V_{j_1})_{\pm}^{F_{12}}$.

Note that, if $j_1 = j_2 \equiv j$ and we set $N^{-1} = 2$ in formula (5.13), then the vectors $\|J, M\rangle_{+}^q$ will make up a basis of $V_j \otimes V_j$ (they will have twisted symmetry $(-1)^{J-2j}$, see the previous case) whereas the vectors $\|J, M\rangle_{-}^q$ will vanish.

We are now ready to find, as announced in sections 2, 4, the "universal twisted permutator" $P_{12}^{\mathcal{F}_{12}}$ of $U_q(su(2))$, defined by the property that the twisted permutation operator $P_{12}^{F_{12}}$ on any tensor product $V \otimes V$ [V being the carrier space of a representation ρ whatever of $U_q(su(2))$] can be obtained by $P_{12}^{F_{12}} = \rho^{\otimes^2}(P_{12}^{\mathcal{F}_{12}})$.

We decompose $V \otimes V$ as in formula (5.10). The casimir of $U_q(su(2))$

$$C_q = X^- X^+ + \left(\frac{q^{\frac{h+1}{2}} - q^{\frac{-h-1}{2}}}{q - q^{-1}}\right)^2 \tag{5.15}$$

has eigenvalues $([j+\frac{1}{2}]_q)^2$, where $[x]_q := \frac{q^x-q^{-x}}{q-q^{-1}}$; in the limit $q\to 1$ $\lim_{q\to 1} C_q = C_c + \frac{1}{4}$, where C_c is the usual casimir of $U(su(2))$ with eigenvalues $j(j+1)$. Defining $f(z)$ by

$$\log_q[f(z)] := \left\{\frac{1}{\ln(q)}\arcsin\left[\frac{(q-q^{-1})\sqrt{z}}{2}\right]\right\}^2 - \frac{1}{4}, \tag{5.16}$$

it is easy to verify that $f(C_q)$ has eigenvalues $q^{j(j+1)}$. Let $\hat{R} := P_{12}[\rho^{\otimes^2}(\mathcal{R})]$. Recalling the formula $\mathcal{R} = \mathcal{F}_{21} q^{\frac{t}{2}} \mathcal{F}_{12}^{-1}$, we realize that the vectors $\|J,M\rangle_\pm^q \in (V_{j_1}\otimes V_{j_2} \oplus V_{j_2}\otimes V_{j_1})_\pm^{F_{12}}$ $(j_1\neq j_2)$ are eigenvectors of $\rho^{\otimes^2}\left[f(\mathbf{1}\otimes C_q)f(C_q\otimes\mathbf{1})\left[f(\Delta(C_q))\right]^{-1}\right]\hat{R}$ and $P_{12}^{F_{12}}$ with the same eigenvalue ± 1. If $j_1=j_2=j$, the same holds for the vectors $\|J,M\rangle_+^q$ (which also form a basis of $V_j\otimes V_j$). Since this holds for all j_1,j_2 appearing in the decomposition (5.10), and if we let j_1,j_2 range on $\mathcal{J}$ the above vectors make up a basis of $V\otimes V$, then

$$P_{12}^{F_{12}} = f(\mathbf{1}\otimes\rho(C_q))f(\rho(C_q)\otimes\mathbf{1})\left[f\left(\rho^{(2)}(C_q)\right)\right]^{-1}\hat{R} \tag{5.17}$$

on $V\otimes V$. We prefer to rewrite $\hat{R}$ as $\hat{R} = [\rho^{\otimes^2}(\mathcal{R}_{21})]P_{12}$, where $\mathcal{R}_{21} = \tau(\mathcal{R})$ and τ is the abstract permutator. Since this equation holds for an arbitrary representation ρ, we can drop the latter and obtain the

Universal expression for the twisted permutation operator of $U_q(su(2))$:

$$P_{12}^{\mathcal{F}_{12}} = f(\mathbf{1}\otimes C_q)f(C_q\otimes\mathbf{1})\left[f(\Delta(C_q))\right]^{-1}\mathcal{R}_{21}\circ\tau \tag{5.18}$$

We omit here the well known expression for the universal $\mathcal{R}$ [1].

5.2 $n\geq 3$ particles

When $n\geq 3$, for any given space V the decomposition of $\bigotimes^n V$ into irreducible representations of the permutation group contains components with partial/mixed symmetry, beside the completely symmetric and the completely antisymmetric ones [8]. If $n=3$, for instance, some components can be diagonalized *either* w.r.t. to P_{12} *or* w.r.t. P_{23} (but not w.r.t. both of them simultaneously). If $n=4$, all components can be diagonalized simultaneously w.r.t. P_{12} and P_{34}, and some will have mixed symmetry (e.g. will be symmetric in the first pair and antisymmetric in the second, or viceversa). We recall that the explicit knowledge of components

[8]The Young tableaux provide the rules for finding the complete decomposition for any n.

with mixed/partial symmetry is required to build $(\bigotimes^n \mathcal{H})_\pm$ if the Hilbert space $\mathcal{H}$ of one particle is the tensor product of different spaces, $\mathcal{H} = V \otimes V'$, as in example 2. in subsection 4.2.1.

It is easy to realize that similar statements hold in the case of the twisted symmetry.

Let us consider again the case V_j, and let $n = 3$ for the sake of simplicity. We show how to construct two different orthonormal bases of $V_j \otimes V_j \otimes V_j$ with (partial) symmetry, and a continuous family of F_{123} on $V_j \otimes V_j \otimes V_j$.

There is evidently only one irreducible representation with highest weight $J = 3j$, the highest weight vector being $|j,j\rangle|j,j\rangle|j,j\rangle$. But there are two independent irreducible representations with highest weight $J = 3j-1$, e.g. those having highest weight vectors $\frac{1}{\sqrt{2}}\left(|j,j-1\rangle|j,j\rangle \pm |j,j\rangle|j,j-1\rangle\right)|j,j\rangle$. The latter are symmetric and antisymmetric respectively w.r.t. P_{12}, but are mixed into each other by the action of P_{23}; alternatively, one can combine these two representations into two new ones, having highest weight vectors $\frac{1}{\sqrt{2}}|j,j\rangle\left(|j,j-1\rangle|j,j\rangle \pm |j,j\rangle|j,j-1\rangle\right)$, which are symmetric and antisymmetric respectively w.r.t. P_{23}, but are mixed into each other by the action of P_{12}. One can easily verify that the first two representations are eigenspaces of $\rho_c^{(2)}(C_c) \otimes id$ with eigenvalues $(2j \pm \frac{1}{2})^2$, the latter two are eigenspaces of $id \otimes \rho_c^{(2)}(C_c)$ with the same eigenvalues. The operators $\rho_c^{(3)}(C_c), \rho_c^{(3)}(h)$ and either $\rho_c^{(2)}(C_c) \otimes id$ or $id \otimes \rho_c^{(2)}(C_c)$ make up a complete set of commuting observables over $V_j \otimes V_j \otimes V_j$. Let

$$\{\|J,M,r\rangle_{12}\}_{J,M,r}, \qquad (resp. \{\|J,M,s\rangle_{23}\}_{J,M,s}), \tag{5.19}$$

with

$$j \leq J \leq 3j, \qquad -J \leq M \leq J, \qquad max\{0, j-J\} \leq r, s \leq min\{2j, j+J\} \tag{5.20}$$

denote an orthonormal basis of eigenvectors of $\rho_c^{(3)}(C_c)$, $\rho^{(3)}(h)$ and $\rho_c^{(2)}(C_c) \otimes id$ [resp. $id \otimes \rho_c^{(2)}(C_c)$] with eigenvalues $J(J+1), M$ and $r(r+1)$ (resp. $s(s+1)$). In particular,

$$\begin{aligned} \|3j-1, 3j-1, 2j - \frac{1}{2} \pm \frac{1}{2}\rangle_{12} &= \frac{1}{\sqrt{2}}\left(|j,j-1\rangle|j,j\rangle \pm |j,j\rangle|j,j-1\rangle\right)|j,j\rangle \\ \|3j-1, 3j-1, 2j - \frac{1}{2} \pm \frac{1}{2}\rangle_{23} &= \frac{1}{\sqrt{2}}|j,j\rangle\left(|j,j-1\rangle|j,j\rangle \pm |j,j\rangle|j,j-1\rangle\right) \end{aligned} \tag{5.21}$$

It is easy to verify that in general the subspace of $V_j \otimes V_j \otimes V_j$ which is (anti)-symmetric w.r.t. P_{12} is spanned by the vectors $\|J,M,r\rangle_{12}$ with $r - min\{2j, j+J\}$ (odd) even (and similarly for P_{23}).

For fixed J, M, there exists a unitary matrix $U(J)$ such that

$$\|J, M, s\rangle_{23} = U(J)_{sr}\|J, M, r\rangle_{12} \tag{5.22}$$

Formulae formally identical to eqs. (5.20), (5.22) hold when $q \neq 1$; we will introduce an additional index q in all objects to denote this dependence.

The elements of $\rho^{(3)}(U_q(su(2)))$ in these two bases read

$$\rho^{(3)}(X) = \begin{cases} \sum_J \sum_r \sum_{M,M'} X_{M,M'}(J)\|J, M, r, q\rangle_{12}\ {}_{12}\langle J, M, r, q\| \\ \sum_J \sum_s \sum_{M,M'} X_{M,M'}(J)\|J, M, s, q\rangle_{23}\ {}_{23}\langle J, M, s, q\|, \end{cases} \tag{5.23}$$

and the matrix elements $X_{M,M'}(J)$ do not depend on r, s.

Now it is easy to check that we can find many-parameter continuous families of matrices F_{123} satisfying eq. (4.20), in the form

$$F_{123} = \begin{cases} \sum_J \sum_M \sum_r A_{r,r'}(J)\|J, M, r, q\rangle_{12}\ {}_{12}\langle J, M, r', 1\rangle\| \\ \sum_J \sum_M \sum_s B_{s,s'}(J)\|J, M, s, q\rangle_{23}\ {}_{23}\langle J, M, s', 1\|, \end{cases} \tag{5.24}$$

where $A(J)$'s are arbitrary unitary matrices and $B(J) = [U(J,q)]^* A(J)[U(J, q = 1)]^T$. The key point is that the matrix elements $A_{r,r'}$ do not depend on M, whereas the matrix elements $X_{M,M'}$ do not depend on r.

It is easy to realize that the family (5.24) interpolates between the two F matrix given in subsection 4.2.1, F'_{123} (if we set $A_{r,r'} = \delta_{r,r'}$) and F''_{123} (if we set $B_{r,r'} = \delta_{r,r'}$).

Considerations analogous to those of subsection 5.1 can be done for $n \geq 3$ when V is areducible representation of $U_q(su(2))$.

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