

Estimation of parameters of gravitational waves from coalescing binaries

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In this paper we deal with the measurement of the parameters of the gravitational wave signal emitted by a coalescing binary signal. We present the results of Monte Carlo simulations carried out for the case of the initial LIGO, incorporating the first post-Newtonian corrections into the waveform. Using the parameters so determined, we estimate the direction to the source. We stress the use of the time-of-coalescence rather than the time-of-arrival of the signal to determine the direction of the source. We show that this can considerably reduce the errors in the determination of the direction of the source.

A major source for the planned interferometric gravitational wave detectors like LIGO [1] and VIRGO [2] is the radiation emitted by a binary system of stars during the last few minutes of its evolution just before the two stars coalesce. The interferometers are expected to have high enough sensitivity to detect sources at cosmological distances. Since the waveform emitted during a binary coalescence can be modeled fairly accurately it is possible to enhance the visibility or the ‘signal-to-noise’ ratio¹, (defined later in the text in eq. 15) of the signal in noisy data by employing such powerful techniques such as Weiner filtering [3, 4]. With the aid of such methods one essentially compares signal energy to that of the noise as opposed to the case of a burst signal where one has to contend with the signal power vis-a-vis noise power [4]. The fact that the signal waveform can be predicted very well also implies that the parameters of the waveform can be estimated to a high accuracy. For instance, by tracking the radiation emitted by a neutron star-neutron star² (NS-NS) binary starting from 10 Hz. to say 1kHz., the masses can be determined to better than a few percent [5, 6, 7, 8, 9, 10]. Observation of several coalescence events could be potentially used to determine Hubble parameter to an accuracy of $\sim 10\%$ [11]. Once in a while, when an event produces a very high SNR (≥ 40) it would be possible to infer the presence of gravitational wave tails in the signal waveform and test general relativity in the strongly non-linear regime [12, 13]. These are, but a modest list of interesting physical and astrophysical information that the observation of gravitational waves is expected to bring forth.

In the recent past the problem of optimal detection and estimation of parameters of the binary has gained considerable attention, thanks to the funding of the American LIGO and Franco-Italian VIRGO projects. The chirp waveform is expected to be detected with the aid of Weiner filtering, wherein the detector output is correlated with a set of templates which together span the relevant range of the parameters of the binary waveform. When a signal is present in the detector output, the filter matched to that particular signal will on the average obtain the largest possible correlation with the detector output amongst all other filters: thus enabling detection as well as estimation of parameters. The parameters of the template that obtains the maximum correlation is an unbiased estimate of the actual signal parameters. Of course, the detector output also contains noise which can effect the correlation thereby giving rise to spurious maxima even when the parameters of the template and those of the signal are mismatched. Thus, the measured parameters are in error. These errors tend to reduce with increasing SNR.

Based on analytical computation of the covariance matrix of the errors in the estimation of parameters and the covariances among them (for the advanced LIGO), it is now generally agreed that the chirp mass (which is a combination of the masses of the binary stars) can be determined at a relative accuracy $\sim .1\%$ - 1% at a SNR of 10. The reduced mass, as it turns out, will have a much larger error especially if the spins are large. In the computation of the covariance matrix one makes the crucial assumption of high SNR. As a result the covariance matrix only gives a bound on the errors in the estimation of parameters and a more rigorous or a different method of computing the variances and covariances is in order. Alternatively one can determine the errors through the Monte Carlo simulation method. The basic idea here is to mimic the actual detection problem on a computer by adding numerous realizations of the noise to the signal and then passing the resultant time series through a bank of Weiner filters each matched to a gravitational wave signal from a coalescing binary system with a particular set of parameters. The covariance matrix

¹ henceforth denoted as SNR

² In this paper a neutron star is considered to have a mass of $1.4 M_{\odot}$, and a black hole will be considered to have a mass of $10. M_{\odot}$

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can be computed by using the ensemble of measured values of the signal parameters. Such a method is robust in the sense that it does not make any assumptions about the SNR and therefore is much closer to the actual detection problem. It also squarely addresses the difficulties faced in a realistic data analysis exercise such as the finite sampling rate, the finiteness of spacing in between the filters, *etc.* For details on the Monte Carlo method see Balasubramanian, Sathyaprakash and Dhurandhar [9].

In this paper we summarize the outcome, of the first in a series, of Monte Carlo simulations which suggests that the covariance matrix grossly underestimates the errors in the estimation of the parameters by over a factor of 3 at an SNR of 10.

The standard method to deduce the direction to the source of the gravitational radiation is to use the differences in the times-of-arrival in a network of three or more detectors. We point out that since the instant of coalescence of a binary system can be measured to a much greater accuracy than that of the time-of-arrival, the former can be employed to infer the direction of the source at a much lower uncertainty than has been so far thought.

In the point mass approximation the restricted first-order post-Newtonian gravitational radiation emitted by a binary system of stars induces a strain in the detector which can be written as

$$h(t) = A(\pi f(t))^{2/3} \cos[\varphi(t)], \quad (1)$$

where $f(t)$ is the instantaneous gravitational wave frequency, which at this level of approximation is equal to twice the orbital frequency and is implicitly given by:

$$t - t_a = \tau_0 \left[1 - \left(\frac{f}{f_a} \right)^{-8/3} \right] + \tau_1 \left[1 - \left(\frac{f}{f_a} \right)^{-2} \right] \quad (2)$$

and $\varphi(t)$ is the phase of the signal given by

$$\varphi(t) = \frac{16\pi f_a \tau_0}{5} \left[1 - \left(\frac{f}{f_a} \right)^{-5/3} \right] + 4\pi f_a \tau_1 \left[1 - \left(\frac{f}{f_a} \right)^{-1} \right] + \Phi \quad (3)$$

where t_a and Φ are, respectively, the so called time-of-arrival and phase at which the signal reaches a frequency f_a ; Φ is a constant depending on the relative orientations of the binary orbit and the arms of the interferometer; τ_0 and τ_1 are constants, having dimensions of time, depending on the masses m_1 and m_2 of the binary system. They are referred to as chirp times and are given by,

$$\tau_0 = \frac{5}{256} \mathcal{M}^{-5/3} (\pi f_a)^{-8/3}, \quad (4)$$

$$\tau_1 = \frac{5}{192\mu(\pi f_a)^2} \left(\frac{743}{336} + \frac{11}{4}\eta \right), \quad (5)$$

where μ is the reduced mass, η is the reduced mass divided by the total mass M , and $\mathcal{M} = \mu^{3/5} M^{2/5}$ is the chirp mass. In this level of approximation the post-Newtonian waveform is thus characterised by a set of five parameters: The amplitude parameter A , the time-of-arrival t_a , the phase at the time of arrival Φ , and the two masses m_1 and m_2 . We shall find below a more convenient parameterization of the waveform.

The Fourier transform of the waveform in the stationary phase approximation is given by

$$\tilde{h}(f) = \mathcal{A} f^{-7/6} \exp \left[i \sum_{\mu=1}^4 \psi_{\mu}(f) \lambda^{\mu} - i \frac{\pi}{4} \right] \quad (6)$$

where \mathcal{A} is a normalization constant and is fixed by specifying the SNR, and $\psi_{\mu}(f)$ are functions only of frequency given by,

$$\psi_1 = 2\pi f, \quad (7)$$

$$\psi_2 = -1, \quad (8)$$

$$\psi_3 = 2\pi f - \frac{16\pi f_a}{5} + \frac{6\pi f_a}{5} \left(\frac{f}{f_a} \right)^{-5/3}, \quad (9)$$

$$\psi_4 = 2\pi f - 4\pi f_a + 2\pi f_a \left(\frac{f}{f_a} \right)^{-1}. \quad (10)$$

Here λ^μ are related to the set of parameters introduced earlier and are given by,

$$\lambda^0 = \mathcal{A} \quad \text{and} \quad \lambda^\mu = \{t_C, \Phi_C, \mathcal{M}, \eta\}, \quad \mu = 1, \dots, 4, \quad (11)$$

where,

$$t_C = t_a + \tau_0 + \tau_1 \quad (12)$$

$$\Phi_C = \Phi - 2\pi f_a t_a + \frac{16\pi f_a \tau_0}{5} + 4\pi f_a \tau_1 \quad (13)$$

For $f < 0$, the Fourier transform can be obtained by the relation $\tilde{h}(-f) = \tilde{h}^*(f)$, obeyed by all real functions $h(t)$.

Central to the computation of the covariance matrix is the scalar product defined over the space of the signal waveforms. Given waveforms $h(t; \boldsymbol{\lambda})$ and $g(t; \boldsymbol{\lambda})$ their scalar product is defined by,

$$\langle h, g \rangle = \int_{f_a}^{\infty} \frac{\tilde{h}(f; \lambda^\mu) \tilde{g}^*(f; \lambda^\mu)}{S_n(f)} df + \text{c.c.} \quad (14)$$

where $S_n(f)$ is the two sided detector noise power spectral density. The scalar product defines a norm on the vector space. The SNR can now be defined for a signal $h(t; \boldsymbol{\lambda})$ when it is passed through the matched filter as the norm of the signal *i.e.*,

$$\rho = \langle h, h \rangle^{1/2}. \quad (15)$$

The covariance matrix $C_{\mu\nu}$ is the inverse of the Fisher information matrix $\Gamma_{\mu\nu}$ given by,

$$\Gamma_{\mu\nu} = \left\langle \frac{\partial \mathbf{s}}{\partial \lambda^\mu}, \frac{\partial \mathbf{s}}{\partial \lambda^\nu} \right\rangle; \quad C_{\mu\nu} = \Gamma^{-1}{}_{\mu\nu}. \quad (16)$$

The diagonal elements of the covariance matrix provide us an estimate of the variances to be expected in the measured values of the parameters in a given experiment. In Fig. 1 we have shown the behaviour of the 1σ uncertainties (square root of the variances) in the estimation of parameters t_a, τ_0, τ_1 and τ_C , as a function of the SNR. As is well known the covariance matrix predicts that the errors fall off in inverse proportion to the SNR. It is to be noted that this is valid only in the ‘high’ SNR limit.

In the actual detection problem the detector output $x(t)$ is filtered through a host of search templates corresponding to *test* parameters, ${}_t\boldsymbol{\lambda}$ and the template that obtains the maximum correlation with the output gives us the measured values ${}_m\boldsymbol{\lambda}$ of the signal. These measured values will in general differ from the actual signal parameters. With the aid of a large number of detectors one obtains an ensemble of measured values ${}_m\boldsymbol{\lambda}$. The average of such an ensemble provides us with an estimate ${}_e\boldsymbol{\lambda}$ and the variances and covariances σ_λ computed using ${}_m\boldsymbol{\lambda}$ helps us in deducing the errors in the estimation and how the different parameters are correlated:

$${}_e\boldsymbol{\lambda} = \overline{{}_m\boldsymbol{\lambda}}, \quad \sigma_\lambda^2 = \overline{({}_m\boldsymbol{\lambda} - \overline{{}_m\boldsymbol{\lambda}})^2}, \quad \Sigma^{\mu\nu} = \frac{\overline{{}_m\lambda^\mu \cdot {}_m\lambda^\nu}}{\sigma_\mu \sigma_\nu}, \quad (17)$$

where an overbar denotes an average over an ensemble and $\Sigma_{\mu\nu}$ the correlation coefficients. In reality we will have only a few detectors and hence a numerical simulation needs to be carried out to deduce the errors in the estimation of parameters. We have carried out such a numerical simulation by using in excess of 5000 realizations of detector noise. Thus, the results of our simulations are statistically significant, the errors in the estimation of various statistical quantities such as the mean and the variance are negligible and hence we do not plot the error bars in our curves. In Figure 1 the dotted line corresponds to the behaviour of the errors in the estimation of various parameters computed at several values of the SNR. At low values of the SNR ($\rho \sim 10$) there is a significant departure of the observed errors from those predicted by the covariance matrix. However at an SNR of ≥ 30 the two curves merge together indicating the validity of the covariance matrix at high enough SNRs. Since most of the events which the interferometric detectors will observe are expected to have an SNR less than 10, we conclude that the accuracy in the determination of the parameters is not as high as was thought to be, based on the values of the covariance matrix. Detailed analysis suggests that this discrepancy is larger when higher post-Newtonian corrections are taken into account. Consequently a more exhaustive analysis than has been reported here or elsewhere [9] is in order. We are in the process of carrying out simulations by taking a second order post-Newtonian waveform and including other physical effects such as the eccentricity of the binary [10].

With reference to Figure 1 we point out that the instant of coalescence t_C can be determined to an accuracy much better than the time-of-arrival t_a . Typically σ_{t_C} is a factor of 50 less than σ_{t_a} . Consequently with the aid of t_C we can fix the direction to the source at a much higher accuracy than with t_a as can be seen from equation (12), t_C is the sum of τ_0 , τ_1 and t_a , and the errors in these parameters tend to cancel because of the presence of negative covariances. It is to be noted, however, that as of now we do not know the orbit of the binary accurately enough, to predict the exact instant of the coalescence. In fact the frequency cutoff imposed by the onset of the plunge orbit will make it difficult to calculate t_C . Moreover, for a realistic detector the noise increases with the frequency beyond about 150 Hz. and the frequency which will be of more interest to us is the one where the power spectrum of the signal divided by the power spectrum of the noise in the detector reaches a maximum. This of course assumes that all the detectors to used for direction measurement are identical.

Let t denote a convenient time parameter which if measured at three detectors gives the direction to the source. For our purpose we take this parameter to be either t_C or t_a . We will assume a very simple configuration of three identical detectors placed at the vertices of an equilateral triangle on the equatorial plane of the earth. We will take the separation between any two detectors as L and consequently the maximum time delay induced by L as $\Delta = L/c$ where c is the velocity of light. Figure 2 illustrates such a network and the coordinate system used. The X axis passes through the detectors labelled 1 and 2 and has its origin at the former. The positive Y axis is chosen as shown and the Z axis is perpendicular to this plane and forms a right handed coordinate system with the other axes. Given the values t_1 , t_2 and t_3 as the measured values of t in the three detectors respectively we can deduce the direction to the source via the time delays $\gamma = t_2 - t_1$ and $\delta = t_3 - t_1$. As the noise in each detector is uncorrelated with the noise in the others t_1, t_2 and t_3 are uncorrelated and as we assume that all the detectors are identical then the errors in t in all the detectors will be the same. However, γ and δ will have non-zero covariances. Thus,

$$\sigma_{t_1} = \sigma_{t_2} = \sigma_{t_3} = \sigma; \quad \sigma_\gamma = \sigma_\delta = \sqrt{2}\sigma. \quad (18)$$

The angles ϕ and θ are related to γ and δ by,

$$\phi = \tan^{-1} \left[\frac{2\delta - \gamma}{\sqrt{3}\gamma} \right], \quad (19)$$

and,

$$\theta = \sin^{-1} \left[\frac{2\gamma\sqrt{\gamma^2 + \delta^2 - \gamma\delta}}{\Delta(2\delta - \gamma)} \right]. \quad (20)$$

The errors in the measurement of the time parameter will induce errors in the measurement of the angles. Assuming the errors in the time parameter to be small we can write the expressions for σ_ϕ and σ_θ as,

$$\sigma_\phi = \frac{\sigma}{\Delta} g_\phi(\theta, \phi) \quad \text{and} \quad \sigma_\theta = \frac{\sigma}{\Delta} g_\theta(\theta, \phi), \quad (21)$$

where,

$$g_\phi(\theta, \phi) = \frac{1}{\sin \theta} \quad \text{and} \quad g_\theta(\theta, \phi) = \sqrt{\frac{\cos^4 \phi - \cos^2 \phi - 1}{-1 + 3 \cos^2 \phi - 2 \cos^4 \phi - \cos^2 \phi \cos^2 \theta \sin^2 \theta}}. \quad (22)$$

The factors $g_\phi(\theta, \phi)$ and $g_\theta(\theta, \phi)$ are typically of order unity for most directions. For earth based detectors the value of Δ is ≈ 15 ms. This certainly rules out the use of t_a to determine the direction as the error in this parameter even for high SNRs is much more than Δ . Though for the initial LIGO we can use t_C , the errors at an SNR of 10 will be around two degrees which is too large to make an optical identification of the source. In order to determine the direction to arcsecond resolution one needs the sensitivity of the advanced LIGO.

We would like to end this paper with the remark that though the inclusion of higher order post-Newtonian corrections is expected to bring down the precision with which the astrophysical parameters can be determined, they are not a cause for worry for the detection problem. Even at the second post-Newtonian correction the signal can essentially be detected with the aid of a one-dimensional lattice of templates [9]. However, more extensive numerical Monte Carlo simulations need to be performed to gain further insight into the estimation problem.

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FIG. 1. Dependence of the errors in the estimation of parameters of the post-Newtonian waveform *i.e.* $\{\sigma_{\tau_0}, \sigma_{\tau_1}, \sigma_{t_a}, \sigma_{t_C}\}$ as a function of SNR. The solid line represents the theoretically computed errors whereas the dotted line represents the errors obtained through Monte Carlo simulations. The simulations have been carried out for case of a $10M_{\odot} - 1.4M_{\odot}$ binary system. The errors in the parameters are expressed in ms.

FIG. 2. This figure illustrates the network of three detectors and the choice of the coordinate system employed. The Z axis is perpendicular to the plane of the paper and points upward. The angle θ is defined as the angle which the direction vector makes with the positive Z axis and ϕ is defined on the $X - Y$ plane as shown in the figure.