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SO(3) vortices and disorder in the 2d SU(2) chiral model[1]

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Abstract

We study the correlation function of the 2d SU(2) principal chiral model on the lattice. By rewriting the model in terms of Z(2) degrees of freedom coupled to SO(3) vortices we show that the vortices play a crucial role in disordering the correlations at low temperature. Using a series of exact transformations we prove that, if satisfied, certain inequalities between vortex correlations imply exponential fall-off of the correlation function at arbitrarily low temperatures. We also present some Monte Carlo evidence that these correlation inequalities are indeed satisfied. Our method can be easily translated to the language of 4d SU(2) gauge theory to establish the role of corresponding SO(3) monopoles in maintaining confinement at small couplings.

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It is commonly believed that two-dimensional lattice spin models with a continuous nonabelian symmetry exhibit no phase transitions at finite temperature. Indeed, according to the Mermin-Wagner theorem, these models do not have an ordered low temperature phase with spontaneous breakdown of the continuous symmetry. However, this does not in itself rule out the possibility of a Kosterlitz-Thouless (KT) type phase transition. The KT transition is characterised by power-law decay of the correlation function in the low temperature phase as opposed to an exponential decay above the critical temperature. The absence of this KT-type phase transition has not been proved rigorously for nonabelian spin models.

Two dimensional spin models are well known to have properties analogous to that of four dimensional gauge theories. In 4d nonabelian gauge theories the corresponding problem is whether the system remains confining down to arbitrarily small finite couplings. A 4d SU(2) lattice gauge theory can be rewritten in terms of SO(3) and Z(2) variables. This exhibits SO(3) monopoles in the measure, and recently a program has been developed for establishing the presence of these monopoles, and their associated strings, as a sufficient mechanism for confinement at arbitrarily small positive coupling [1]. Completely analogous considerations can be developed for the SU(2)$\times$SU(2) chiral spin model in 2d. In a previous paper [2] we have rewritten the partition function and the two-point correlation function of this model in terms of SO(3) and Z(2) variables, and presented some arguments supporting the role that SO(3) vortices [3] (the analogues of gauge monopoles) and strings connecting them can play in disordering the system.

In the present paper we use this picture to derive an exponentially falling upper bound on the correlation function of the SU(2)$\times$SU(2) model in terms of correlations between SO(3) "stringy" vortices. The main idea is the following. After rewriting the SU(2) degrees of freedom in terms of Z(2) and SO(3) variables and performing a duality transformation on the Z(2) variables the resulting system is an Ising model on the dual lattice coupled to SO(3) vortices. The effect of the vortices on the Ising spins is similar to that of an external magnetic field, provided that appropriate expectations of vortices pairwise connected by long strings winding around the lattice satisfy certain positivity conditions and remain nonvanishing in the large lattice limit. It is crucial that these expectations are defined with respect to an SO(3) (rather than an SU(2)) measure (see below). One thus may bound the correlation function of the original SU(2) model from above by a corresponding expectation in an effective Ising model with a nonzero external magnetic field. This correlation in turn is well known to decay exponentially thus giving an exponentially falling upper bound to the two-point function of the SU(2) principal chiral model.

We shall work on a finite two dimensional square lattice Λ with periodic boundary conditions. The degrees of freedom ("spins") are SU(2) group elements U_s attached to lattice sites with nearest neighbour ferromagnetic couplings. The partition function of the system is

$$Z = \prod_{s\in\Lambda} \int dU_s \;\; \exp\left[\beta \sum_{l\in\Lambda} \mathrm{tr} U_l\right], \tag{1}$$

where β is the inverse temperature, sites, links and plaquettes are labelled with s, l, p respectively, and $U_l = U_s^\dagger U_{s'}$ with $[ss']$ being ∂l, the ordered boundary of the link l. Throughout the paper all group integrations (discrete and continuous) will be performed using the Haar-measure normalised to unity.

Let $\{\tau_1, \tau_2\}$ be a pair of Z(2) elements, Z(2)=$\{\pm 1\}$ being the centre of the symmetry group SU(2). We can introduce a twist τ_1 in the '1' direction and τ_2 in the other and denote by $Z(\tau_1, \tau_2)$ the partition sum in the presence of these twists. By a twist in the direction i we mean that on a stack of links winding around the lattice along the i direction the couplings are changed from β to $\tau_i\beta$. Physically $\tau_i = -1$ means that a topologically nontrivial "domain-wall" was created along the affected links. This domain-wall winds around the lattice, it is closed but not a boundary of any region. The order parameter that we shall consider is the expectation of such a twist defined as

$$G(L) = \frac{\int d\tau_1 \int d\tau_2 \, \tau_2 Z(\tau_1, \tau_2)}{\int d\tau_1 \int d\tau_2 \, Z(\tau_1, \tau_2)} = \frac{Z_+(L) - Z_-(L)}{Z_+(L) + Z_-(L)}, \tag{2}$$

where $Z_\pm(L) = \int d\tau_2 \;\, Z(\pm 1, \tau_2)$ and L is the linear size of the lattice. (2) is the spin analog of the electric-flux free-energy order parameter of gauge theory [5]. If by sending the lattice size to infinity, $G(L)$ goes to zero exponentially, the system is essentially disordered and it does not "feel" the presence of the enforced domain-wall even when the lattice becomes infinitely large.

$G(L)$ has qualitatively the same asymptotic behaviour as the spin-spin correlation function, and it can indeed be rigorously proved [4] that the two-point correlation function for separation L is bounded from above by a constant times $G(L)$. This means that the exponential fall-off of $G(L)$ would imply the same asymptotic behaviour of the spin-spin correlation function.

For a quantitative study of $G(L)$ we rewrite the theory (1) by means of a decomposition into Z(2) and SU(2)/Z2 $\approx$ SO(3) variables as was done in [2], to which we refer for details. After performing an additional duality transformation on the Z(2) degrees of freedom the order parameter assumes the form

$$G(L) = \frac{1}{Z_+(L) + Z_-(L)} \prod_{s\in\Lambda} \int dU_s \, \exp\left(\sum_{l\in\Lambda} M(U_l)\right) \times \prod_{p\in\Lambda} \int d\omega_p \; \chi_{d\eta_p}(\omega_p) \; \eta_C \; \exp\left[\sum_{l\notin C} K(U_l)\delta\omega_l - \sum_{l\in C} K(U_l)\delta\omega_l\right], \quad (3)$$

where $\eta_l = \operatorname{sign}\operatorname{tr}U_l$, $\mathrm{d}\eta_p = \prod_{l\in\partial p}\eta_l$, the χ's are characters of Z(2) and $\eta_C = \prod_{l\in C}\eta_l$ with C being a loop winding around the lattice in the direction perpendicular to the twist. The functions $M(U_l)$ and $K(U_l)$ are given by

$$K(U_l) = \frac{1}{2}\ln\coth\beta|\mathrm{tr}U_l|, \qquad M(U_l) = \frac{1}{2}\ln\left(\cosh\beta|\mathrm{tr}U_l|\sinh\beta|\mathrm{tr}U_l|\right) \quad (4)$$

Now the integrand in (3) depends on the SU(2) variables U_s only through the SU(2)/Z(2) cosets since it is clearly invariant under a local transformation of the U_s's by elements of Z(2). Thus the integration is effectively over SO(3) degrees of freedom. In compensation, (3) also contains Z(2) spins ω_p attached to plaquettes. Spins on neighbouring plaquettes sharing the link l interact via the fluctuating coupling $K(U_l)$. The Z(2) part of the system is now essentially a ferromagnetic Ising model but with couplings depending on the SO(3) degrees of freedom. Because of the duality transformation, low temperature in the original model corresponds to high temperature i.e. small $K(U_l)$'s in this Ising model. The Ising and the SO(3) variables are further coupled through the Z(2) characters. By definition $\chi_{d\eta_p}(\omega_p) = 1$ if $d\eta_p = 1$: and $\chi_{d\eta_p}(\omega_p) = \omega_p$ if $d\eta_p = -1$, which means that there is an SO(3) vortex on the plaquette p. The Z(2) characters thus couple the Ising spins to SO(3) vortices. It is also obvious from the construction that vortices only occur pairwise connected by "η-strings", i.e. stacks of links on which $\eta_l = -1$ (Fig. 1). This terminology is motivated by the analogy with Dirac strings in gauge theories. The strings are not gauge invariant but their endpoints, the vortices (monopoles in gauge theories), are. Finally, the $M(U_l)$ part of the action depends only on the SO(3) variables.

In this representation $G(L)$ involves a twist along C in the *dual* Ising model of ω spins. Notice that while in the original system the twist was along a stack of links winding around the lattice in the "1" direction, in the dual system the twisted links form a loop C going around the lattice in the direction perpendicular to "1". Furthermore, there are two additional crucial factors in the measure: η_C, and the product of Z(2) characters that couple SO(3) vortices to Z(2) spins. For a fixed SO(3) configuration with vortices at $(p_1, ...p_{2n})$ and S η-strings crossing C, these give an overall factor of $(-1)^S \prod_{i=1}^{2n} \omega_{p_i}$. Were it not for this additional factor depending on the SO(3) configuration, the system would essentially be an ordinary ferromagnetic Ising model at high temperature with unbroken Z(2) symmetry and $G(L)$ going to unity on large lattices.

Let us then look at the Z(2) characters in the measure. In a fixed "background" SO(3) configuration containing $2n$ vortices, the sum over the Ising configurations gives a $2n$-point function of the Ising system with couplings $K(U_l)$. When finally the SO(3) variables are integrated out, we get a sum over all possible (even) numbers and locations of vortices which translates into a sum of all possible correlations in the Ising system taken with additional weights coming from the SO(3) part of the measure. This is very much reminescent of the expansion of the partition function of an Ising model with respect to an external magnetic field, where the same type of sum appears. This motivates the following physical picture. The high temperature Ising system of ω spins would be in its symmetric phase by itself but the coupling to the vortices breaks the symmetry by creating an effective external magnetic field. In this broken phase the free energy of the twist (3) along C grows exponentially with the lattice size implying exponential decay for $G(L)$.

To make this argument quantitative we proceed as follows. We first insert a delta function in the measure in (3) to constrain η_C to unity. Let $G(L, C_+)$ denote $G(L)$ computed in the presence of this constraint. It can be rigorously proven that $G(L) \leq G(L, C_+)$, so it is enough to verify exponential fall-off for $G(L, C_+)$. We next compare $G(L, C_+)$ to the quantity

$$G_{\text{eff}}(h, L) = \frac{1}{Z_{\text{eff}}(h, L)} \prod_{p \in \Lambda} \int d\omega_p \, \exp\left[\sum_{l \notin C} K(\mathbf{1}) \delta\omega_l - \sum_{l \in C} K(\mathbf{1}) \delta\omega_l + \sum_{p \in \Lambda} h\omega_p \right], \qquad (5)$$

i.e. the expectation of a twist along C in an effective Ising system of ω spins coupled to a magnetic field h. Now using a method similar to Ginibre's proof of Griffiths-type inequalities [6], the difference $G_{\text{eff}}(h, L) - G(L, C_+)$ can be expressed as a sum of terms, each term being of the form

$$\Gamma_{L,\bar{\theta}}[P_1, P_2, ..., P_n] \equiv \langle \prod_{i=1}^{n} (\theta^-_{P_i} - \bar{\theta}\theta^+_{P_i}) \prod_{p \neq \{P_i\}} \theta^+_p \rangle_L, \qquad (6)$$

times positive numerical coefficients. Here, $P_1, ..., P_n$ is a set of n vortex pairs, $\theta^{\pm}_{P_i} = \theta^{\pm}_{p_{i1}} \theta^{\pm}_{p_{i2}}$, where p_{i1}, p_{i2} denote the locations of the two vortices of the i-th pair, and $\theta^{\pm}_p$ constrains dη_p to be ± 1. $\langle \; - \; \rangle_L$ means integration with respect *only to a pure SO(3) measure* defined simply by the action $M(U_l)$, 4, and including the constraint $\eta_C = 1$. Also, we defined $\bar{\theta} \equiv \tanh h$. Since the integration measure is positive, at $\bar{\theta} = 0$ expression (6) is positive, hence $G_{\text{eff}}(0, L) \leq G(L)$. By continuity the same then holds in a neighbourhood of $\bar{\theta} = 0$ ($h = 0$). If this neighbourhood does not shrink to zero when the lattice size is sent to infinity, $G(L)$ must obey exponential decay since $G_{\text{eff}}(h, L)$ does for any non-zero h, the mass-gap being proportional to $\tanh h$. We have thus rigorously reduced the existence of a mass gap to a condition on vortex pair correlations.

Next consider the assertion

$$\Gamma_{L,\bar{\theta}}(P_1, P_2, ..., P_n) \;\geq\; \text{const.} \times \Gamma_{L,\bar{\theta}}(P_1) \times ... \times \Gamma_{L,\bar{\theta}}(P_n), \tag{7}$$

i.e. that the correlators Γ of vortex pairs (6) are bounded by products of vortex pair correlations. This (highly nontrivial) inequality can be rigorously proven for $\bar{\theta} = 0$ by an argument that reduces (7) to an application of the FKG inequalities [7]. It is then very plausible that it also holds for sufficiently small values of $\bar{\theta}$. Indeed, preliminary results indicate that (7) holds for $\bar{\theta}$ such that the r.h.s. remains nonnegative, i.e. for $\bar{\theta} \leq k\bar{\theta}_0$, where k is a numerical constant of order unity, and $0 \leq \bar{\theta}_0$ such that

$$\Gamma_{L,\bar{\theta}}(P) = \langle \theta^-_{p_1} \theta^-_{p_2} \prod_{p \neq p_i} \theta^+_p \rangle_L \;-\; \bar{\theta}_0 \langle \prod_{p \in \Lambda} \theta^+_p \rangle_L \geq 0 \tag{8}$$

holds on arbitrarily large lattices regardless the location (p_1 and p_2) of the two members of the vortex pair P. This is equivalent to the statement that the free energy cost of a pair of vortices

$$F_L(p_1, p_2) = -\frac{1}{\beta} \ln \frac{\langle \theta^-_{p_1} \theta^-_{p_2} \prod_{p \neq p_i} \theta^+_p \rangle_L}{\langle \prod_{p \in \Lambda} \theta^+_p \rangle_L} \tag{9}$$

is bounded for any lattice size. The only case when this free energy can in principle be sensitive to the lattice size is when p_1 and p_2 are on opposite sides of C and the constraint $\eta_C = 1$ forces the η-string connecting them to go around the lattice (Fig. 1).

Now *in two dimensions* the energy of such an η-string stays finite as $L \to \infty$ in the semiclassical approximation, where one obtains $F_L(P) =$ constant, whereas it diverges with L in higher dimensions. This is due to flux spreading [8] that allows the cost of the string creation to be spread laterally in the direction perpendicular to the string. As explained above, constant $F_L(P)$ implies h and hence a mass gap proportional to $\exp -(\text{const})\beta$. In the remainder of the paper we shall briefly present the results of a Monte Carlo measurement of $F_L(p_1, p_2)$ that indicate that this behavior of $F_L(P)$ holds for the exact expectation (9) . Details of this Monte Carlo calculation will appear elsewhere.

We measured the quantity $\exp(-\beta F_L(p_1, p_2))$ by Monte Carlo using the SO(3) action $|\text{tr}U_l|$ appearing in (9). The simulations were performed on lattices $5 \leq L \leq 13$

at $\beta = 2.0$. As can be seen from the location of the specific heat peak, this is already on the weak coupling side of the crossover in the SO(3) model. The measurement was done by simply counting in a long Monte Carlo run the number of configurations having exactly two vortices at the fixed locations p_1 and p_2 and in addition satisfying the constraint $\eta_C = 1$ (Fig. 1). This constraint forces the eta string connecting the two nearby vortices to run all the way around the lattice. Finally the number of these configurations was divided by the number of configurations containing no vortices at all. Our results are summarised in Figure 2. We can see that the probabilty of having a vortex pair at a given location with a long eta string decreases on small lattices until it stabilises on moderate sized lattices ($L \approx 8 - 9$) at a nonzero constant value. Recall that for our purposes it is enough that this quantity remains nonzero in the $L \to \infty$ limit. Figure 3 illustrates the pronounced effect on (9) of flux spreading in the lateral direction.

To summarise, we have seen how vortices can disorder the correlation function in the two dimensional SU(2)$\times$SU(2) chiral spin model at arbitrarily low temperatures. By an exact rewriting of the original model to separate SO(3) and Z(2) degrees of freedom, we derived sufficient conditions on certain SO(3) vortex correlations for the existence of an exponentiallyfalling upper bound to the order paramete. To complete the argument we made two interrelated assumptions concerning the behavior of the correlations (7), (9). We presented the results of a Monte Carlo calculation that confirm the expected behavior of (9). Clearly it would be very worthwile to find an analytic proof of these two assumptions thus completing a rigorous demonstration of the absence of a KT-type phase transition in this model. It would be also interesting to extend these arguments to similar models with different symmetry groups. There are two obvious possible ways of generalisation. One is to SU(n) principal chiral models and the other to O(n) vector models by noting that the SU(2) principal chiral model is equivalent to the O(4) vector model.

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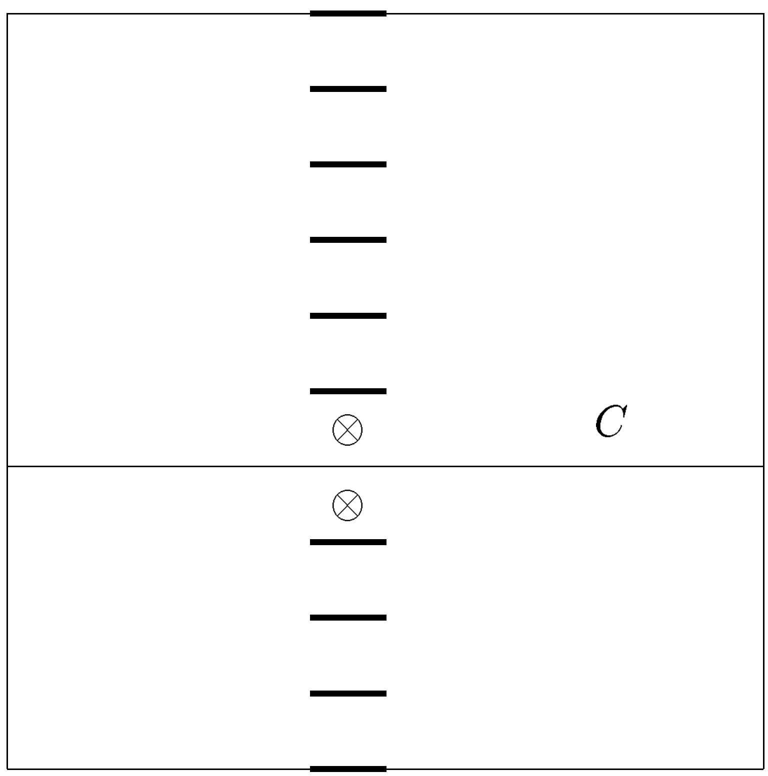
Figure captions

Figure 1 Two vortices connected by an η-string winding around the lattice.

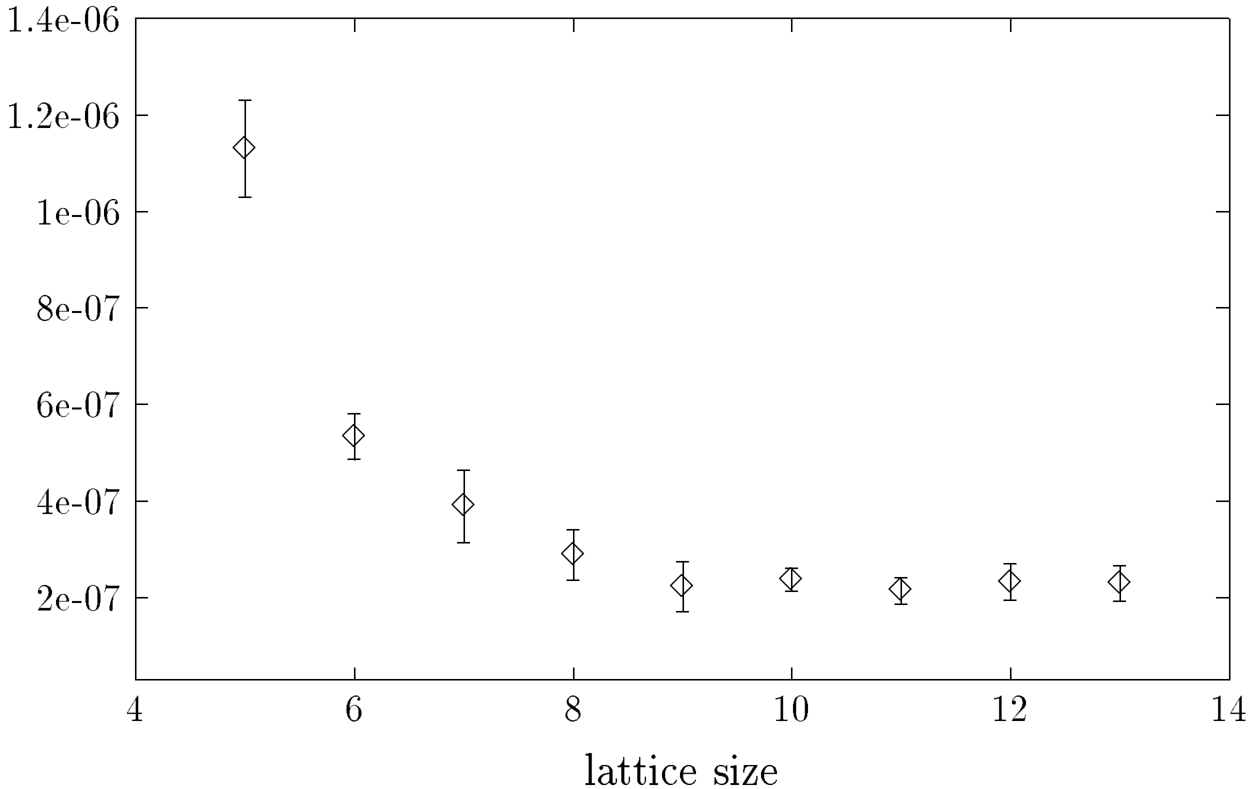
Figure 2 The probability of the vortex pair shown in Figure 1 as a function of the lattice size at inverse temperature $\beta = 2.0$.

Figure 3 The probability of a vortex pair with η-string going around the lattice at $\beta = 2.0$. The length of the side of the lattice parallel to the string is fixed to 9 and only the "transverse" size (perpendicular to the string) is changed.

Figure 1



$\otimes$ vortex $\qquad$ — $\eta_l = -1$


1.4e-06
1.2e-06
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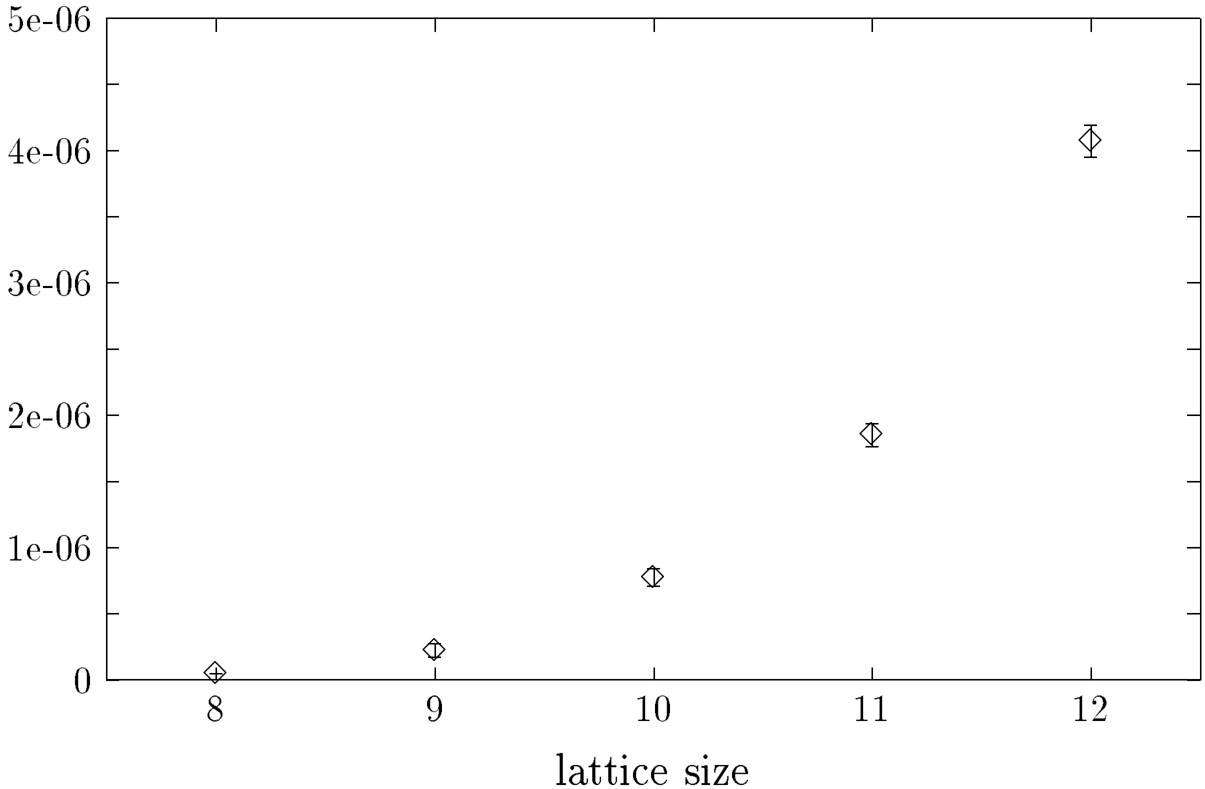
8e-07
6e-07
4e-07
2e-07
4
6
8
10
12
14
lattice size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3e-06
2e-06
1e-06
0
8
9
10
11
12
lattice size