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Quantizing the (0,4) Supersymmetric ADHM Sigma Model

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ABSTRACT

We discuss the quantization of the ADHM sigma model. We show that the only quantum contributions to the effective theory come from the chiral anomalies and compute the first and second order terms. Finally the limit of vanishing instanton size is discussed.

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1. Introduction

Recently there has been considerable interest in the use of massive linear sigma models to construct string vacua in the infrared limit of the renormalization group flow. In an interesting paper [1] Witten discusses a class of massive linear sigma models possessing (0,4) supersymmetry which flow in the infrared to conformally invariant theories describing ADHM instantons. Previous work on the ADHM sigma model has focused on classical aspects of the $(0,4)$ supersymmetry multiplet used and in particular the construction of off-shell superfield formalisms [2,3,4]. Here we will study the models quantum properties and its rich interplay between geometry and field theory in detail. The general (p,q) supersymmetric massive sigma model has been constructed before [5] and their quantization is discussed to two loop order in [6]. We will show here that the ADHM sigma Model is ultra violet finite to all orders of perturbation theory and integrate out the massive fields to obtain the low energy effective theory. Due to anomalies this theory has interesting non trivial properties and we obtain the quantum corrections to order α'^2 by requiring that the anomalies are appropriately canceled. We conclude by making some comments about the case where the instanton size vanishes.

2. The ADHM Sigma Model

In [1] Witten constructs an on-shell (0,4) supersymmetric linear sigma model which parallels the ADHM construction of instantons [7]. The model consists of $4k$ bosons X^{AY}, $A=1,2$, $Y=1,2...,2k$ with right handed superpartners $\psi_-^{A'Y}$, $A'=1,2$. There is also a similar multiplet of fields $\phi^{A'Y'}$, $\chi_-^{AY'}$ $Y'=1,2...,2k'$. In addition there are n left handed fermions λ_+^a, $a=1,2...,n$. The $A,B...$ and $A',B'...$ indices are raised (lowered) by the two by two antisymmetric tensor ϵ^{AB} (ϵ_{AB}), $\epsilon^{A'B'}$ ($\epsilon_{A'B'}$). The $Y,Z...$ and $Y',Z'...$ indices are raised (lowered) by the invariant tensor of $Sp(k)$, $Sp(k')$ respectively which are also denoted by ϵ^{YZ} (ϵ_{YZ}), $\epsilon^{Y'Z'}$ ($\epsilon_{Y'Z'}$)

The interactions are provided for by a tensor $C^a_{AA'}(X,\phi)$ in a manner similar to the construction of off-shell models [5]. The action for the theory is given by

$$
\begin{aligned}
S = \int d^2x \Big\{ & \epsilon_{AB}\epsilon_{YZ}\partial_= X^{AY}\partial_{\neq} X^{BZ} + i\epsilon_{A'B'}\epsilon_{YZ}\psi_-^{A'Y}\partial_{\neq}\psi_-^{B'Z} \\
& + \epsilon_{A'B'}\epsilon_{Y'Z'}\partial_=\phi^{A'Y'}\partial_{\neq}\phi^{B'Z'} + i\epsilon_{AB}\epsilon_{Y'Z'}\chi_-^{AY'}\partial_{\neq}\chi_-^{BZ'} \\
& + i\lambda_+^a\partial_=\lambda_+^a - \frac{im}{2}\lambda_+^a\left(\epsilon^{BD}\frac{\partial C^a_{BB'}}{\partial X^{DY}}\psi_-^{B'Y} + \epsilon^{B'D'}\frac{\partial C^a_{BB'}}{\partial \phi^{D'Y'}}\chi_-^{BY'}\right) \\
& - \frac{m^2}{8}\epsilon^{AB}\epsilon^{A'B'}C^a_{AA'}C^a_{BB'}\Big\} \ ,
\end{aligned}
\tag{2.1}
$$

where m is an arbitrary mass parameter. Note the twisted form of the Yukawa interactions in (2.1) in comparison to the models of [5].

Provided that $C^a_{AA'}$ takes the simple form

$$
C^a_{AA'} = M^a_{AA'} + \epsilon_{AB}N^a_{A'Y}X^{BY} + \epsilon_{A'B'}D^a_{AY'}\phi^{B'Y'} + \epsilon_{AB}\epsilon_{A'B'}E^a_{YY'}X^{BY}\phi^{B'Y'} \ , \tag{2.2}
$$

subject to the constraint

$$
C^a_{AA'}C^a_{BB'} + C^a_{BA'}C^a_{AB'} = 0 \ , \tag{2.3}
$$

then the action (2.1) has the on-shell (0,4) supersymmetry

$$
\begin{aligned}
\delta X^{AY} &= i\epsilon_{A'B'}\eta_+^{AA'}\psi_-^{B'Y} \\
\delta\psi_-^{A'Y} &= \epsilon_{AB}\eta_+^{AA'}\partial_= X^{BY} \\
\delta\phi^{A'Y'} &= i\epsilon_{AB}\eta_+^{AA'}\chi_-^{BY'} \\
\delta\chi_-^{AY'} &= \epsilon_{A'B'}\eta_+^{AA'}\partial_=\phi^{B'Y'} \\
\delta\lambda_+^a &= \eta_+^{AA'}C^a_{AA'} \ ,
\end{aligned}
\tag{2.4}
$$

where $\eta_+^{AA'}$ is an infinitesimal spinor parameter. As is discussed by Witten [1], the above construction of models with (0,4) supersymmetry can be interpreted as a string theory analogue of the ADHM construction of instantons with instanton number k' in a spacetime dimension of $4k$.

The general form of massive (p,q) supersymmetric sigma models has been discussed in terms of $(0,1)$ superfields in [5] and we now provide such a formulation of the ADHM model. To this end we introduce a tensor $I^A_{A'}$ satisfying

$$\epsilon_{AB} I^A_{A'} I^B_{B'} = \epsilon_{A'B'} \tag{2.5}$$

which can be interpreted as a complex structure in the sense that $I^{AB'} I_{AC'} = -\delta^{B'}_{C'}$, $I^{BA'} I_{CA'} = -\delta^B_C$. The 'twisted' superfields are

$$\begin{aligned} \mathcal{X}^{AY} &= X^{AY} + \theta^- I^A_{A'} \psi_-^{A'Y} \\ \Phi^{A'Y'} &= \phi^{A'Y'} + \theta^- I_A^{A'} \chi_-^{AY'} \\ \Lambda^a_+ &= \lambda^a_+ + \theta^- F^a \ , \end{aligned} \tag{2.6}$$

where θ^- is the (0,1) superspace coordinate with the superspace covariant derivative

$$D_- = \frac{\partial}{\partial \theta^-} + i\theta^- \partial_=$$

and F^a an auxiliary field. After removing F^a by its equation of motion and using the constraint (2.3), the action (2.1) can be seen to have the superspace form

$$\begin{aligned} S_{effective} = -i \int d^2x d\theta^- \Big\{ & \epsilon_{AB}\epsilon_{YZ} D_- \mathcal{X}^{AY} \partial_{\neq} \mathcal{X}^{BZ} + \epsilon_{A'B'}\epsilon_{Y'Z'} D_- \Phi^{A'Y'} \partial_{\neq} \Phi^{B'Z'} \\ & -i\delta_{ab} \Lambda^a_+ D_- \Lambda^b_+ - m C_a \Lambda^a_+ \Big\} \ , \end{aligned} \tag{2.7}$$

where $C^a = I^{AA'} C^a_{AA'}$. The inclusion of the auxiliary field allows one to close a $(0,1)$ part of the supersymmetry algebra off-shell. As with the component field formulation (2.1) the full $(0,4)$ supersymmetry is only on-shell. A manifestly off-shell form requires harmonic superfields with an infinite number of auxiliary fields [3].

Lastly we outline the $k = k' = 1$, $n = 8$ case (ie. a single instanton in $\mathbf{R}^4$) analyzed by Witten which will be of primary interest here. The right handed fermions are taken to be $\lambda_+^a = (\lambda_+^{AY'}, \lambda_+^{YY'})$ and the tensor $C^a_{AA'}$ takes the form

$$
\begin{aligned}
C^{YY'}_{BB'} &= \epsilon_{BC}\epsilon_{B'C'}X^{CY}\phi^{C'Y'} \\
C^{AY'}_{BB'} &= \frac{\rho}{\sqrt{2}}\epsilon_{B'C'}\delta^A_B\phi^{C'Y'} \ ,
\end{aligned}
$$

where ρ is an arbitrary constant interpreted as the instanton size. The bosonic potential for this theory is easily worked out as

$$
V = \frac{m^2}{8}(\rho^2 + X^2)\phi^2 \ , \tag{2.8}
$$

where $X^2 = \epsilon_{AB}\epsilon_{YZ}X^{AY}X^{BZ}$ and similarly for ϕ^2. Thus, for $\rho \neq 0$, the vacuum states of the theory are defined by $\phi^{A'Y'} = 0$, and parameterize $\mathbf{R}^4$. The X^{AY} and $\psi_-^{A'Y}$ are massless fields while $\phi^{A'Y'}$ and $\chi_-^{AY'}$ are massive. This yields exactly 4 of the λ_+^a massive and 4 massless.

3. Quantization

3.0.1 Renormalization

It is not hard to see that the model described above is superrenormalizable in two dimensions as the interaction vertices do not carry any momentum factors. In fact a little inspection reveals that the only possible divergence of the theory are the one loop graphs contributing to the potential. Using dimensional regularization in $D = 2 + \epsilon$ dimensions the bosonic graphs are readily calculated to be

$$
\begin{aligned}
\Gamma_{Div}(\text{bosons}) = -\frac{m^2}{8\pi\epsilon}\Bigg[&\epsilon^{AB}\epsilon^{CD}\epsilon^{C'D'}\epsilon^{YZ}\frac{\partial C^a_{CC'}}{\partial X^{AY}}\frac{\partial C^a_{DD'}}{\partial X^{BZ}} \\
&+\epsilon^{A'B'}\epsilon^{C'D'}\epsilon^{CD}\epsilon^{Y'Z'}\frac{\partial C^a_{CC'}}{\partial \phi^{A'Y'}}\frac{\partial C^a_{DD'}}{\partial \phi^{B'Z'}}\Bigg] \ ,
\end{aligned} \tag{3.1}
$$

while the fermionic graphs are

$$
\begin{aligned}
\Gamma_{Div}(\text{fermions}) = \frac{m^2}{8\pi\epsilon}\Bigg[&\frac{1}{2}\epsilon^{AC}\epsilon^{BD}\epsilon^{C'D'}\epsilon^{YZ}\frac{\partial C^a_{CC'}}{\partial X^{AY}}\frac{\partial C^a_{DD'}}{\partial X^{BZ}} \\
&+\frac{1}{2}\epsilon^{A'C'}\epsilon^{B'D'}\epsilon^{CD}\epsilon^{Y'Z'}\frac{\partial C^a_{CC'}}{\partial \phi^{A'Y'}}\frac{\partial C^a_{DD'}}{\partial \phi^{B'Z'}}\Bigg] \ .
\end{aligned} \tag{3.2}
$$

One can see that the epsilon tensor terms in (3.1) and (3.2) are different as a result of the twisted form of the Yukawa interactions. It is not immediately obvious then that the bosonic and fermionic divergences cancel. However using (2.2) it is not much trouble to see that they do and hence $\Gamma_{Div} = 0$. Thus Witten's ADHM model is ultraviolet finite to all orders of perturbation theory. Therefore there is no renormalization group flow in these models. This result may be expected, but is not guaranteed by supersymmetry, as there is a general argument for finiteness only for off-shell $(0,4)$ sigma models, with some modifications required due to anomalies [8].

3.0.2 Integrating the Massive Modes

In this section we will integrate out the massive modes. We shall postpone the problem of anomalies in chiral supersymmetric models until the next section. We assume for simplicity here that $M^a_{AA'} = N^a_{AY} = 0$, $D^a_{A'Y'} \neq 0$ so that the X^{AY} and $\psi_-^{A'Y}$ fields are massless, the $\phi^{A'Y'}$ and $\chi_-^{AY'}$ fields massive and the vacuum is at $\phi^{A'Y'} = 0$. The theory is then only quadratic in the massive fields and integrating over them is therefore exact at the one loop level. This assumption also ensures that the interacting theory breaks the $SU(2) \times Sp(k) \times SU(2) \times Sp(k')$ symmetry of the free theory down to $SU(2) \times Sp(k')$ [1]. We may therefore write

$$C^a_{AA'} = \epsilon_{A'B'} D^a_{AY'} \phi^{B'Y'} + \epsilon_{AB}\epsilon_{A'B'} E^a_{YY'} X^{BY} \phi^{B'Y'} \equiv \epsilon_{A'B'} B^a_{AY'}(X) \phi^{B'Y'} \ .$$

At this point it is necessary to split up the left handed fermions into there massive and massless parts. If we introduce the zero modes $v^a_i(X)$, $i = 1, 2..., n - 4k'$ of the fermion mass matrix, defined such that

$$v^a_i B^a_{AY'} = 0, \quad v^a_i v^a_j = \delta_{ij}$$

and a similar set of massive modes $u^a_I(X)$, $I = 1, 2..., 2k'$ satisfying

$$u^a_I u^a_J = \delta_{IJ}, \quad u^a_I v^a_j = 0$$

then we may separate the λ_+^a as

$$\lambda_+^a = v_i^a \zeta_+^i + u_I^a \zeta_+^I \ . \tag{3.3}$$

so that the ζ_+^i are massless and the ζ_+^I massive. We now rewrite the action (2.1) in terms of the massless and massive fields

$$S = S_{massless} + S_{massive} \tag{3.4}$$

where $S_{massless}$ is the part of (2.1) which only contains the massless fields. Explicitly

$$\begin{aligned} S_{massless} = \int d^2x \Big\{ & \epsilon_{AB}\epsilon_{YZ}\partial_= X^{AY}\partial_{\neq}X^{BZ} + i\epsilon_{A'B'}\epsilon_{YZ}\psi_-^{A'Y}\partial_{\neq}\psi_-^{B'Z} \\ & + i\zeta_+^i(\delta_{ij}\partial_=\zeta_+^j + A_{ijAY}\partial_= X^{AY}\zeta_+^j)\Big\} \ , \end{aligned} \tag{3.5}$$

where

$$A_{ijAY} = v_i^a \frac{\partial v_j^a}{\partial X^{AY}} \tag{3.6}$$

is the induced $SO(n-4k')$ connection and

$$\begin{aligned} S_{massive} = \int d^2x \Big\{ & \epsilon_{A'B'}\epsilon_{Y'Z'}\partial_=\phi^{A'Y'}\partial_{\neq}\phi^{B'Z'} + i\epsilon_{AB}\epsilon_{Y'Z'}\chi_-^{AY'}\partial_{\neq}\chi_-^{BZ'} \\ & + i\delta_{IJ}\zeta_+^I\partial_=\zeta_+^J + iA_{IJAY}\partial_= X^{AY}\zeta_+^I\zeta_+^J + 2iA_{iJAY}\partial_= X^{AY}\zeta_+^i\zeta_+^J \\ & - im\epsilon_{B'C'}v_i^a E_{YY'}^a\zeta_+^i\phi^{C'Y'}\psi_-^{B'Y} - im\epsilon_{B'C'}u_I^a E_{YY'}^a\zeta_+^I\phi^{C'Y'}\psi_-^{B'Y} \\ & - imu_I^a B_{BY'}^a\zeta_+^I\chi_-^{BY'} - \frac{m^2}{8}\epsilon^{AB}\epsilon_{C'D'}B_{AY'}^a B_{BZ'}^a\phi^{C'Y'}\phi^{D'Z'}\Big\} \ , \end{aligned} \tag{3.7}$$

where $A_{IJAY} = u_I^a\partial u_J^a/\partial X^{AY}$ and $A_{iJAY} = v_i^a\partial u_J^a/\partial X^{AY}$.

The classical low energy effective action is simply obtained by considering the most general action possible which is compatible with all of the symmetries of the theory. To calculate the effective action quantum mechanically we will integrate over the massive fields and discard higher derivative terms. First we notice that because of the nontrivial definition of the massless left handed fermions (3.3), $S_{massless}$ is not (0,4) supersymmetric by itself as it is missing a four fermion interaction term. The problem is rectified by noting that there is a tree graph, with a single internal $\phi^{A'Y'}$ field propagating, which contributes to the low energy effective action. In order to avoid the singular behaviour of the propagator at zero momentum, when calculating this graph it is helpful to use the massive propagator for $\phi^{A'Y'}$, obtained from the last term in (3.7) .

At this point it is useful to write, using (2.3) ,

$$B^a_{AY'} B^a_{BZ'} = \epsilon_{AB} \epsilon_{Y'Z'} \Omega$$

where

$$\Omega(X) = \frac{1}{4k'} \epsilon^{AB} \epsilon^{Y'Z'} B^a_{AY'} B^a_{BZ'} \ . \tag{3.8}$$

The last term in (3.7) becomes

$$-\frac{m^2}{4} \epsilon_{A'B'} \epsilon_{Y'Z'} \Omega \phi^{A'Y'} \phi^{B'Z'} \ ,$$

and hence Ω can be interpreted as the X^{AY} dependent mass for $\phi^{A'Y'}$. The tree graph can then be seen to contribute the four fermion term

$$-\frac{1}{2} \zeta^i_+ \zeta^j_+ F^{ij}_{A'YB'Z} \psi^{A'Y}_- \psi^{B'Z}_-$$

where

$$F^{ij}_{A'YB'Z} = 4 \epsilon_{A'B'} \epsilon^{Y'Z'} \Omega^{-1} v^a_i E^a_{(Y|Y'} v^b_j E^b_{|Z)Z'} \ , \tag{3.9}$$

which we will later relate to the field strength tensor.

We may now discard all vertices with only one massive field in (3.7) and examine the one loop contributions to the effective action. Inspection of the quadratic terms in $S_{massive}$ shows there are no contributions to the gauge connection in (3.5).

Furthermore, of all the other possible contributions, only those corresponding to the effective potential do not involve higher derivatives. To calculate this we simply set $\partial_= X^{AY} = \partial_{\neq} X^{AY} = \psi_-^{A'Y} = 0$. Thus only the last two terms in (3.7) need be considered (we no longer use a massive propagator for $\phi^{A'Y'}$). The calculation of the effective potential then receives the standard bosonic and fermionic contributions (in Euclidean momentum space)

$$V_{eff}(\text{bosons}) = \frac{\alpha'}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n}\int d^2p \ \mathrm{Tr}\left[\frac{\epsilon_{C'D'}\epsilon^{AB}B^a_{AY'}B^a_{BZ'}}{4p^2/m^2}\right]^n \tag{3.10}$$

and

$$V_{eff}(\text{fermions}) = -\frac{\alpha'}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n}\int d^2p \ \mathrm{Tr}\left[\frac{u^a_I u^{bI} B^a_{CY'}B^b_{DZ'}}{2p^2/m^2}\right]^n \ . \tag{3.11}$$

Now the definition (3.8) yields the following expressions

$$\epsilon_{C'D'}\epsilon^{AB}B^a_{AY'}B^a_{BZ'} = 2\epsilon_{C'D'}\epsilon_{Y'Z'}\Omega$$

and

$$u^a_I u^{bI} B^a_{CY'}B^b_{DZ'} = \epsilon_{CD}\epsilon_{Y'Z'}\Omega \ .$$

Therefore (3.10) is completely canceled by (3.11) and there is no contribution to the effective potential. This is in accord with the full one loop effective potential calculation performed in [6]. There it was found that the bosonic and fermionic contributions cancel when the tensor $C^a_{AA'}$ is linear in the fields, as it is here for the massive fields.

From the above analysis we conclude that the effective quantum action of the massless fields is

$$\begin{aligned} S_{effective} = \int d^2x \Big\{ & \epsilon_{AB}\epsilon_{YZ}\partial_= X^{AY}\partial_{\neq}X^{BZ} + i\epsilon_{A'B'}\epsilon_{YZ}\psi_-^{A'Y}\partial_{\neq}\psi_-^{B'Z} \\ & + i\zeta_+^i(\delta_{ij}\partial_=\zeta_+^j + A_{ijAY}\partial_= X^{AY}\zeta_+^j) - \frac{1}{2}\zeta_+^i\zeta_+^j F^{ij}_{A'YB'Z}\psi_-^{A'Y}\psi_-^{B'Z}\Big\} \ . \end{aligned} \tag{3.12}$$

This is simply the action of the general (0,4) supersymmetric nonlinear sigma model [5], although the right handed superpartners of X^{AY} are 'twisted'. As with the original theory (2.1), the low energy effective theory (3.12) admits a $(0,1)$ superfield form. Introducing the superfield $\Lambda^i_+ = \zeta^i_+ + \theta^- F^i$ then allows us (after removing F^i by its equation of motion) to express (3.12) as

$$
\begin{aligned}
S_{effective} = -i \int d^2x d\theta^- \Big\{ & \epsilon_{AB}\epsilon_{YZ} D_- \mathcal{X}^{AY} \partial_{\neq} \mathcal{X}^{BZ} \\
& -i\Lambda^i_+ (\delta_{ij} D_- \Lambda^j_+ + A_{ijAY} D_- \mathcal{X}^{AY} \Lambda^j_+) \Big\} \ ,
\end{aligned}
\tag{3.13}
$$

provided that $F^{ij}_{A'YB'Z}$ satisfies

$$
F^{ij}_{A'YB'Z} = I^A{}_{A'} I^B{}_{B'} F^{ij}_{AYBZ} \tag{3.14}
$$

where F^{ij}_{AYBZ} is the curvature of the connection (3.6),

$$
F^{ij}_{AYBZ} = \partial_{AY} A_{ijBZ} - \partial_{BZ} A_{ijAY} + A_{ikAY} A_{kjBZ} - A_{ikBZ} A_{kjAY} \ .
$$

This is just the familiar constraint on (0,4) models that the field strength be compatible with the complex structure [5]. Furthermore it is not hard to check that $S_{effective}$ does indeed possess the full on-shell (0,4) supersymmetry (the superspace formulation (3.13) only ensures (0,1) supersymmetry) precisely when (3.14) is satisfied.

For the $k = k' = 1$, $n = 8$ model above it is straightforward to determine the non zero components of v^a_i and u^a_I as

$$
\begin{aligned}
v^{YY'}_{ZZ'} &= \frac{\rho}{\sqrt{\rho^2 + X^2}} \delta^Y_Z \delta^{Y'}_{Z'} \qquad & v^{AY'}_{ZZ'} &= -\frac{\sqrt{2}}{\sqrt{\rho^2 + X^2}} X^A{}_Z \delta^{Y'}_{Z'} \\
u^{YY'}_{BZ'} &= \frac{\sqrt{2}}{\sqrt{\rho^2 + X^2}} X_B{}^Y \delta^{Y'}_{Z'} \qquad & u^{AY'}_{BZ'} &= \frac{\rho}{\sqrt{\rho^2 + X^2}} \delta^A_B \delta^{Y'}_{Z'} \ ,
\end{aligned}
\tag{3.15}
$$

and hence the mass term (3.8) is

$$
\Omega = (X^2 + \rho^2) \ .
$$

The gauge field A_{ijAY} obtained from (3.15) is simply that of a single instanton on the manifold $\mathbf{R}^4$

$$A_{AX}^{YY'ZZ'} = -\epsilon^{Y'Z'} \frac{(\delta_X^Z X_A{}^Y + \delta_X^Y X_A{}^Z)}{(\rho^2 + X^2)} \,, \tag{3.16}$$

and the four fermion vertex (3.9) is

$$F_{A'YB'Z}^{TT'UU'} = \frac{4\rho^2}{(X^2+\rho^2)^2} \epsilon_{A'B'} \epsilon^{T'U'} \delta_{(Y}^T \delta_{Z)}^U \,, \tag{3.17}$$

which is precisely the field strength of an instanton, justifying our presumptuous notation, and can be easily seen to satisfy (3.14).

3.0.3 Anomalies

So far we have ignored the possibility of anomalies in the quantum theory. While the original theory (2.1) is simply a linear sigma model and therefore possesses no anomalies, this is not the case for the effective theory (3.12). It is well known that off-shell $(0,4)$ supersymmetric sigma models suffer from chiral anomalies which break spacetime gauge and coordinate invariance, unless the gauge field can be embedded in the spin connection of the target space. In addition, working in $(0,1)$ superspace only ensures that $(0,1)$ supersymmetry is preserved and there are also extended supersymmetry anomalies where the $(0,4)$ supersymmetry is not preserved. We therefore expect that we will have to add finite local counter terms to (3.12) at all orders of perturbation theory so as to cancel these anomalies. This requires that the spacetime metric and antisymmetric tensor fields become non trivial at higher orders of α', while on the other hand the gauge connection is unaffected [9].

An alternative way of viewing this is to note that although the action (3.12) is classically conformally invariant, when quantized it may not be ultraviolet finite and hence break scale invariance. There is a power counting argument which asserts that off-shell $(0,4)$ supersymmetric models are ultraviolet finite [8]. This argument is further complicated by sigma model anomalies and it has been stated that only the non chiral models are ultraviolet finite. Indeed while the $(0,4)$ theory is one loop finite there is a two loop contribution of the form $\mathrm{Tr}(R^2 - F^2)$ [6] which certainly does not vanish for the model (3.13). In general this leads to non vanishing β-functions and a renormalization group flow which we must then take into account when determining the conformal fixed point. However, in models with $(0,4)$ supersymmetry the non vanishing β-functions can be canceled by redefining the spacetime fields at each order of α', in such a way as to ensure that supersymmetry is preserved in perturbation theory [9]. This has been well studied and verified up to three loops. Thus the ultraviolet divergences which arise in the quantization of off-shell $(0,4)$ models are really an artifact of the use of a renormalization scheme which does not preserve the supersymmetry. The off-shell $(0,4)$ models are ultraviolet finite in an appropriate renormalization scheme.

However the model here has only on-shell $(0,4)$ supersymmetry and these arguments do not immediately apply. At least in the $k = k' = 0$ case however the gauge group $SO(4) \cong SU(2) \times SU(2)$ contains the subgroup $Sp(1) \cong SU(2)$ which admits three complex structures obeying the algebra of the quaternions. This endows the target space of the left handed fermions with a hyper Kahler structure and facilitates an off-shell formulation using constrained superfields [10]. We may therefore expect that it is ultraviolet finite in the same manner as the off-shell models described above.

In [9] the necessary field redefinitions were derived to order α'^2 for $(0,4)$ supersymmetric sigma models. Both the target space metric and antisymmetric tensor field strength receive corrections to all orders in α'. Howe and Papadopoulos found that in order to maintain $(0,4)$ supersymmetry in perturbation theory the target space metric (which is flat here at the classical level) must receive corrections in the form of a conformal factor

$$\epsilon_{AB}\epsilon_{YZ} \rightarrow \left(1 - \frac{3}{2}\alpha' f - \frac{3}{16}\alpha'^2 \triangle f + ...\right)\epsilon_{AB}\epsilon_{YZ} \ . \tag{3.18}$$

They also showed, up to three loop order, that these redefinitions cancel the ultraviolet divergences which arise when one renormalizes (3.12) using standard $(0,1)$ superspace methods, which do not ensure (0,4) supersymmetry is preserved perturbatively. In addition, the antisymmetric field strength tensor becomes $H = -\frac{3}{4}\alpha' * df$ so as to cancel the gauge anomaly $dH = -\frac{3\alpha'}{4}\mathrm{Tr}F \wedge F$. Furthermore there will be no corrections to the instanton gauge field.

For the instanton number one model considered here, Howe and Papadopoulos give the function f as

$$f = -\triangle \ln(X^2 + \rho^2) \ .$$

It is a simple matter to calculate the conformal factor (3.18) and hence the Target space metric as

$$g_{ABYZ} = \left(1 + 6\alpha' \frac{X^2 + 2\rho^2}{(X^2+\rho^2)^2} - 18\alpha'^2 \frac{\rho^4}{(X^2+\rho^2)^4} + \ldots\right) \epsilon_{AB}\epsilon_{YZ} \ . \qquad (3.19)$$

To order α' this is the solution of Callan, Harvey and Strominger [11] obtained by solving the first order equations of motion of the 10 dimensional heterotic string (although with $n = 6$ rather than $n = 8$ in their notation). Thus the target space has been curved around the instanton by stringy effects but remains non singular so long as $\rho \neq 0$. The case $\rho = 0$ is of great interest as it may provide a string theoretic compactification of instanton moduli space. We will briefly discuss this in the next section.

4. Concluding Remarks

In the above we found the order α'^2 corrections to the low energy effective action of the ADHM sigma model. Such solutions have been discussed before [11] and we agree with their solution to first order. In our calculations we have expanded in the parameter

$$\alpha' \Omega^{-1} = \frac{\alpha'}{X^2 + \rho^2}$$

and hence our approximations are valid for all X if $\rho^2 \gg \alpha'$ and for large X even if ρ is small. An interesting question raised is what are the stringy corrections to the classical instanton in the extreme case that its size vanishes? One can see from (3.19) that the order α' corrections persist when $\rho = 0$ so the effective theory is non trivial. It has been conjectured that there should be a $(4,4)$ supersymmetric sigma model for instantons of zero size [3,4] which could be constructed from a massive linear $(4,4)$ supersymmetric model. In [4] the conditions for the ADHM model to possess full (4,4) supersymmetry in the infrared limit were derived. There it was found that the metric must be conformally flat, with the metric satisfying Laplace's equation. This is in agreement with what we have found here in the $\rho = 0$ case above (see (3.18) and (3.19)) and lends some additional support to the conjecture.

If we start with the linear sigma model (2.1) with $\rho = 0$ the λ_+^{AY} fields are massless and decouple from the theory. The vacuum states are defined by $X^{AY} = 0$ or $\phi^{A'Y'} = 0$ and there is a symmetry between X^{AY} and $\phi^{A'Y'}$. Let us assume we choose the $\phi^{A'Y'} = 0$ vacuum. Then as before the fields X^{AY} and $\psi_-^{AY'}$ are massless and the $\phi^{A'Y'}$, $\chi_-^{AY'}$ and $\lambda_+^{YY'}$ fields all have masses $m\sqrt{X^2}$. Upon integrating out the massive fields we would simply obtain a free field theory, which trivially possesses $(4,4)$ supersymmetry. At the degenerate vacuum $X^{AY} = 0$ however, all fields are massless and there is a single interaction term $m\lambda_{+YY'}\phi_{A'}{}^{Y'}\psi_-^{A'Y}$. Thus the moduli space of vacua does not have a manifold structure. For $X^{AY} \neq 0$ the vacuum states are simply $\mathbf{R}^4$ but at the point $X^{AY} = 0$ lies another entire copy of $\mathbf{R}^4$ (parameterized by the $\phi^{A'Y'}$). This odd state of affairs is smoothly resolved if we first construct the effective theory and then take the limit of vanishing instanton size.

Let us now take the limit $\rho \to 0$ of the effective action (3.12). It should be noted that the Yang-Mills instanton has shrunk to zero size but it has not disappeared in the sense that the topological charge remains equal to one. Unfortunately our expressions are not a priori valid near $X = 0$. Nevertheless we will try to shed some some light about what the complete string theory solution could be in that region. When ρ vanishes both the field strength (3.9) and the $O(\alpha')$ sigma model anomaly vanish. We are however, still left with a non trivial metric and anti symmetric tensor. It is reasonable to assume then that all the anomalies are canceled by these. The metric then has the exact conformal factor

$$g_{\mu\nu} = \left(1 - \frac{3\alpha'}{2} f\right) \delta_{\mu\nu} \ , \tag{4.1}$$

and antisymmetric field

$$H = -\frac{3\alpha'}{4} \epsilon_{\mu\nu\rho\lambda} \partial^\lambda f \ , \tag{4.2}$$

where $f = -4/X^2$ and we have switched to a more convenient notation. This geometry is similar to the one discussed by Callan, Harvey and Strominger [11], although the anti symmetric field is not the same and leads to a different interpretation in the limit $X^2 \to 0$ as we will shortly see. the target space is non singular, asymptotically flat and has a semi-infinite tube with asymptotic radius $\sqrt{6\alpha'}$, centered around the instanton. That is to say the apparent singularity at $X = 0$ in (3.19) is pushed off to an "internal infinity" down the infinite tube. Thus the problematic $X^{AY} = 0$ vacua are pushed an infinite distance away and the manifold structure is preserved. The resolution of this description with the non manifold picture described above has been discussed by Witten [12].

In the limit $X^2 \ll \alpha'$ the modified spin connection with torsion becomes, where α, β are vierbein indices,

$$\begin{aligned} \omega_\mu^{(-)\alpha\beta} &\equiv \omega_\mu^{\alpha\beta} + H_\mu{}^{\alpha\beta} \\ &= -(\delta_\mu^\alpha \delta_\nu^\beta - \delta_\nu^\alpha \delta_\mu^\beta + \epsilon_{\mu\nu}{}^{\alpha\beta}) \frac{X^\nu}{X^2} \ , \end{aligned} \tag{4.3}$$

which is a flat connection! That is to say far down the infinite tube the torsion parallelizes the manifold (which is asymptotically S^3). The gauge connection is also flat (for $X^{AY} \neq 0$) and can therefore be embedded into the generalized spin connection (4.3). The low energy effective theory therefore possesses $(4,4)$ supersymmetry in the region $X^2 \to 0$. This supports our assumption that the anomalies are canceled and the expressions (4.1) and (4.2) are exact, at least in this region.

For the region $X^2 \gg \alpha'$ our perturbative expansion is valid and the the theory only possesses $(0,4)$ supersymmetry since the gauge connection can not be embedded into the spin connection. Although in a similar spirit in the limit $X^2 \to \infty$ the curvatures vanish and the theory is free and again has $(4,4)$ supersymmetry. In a sense then the $\rho = 0$ ADHM instanton can be viewed as a soliton in the space of string vacua interpolating between two $(4,4)$ supersymmetric sigma models, just as the target space can be viewed as interpolating between two supersymmetric ground states of supergravity [13].

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