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Quantum Cosmology and the Structure of Inflationary Universe

Andrei Linde [1]

Department of Physics, Stanford University,
Stanford, CA 94305–4060, USA

Abstract

In this review I will consider several different issues related to inflation. I will begin with the wave function of the Universe. This issue is pretty old, but recently there were some new insights based on the theory of the self-reproducing inflationary universe. Then we will discuss stationarity of inflationary universe and the possibility to make predictions in the context of quantum cosmology using stochastic approach to inflation. Returning to more pragmatic aspects of inflationary theory, we will discuss inflationary models with $\Omega \neq 1$. Finally, we will describe several aspects of the theory of reheating of the Universe based on the effect of parametric resonance.

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1 Wave function of the Universe

Investigation of the wave function of the Universe goes back to the fundamental papers by Wheeler and DeWitt [1]. However, for a long time it seemed almost meaningless to apply the notion of the wave function to the Universe itself, since the Universe is not a microscopic object. Only with the development of inflationary cosmology [2]–[7] it became clear that the whole Universe could appear from a tiny part of space of the Planck length (at least in the chaotic inflation scenario [5]). Such a tiny piece of space can appear as a result of quantum fluctuations of metric, which should be studied in the context of quantum cosmology.

Unfortunately, quantum cosmology is not a well developed science. This theory is based on the Wheeler-DeWitt equation, which is the Schrödinger equation for the wave function of the Universe. However, Bryce DeWitt, one of the authors of this equation, in some of his talks emphasized that he is not particularly fond of it. This equation has many solutions, and at the present time the best method to specify preferable solutions of this equation (i.e. its boundary conditions) is based on the Euclidean approach to quantum gravity. This method is very powerful, but some of its applications are not well justified. In such cases this method may give incorrect answers, but rather paradoxically sometimes these answers appear to be correct in application to some other questions. Therefore it becomes necessary not only to solve the problem in the Euclidean approach, but also to check, using one's best judgement, whether the solution is related to the original problem or to something else. An alternative approach is based on the use of stochastic methods in inflationary cosmology. These methods allows one to understand such effects as creation of inflationary density perturbations, the theory of tunneling, and even the theory of self-reproduction of inflationary universe. Both Euclidean approach and stochastic approach to inflation have their limitations. However, despite all its problems, quantum cosmology is a very exciting science which changed dramatically our point of view on the structure and evolution of the Universe.

We will begin our discussion with the issue of the Universe creation. According to classical cosmology, the Universe appeared from the singularity in a state of infinite density. Of course, when the density was greater than the Planck density $M_{\rm P}^4$ one could not trust classical Einstein equations, but in many cases there is no demonstrated need to study the Universe creation using the methods of quantum theory. For example, in the simplest versions of chaotic inflation scenario [5] inflation, at the classical level, could begin directly in the initial singularity. However, in certain models, such as the Starobinsky model [2] or the new inflationary universe scenario [4], inflation cannot start in a state of infinite density. In such cases one may speculate about the possibility that inflationary universe appears due to quantum tunneling "from nothing."

The first idea how one can describe creation of inflationary universe "from nothing" was suggested in 1981 by Zeldovich [8] in application to the Starobinsky model [2]. His idea was qualitatively correct, but he did not propose any quantitative description of this process. A very important step in this direction was made in 1982 by Vilenkin [9]. He suggested to calculate the

Euclidean action on de Sitter space with the energy density $V(\phi)$,

$$S_E = \int d^4x\sqrt{-g}\left(-\frac{R}{16\pi G} + V(\phi)\right) = -\frac{3M_P^4}{8V(\phi)} \ . \tag{1}$$

This action was interpreted as the action on the tunneling trajectory describing creation of the Universe with the scale factor $a = H^{-1} = \sqrt{\frac{3M_P^2}{8\pi V}}$ from the state with $a = 0$. This would imply that the probability of quantum creation of the Universe is given by

$$\mathcal{P} \propto \exp(-S_E) = \exp\left(\frac{3}{8V(\phi)}\right) . \tag{2}$$

(In the first three sections of this review we use the system of units with the Planck mass $M_P = 1$.) A year later this result received a strong support when Hartle and Hawking reproduced it by a different though closely related method [10]. They argued that the wave function of the "ground state" of the Universe with a scale factor a filled with a scalar field ϕ in the semi-classical approximation is given by

$$\Psi_0(a,\phi) \sim \exp\left(-S_E(a,\phi)\right) \ . \tag{3}$$

Here $S_E(a,\phi)$ is the Euclidean action corresponding to the Euclidean solutions of the Lagrange equation for $a(\tau)$ and $\phi(\tau)$ with the boundary conditions $a(0) = a, \phi(0) = \phi$. The reason for choosing this particular wave function was explained as follows. Let us consider the Green's function of a particle which moves from the point $(0, t')$ to the point $\mathbf{x}, t$:

$$< \mathbf{x}, 0|0, t' >= \sum_n \Psi_n(\mathbf{x})\Psi_n(0)\exp\left(iE_n(t-t')\right) = \int d\mathbf{x}(t)\exp\left(iS(\mathbf{x}(t))\right) \ , \tag{4}$$

where Ψ_n is a complete set of energy eigenstates corresponding to the energies $E_n \geq 0$.

To obtain an expression for the ground-state wave function $\Psi_0(\mathbf{x})$, one should make a rotation $t \to -i\tau$ and take the limit as $\tau \to -\infty$. In the summation (4) only the term $n = 0$ with the lowest eigenvalue $E_0 = 0$ survives, and the integral transforms into $\int dx(\tau)\exp(-S_E(\tau))$. Hartle and Hawking have argued that the generalization of this result to the case of interest in the semiclassical approximation would yield (3).

The gravitational action corresponding to one half of the Euclidean section S_4 of de Sitter space with $a(\tau) = H^{-1}(\phi)\cos H\tau$ $(0 \leq \tau \leq H^{-1})$ is negative,

$$S_E(a,\phi) = -\frac{3\pi}{4}\int d\eta\left[\left(\frac{da}{d\eta}\right)^2 - a^2 + \frac{8\pi V}{3}a^4\right] = -\frac{3}{16V(\phi)} \ . \tag{5}$$

Here η is the conformal time, $\eta = \int \frac{d\tau}{a(\tau)}$. Therefore, according to [10],

$$\Psi_0(a,\phi) \sim \exp\left(-S_E(a,\phi)\right) \sim \exp\left(\frac{3}{16V(\phi)}\right) \ . \tag{6}$$

By taking a square of this wave function one again obtains eq. (2). The corresponding expression has a very sharp maximum as $V(\phi) \to 0$. This suggests that the probability of finding the

Universe in a state with a large field ϕ and having a long stage of inflation should be strongly suppressed. Some authors consider it as a strong argument against the Hartle-Hawking wave function. However, nothing in the 'derivation' of the Hartle-Hawking wave function tells that it describes initial conditions for inflation. The point of view of the authors of this wave function was not quite clear. They have written that their wave function gives the amplitude for the Universe to appear from nothing. On the other hand, Hawking emphasized [11] that "instead of talking about the Universe being created one should just say: the Universe is." This seems to imply that the Hartle-Hawking wave function was not designed to describe initial conditions at the moment of the Universe creation.

Indeed, eq. (2) from the very beginning did not seem to apply to the probability of the Universe creation. The total energy of matter in a closed de Sitter space with $a(t) = H^{-1}\cosh Ht$ is greater than its minimal volume $\sim H^{-3}$ multiplied by $V(\phi)$, which gives the total energy of the Universe $E \gtrsim M_{\rm P}^3/\sqrt{V}$. Thus the minimal value of the total energy of matter contained in a closed de Sitter universe *grows* when V decreases. For example, in order to create the Universe at the Planck density $V \sim M_{\rm P}^4$ one needs no more than the Planckian energy $M_{\rm P} \sim 10^{-5}$ g. For the Universe to appear at the GUT energy density $V \sim M_X^4$ one needs to create from nothing the Universe with the total energy of matter of the order of $M_{\rm Schwarzenegger} \sim 10^2$ kg, which is obviously much more difficult. Meanwhile, eq. (2) suggests that it should be much easier to create a huge Universe with small V but enormously large total energy rather than a small Universe with large V. This seems very suspicious.

There is one particular place where the derivation (or interpretation) of eq. (2) could go wrong. The effective Lagrangian of the scale factor a in (5) has a wrong sign. The Lagrange equations do not know anything about the sign of the Lagrangian, so we may simply change the sign before studying the tunneling. Only after representing the theory in a conventional form can we consider tunneling of the scale factor. But this then gives us the probability of quantum creation of the Universe

$$\mathcal{P} \propto \exp(-|S_E|) = \exp\left(-\frac{3}{8V(\phi)}\right) . \tag{7}$$

This equation predicts that a typical initial value of the field ϕ is given by $V(\phi) \sim 1$ (if one does not speculate about the possibility that $V(\phi) \gg 1$), which leads to a very long stage of inflation.

Originally I obtained this result by the method described above. However, because of the ambiguity of the notion of tunneling from the state $a = 0$, I was not quite satisfied and decided to look at it from the perspective of derivation of the Hartle-Hawking wave function. In this approach the problem of the wrong sign of the Lagrangian appears again, though in a somewhat different form. Indeed, the total energy of a closed Universe is zero, being a sum of the positive energy of matter and the negative energy of the scale factor a.) Thus, the energy E_n of the scale factor is negative, and in order to suppress terms with large negative E_n and to obtain Ψ_0 from (5) one should rotate t not to $-i\tau$, but to $+i\tau$. This gives [12]

$$\Psi_0(a,\phi) \sim \exp\Big(-|S_E(a,\phi)|\Big) \sim \exp\left(-\frac{3}{16V(\phi)}\right) , \tag{8}$$

and

$$P(\phi) \sim |\Psi_0(a,\phi)|^2 \sim \exp\Big(-2|S_E(a,\phi)|\Big) \sim \exp\left(-\frac{3}{8V(\phi)}\right) . \tag{9}$$

Few months later this equation was also derived by Zeldovich and Starobinsky [13], Rubakov [14], and Vilenkin [15] using the methods similar to the first method mentioned above (investigation of tunneling in the theory with the wrong sign of the Lagrangian). The corresponding wave function (8) was called "the tunneling wave function."

An obvious objection against this result is that it may be incorrect to use different ways of rotating t for quantization of the scale factor and of the scalar field. However, the idea that a consistent quantization of matter coupled to gravity can be accomplished by a proper choice of a complex contour of integration may be too optimistic. We know, for example, that despite many attempts to suggest Euclidean formulation for nonequilibrium quantum statistics or for the field theory in a nonstationary background, such formulation does not exist. It is quite clear from (4) that the $t \to -i\tau$ trick would not work if the spectrum E_n were not bounded from below. Absence of equilibrium, of any simple stationary ground state seems to be a typical situation in quantum cosmology. In some cases where a stationary or quasistationary ground state does exist, eq. (2) may be correct, see the next Section. In a more general situation it may be very difficult to obtain any simple expression for the wave function of the Universe. However, in certain limiting cases this problem is relatively simple. For example, at present the scale factor a is very big and it changes very slowly, so one can consider it to be a C-number and quantize matter fields only by rotating $t \to -i\tau$. On the other hand, in the inflationary universe the evolution of the scalar field is very slow; during the typical time intervals $O(H^{-1})$ it behaves essentially as a classical field, so one can describe the process of the creation of an inflationary universe filled with a homogeneous scalar field by quantization of the scale factor a only and by rotation $t \to i\tau$.

Derivation of equations (2), (9) and their interpretation is far from being rigorous, and therefore even now it remains the subject of debate. In our opinion, the Hartle-Hawking wave function describes not the Universe creation, but the fluctuations of the Universe near its de Sitter ground state, under the condition that such a state exists, see next section. Meanwhile the distribution (9) is related to the probability of creation of inflationary universe from nothing (or from the space-time foam). However, the two different derivations of this probability distribution emphasize two slightly different features of the process. Investigation of tunneling should give the probability of quantum creation of the Universe of a size H^{-1} from the Universe with $a = 0$. Meanwhile wave function of the "ground state" should give information about some kind of probability distribution of various Universes in the space-time foam. We will not concentrate here on this subtle difference since we believe that it would bring us far away from the domain of applicability of our approach. Also, we should emphasize again that quantum tunneling is necessary only if one cannot use the classical trajectory. In the Starobinsky model [2], as well as in the new inflationary universe scenario [4], creation of the Universe "from nothing" appears to be one of the most natural mechanisms for inflation to begin. Meanwhile, in the simplest version of chaotic inflation scenario the process of inflation formally may begin directly in the singularity, in a state with infinitely large $V(\phi)$, without any need for quantum tunneling. However, quantum tunneling in that case is possible as well, since for $V(\phi) \sim 1$ the probability of quantum creation of inflationary universe is not exponentially suppressed.

In the next section we will discuss stochastic approach to quantum cosmology. Within this approach equations (2) and (9) can be derived in a much more clear and rigorous way, but they will have somewhat different interpretation.

2 Wave function of the Universe and stochastic approach to inflation

The first models of inflation were based on the standard assumption of the big bang theory that the Universe was created at a single moment of time in a state with the Planck density, and that it was hot and large (much larger than the Planck scale $M_{\rm P}^{-1} = 1$) from the very beginning. The success of inflation in solving internal problems of the big bang theory apparently removed the last doubts concerning the big bang cosmology. Even in our quantum mechanical treatment of the Universe production we still used the standard idea that the Universe as a whole can be described by one scale factor a, and its creation should be considered as a process beginning from $a = 0$. Meanwhile during the last ten years the inflationary theory has broken the umbilical cord connecting it with the old big bang theory, and acquired an independent life of its own. For the practical purposes of description of the observable part of our Universe one may still speak about the big bang. However, if one tries to understand the beginning of the Universe, or its end, or its global structure, then some of the notions of the big bang theory become inadequate.

For example, in the chaotic inflation scenario [5] even without taking into account quantum effects there was no need to assume that the whole Universe appeared from nothing at a single moment of time associated with the big bang, that the Universe was hot from the very beginning and that the inflaton scalar field ϕ which drives inflation originally occupied the minimum of its potential energy. On the other hand, it was found that if the Universe in this scenario contains at least one inflationary domain of a size of horizon ('h-region') with a sufficiently large and homogeneous scalar field ϕ, then this domain will permanently produce new h-regions of a similar type due to quantum fluctuations [16, 17]. This process occurs in the old, new and extended inflation scenario as well [18]–[21]. Thus, instead of a single big bang producing a one-bubble Universe, we are speaking now about inflationary bubbles producing new bubbles, producing new bubbles, *ad infinitum*. In this sense, inflation is not a short intermediate stage of duration $\sim 10^{-35}$ seconds, but a self-regenerating process, which occurs in some parts of the Universe even now, and which will continue without end.

It is extremely complicated to describe an inhomogeneous self-reproducing Universe. Fortunately, there is a particular kind of stationarity of the process of the Universe self-reproduction which makes things more regular. Due to the no-hair theorem for de Sitter space, the process of production of new inflationary domains occurs independently of any processes outside the horizon. This process depends only on the values of the fields inside each h-region of radius H^{-1}. Each time a new inflationary h-region is created during the Universe expansion, the physical processes inside this region will depend only on the properties of the fields inside it, but not on the 'cosmic time' at which it was created.

In addition to this most profound stationarity, which we will call *local stationarity*, there may also exist some simple stationary probability distributions which may allow us to say, for example, what the probability is of finding a given field ϕ at a given point. To examine this possibility one should consider the probability distribution $P_c(\phi, \chi, t)$, which describes the probability of finding the field ϕ at a given point at a time t, under the condition that at the time $t = 0$ the field ϕ at this point was equal to χ. The same function also describes the probability that the scalar field which at time t was equal to ϕ, at some earlier time $t = 0$ was equal to χ.

The probability distribution P_c is in fact the probability distribution per unit volume in *comoving coordinates* (hence the index c in P_c), which do not change during expansion of the Universe. By considering this probability distribution we neglect the main source of the self-reproduction of inflationary domains, which is the exponential growth of their volume. Therefore, in addition to P_c, we introduced the probability distribution $P_p(\phi, \chi, t)$, which describes the probability to find a given field configuration in a unit physical volume [16]. In the situations where one of these distributions can be stationary, we will speak about *global stationarity.*

Let us remember some details of stochastic approach to inflation. Consider the simplest model of chaotic inflation based on the theory of a scalar field ϕ minimally coupled to gravity, with the effective potential $V(\phi)$. If the classical field ϕ is sufficiently homogeneous in some domain of the Universe, then its behavior inside this domain is governed by the equation $3H\dot{\phi} = -dV/d\phi$, where $H^2 = \frac{8\pi V(\phi)}{3}$. Investigation of these equations has shown that in all power-law potentials $V(\phi) \sim \phi^n$ inflation occurs at $\phi > \phi_e \sim n/6$. In the theory with an exponential potential $V(\phi) \sim e^{\alpha\phi}$ inflation ends only if we bend down the potential at some point ϕ_e; for definiteness we will take $\phi_e = 0$ in this theory.

Inflation stretches all initial inhomogeneities. Therefore, if the evolution of the Universe were governed solely by classical equations of motion, we would end up with an extremely smooth Universe with no primordial fluctuations to initiate the growth of galaxies. Fortunately, new density perturbations are generated during inflation due to quantum effects. The wavelengths of all vacuum fluctuations of the scalar field ϕ grow exponentially in the expanding Universe. When the wavelength of any particular fluctuation becomes greater than H^{-1}, this fluctuation stops oscillating, and its amplitude freezes at some nonzero value $\delta\phi(x)$ because of the large friction term $3H\dot{\phi}$ in the equation of motion of the field ϕ. The amplitude of this fluctuation then remains almost unchanged for a very long time, whereas its wavelength grows exponentially. Therefore, the appearance of such a frozen fluctuation is equivalent to the appearance of a classical field $\delta\phi(x)$ that does not vanish after averaging over macroscopic intervals of space and time.

Because the vacuum contains fluctuations of all wavelengths, inflation leads to the creation of more and more perturbations of the classical field with wavelengths greater than H^{-1}. The average amplitude of such perturbations generated during a time interval H^{-1} (in which the

Universe expands by a factor of e) is given by

$$|\delta\phi(x)| \approx \frac{H}{2\pi} \ . \tag{10}$$

The phases of each wave are random. Therefore, the sum of all waves at a given point fluctuates and experiences Brownian jumps in all directions in the space of fields.

One can describe the stochastic behavior of the inflaton field using diffusion equations for the probability distribution $P_c(\phi,t|\chi)$. The first equation is called the backward Kolmogorov equation,

$$\frac{\partial P_c(\phi,t|\chi)}{\partial t} = \frac{H^{3/2}(\chi)}{8\pi^2}\frac{\partial}{\partial\chi}\left(H^{3/2}(\chi)\frac{\partial}{\partial\chi}P_c(\phi,t|\chi)\right) - \frac{V'(\chi)}{3H(\chi)}\frac{\partial}{\partial\chi}P_c(\phi,t|\chi) \ . \tag{11}$$

In this equation one considers the value of the field ϕ at the time t as a constant, and finds the time dependence of the probability that this value was reached during the time t as a result of diffusion of the scalar field from different possible initial values $\chi \equiv \phi(0)$.

The second equation is the adjoint to the first one; it is called the forward Kolmogorov equation, or the Fokker-Planck equation [22],

$$\frac{\partial P_c(\phi,t|\chi)}{\partial t} = \frac{\partial}{\partial\phi}\left(\frac{H^{3/2}(\phi)}{8\pi^2}\frac{\partial}{\partial\phi}\Big(H^{3/2}(\phi)P_c(\phi,t|\chi)\Big) + \frac{V'(\phi)}{3H(\phi)}P_c(\phi,t|\chi)\right) \ . \tag{12}$$

One may try to find a stationary solution of equations (11), (12), assuming that $\frac{\partial P_c(\phi,t|\chi)}{\partial t} = 0$. The simplest stationary solution (subexponential factors being omitted) would be [22, 24, 25]

$$P_c(\phi,t|\chi) \sim \exp\left(\frac{3}{8V(\phi)}\right)\cdot\exp\left(-\frac{3}{8V(\chi)}\right) \ . \tag{13}$$

This looks like a miracle: The first term in this expression is equal to the square of the Hartle-Hawking wave function of the Universe (2), whereas the second one gives the square of the tunneling wave function (9)! And we obtained this result without any ambiguous considerations based on Euclidean approach to quantum cosmology!

At first glance, this result gives a direct confirmation and a simple physical interpretation of both the Hartle-Hawking wave function of the Universe *and* the tunneling wave function. First of all, we see that the distribution of the probability to find the Universe in a state with the field ϕ is proportional to $\exp\left(\frac{3}{8V(\phi)}\right)$. Note that we are speaking here about the state of the Universe rather than the probability of its creation. Meanwhile, the probability that the Universe emerged from the state with the field χ is proportional to $\exp\left(-\frac{3}{8V(\chi)}\right)$. Now we are speaking about the probability that a given part of the Universe was created from the state with the field χ, and the result coincides with our result for the probability of the quantum creation of the Universe, eq. (9).

This would be a great peaceful resolution of the conflict between the two wave functions. However, the situation is much more complicated. In all realistic cosmological theories, in which

$V(\phi) = 0$ at its minimum, the Hartle-Hawking distribution $\exp\left(\frac{3}{8V(\phi)}\right)$ is not normalizable. The source of this difficulty can be easily understood: any stationary distribution may exist only due to compensation of the classical flow of the field ϕ downwards to the minimum of $V(\phi)$ by the diffusion motion upwards. However, diffusion of the field ϕ discussed above exists only during inflation. Thus, there is no diffusion motion upwards from the region $\phi < \phi_e$. Therefore expression (13) is not a true solution of equation (12); all solutions with the proper boundary conditions at $\phi = \phi_e$ (i.e. at the end of inflation) are non-stationary (decaying) [16].

It is possible to use the solution (13) in the cases where the state can be quasistationary. For example, in the case when the effective potential has a local minimum with a sufficiently large V, this distribution gives a correct expression for the probability of the Hawking-Moss tunneling [22]. We were unable to find a situation in the context of inflationary cosmology where one could ascribe a more fundamental meaning to the Hartle-Hawking wave function, but of course this might be a result of our own limitations.

One can get an additional insight by investigation of the probability distribution P_p. In order to do so, one should write stochastic equations for $\mathcal{V}(\phi, t|\chi)$, where $\mathcal{V}(\phi, t|\chi)$ is the total volume of domains with the field ϕ originated from the domains containing field χ. The system of stochastic equations for $\mathcal{V}(\phi, t|\chi)$ can be obtained from eqs. (11), (12) by adding the term $3H\mathcal{V}$, which appears due to the growth of physical volume of the Universe by the factor $1 + 3H(\phi)\,dt$ during each time interval dt [23]–[25]:

$$\frac{\partial \mathcal{V}}{\partial t} = \frac{H^{3/2}(\chi)}{8\pi^2}\,\frac{\partial}{\partial \chi}\left(H^{3/2}(\chi)\,\frac{\partial \mathcal{V}}{\partial \chi}\right) - \frac{V'(\chi)}{3H(\chi)}\,\frac{\partial \mathcal{V}}{\partial \chi} + 3H(\chi)\mathcal{V}\ , \tag{14}$$

$$\frac{\partial \mathcal{V}}{\partial t} = \frac{\partial}{\partial \phi}\left(\frac{H^{3/2}(\phi)}{8\pi^2}\,\frac{\partial}{\partial \phi}\Big(H^{3/2}(\phi)\mathcal{V}\Big) + \frac{V'(\phi)}{3H(\phi)}\,\mathcal{V}\right) + 3H(\phi)\mathcal{V}\ . \tag{15}$$

To find solutions of these equations one must specify boundary conditions. Behavior of solutions of these equations typically is not very sensitive to the boundary conditions at the boundary $\phi = \phi_e$ where inflation ends; it is sufficient to assume that the diffusion coefficient (and, correspondingly, the first terms in the r.h.s. of equations (14), (15)) vanish for $\phi < \phi_e$. The conditions at the Planck boundary $\phi = \phi_p$ play a more important role. In what follows we will assume that inflation ceases to exist at $\phi > \phi_p$ [25]. This leads to the boundary condition

$$\mathcal{V}(\phi_p, t|\chi) = \mathcal{V}(\phi, t|\chi_p) = 0\ , \tag{16}$$

where $V(\phi_p) \equiv V(\chi_p) = O(1)$.

One may try to obtain solutions of equations (14), (15) in the form of the following series of system of eigenfunctions of the pair of adjoint linear operators defined by the left hand sides of the equations below:

$$\mathcal{V}(\phi, t|\chi) = \sum_{s=1}^{\infty} e^{\lambda_s t}\,\psi_s(\chi)\,\pi_s(\phi)\ . \tag{17}$$

Indeed, this gives us a solution of eq. (15) if

$$\frac{H^{3/2}}{8\pi^2}\frac{\partial}{\partial \chi}\left(H^{3/2}\frac{\partial}{\partial \chi}\psi_s(\chi)\right) - \frac{V'}{3H}\frac{\partial}{\partial \chi}\psi_s(\chi) + 3H\cdot\psi_s(\chi) = \lambda_s\,\psi_s(\chi)\ , \tag{18}$$

and

$$\frac{\partial}{\partial\phi}\left(\frac{H^{3/2}}{8\pi^2}\frac{\partial}{\partial\phi}\left(H^{3/2}\pi_j(\phi)\right)\right)+\frac{\partial}{\partial\phi}\left(\frac{V'}{3H}\pi_j(\phi)\right)+3H\cdot\pi_j(\phi)=\lambda_j\,\pi_j(\phi)\ . \tag{19}$$

In our case (with regular boundary conditions) one can easily show that the spectrum of λ_j is discrete and bounded from above. Therefore the asymptotic solution for $\mathcal{V}(\phi,t|\chi)$ in the limit $t\to\infty$ is given by

$$\mathcal{V}(\phi,t|\chi)=e^{\lambda_1 t}\,\psi_1(\chi)\,\pi_1(\phi)\cdot\left(1+O\left(e^{-(\lambda_1-\lambda_2)t}\right)\right)\ . \tag{20}$$

Here $\psi_1(\chi)$ is the only positive eigenfunction of eq. (18), λ_1 is the corresponding (real) eigenvalue, and $\pi_1(\phi)$ is the eigenfunction of the conjugate operator (19) with the same eigenvalue λ_1. Note, that λ_1 is the highest eigenvalue, $\mathrm{Re}\,(\lambda_1-\lambda_2)>0$. This means that the distribution

$$P_p(\phi,t|\chi)=e^{-\lambda_1 t}\,\mathcal{V}(\phi,t|\chi) \tag{21}$$

gradually converges to the time-independent normalized distribution

$$P_p(\phi,\chi)\equiv P_p(\phi,t\to\infty|\chi)=\psi_1(\chi)\,\pi_1(\phi)\ . \tag{22}$$

It is this stationary distribution that we were looking for. $P_p(\phi,\chi)$ gives the fraction of the volume of the Universe occupied by the field ϕ, under the condition that the corresponding part of the Universe at some time in the past contained the field χ. The remaining problem is to find the functions $\psi_1(\chi)$ and $\pi_1(\phi)$, and to check that all assumptions about the boundary conditions which we made on the way to eq. (20) are actually satisfied.

We have solved this problem for chaotic inflation in a wide class of theories including the theories with polynomial and exponential effective potentials $V(\phi)$ and found the corresponding stationary distributions [25]. Here we will present some of our results for the theory $\frac{\lambda}{4}\phi^4$.

Solution of equations (18) and (19) for $\psi_1(\chi)$ and $\pi_1(\phi)$ shows that these functions are extremely small at $\phi\sim\phi_e$ and $\chi\sim\chi_e$, where $\phi_e\sim\chi_e\sim 1$ correspond to the end of inflation. These functions grow at large ϕ and χ, then rapidly decrease, and vanish at $\phi=\chi=\phi_p$. With a decrease of λ the solutions become more and more sharply peaked near the Planck boundary. To give a physical interpretation to our solutions, it will be convenient to parametrize λ_1 in the following form: $\lambda_1=d(\lambda)H_{\max}(\lambda)$. Here d is the so-called fractal dimension of inflationary universe [26, 25], and $H_{\max}$ is the maximal value of the Hubble constant in the model under consideration. For example, in the models where inflation ceases to exist at the Planck density $V(\phi)=1$ the maximal value of the Hubble constant is given by $2\sqrt{\frac{2\pi}{3}}$. The eigenvalues $d(\lambda)$ corresponding to different coupling constants λ are given by the following table:

λ	1	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
d	0.9719	1.526	1.915	2.213	2.438	2.604	2.724

As we see, in the limit $\lambda\to 0$ the fractal dimension $d(\lambda)$ grows toward the usual space dimension 3.

It is very interesting to study the behavior of P_p at small ϕ and χ, i.e. at the stage which determines the structure of the observable part of the Universe. One could expect to find a dependence similar to the one given by eq. (13), i.e. $P_p \sim \exp\left(\frac{3}{8V(\phi)}\right) \cdot \exp\left(-\frac{3}{8V(\chi)}\right)$. Indeed, this remains true for the dependence of P_p on χ. Meanwhile, since there is no diffusion term at $\phi < \phi_e$, the solution at small $\phi > \phi_e$ should match the solution obtained by neglecting the first (diffusion) term at $\phi < \phi_e$. As a result, instead of the product of the Hartle-Hawking and the tunneling solution for the theory $\frac{\lambda}{4}\phi^4$ for small ϕ and χ (for $\phi, \chi < \lambda^{-1/8}$) we obtain

$$\mathcal{V}(\phi,\chi,t) \;=\; e^{d(\lambda)H_{\max}t}\; P_p(\phi,\chi) \;\sim\; e^{d(\lambda)H_{\max}t}\; \phi^{\sqrt{\frac{6\pi}{\lambda}}\lambda_1}\; \exp\left(-\frac{3}{8V(\chi)}\right) \;. \tag{23}$$

Thus, the square of the tunneling wave function is here, but the square of the Hartle-Hawking wave function dropped away. The dependence of $\mathcal{V}(\phi,\chi,t)$ on χ and ϕ is extremely sharp. For example, for the realistic value $\lambda \sim 10^{-13}$ one has $P_p(\phi,\chi) \sim e^{10^{13}\chi^{-4}}\,\phi^{10^8}$.

The factor $e^{d(\lambda)H_{\max}t}$ controls the speed of exponential expansion of the volume filled by a given field ϕ. *This speed does not depend on the field* ϕ, and has the same order of magnitude as the speed of expansion at the Planck density. One should emphasize that the factor $e^{d(\lambda)H_{\max}t}$ gives the rate of growth of the combined volume of all domains with a given field ϕ (or of all domains containing matter with a given density) *not only at very large ϕ, where quantum fluctuations are large, but at small ϕ as well, and even after inflation* [25]. This result may seem absolutely unexpected, since the volume of each particular inflationary domain grows like $e^{3H(\phi)t}$, and after inflation the law of expansion becomes completely different. One should distinguish, however, between the growth of each particular domain, accompanied by a decrease of density inside it, and the growth of the total volume of all domains containing matter with a given (constant) density. In the standard big bang theory the second possibility did not exist, since the energy density was assumed to be the same in all parts of the Universe ("cosmological principle"), and it was not constant in time.

The reason why there is a universal expansion rate $e^{\lambda_1 t}$ can be understood as follows. Because of the self-reproduction of the Universe there always exist many domains with $\phi \sim \phi_p$, and their combined volume grows almost as fast as $e^{3H(\phi_p)t}$. Then the field ϕ inside some of these domains decreases. The total volume of domains containing some small field ϕ grows not only due to expansion $\sim e^{3H(\phi_p)t}$, but mainly due to the unceasing process of expansion of domains with large ϕ and their subsequent rolling (or diffusion) towards small ϕ.

The distribution $P_p(\phi,\chi) = \psi_1(\chi)\,\pi_1(\phi)$ which we have obtained does not depend on time t. However, in general relativity one may use many different time parametrizations, and the same physics can be described differently in different 'times'. One of the most natural choices of time in the context of stochastic approach to inflation is the time $\tau = \ln\frac{a(x,t)}{a(x,0)} = \int H(\phi(x,t),t)\,dt$ [22, 27]. Here $a\,(x,t)$ is a local value of the scale factor in the inflationary universe. By using this time variable, we were able to obtain not only numerical solutions to the stochastic equations, but also simple asymptotic expressions describing these solutions. For example, for the theory $\frac{\lambda}{4}\phi^4$ both the eigenvalue λ_1 and the 'fractal dimension' d_f (which in this case refers both to the Planck boundary at ϕ_p and to the end of inflation at ϕ_e) are given by $d_f = \lambda_1 \sim 3 - 1.1\sqrt{\lambda}$, and the stationary distribution is [25]

$$P_p(\phi,\chi) \sim \exp\Big(-\frac{3}{8V(\chi)}\Big)\Big(\frac{1}{V(\chi)+0.4}-\frac{1}{1.4}\Big)\cdot\phi\exp\Big(-\pi(3-\lambda_1)\phi^2\Big)$$
$$\sim \exp\Big(-\frac{3}{2\lambda\chi^4}\Big)\Big(\frac{4}{\lambda\chi^4+1.6}-\frac{1}{1.4}\Big)\cdot\phi\exp\Big(-3.5\sqrt{\lambda}\phi^2\Big)\ . \qquad (24)$$

The first factor again coincides with the square of the tunneling wave function, and again there is no trace of the Hartle-Hawking wave function. This expression is valid in the whole interval from ϕ_e to ϕ_p and it correctly describes asymptotic behavior of $P_p(\phi,\chi)$ both at $\chi\sim\chi_e$ and at $\chi\sim\chi_p$.

A similar investigation can be carried out for the theory $V(\phi)=V_o\ e^{\alpha\phi}$. The corresponding solution is

$$P_p(\phi,\chi) \sim \exp\Big(-\frac{3}{8V(\chi)}\Big)\Big(\frac{1}{V(\chi)}-1\Big)\cdot\Big(\frac{1}{V(\phi)}-1\Big)V^{-1/2}(\phi)\ . \qquad (25)$$

This expression gives a rather good approximation for $P_p(\phi,\chi)$ for all ϕ and χ.

The main result is that under certain conditions the properties of our Universe can be described by a time-independent probability distribution, which we have found for theories with polynomial and exponential effective potentials. Thus, inflation solves many problems of the big bang theory and ensures that this theory provides an excellent description of the local structure of the Universe. However, after making all kinds of improvements of this theory, we are now winding up with a model of a stationary Universe, in which the notion of the big bang loses its dominant position, being removed to the indefinite past.

3 Predictions in quantum cosmology

3.1 Moderate approach: comparing probabilities within the same Universe

When inflationary theory was first formulated, we did not know how much it was going to influence our understanding of the structure of the Universe. We were happy that inflation provided an easy explanation of the homogeneity of the Universe. However, we did not know that the same theory simultaneously predicts that on the extremely large scale the Universe becomes entirely inhomogeneous, and that this inhomogeneity is good, since it is one of the manifestations of the process of self-reproduction of inflationary Universe.

The new picture of the Universe which emerges now is very unusual, and we are still in the process of learning how to ask proper questions in the context of the new cosmological paradigm. Previously we assumed that we live in a Universe which has the same properties everywhere ("cosmological principle"). Then one could make a guess about the most natural initial conditions in the Universe, and all the rest followed almost automatically. Now we learned that even if one begins with a non-uniform Universe, later it becomes extremely homogeneous on a very large scale. However, simultaneously it becomes absolutely non-uniform on a much greater scale. the Universe becomes divided into different exponentially large regions where the laws of low-energy physics can be different. In certain cases the relative fraction of volume of the Universe in a state with given fields or with a given density does not depend on time, whereas the total volume of all parts of the Universe continues growing exponentially.

This change of the picture of the world is important by itself. However, it would be even better if we could use it to make certain predictions based on this picture. In this situation the problem of introduction of a proper measure of probability becomes most important. One of the most natural choices of such measure is given by the probability distribution $P_p(\phi, \chi, t)$. The hypothesis behind this proposal is that we are typical, and therefore we live in those parts of the Universe where most other people do. The total number of people which can live in domains with given properties should be proportional to the total volume of these domains. There are two versions of this hypothesis, the moderate and the radical ones. The moderate version is based on investigation of $P_p(\phi, \chi, t)$ [16, 25, 28]. If this distribution is stationary, then it seems reasonable to use it as a measure of the total volume of domains with any particular properties at any given moment of time t.

The first example of this approach is given by the consideration of the axion problem. In the non-inflationary cosmology it was shown that the axion mass should be greater than 10^{-5} eV in order to avoid having too much energy stored in the axion field [29]. However, the derivation of this constraint fails in inflationary cosmology if one takes into account quantum fluctuations of the axion field and eternal production of domains where this field takes all its possible values. Then it can be shown that life of our type can appear only in those domains where the axion field is sufficiently small and under certain conditions discussed in [30] the standard constraint $m_a > 10^{-5}$ eV disappears.

Another interesting example is given by the probability distribution for finding the most probable values of the effective gravitational constant in the Brans-Dicke inflationary cosmology [28]. We have shown there that inflation in the Brans-Dicke theory leads to division of the Universe into different exponentially large domains with different values of the gravitational constant, and, correspondingly, with different values of density perturbations. Then one can use the probability distribution $P_p(\phi, \chi, t)$ to find most probable values of the gravitational constant. In this approach it is possible either to explain the anomalously large value of the Planck mass, or at least relate it to certain small parameters in the theory, e.g. to the small anisotropy of the microwave background radiation. Note, that the very language which we are using may sound somewhat strange. Indeed, typically the purpose is to express the anisotropy of the microwave background radiation via some fundamental parameters of the theory. In our case the Planck mass is not fundamental, and its value is anomalously large in those domains where the microwave background radiation is anomalously small.

In what follows I will briefly describe some nonperturbative effects which may lead to a considerable local deviation of density from the critical density of a flat Universe [31].

Let us consider all parts of inflationary universe which contain a given field ϕ at a given moment of time t. One may wonder, what was the value of this field in those domains at the moment $t - H^{-1}$? The answer is simple: One should add to ϕ the value of its classical drift $\Delta\phi$ during the time H^{-1}, $\Delta\phi = \dot{\phi}H^{-1}$. One should also add the amplitude of a quantum jump $\delta\phi$. The typical jump is given by $\delta\phi = \pm\frac{H}{2\pi}$. At the last stages of inflation this quantity is by many orders of magnitude smaller than $\Delta\phi$. However, in which sense jumps $\pm\frac{H}{2\pi}$ are typical? If we consider any particular initial value of the field ϕ, then the typical jump from this point is indeed given by $\pm\frac{H}{2\pi}$. However, if we are considering all domains with a given ϕ and trying to find all those domains from which the field ϕ could originate back in time, the answer may be quite different. Indeed, the total volume of all domains with a given field ϕ at any moment of time t strongly depends on ϕ: $P_p(\phi) \sim \phi^{\sqrt{\frac{6\pi}{\lambda}}\lambda_1} \sim \phi^{10^8}$, see eq. (23). This means that the total volume of all domains which could jump towards the given field ϕ from the value $\phi + \delta\phi$ will be enhanced by a large additional factor $\frac{P_p(\phi+\delta\phi)}{P_p(\phi)} \sim \left(1 + \frac{\delta\phi}{\phi}\right)^{\sqrt{\frac{6\pi}{\lambda}}\lambda_1}$. On the other hand, the probability of large jumps $\delta\phi$ is suppressed by the Gaussian factor $\exp\left(-\frac{2\pi^2\delta\phi^2}{H^2}\right)$. One can easily verify that the product of these two factors has a sharp maximum at $\delta\phi = \lambda_1\phi\cdot\frac{H}{2\pi}$, and the width of this maximum is of the order $\frac{H}{2\pi}$. In other words, most of the domains of a given field ϕ are formed due to jumps which are greater than the "typical" ones by a factor $\lambda_1\phi \pm O(1)$.

Our part of the Universe in the inflationary scenario with $V(\phi) = \frac{\lambda}{4}\phi^4$ is formed at $\phi \sim 5$ (in the units $M_{\rm P} = 1$), and the constant $\lambda_1 \approx 2\sqrt{6\pi} \sim 8.68$ for our choice of boundary conditions. This means that our part of the Universe should be created as a result of a jump down which is about $\lambda_1\phi \sim 40$ times greater than the standard jump. The standard jumps lead to density perturbations of the amplitude $\delta\rho \sim 5\cdot 10^{-5}\rho_c$ (in the normalization of [7]). Thus, according to our nonperturbative analysis, we should live inside a region where density is smaller than the critical density by about $\delta\rho \sim 2\cdot 10^{-3}\rho_c$. As we already mentioned, the probability of such fluctuations should be suppressed by $\exp\left(-\frac{2\pi^2\delta\phi^2}{H^2}\right)$, which in our case gives the suppression factor $\sim \exp(-10^3)$. It is well known that exponentially suppressed perturbations typically give rise to spherically symmetric bubbles. Note also, that the Gaussian distribution suppressing the amplitude of the perturbations refers to the amplitude of a perturbation in its maximum. It is possible that we live not in the place corresponding to the maximum of the fluctuation. However, this could only happen if the nonperturbative jump down was even greater in the amplitude that we expected. Meanwhile, as we already mentioned, the distribution of the amplitudes of such jumps has width of only about $\frac{H}{2\pi}$. This means that we should live very close to the center of the giant fluctuation, and the difference of energy densities between the place where we live and the center of the "bubble" should be only about the same amplitude as the typical perturbative fluctuation $\delta\rho \sim 5\cdot 10^{-5}\rho_c$. In other words, we should live very close to the center of the nearly perfect spherically symmetric bubble, which contains matter with a smaller energy density than the matter outside it.

It is very tempting to interpret this effect in such a way that the Universe around us becomes locally open. The true description of this effect is, of course, much more complicated; perhaps we should see the Hubble constant decreasing at large distances. This effect is extremely unusual. We became partially satisfied by our understanding of this effect only after we confirmed its existence by four different methods, including computer simulations [31]. However, it may happen that what we have found is simply a mathematical property of some particular hypersurfaces in inflationary universe, and it does not have any implications for the part of the Universe where we live.

Indeed, it is quite legitimate to use the distributions like P_p for descriptions of the structure of inflationary universe. However, it is not quite clear whether one can use them to evaluate probabilities. For example, instead of using the distribution $P_p(\phi, \chi, t)$ one may use the distribution $P_p(\phi, \chi\tau)$, where $\tau \sim \log a$, and many of our result (though not all of them) will change dramatically [25, 28]. Still another answer will be obtained if one uses some other cut-off procedure, see [32]. The source of this ambiguity can be easily understood. The total volume of all parts of an eternally inflating Universe is infinite in the limit $t \to \infty$ (or $\tau \to \infty$). Therefore when we are trying to compare volumes of domains with different properties, we are comparing infinities. This leads to answers depending on the way we are making this comparison.

It is possible that eventually we will resolve this problem. Still it will not guarantee that we are on the right track. Our use of P_p as a probability measure was based on two hidden assumptions. The first assumption is that we are typical observers. The second assumption is that the number of typical observers is directly proportional to the volume of the Universe. If this is correct, then we should live in the place where most observers live, which should correspond to a maximum of P_p.

However, is it absolutely clear that the probability for an observer to be born in a particular part of the Universe is directly proportional to its volume, or one should take into account something else? One cannot get any crop even from a very large field without having seeds first. The idea that life appears automatically once there is enough space to be populated may be too primitive. It is based on the assumption that one can describe emergence of life solely in terms of physics. It is certainly a most economical approach, and one should try to go as far as possible without invoking additional hypotheses. However, one should keep in mind that this approach may happen to be incomplete, especially if consciousness has its own degrees of freedom [7, 33].

Another related question is whether we are actually typical? Does it make any sense for each of us to calculate *a posteriori* what was the probability to be born Russian, Italian or Chinese? Should we insist on our own mediocrity, or, *vice versa*, should we try to explain why are we so special? After all, for a long time we thought that we had the aristocratic privilege to be the most intelligent species in the Universe. This, of course, may be wrong. Still, before using probabilities to calculate the likelihood of our existence in a particular part of the Universe, it may be a good idea to learn more about ourselves. I would take a certain risk to make a conjecture that until we understand what is our life and what is the nature of consciousness our understanding of quantum cosmology will remain fundamentally incomplete.

3.2 A more radical approach: comparing Universes with different coupling constants

Previously we compared volume of the parts of the Universe with some particular properties within one Universe. A more ambiguous program is based on a combination of the baby Universe theory and stochastic approach to inflation. The idea is that the coupling constants may take different values in different Universes, or, more precisely, in different quantum states of the Universe [34]. If this is the case, then perhaps we should live in those Universes where conditions are better and the total volume suitable for life is greater [35]–[38].

The total volume is given by $\mathcal{V}(\phi,\chi,t) = e^{d(\lambda)H_{\max}(\lambda)t}\, P_p(\phi,\chi,t)$. The first term in this expression is especially important. If (and this is a big "if"!) one can compare the volumes of different Universes with different coupling constants at the same time t, the greatest volume will be occupied by the Universes with the largest product $d(\lambda)H_{\max}(\lambda)$. For stationary $P_p(\phi,\chi,t) = P_p(\phi,\chi)$ the exponential growth of $\mathcal{V}(\phi,\chi,t)$ in the state with the largest $d(\lambda)H_{\max}(\lambda)$ eventually beats all anthropic considerations. This may lead to a very sharp prediction of the coupling constants which maximize $d(\lambda)H_{\max}(\lambda)$.

Unfortunately, this immediately leads to a trouble. For example, in our investigation of the theory $\frac{\lambda}{4}\phi^4$ we have found that $H_{\max} = 2\sqrt{\frac{2\pi}{3}}$ does not depend on λ, whereas the fractal dimension $d(\lambda)$ has its maximum $d = 3$ in the limit $\lambda = 0$. This is a rather general conclusion which seems to suggest that the inflationary effective potential should be absolutely flat. But then there will be no density perturbations which are necessary for galaxy formation. One may try to avoid the problem with density perturbations assuming that they will be produced by cosmic strings [36, 37], but in the theory with absolutely flat potentials there will be no reheating and no cosmic strings. One may argue that this means that the potential should be *almost* flat, i.e. that it should be curved just enough to allow baryons and strings to be produced and life to appear. In fact, in such a case strings are not necessary. For example, one may consider the hybrid inflation model [39]. In this model one can have good inflation and sufficiently large density perturbations without any need for cosmic strings even if the potential is extremely (though not exactly) flat. But the problem is that the gain in $e^{d(\lambda)H_{\max}(\lambda)t}$ eventually always beats the anthropic considerations, which pushes us towards the models with *exactly* flat potentials. If the effective potential is exactly flat, we have no reheating and no regular density perturbations, but even in this case life may appear in an infinite empty Universe with a very small but finite probability due to extremely improbable quantum fluctuations. Even though such conditions are extremely improbable, eventually we will be compensated by the indefinitely large growth of volume due to the term $e^{d(\lambda)H_{\max}(\lambda)t}$. However, in such a scenario there is no reason for our part of the Universe to be homogeneous on the scale 10^{28} cm, which is much greater than what is needed for our existence. One may also argue that if quantum cosmology pushes us outside of the limits of our normal existence, it probably puts us at the verge of being immediately extinct.

Another example is related to the cosmological constant problem. Adding it to the Lagrangian also tends to increase $d(\lambda)$. Thus the considerations based on the investigation of the factor $e^{d(\lambda)H_{\max}(\lambda)t}$ may push us towards very large values of the vacuum energy density [36]. Of course, one cannot go too far since our life cannot exist if the vacuum energy density is too large. However, anthropic considerations allow vacuum energy density V_0 two orders of magnitude greater than the critical density $\sim 10^{-29}\text{g}\cdot\text{cm}^{-3}$, i.e. two orders of magnitude greater than the present observational constraints on V_0 [40]. Moreover, as we just mentioned, the rapidly growing factor $e^{d(\lambda)H_{\max}(\lambda)t}$ should beat all anthropic considerations and should push V_0 even higher, which would be in a definite contradiction with the observational data.

This indicates that something should be modified either in the radical approach described in this subsection or in our choice of the theories to which we applied this approach [38]. Each of these possibilities can be true. First of all, it is not quite clear whether it makes any sense at all to compare volume of different Universes (rather than volume of different parts of the same Universe) at the same time. Then, in certain theories the probability distribution $\mathcal{V}(\phi,\chi,t)$ is not stationary [28], so it cannot be represented as $e^{d(\lambda)H_{\max}(\lambda)t}\, P_p(\phi,\chi)$. Finally, under certain conditions the fastest growth of $\mathcal{V}(\phi,\chi,t)$ appears in the theories where the effective potential is not flat and the cosmological constant is not large [38]. For example, adding a positive cosmological constant in the Starobinsky model *decreases* the rate of expansion. This pushes the cosmological constant to zero [38]. Unfortunately, this cannot be considered as a possible solution of the cosmological constant problem since the same mechanism may push the cosmological constant even further, toward its negative values. To solve the cosmological constant problem it would be necessary to find a mechanism which pushes it to zero from both sides.

It would be premature to make any final conclusions about the radical approach described above. The idea to use stochastic approach to inflation in order to understand our place in the world is extremely attractive. However, this powerful weapon should be used with caution, especially when one tries to extend its limits of applicability and use it in the context of the baby Universe theory. A possible attitude towards this approach is to consider it as a kind of "theoretical experiment." We may try to use probabilistic considerations in our trial-and-error approach to quantum cosmology. If we get unreasonable results, this may serve as an indication that we are using quantum cosmology incorrectly. However, if we solve some problems which could not be solved in any other way, then we will have a reason to believe that we are moving in the right direction. In our opinion, at the present moment we do not have sufficient reasons to believe that the effective potential should be exactly flat, that the density perturbations should be produced by strings appearing after inflation, and that the cosmological constant should be as large as possible. On the other hand, it is not excluded that the stochastic approach to inflation, or some of its generalizations, will help us to solve the cosmological constant problem. This possibility certainly deserves further investigation. We will return to the possibility of making predictions and calculating probabilities in quantum cosmology in the next section, where we will consider the model of an open inflationary universe.

4 Inflation with $\Omega \neq 1$

4.1 Inflation and flatness of the Universe

One of the most robust predictions of inflationary cosmology is that the Universe after inflation becomes extremely flat, which corresponds to $\Omega = 1$. Here $\Omega = \frac{\rho}{\rho_c}$, ρ_c being the energy density of a flat Universe. There were many good reasons to believe that this prediction was quite generic. The only way to avoid this conclusion is to assume that the Universe inflated only by about e^{60} times. Exact value of the number of e-foldings N depends on details of the theory and may somewhat differ from 60. It is important, however, that in any particular theory inflation by extra 2 or 3 e-foldings would make the Universe with $\Omega = 0.5$ or with $\Omega = 1.5$ almost exactly flat. Meanwhile, the typical number of e-foldings in chaotic inflation scenario in the theory $\frac{m^2}{2}\phi^2$ is not 60 but rather 10^{12}.

One can construct models where inflation leads to expansion of the Universe by the factor e^{60}. However, in most of such models small number of e-foldings simultaneously implies that density perturbations are extremely large. It may be possible to overcome this obstacle by a specific choice of the effective potential. However, this would be only a partial solution. If the Universe does not inflate long enough to become flat, then by the same token it does not inflate long enough to become homogeneous and isotropic. Thus, the main reason why it is difficult to construct inflationary models with $\Omega \neq 1$ is not the issue of fine tuning of the parameters of the models, which is necessary to obtain the Universe inflating exactly e^{60} times, but the problem of obtaining a homogeneous Universe after inflation.

Fortunately, it is possible to solve this problem, both for a closed Universe [42] and for an open one [43]–[48]. The main idea is to use the well known fact that the region of space created in the process of a quantum tunneling tends to have a spherically symmetric shape, and homogeneous interior, if the tunneling process is suppressed strongly enough. Then such bubbles of a new phase tend to evolve (expand) in a spherically symmetric fashion. Thus, if one could associate the whole visible part of the Universe with an interior of one such region, one would solve the homogeneity problem, and then all other problems will be solved by the subsequent relatively short stage of inflation.

For a closed Universe the realization of this program is relatively straightforward [42, 47]. One should consider the process of quantum creation of a closed inflationary universe from "nothing." If the probability of such a process is exponentially suppressed (and this is indeed the case if inflation is possible only at the energy density much smaller than the Planck density [12]), then the Universe created that way will be rather homogeneous from the very beginning.

The situation with an open Universe is much more complicated. Indeed, an open Universe is infinite, and it may seem impossible to create an infinite Universe by a tunneling process. Fortunately, this is not the case: any bubble formed in the process of the false vacuum decay looks from inside like an infinite open Universe [49]. If this Universe continues inflating inside the bubble [43]–[48], then we obtain an open inflationary Universe.

These possibilities became a subject of an active investigation only very recently, and there are still many questions to be addressed. First of all, the bubbles created by tunneling are not *absolutely* uniform even if the probability of tunneling is very small. This may easily spoil the whole scenario since in the end of the day we need to explain why the microwave background radiation is isotropic with an accuracy of about 10^{-5}. Previously we did not care much about initial homogeneities, but if the stage of inflation is short, we will the see original inhomogeneities imprinted in the perturbations of the microwave background radiation.

The second problem is to construct realistic inflationary models where all these ideas could be realized in a natural way. Whereas for the closed Universe this problem can be easily solved [42, 47], for an open Universe we again meet complications. It would be very nice to to obtain an open Universe in a theory of just one scalar field [45]. However, in practice it is not very easy to obtain a satisfactory model of this type. Typically one is forced either to introduce very complicated effective potentials, or consider theories with nonminimal kinetic terms for the inflaton field [46]. This makes the models not only fine-tuned, but also rather complicated. It is very good to know that the models of such type in principle can be constructed, but it is also very tempting to find a more natural realization of the inflationary universe scenario which would give inflation with $\Omega < 1$.

This goal can be achieved if one considers models of two scalar fields [47]. One of them may be the standard inflaton field ϕ with a relatively small mass, another may be, e.g., the scalar field responsible for the symmetry breaking in GUTs. The presence of two scalar fields allows one to obtain the required bending of the inflaton potential by simply changing the definition of the inflaton field in the process of inflation. At the first stage the role of the inflaton is played by a heavy field with a steep barrier in its potential, while on the second stage the role of the inflaton is played by a light field, rolling in a flat direction "orthogonal" to the direction of quantum tunneling. This change of the direction of evolution in the space of scalar fields removes the naturalness constraints for the form of the potential, which are present in the case of one field.

Inflationary models of this type are quite simple, yet they have many interesting features. In these models the Universe consists of infinitely many expanding bubbles immersed into exponentially expanding false vacuum state. Each of these bubbles inside looks like an open Universe, but the values of Ω in these Universes may take any value from 1 to 0. In some of these models the situation is even more complicated: Interior of each bubble looks like an infinite Universe with an effective value of Ω slowly decreasing to $\Omega = 0$ at an exponentially large distance from the center of the bubble. We will call such Universes quasiopen. Thus, rather unexpectedly, we are obtaining a large variety of interesting and previously unexplored possibilities. Our discussion of these possibilities will follow our recent paper with Arthur Mezhlumian [48].

4.2 Tunneling probability and spherical symmetry

Typically it is assumed that the bubbles containing open Universes are exactly spherically symmetric (or, to be more accurate, $O(3,1)$-symmetric [49]). Meanwhile in realistic situations this condition may be violated for several reasons. First of all, the bubble may be formed not quite symmetric. Then its shape may change even further due to growth of its initial inhomogeneities and due to quantum fluctuations which appear during the bubble wall expansion. As we will see, this may cause a lot of problems if one wishes to maintain the degree of anisotropy of the microwave background radiation inside the bubble at the level of 10^{-5}.

First of all, let us consider the issue of symmetry of a bubble at the moment of its formation. For simplicity we will investigate the models where tunneling can be described in the thin wall approximation. We will neglect gravitational effects, which is possible as far as the initial radius r of the bubble is much smaller than H^{-1}. In this approximation (which works rather well for the models to be discussed) euclidean action of the $O(4)$-symmetric instanton describing bubble formation is given by

$$S = -\frac{\epsilon}{2}\pi^2 r^4 + 2\pi^2 r^3 s \ . \tag{26}$$

Here r is the radius of the bubble at the moment of its formation, ϵ is the difference of $V(\phi)$ between the false vacuum $\phi_{\rm initial}$ and the true vacuum $\phi_{\rm final}$, and s is the surface tension,

$$s = \int_{\phi_{\rm initial}}^{\phi_{\rm final}} \sqrt{2(V(\phi) - V(\phi_{\rm final}))}\, d\phi \ . \tag{27}$$

The radius of the bubble can be obtained from the extremum of (26) with respect to r:

$$r = \frac{3s}{\epsilon} \ . \tag{28}$$

Let us check how the action S will change if one consider a bubble of a radius $r + \Delta r$. Since the first derivative of S at its extremum vanishes, the change will be determined by its second derivative,

$$\Delta S = \frac{1}{2} S''(\Delta r)^2 = 9\pi^2\, \frac{s^2}{\epsilon}\, (\Delta r)^2 \ . \tag{29}$$

Now we should remember that all trajectories which have an action different from the action at extremum by no more than 1 are quite legitimate. Thus the typical deviation of the radius of the bubble from its classical value (28) can be estimated from the condition $\Delta S \sim 1$, which gives

$$|\Delta r| \sim \frac{\sqrt{\epsilon}}{3\pi\, s} \ . \tag{30}$$

Note, that even though we considered spherically symmetric perturbations, our estimate is based on corrections proportional to $(\delta r)^2$, and therefore it should remain valid for perturbations which have an amplitude Δr, but change their sign in different parts of the bubble surface. Thus, eq. (30) gives an estimate of a typical degree of asymmetry of the bubble at the moment of its creation:

$$A(r) \equiv \frac{|\Delta r|}{r} \sim \frac{\epsilon\sqrt{\epsilon}}{3\pi\, s^2} \ . \tag{31}$$

This simple estimate exactly coincides with the corresponding result obtained by Garriga and Vilenkin [52] in their study of quantum fluctuations of bubble walls. It was shown in [52] that when an empty bubble begins expanding, the typical deviation Δr remains constant. Therefore the asymmetry given by the ratio $\frac{|\Delta r|}{r}$ gradually vanishes. This is a pretty general result: Waves produced by a brick falling to a pond do not have the shape of a brick, but gradually become circles.

However, in our case the situation is somewhat more complicated. The wavefront produced by a brick in inflationary background preserves the shape of the brick if its size is much greater than H^{-1}. Indeed, the wavefront moves with the speed approaching the speed of light, whereas the distance between different parts of a region with initial size greater than H^{-1} grows with a much greater (and ever increasing) speed. This means that inflation stretches the wavefront without changing its shape on scale much greater than H^{-1}. Therefore during inflation which continues inside the bubble the symmetrization of its shape occurs only in the very beginning, until the radius of the bubble approaches H^{-1}. At this first stage expansion of the bubble occurs mainly due to the motion of the walls rather than due to inflationary stretching of the Universe, and our estimate of the bubble wall asymmetry as well as the results obtained by Garriga and Vilenkin for the empty bubble remain valid. At the moment when the radius of the bubble becomes equal to H^{-1} its asymmetry becomes

$$A(H^{-1}) \sim |\Delta r| H \sim \frac{\sqrt{\epsilon} H}{3\pi\, s} \ , \qquad (32)$$

and the subsequent expansion of the bubble does not change this value very much. Note that the Hubble constant here is determined by the vacuum energy *after* the tunneling, which may differ from the initial energy density ϵ.

The deviation of the shape of the bubble from spherical symmetry implies that the beginning of the second stage of inflation inside the bubble will be not exactly synchronous, with the delay time $\Delta t \sim \Delta r$. This, as usual, may lead to adiabatic density perturbations on the horizon scale of the order of $H\Delta t$, which coincides with the bubble asymmetry A after its size becomes greater than H^{-1}, see Eq. (32).

To estimate this contribution to density perturbations, let us consider again the simplest model with the effective potential

$$V(\phi) = \frac{m^2}{2}\phi^2 - \frac{\delta}{3}\phi^3 + \frac{\lambda}{4}\phi^4 \ . \qquad (33)$$

Now we will consider it in the limit where the two minima of this potential have almost the same depth, which is necessary for validity of the thin wall approximation. In this case $2\delta^2 = 9M^2\lambda$, and the effective potential (33) looks approximately like $\frac{\lambda}{4}\phi^2(\phi-\phi_0)^2$, where $\phi_0 = \frac{2\delta}{3\lambda} = \sqrt{\frac{2}{\lambda}}M$ is the position of the local minimum of the effective potential. The surface tension in this model is given by $s = \sqrt{\frac{\lambda}{2}}\frac{\phi_0^3}{6} = \frac{M^3}{3\lambda}$ [53]. We will also introduce a phenomenological parameter μ, such that $\mu\frac{M^4}{16\lambda} = \epsilon$. The smallness of this parameter controls applicability of the thin-wall approximation, since the value of the effective potential near the top of the potential barrier at $\phi = \phi_0/2$ is given by $\frac{M^4}{16\lambda}$. Then our estimate of density perturbations associated with the bubble wall (32) gives

$$\left.\frac{\delta\rho}{\rho}\right|_{\rm bubble} \sim A(H^{-1}) \sim \frac{\sqrt{\mu\lambda} H}{4\pi M} \ . \qquad (34)$$

Here H is the value of the Hubble constant at the beginning of inflation inside the bubble.

In order to have $\frac{\delta\rho}{\rho}\Big|_{\rm bubble} \lesssim 5\times 10^{-5}$ (the number 5×10^{-5} corresponds to the amplitude of density perturbations in the COBE normalization) one should have

$$\frac{\delta\rho}{\rho}\bigg|_{\rm bubble} \sim \frac{\sqrt{\mu\lambda}H}{4\pi M} \lesssim 5\times 10^{-5}\ . \qquad (35)$$

For $H \ll M$ perturbations produced by the bubble walls may be sufficiently small even if the coupling constants are relatively large and the bubbles at the moment of their formation are very inhomogeneous.

There is a long way from our simple estimates to the full theory of anisotropies of cosmic microwave background induced by fluctuations of the domain wall. In particular, the significance of this effect will clearly depend on the value of Ω [55]. The constraint (35) may appear only if one can "see" the scale at which the bubble walls have imprinted their fluctuations. If inflation is long enough, this scale becomes exponentially large, we do not see the fluctuations due to bubble walls, but then we return to the standard inflationary scenario of a flat inflationary universe. However, for $\Omega \ll 1$ inflation is short, and it does not preclude us from seeing perturbations in a vicinity of the bubble walls. In such a case one should take the constraint (35) very seriously.

4.3 The simplest model of a (quasi)open inflationary Universe

In this section we will explore an extremely simple model of two scalar fields, where the Universe after inflation becomes open (or quasiopen, see below) in a very natural way [47].

Consider a model of two noninteracting scalar fields, ϕ and σ, with the effective potential

$$V(\phi,\sigma) = \frac{m^2}{2}\phi^2 + V(\sigma)\ . \qquad (36)$$

Here ϕ is a weakly interacting inflaton field, and σ, for example, can be the field responsible for the symmetry breaking in GUTs. We will assume that $V(\sigma)$ has a local minimum at $\sigma = 0$, and a global minimum at $\sigma_0 \neq 0$, just as in the old inflationary theory. For definiteness, we will assume that this potential is given by $\frac{M^2}{2}\sigma^2 - \alpha M\sigma^3 + \frac{\lambda}{4}\sigma^4 + V(0)$, with $V(0) \sim \frac{M^4}{4\lambda}$, but it is not essential; no fine tuning of the shape of this potential will be required.

Note that so far we did not make any unreasonable complications to the standard chaotic inflation scenario; at large ϕ inflation is driven by the field ϕ, and the GUT potential is necessary in the theory anyway. In order to obtain density perturbations of the necessary amplitude the mass m of the scalar field ϕ should be of the order of $10^{-6}M_{\rm P} \sim 10^{13}$ GeV [7].

Inflation begins at $V(\phi,\sigma) \sim M_{\rm P}^4$. At this stage fluctuations of both fields are very strong, and the Universe enters the stage of self-reproduction, which finishes for the field ϕ only when it becomes smaller than $M_{\rm P}\sqrt{\frac{M_{\rm P}}{m}}$ and the energy density drops down to $mM_{\rm P}^3 \sim 10^{-6}M_{\rm P}^4$ [7].

Quantum fluctuations of the field σ in some parts of the Universe put it directly to the absolute minimum of $V(\sigma)$, but in some other parts the scalar field σ appears in the local minimum of $V(\sigma)$ at $\sigma = 0$. We will follow evolution of such domains. Since the energy density in such domains will be greater, their volume will grow with a greater speed, and therefore they will be especially important for us.

One may worry that all domains with $\sigma = 0$ will tunnel to the minimum of $V(\sigma)$ at the stage when the field ϕ was very large and quantum fluctuations of the both fields were large too. This may happen if the Hubble constant induced by the scalar field ϕ is much greater than the curvature of the potential $V(\sigma)$:

$$\frac{m\phi}{M_{\rm P}} \gtrsim M \ . \tag{37}$$

This decay can be easily suppressed if one introduces a small interaction $g^2\phi^2\sigma^2$ between these two fields, which stabilizes the state with $\sigma = 0$ at large ϕ. Another possibility is to add a nonminimal interaction with gravity of the form $-\frac{\xi}{2}R\phi^2$, which makes inflation impossible for $\phi > \frac{M_{\rm P}}{8\phi\xi}$. In this case the condition (37) will never be satisfied. However, there is a much simpler answer to this worry. If the effective potential of the field ϕ is so large that the field σ can easily jump to the true minimum of $V(\sigma)$, then the Universe becomes divided into infinitely many domains with all possible values of σ distributed in the following way [22, 7]:

$$\frac{P(\sigma=0)}{P(\sigma=\sigma_0)} \sim \exp\left(\frac{3M_{\rm P}^4}{8V(\phi,0)} - \frac{3M_{\rm P}^4}{8V(\phi,\sigma)}\right) = \exp\left(\frac{3M_{\rm P}^4}{4(m^2\phi^2+2V(0))} - \frac{3M_{\rm P}^4}{4m^2\phi^2}\right) \ . \tag{38}$$

One can easily check that at the moment when the field ϕ decreases to $\frac{MM_{\rm P}}{m}$ and the condition (37) becomes violated, we will have

$$\frac{P(0)}{P(\sigma_0)} \sim \exp\left(-\frac{C}{\lambda}\right) \ , \tag{39}$$

where C is some constant, $C = O(1)$. After this moment the probability of the false vacuum decay typically becomes much smaller. Thus the fraction of space which survives in the false vacuum state $\sigma = 0$ until this time typically is very small, but finite (and calculable). It is important, that these rare domains with $\sigma = 0$ eventually will dominate the volume of the Universe since if the probability of the false vacuum decay is small enough, the volume of the domains in the false vacuum will continue growing exponentially without end.

The main idea of our scenario can be explained as follows. Because the fields σ and ϕ do not interact with each other, and the dependence of the probability of tunneling on the vacuum energy at the GUT scale is negligibly small [49], tunneling to the minimum of $V(\sigma)$ may occur with approximately equal probability at all sufficiently small values of the field ϕ (see, however, below). The parameters of the bubbles of the field σ are determined by the mass scale M corresponding to the effective potential $V(\sigma)$. This mass scale in our model is much greater than m. Thus the duration of tunneling in the Euclidean "time" is much smaller than m^{-1}. Therefore the field ϕ practically does not change its value during the tunneling. If the probability of decay at a given ϕ is small enough, then it does not destroy the whole vacuum state $\sigma = 0$ [50]; the bubbles of the new phase are produced all the way when the field ϕ rolls down to $\phi = 0$. In this process the Universe becomes filled with (nonoverlapping) bubbles immersed in the false vacuum state with $\sigma = 0$. Interior of each of these bubbles represents an open Universe. However, these bubbles contain *different* values of the field ϕ, depending on the value of this field at the moment when the bubble formation occurred. If the field ϕ inside a bubble is smaller than $3M_{\rm P}$, then the Universe inside this bubble will have a vanishingly small Ω, at the age 10^{10} years after the end of inflation it will be practically empty, and life of our type would not exist there. If the field ϕ is much greater than $3M_{\rm P}$, the Universe inside the bubble will be almost exactly flat, $\Omega = 1$, as in the simplest version of the chaotic inflation scenario. It is important, however, that *in an eternally existing self-reproducing Universe there will be infinitely many Universes containing any particular value of* Ω*, from* $\Omega = 0$ *to* $\Omega = 1$, and one does not need any fine tuning of the effective potential to obtain a Universe with, say, $0.2 < \Omega < 0.3$

Of course, one can argue that we did not solve the problem of fine tuning, we just transformed it into the fact that only a very small percentage of all Universes will have $0.2 < \Omega < 0.3$. However, first of all, we achieved our goal in a very simple theory, which does not require any artificial potential bending and nonminimal kinetic terms. Then, there may be some reasons why it is preferable for us to live in a Universe with a small (but not vanishingly small) Ω.

The simplest way to approach this problem is to find how the probability for the bubble production depends on ϕ. As we already pointed out, for small ϕ this dependence is not very strong. On the other hand, at large ϕ the probability rapidly grows and becomes quite large at $\phi > \frac{MM_{\rm P}}{m}$. This may suggest that the bubble production typically occurs at $\phi > \frac{MM_{\rm P}}{m}$, and then for $\frac{M}{m} \gg 3$ we typically obtain flat Universes, $\Omega = 1$. This is another manifestation of the problem of premature decay of the state $\sigma = 0$ which we discussed above. Moreover, even if the probability to produce the Universes with different ϕ were entirely ϕ-independent, one could argue that the main volume of the habitable parts of the Universe is contained in the bubbles with $\Omega = 1$, since the interior of each such bubble inflated longer. Indeed, the total volume of each bubble created in a state with the field ϕ during inflation in our model grows by the factor of $\exp \frac{6\pi\phi^2}{M_{\rm P}^2}$ [7]. It seems clear that the bubbles with greater ϕ will give the largest contribution to the total volume of the Universe after inflation. This would be the simplest argument in favor of the standard prediction $\Omega = 1$ even in our class of models.

However, there exist several ways of resolving this problem: involving coupling $g^2\phi^2\sigma^2$, which stabilizes the state $\sigma = 0$ at large ϕ, or adding nonminimal interaction with gravity of the form $-\frac{\xi}{2}R\phi^2$. In either way one can easily suppress production of the Universes with $\Omega = 1$. Then the maximum of probability will correspond to some value $\Omega < 1$, which can be made equal to any given number from 1 to 0 by changing the parameters g^2 and ξ.

For example, let us add to the Lagrangian the term $-\frac{\xi}{2}R\phi^2$. This term makes inflation impossible for $\phi > \phi_c = \frac{M_{\rm P}}{\sqrt{8\pi\xi}}$. If initial value of the field ϕ is much smaller than ϕ_c, the size of the Universe during inflation grows $\exp \frac{2\pi\phi^2}{M_{\rm P}^2}$ times, and the volume grows $\exp \frac{6\pi\phi^2}{M_{\rm P}^2}$ times, as in the theory $\frac{m^2}{2}\phi^2$ with $\xi = 0$. For initial ϕ approaching ϕ_c these expressions somewhat change, but in order to get a very rough estimate of the increase of the size of the Universe in this model (which is sufficient to get an illustration of our main idea) one can still use the old expression $\exp \frac{2\pi\phi^2}{M_P^2}$. This expression reaches its maximum near $\phi = \phi_c$, at which point the effective gravitational constant becomes infinitely large and inflationary regime ceases to exist [54, 38]. Thus, one may argue that in this case the main part of the volume of the Universe will appear from the bubbles with initial value of the field ϕ close to ϕ_c. For $\xi \ll 4.4 \times 10^{-3}$ one has $\phi_c \gg 3M_P$. In this case one would have typical Universes expanding much more than e^{60} times, and therefore $\Omega \approx 1$. For $\xi \gg 4.4 \times 10^{-3}$ one has $\phi_c \ll 3M_P$, and therefore one would have $\Omega \ll 1$ in all inflationary bubbles. It is clear that by choosing particular values of the constant ξ in the range of $\xi \sim 4.4 \times 10^{-3}$ one can obtain the distribution of the Universes with the maximum of the distribution concentrated near any desirable value of $\Omega < 1$. Note that the position of the peak of the distribution is very sensitive to the value of ξ: to have the peak concentrated in the region $0.2 < \Omega < 0.3$ one would have to fix ξ (i.e. ϕ_c) with an accuracy of few percent. Thus, in this approach to the calculation of probabilities to live in a Universe with a given value of Ω we still have the problem of fine tuning.

However, calculation of probabilities in the context of the theory of a self-reproducing Universe is a very ambiguous process, and it is even not quite clear that this process makes any sense at all. For example, we may formulate the problem in a different way. Consider a domain of the false vacuum with $\sigma = 0$ and $\phi = \phi_1$. After some evolution it produces one or many bubbles with $\sigma = \sigma_0$ and the field ϕ which after some time becomes equal to ϕ_2. One may argue that the most efficient way this process may go is the way which in the end produces the greater volume. Indeed, for the inhabitants of a bubble it does not matter how much time did it take for this process to occur. The total number of observers produced by this process will depend on the total volume of the Universe at the hypersurface of a given density, i.e. on the hypersurface of a given ϕ. If the domain instantaneously tunnels to the state σ_0 and ϕ_1, and then the field ϕ in this domain slowly rolls from ϕ_1 to ϕ_2, then the volume of this domain grows $\exp\left(\frac{2\pi}{M_P^2}(\phi_1^2 - \phi_2^2)\right)$ times [7]. Meanwhile, if the tunneling takes a long time, then the field ϕ rolls down extremely slowly being in the false vacuum state with $\sigma = 0$. In this state the Universe expands much faster than in the state with $\sigma = \sigma_0$. Since it expands much faster, and it takes the field much longer to roll from ϕ_1 to ϕ_2, the trajectories of this kind bring us much greater volume. This may serve as an argument that most of the volume is produced by the bubbles created at a very small ϕ, which leads to the Universes with very small Ω.

One may use another set of considerations, studying all trajectories beginning at ϕ_1, t_1 and ending at ϕ_2, t_2. This will bring us another answer, or, to be more precise, another set of answers, which will depend on the choice of the time parametrization [25]. Still another answer will be obtained by the method recently proposed by Vilenkin, who suggested to introduce a particular cutoff procedure which (almost) completely eliminates dependence of the final answer on the time parametrization [32]. A more radical possibility would be to integrate over all time parametrizations. This task is very complicated, but it would completely eliminate dependence of the final answer on the time parametrization.

There is a very deep reason why the calculation of the probability to obtain a Universe with a given Ω is so ambiguous. We have discussed this reason in Sect. 3.1 in general terms; let us see how the situation looks in application to the open Universe scenario. For those who lives inside a bubble there is be no way to say at which stage (at which time from the point of view of an external observer) this bubble was produced. Therefore one should compare *all* of these bubbles produced at all possible times. The self-reproducing Universe should exist for indefinitely long time, and therefore it should contain infinitely many bubbles with all possible values of Ω. Comparing infinities is a very ambiguous task, which gives results depending on the procedure of comparison. For example, one can consider an infinitely large box of apples and an infinitely large box of oranges. One may pick up one apple and one orange, then one apple and one orange, over and over again, and conclude that there is an equal number of apples and oranges. However, one may also pick up one apple and two oranges, and then one apple and two oranges again, and conclude that there is twice as many oranges as apples. The same situation happens when one tries to compare the number of bubbles with different values of Ω. If we would know how to solve the problem of measure in quantum cosmology, perhaps we would be able to obtain something similar to an open Universe in the trivial $\lambda\phi^4$ theory without any first order phase transitions [31], see Sect. 3.1. In the meantime, it is already encouraging that in our scenario there are infinitely many inflationary universes with all possible value of $\Omega < 1$. We can hardly live in the empty bubbles with $\Omega = 0$. As for the choice between the bubbles with different nonvanishing values of $\Omega < 1$, it is quite possible that eventually we will find out an unambiguous way of predicting the most probable value of Ω, and we are going to continue our work in this direction. However, as we already discussed in the previous section, it might also happen that this question is as meaningless as the question whether it is more probable to be born as a Chinese rather than as an Italian. It is quite conceivable that the only way to find out in which of the bubbles do we live is to make observations.

Some words of caution are in order here. The bubbles produced in our simple model are not *exactly* open Universes. Indeed, in the models discussed in [49]–[45] the time of reheating (and the temperature of the Universe after the reheating) was synchronized with the value of the scalar field inside the bubble. In our case the situation is very similar, but not exactly. Suppose that the Hubble constant induced by $V(0)$ is much greater than the Hubble constant related to the energy density of the scalar field ϕ. Then the speed of rolling of the scalar field ϕ sharply increases inside the bubble. Thus, in our case the field σ synchronizes the motion of the field ϕ, and then the hypersurface of a constant field ϕ determines the hypersurface of a constant temperature. In the models where the rolling of the field ϕ can occur only inside the bubble (we will discuss such a model shortly) the synchronization is precise, and everything goes as in the models of refs. [49]–[45]. However, in our simple model the scalar field ϕ moves down outside the bubble as well, even though it does it very slowly. Thus, synchronization of motion of the fields σ and ϕ is not precise; hypersurface of a constant σ ceases to be a hypersurface of a constant density. For example, suppose that the field ϕ has taken some value ϕ_0 near the bubble wall when the bubble was just formed. Then the bubble expands, and during this time the field ϕ outside the wall decreases, as $\exp\left(-\frac{m^2 t}{3H_1}\right)$, where $H_1 \approx H(\phi = \sigma = 0)$ is the Hubble constant at the first stage of inflation, $H_1 \approx \sqrt{\frac{8\pi V(0)}{3M_P^2}}$. At the moment when the bubble expands e^{60} times, the field ϕ in the region just reached by the bubble wall decreases to $\phi_o \exp\left(-\frac{20m^2}{H_1^2}\right)$ from its original value ϕ_0. the Universe inside the bubble is a homogeneous open Universe only if this change is negligibly small. This may not be a real problem. Indeed, let us assume that $V(0) = \tilde{M}^4$, where $\tilde{M} = 10^{17}$ GeV.

(Typically the energy density scale $\tilde{M}$ is related to the particle mass as follows: $\tilde{M} \sim \lambda^{-1/4} M$.) In this case $H_1 = 1.7 \times 10^{15}$ GeV, and for $m = 10^{13}$ GeV one obtains $\frac{20m^2}{H_1^2} \sim 10^{-4}$. In such a case a typical degree of distortion of the picture of a homogeneous open Universe is very small.

Still this issue requires careful investigation. When the bubble wall continues expanding even further, the scalar field outside of it eventually drops down to zero. Then there will be no new matter created near the wall. Instead of infinitely large homogeneous open Universes we are obtaining spherically symmetric islands of a size much greater than the size of the observable part of our Universe. We do not know whether this unusual picture is an advantage or a disadvantage of our model. Is it possible to consider different parts of the same exponentially large island as domains of different "effective" Ω? Can we attribute some part of the dipole anisotropy of the microwave background radiation to the possibility that we live somewhere outside of the center of such island? In any case, as we already mentioned, in the limit $m^2 \ll H_1^2$ we do not expect that the small deviations of the geometry of space inside the bubble from the geometry of an open Universe can do much harm to our model.

Our model admits many generalizations, and details of the scenario which we just discussed depend on the values of parameters. Let us forget for a moment about all complicated processes which occur when the field ϕ is rolling down to $\phi = 0$, since this part of the picture depends on the validity of our ideas about initial conditions. For example, there may be no self-reproduction of inflationary domains with large ϕ if one considers an effective potential of the field ϕ which is very curved at large ϕ. However, there will be self-reproduction of the Universe in a state $\phi = \sigma = 0$, as in the old inflation scenario. Then the main portion of the volume of the Universe will be determined by the processes which occur when the fields ϕ and σ stay at the local minimum of the effective potential, $\phi = \sigma = 0$. For definiteness we will assume here that $V(0) = \tilde{M}^4$, where $\tilde{M}$ is the stringy scale, $\tilde{M} \sim 10^{17} - 10^{18}$ GeV. Then the Hubble constant $H_1 = \sqrt{\frac{8\pi V(0)}{3M_{\rm P}^2}} \sim \sqrt{\frac{8\pi}{3}}\frac{\tilde{M}^2}{M_{\rm P}}$ created by the energy density $V(0)$ is much greater than $m \sim 10^{13}$ GeV. In such a case the scalar field ϕ will not stay exactly at $\phi = 0$. It will be relatively homogeneous on the horizon scale H_1^{-1}, but otherwise it will be chaotically distributed with the dispersion $\langle\phi^2\rangle = \frac{3H^4}{8\pi^2 m^2}$ [7]. This means that the field ϕ inside each of the bubbles produced by the decay of the false vacuum can take any value ϕ with the probability

$$P \sim \exp\left(-\frac{\phi^2}{2\langle\phi^2\rangle}\right) \sim \exp\left(-\frac{3m^2\phi^2 M_{\rm P}^4}{16\tilde{M}^8}\right) \ . \tag{40}$$

One can check that for $\tilde{M} \sim 4.3 \times 10^{17}$ GeV the typical value of the field ϕ inside the bubbles will be $\sim 3 \times 10^{19}$ GeV. Thus, for $\tilde{M} > 4.3 \times 10^{17}$ GeV most of the Universes produced during the vacuum decay will be flat, for $\tilde{M} < 4.3 \times 10^{17}$ GeV most of them will be open. It is interesting that in this version of our model the percentage of open Universes is determined by the stringy scale (or by the GUT scale). However, since the process of bubble production in this scenario goes without end, the total number of Universes with any particular value of $\Omega < 1$ will be infinitely large for any value of $\tilde{M}$. Thus this model shows us is the simplest way to resurrect some of the ideas of the old inflationary theory with the help of chaotic inflation, and simultaneously to obtain inflationary Universe with $\Omega < 1$.

Note that this version of our model will not suffer for the problem of incomplete synchronization. Indeed, the average value of the field ϕ in the false vacuum outside the bubble will remain constant until the bubble triggers its decrease. However, this model, just as its previous version, may suffer from another problem. The Hubble constant H_1 before the tunneling in this model was much greater than the Hubble constant H_2 at the beginning of the second stage of inflation. Therefore the fluctuations of the scalar field before the tunneling were very large, $\delta\phi \sim \frac{H_1}{2\pi}$, much greater than the fluctuations generated after the tunneling, $\delta\phi \sim \frac{H_2}{2\pi}$. This may lead to very large density perturbations on the scale comparable to the size of the bubble. For the models with $\Omega = 1$ this effect would not cause any problems since such perturbations would be far away over the present particle horizon, but for small Ω this may lead to unacceptable anisotropy of the microwave background radiation.

Fortunately, this may not be a real difficulty. A possible solution is very similar to the bubble symmetrization described in the previous section.

Indeed, let us consider more carefully how the long wave perturbations produced outside the bubble may penetrate into it. At the moment when the bubble is formed, it has a size (28), which is smaller than H_1^{-1} [49]. Then the bubble walls begin moving with the speed gradually approaching the speed of light. At this stage the comoving size of the bubble (from the point of view of the original coordinate system in the false vacuum) grows like

$$r(t) = \int_0^t dt e^{-H_1 t} = H_1^{-1}(1 - e^{-H_1 t}) \ . \tag{41}$$

During this time the fluctuations of the scalar field ϕ of the amplitude $\frac{H_1}{2\pi}$ and of the wavelength H_1^{-1}, which previously were outside the bubble, gradually become covered by it. When these perturbations are outside the bubble, inflation with the Hubble constant H_1 prevents them from oscillating and moving. However, once these perturbations penetrate inside the bubble, their amplitude becomes decreasing [56, 57]. Indeed, since the wavelength of the perturbations is $\sim H_1^{-1} \ll H_2^{-1} \ll m^{-1}$, these perturbations move inside the bubbles as relativistic particles, their wavelength grow as $a(t)$, and their amplitude decreases just like an amplitude of electromagnetic field, $\delta\phi \sim a^{-1}(t)$, where a is the scale factor of the Universe inside a bubble [56]. This process continues until the wavelength of each perturbation reaches H_2^{-1} (already at the second stage of inflation). During this time the wavelength grows $\frac{H_1}{H_2}$ times, and the amplitude decreases $\frac{H_2}{H_1}$ times, to become the standard amplitude of perturbations produced at the second stage of inflation: $\frac{H_2}{H_1}\frac{H_1}{2\pi} = \frac{H_2}{2\pi}$.

In fact, one may argue that this computation was too naive, and that these perturbations should be neglected altogether. Typically we treat long wave perturbations in inflationary universe like classical wave for the reason that the waves with the wavelength much greater than the horizon can be interpreted as states with extremely large occupation numbers [7]. However, when the new born perturbations (i.e. fluctuations which did not acquire an exponentially large wavelength yet) enter the bubble (i.e. under the horizon), they effectively return to the realm of quantum fluctuations again. Then one may argue that one should simply forget about the waves with the wavelengths small enough to fit into the bubble, and consider perturbations created at the second stage of inflation not as a result of stretching of these waves, but as a new process of creation of perturbations of an amplitude $\frac{H_2}{2\pi}$.

One may worry that perturbations which had wavelengths somewhat greater than H_1^{-1} at the moment of the bubble formation cannot completely penetrate into the bubble. If, for example, the field ϕ differs from some constant by $+\frac{H_1}{2\pi}$ at the distance H_1^{-1} to the left of the bubble at the moment of its formation, and by $-\frac{H_1}{2\pi}$ at the distance H_1^{-1} to the right of the bubble, then this difference remains frozen independently of all processes inside the bubble. This may suggest that there is some unavoidable asymmetry of the distribution of the field inside the bubble. However, the field inside the bubble will not be distributed like a straight line slowly rising from $-\frac{H_1}{2\pi}$ to $+\frac{H_1}{2\pi}$. Inside the bubble the field will be almost homogeneous; the inhomogeneity $\delta\phi \sim -\frac{H_1}{2\pi}$ will be concentrated only in a small vicinity near the bubble wall.

Finally we should verify that this scenario leads to bubbles which are symmetric enough, see eq. (35). Fortunately, here we do not have any problems. One can easily check that for our model with $m \sim 10^{13}$ GeV and $\tilde{M} \sim \lambda^{-1/4} M > 10^{17} GeV$ the condition (35) can be satisfied even for not very small values of the coupling constant λ.

The arguments presented above should be confirmed by a more detailed investigation of the vacuum structure inside the expanding bubble in our scenario. If, as we hope, the result of the investigation will be positive, we will have an extremely simple model of an open inflationary universe. In the meantime, it would be nice to have a model where we do not have any problems at all with synchronization and with large fluctuations on the scalar field in the false vacuum. We will consider such a model in the next section.

4.4 Hybrid inflation and natural inflation with $\Omega < 1$

The model to be discussed below is a version of the hybrid inflation scenario [39], which is a slight generalization (and a simplification) of our previous model (36):

$$V(\phi, \sigma) = \frac{g^2}{2}\phi^2\sigma^2 + V(\sigma) \ . \tag{42}$$

We eliminated the massive term of the field ϕ and added explicitly the interaction $\frac{g^2}{2}\phi^2\sigma^2$, which, as we have mentioned already, can be useful (though not necessary) for stabilization of the state $\sigma = 0$ at large ϕ. Note that in this model the line $\sigma = 0$ is a flat direction in the (ϕ, σ) plane. At large ϕ the only minimum of the effective potential with respect to σ is at the line $\sigma = 0$. To give a particular example, one can take $V(\sigma) = \frac{M^2}{2}\sigma^2 - \alpha M\sigma^3 + \frac{\lambda}{4}\sigma^4 + V_0$. Here V_0 is a constant which is added to ensure that $V(\phi, \sigma) = 0$ at the absolute minimum of $V(\phi, \sigma)$. In this case the minimum of the potential $V(\phi, \sigma)$ at $\sigma \neq 0$ is deeper than the minimum at $\sigma = 0$ only for $\phi < \phi_c$, where $\phi_c = \frac{M}{g}\sqrt{\frac{2\alpha^2}{\lambda} - 1}$. This minimum for $\phi = \phi_c$ appears at $\sigma = \sigma_c = \frac{2\alpha M}{\lambda}$.

The bubble formation becomes possible only for $\phi < \phi_c$. After the tunneling the field ϕ acquires an effective mass $m = g\sigma$ and begins to move towards $\phi = 0$, which provides the mechanism for the second stage of inflation inside the bubble. In this scenario evolution of the scalar field ϕ is exactly synchronized with the evolution of the field σ, and the Universe inside the bubble appears to be open.

Effective mass of the field ϕ at the minimum of $V(\phi,\sigma)$ with $\phi = \phi_c$, $\sigma = \sigma_c = \frac{2\alpha M}{\lambda}$ is $m = g\sigma_c = \frac{2g\alpha M}{\lambda}$. With a decrease of the field ϕ its effective mass at the minimum of $V(\phi,\sigma)$ will grow, but not significantly. For simplicity, we will consider the case $\lambda = \alpha^2$. In this case it can be shown that $V(0) = 2.77\,\frac{M^4}{\lambda}$, and the Hubble constant before the phase transition is given by $4.8\,\frac{M^2}{\sqrt{\lambda}M_{\rm P}}$. One should check what is necessary to avoid too large density perturbations (35). However, one should take into account that the mass M in (35) corresponds to the curvature of the effective potential near $\phi = \phi_c$ rather than at $\phi = 0$. In our case this implies that one should use $\sqrt{2}M$ instead of M in this equation. Then one obtains the following constraint on the mass M: $M\sqrt{\mu} \lesssim 2 \times 10^{15}$ GeV. Note that the thin wall approximation (requiring $\mu \ll 1$) breaks down far away from $\phi = \phi_c$. Therefore in general eq. (35) should be somewhat improved. However for $\phi \approx \phi_c$ it works quite well. To be on a safe side, we will take $M = 5 \times 10^{14}$ GeV. Other parameters may vary; one may consider, e.g., the theory with $g \sim 10^{-5}$, which gives $\phi_c = \frac{M}{g} \sim 5 \times 10^{19}$ GeV $\sim 4M_{\rm P}$. The effective mass m after the phase transition is equal to $\frac{2gM}{\sqrt{\lambda}}$ at $\phi = \phi_c$, and then it grows by only 25% when the field ϕ changes all the way down from ϕ_c to $\phi = 0$. As we already mentioned, in order to obtain the proper amplitude of density perturbations produced by inflation inside the bubble one should have $m \sim 10^{13}$ GeV. This corresponds to $\lambda = \alpha^2 = 10^{-6}$.

The bubble formation becomes possible only for $\phi < \phi_c$. If it happens in the interval $4M_{\rm P} > \phi > 3M_{\rm P}$, we obtain a flat Universe. If it happens at $\phi < 3M_{\rm P}$, we obtain an open Universe. Depending on the initial value of the field ϕ, we can obtain all possible values of Ω, from $\Omega = 1$ to $\Omega = 0$. The value of the Hubble constant at the minimum with $\sigma \neq 0$ at $\phi = 3M_{\rm P}$ in our model does not differ much from the value of the Hubble constant before the bubble formation. Therefore we do not expect any specific problems with the large scale density perturbations in this model. Note also that the probability of tunneling at large ϕ is very small since the depth of the minimum at $\phi \sim \phi_c$, $\sigma \sim \sigma_c$ does not differ much from the depth of the minimum at $\sigma = 0$, and there is no tunneling at all for $\phi > \phi_c$. Therefore the number of flat Universes produced by this mechanism will be strongly suppressed as compared with the number of open Universes, the degree of this suppression being very sensitive to the value of ϕ_c. Meanwhile, life of our type is impossible in empty Universes with $\Omega \ll 1$. This may provide us with a tentative explanation of the small value of Ω in the context of our model.

Another model of inflation with $\Omega < 1$ is the based on a certain modification of the "natural inflation" scenario [41]. The main idea is to take the effective potential of the "natural inflation" model, which looks like a tilted Mexican hat, and make a deep hole it its center at $\phi = 0$ [48]. In the beginning inflation occurs near $\phi = 0$, but then the bubbles with $\phi \neq 0$ appear. Depending on the phase of the complex scalar field ϕ inside the bubble, the next stage of inflation, which occurs just as in the old version of the "natural inflation" scenario, leads to formation of the Universes with all possible values of Ω.

A detailed discussion of this scenario can be found in [48]; we will not repeat it here. What is most important for us is that there exist several rather simple models of an open inflationary universe. Inflationary models with $\Omega = 1$ admittedly are somewhat simpler. Therefore we still hope that several years later we will know that our Universe is flat, which will be a strong experimental evidence in favor of inflationary cosmology in its simplest form. However, if observational data will show, beyond any reasonable doubt, that $\Omega \neq 1$, it will not imply that inflationary theory is wrong. Indeed, now we know that there is a large class of internally consistent cosmological models which may describe creation of large homogeneous Universes with all possible values of Ω, and so far all of these models are based on inflationary cosmology.

5 Reheating after inflation

The theory of reheating of the Universe after inflation is the most important application of the quantum theory of particle creation, since almost all matter constituting the Universe at the subsequent radiation-dominated stage was created during this process [7]. At the stage of inflation all energy was concentrated in a classical slowly moving inflaton field ϕ. Soon after the end of inflation this field began to oscillate near the minimum of its effective potential. Gradually it produced many elementary particles, they interacted with each other and came to a state of thermal equilibrium with some temperature T_r, which was called the reheating temperature.

An elementary theory of reheating was first developed in [58] for the new inflationary scenario. Independently a theory of reheating in the R^2 inflation was constructed in [59]. Various aspects of this theory were further elaborated by many authors, see e.g. [62, 63]. Still, a general scenario of reheating was absent. In particular, reheating in the chaotic inflation theory remained almost unexplored. The present section contains results obtained recently in our work with Kofman and Starobinsky [60]. We have found that the process of reheating typically consists of three different stages. At the first stage, which cannot be described by the elementary theory of reheating, the classical coherently oscillating inflaton field ϕ decays into massive bosons (in particular, into ϕ-particles) due to parametric resonance. In many models the resonance is very broad, and the process occurs extremely rapidly (explosively). Because of the Pauli exclusion principle, there is no explosive creation of fermions. To distinguish this stage from the stage of particle decay and thermalization, we will call it *pre-heating.* Bosons produced at that stage are far away from thermal equilibrium and typically have enormously large occupation numbers. The second stage is the decay of previously produced particles. This stage typically can be described by methods developed in [58]. However, these methods should be applied not to the decay of the original homogeneous inflaton field, but to the decay of particles and fields produced at the stage of explosive reheating. This considerably changes many features of the process, including the final value of the reheating temperature. The third stage is the stage of thermalization, which can be described by standard methods, see e.g. [7, 58]; we will not consider it here. Sometimes this stage may occur simultaneously with the second one. In our investigation we have used the formalism of the time-dependent Bogoliubov transformations to find the density of created particles, $n_{\vec{k}}(t)$. A detailed description of this theory will be given in [61]; here we will outline our main conclusions using a simple semiclassical approach.

We will consider a simple chaotic inflation scenario describing the classical inflaton scalar field ϕ with the effective potential $V(\phi) = \pm\frac{1}{2}m_\phi^2\phi^2 + \frac{\lambda}{4}\phi^4$. Minus sign corresponds to spontaneous symmetry breaking $\phi \to \phi + \sigma$ with generation of a classical scalar field $\sigma = \frac{m_\phi}{\sqrt{\lambda}}$. The field ϕ after inflation may decay into bosons χ and fermions ψ due to the interaction terms $-\frac{1}{2}g^2\phi^2\chi^2$ and $-h\bar{\psi}\psi\phi$. Here λ, g and h are small coupling constants. In case of spontaneous symmetry breaking, the term $-\frac{1}{2}g^2\phi^2\chi^2$ gives rise to the term $-g^2\sigma\phi\chi^2$. We will assume for simplicity that the bare masses of the fields χ and ψ are very small, so that one can write $m_\chi(\phi) = g\phi$, $m_\psi(\phi) = |h\phi|$.

Let us briefly recall the elementary theory of reheating [7]. At $\phi > M_{\rm P}$, we have a stage of inflation. This stage is supported by the friction-like term $3H\dot{\phi}$ in the equation of motion for the scalar field. Here $H \equiv \dot{a}/a$ is the Hubble parameter, $a(t)$ is the scale factor of the Universe. However, with a decrease of the field ϕ this term becomes less and less important, and inflation ends at $\phi \lesssim M_{\rm P}/2$. After that the field ϕ begins oscillating near the minimum of $V(\phi)$. The amplitude of the oscillations gradually decreases because of expansion of the Universe, and also because of the energy transfer to particles created by the oscillating field. Elementary theory of reheating is based on the assumption that the classical oscillating scalar field $\phi(t)$ can be represented as a collection of scalar particles at rest. Then the rate of decrease of the energy of oscillations coincides with the decay rate of ϕ-particles. The rates of the processes $\phi \to \chi\chi$ and $\phi \to \psi\psi$ (for $m_\phi \gg 2m_\chi, 2m_\psi$) are given by

$$\Gamma(\phi \to \chi\chi) = \frac{g^4\sigma^2}{8\pi m_\phi} \ , \qquad \Gamma(\phi \to \psi\psi) = \frac{h^2 m_\phi}{8\pi} \ . \tag{43}$$

Reheating completes when the rate of expansion of the Universe given by the Hubble constant $H = \sqrt{\frac{8\pi\rho}{3M_{\rm P}^2}} \sim t^{-1}$ becomes smaller than the total decay rate $\Gamma = \Gamma(\phi \to \chi\chi) + \Gamma(\phi \to \psi\psi)$. The reheating temperature can be estimated by $T_r \simeq 0.1\sqrt{\Gamma M_{\rm P}}$.

It is interesting to note that in accordance with the elementary theory of reheating the amplitude squared of the oscillating scalar field decays exponentially, as $e^{-\Gamma t}$. Phenomenologically, this can be described by adding the term $\Gamma\dot{\phi}$ to the equation of motion of the scalar field. Unfortunately, many authors took this prescription too seriously and investigated the possibility that the term $\Gamma\dot{\phi}$, just like the term $3H\dot{\phi}$, can support inflation. We should emphasize [60], that adding the term $\Gamma\dot{\phi}$ to the equation of motion is justified only at the stage of oscillations (i.e. after the end of inflation), and only for the description of the *amplitude of oscillations* of the scalar field, rather than for the description of the scalar field itself. Moreover, even at the stage of oscillations this description becomes incorrect as soon as the resonance effects become important.

As we already mentioned, elementary theory of reheating can provide a qualitatively correct description of particle decay at the last stages of reheating. Moreover, this theory is always applicable if the inflaton field can decay into fermions only, with a small coupling constant $h^2 \ll m_\phi/M_{\rm P}$. However, typically this theory is inapplicable to the description of the first stages of reheating, which makes the whole process quite different. In what follows we will develop the theory of the first stages of reheating. We will begin with the theory of a massive scalar field ϕ decaying into particles χ, then we consider the theory $\frac{\lambda}{4}\phi^4$, and finally we will discuss reheating in the theories with spontaneous symmetry breaking.

We begin with the investigation of the simplest inflationary model with the effective potential $\frac{m_\phi^2}{2}\phi^2$. Suppose that this field only interacts with a light scalar field χ ($m_\chi \ll m_\phi$) due to the term $-\frac{1}{2}g^2\phi^2\chi^2$. The equation for quantum fluctuations of the field χ with the physical momentum $\vec{k}/a(t)$ has the following form:

$$\ddot{\chi}_k + 3H\dot{\chi}_k + \left(\frac{k^2}{a^2(t)} + g^2\Phi^2 \, \sin^2(m_\phi t)\right)\chi_k = 0 \ , \tag{44}$$

where $k = \sqrt{\vec{k}^2}$, and Φ stands for the amplitude of oscillations of the field ϕ. As we shall see, the main contribution to χ-particle production is given by excitations of the field χ with $k/a \gg m_\phi$, which is much greater than H at the stage of oscillations. Therefore, in the first approximation we may neglect the expansion of the Universe, taking $a(t)$ as a constant and omitting the term $3H\dot{\chi}_k$ in (44). Then the equation (44) describes an oscillator with a variable frequency $\Omega_k^2(t) = k^2a^{-2} + g^2\Phi^2 \, \sin^2(m_\phi t)$. Particle production occurs due to a nonadiabatic change of this frequency. Equation (44) can be reduced to the well-known Mathieu equation:

$$\chi_k'' + (A(k) - 2q\cos 2z)\,\chi_k = 0 \ , \tag{45}$$

where $A(k) = \frac{k^2}{m_\phi^2 a^2} + 2q$, $q = \frac{g^2\Phi^2}{4m_\phi^2}$, $z = m_\phi t$, prime denotes differentiation with respect to z. An important property of solutions of the equation (45) is the existence of an exponential instability $\chi_k \propto \exp(\mu_k^{(n)} z)$ within the set of resonance bands of frequencies $\Delta k^{(n)}$ labeled by an integer index n. This instability corresponds to exponential growth of occupation numbers of quantum fluctuations $n_{\vec{k}}(t) \propto \exp(2\mu_k^{(n)} m_\phi t)$ that may be interpreted as particle production. As one can show, near the line $A = 2q$ there are regions in the first, the second and the higher instability bands where the unstable modes grow extremely rapidly, with $\mu_k \sim 0.2$. We will show analytically in [61] that for $q \gg 1$ typically $\mu_k \sim \frac{\ln 3}{2\pi} \approx 0.175$ in the instability bands along the line $A = 2q$, but its maximal value is $\frac{\ln(1+\sqrt{2})}{\pi} \approx 0.28$. Creation of particles in the regime of a broad resonance ($q > 1$) with $2\pi\mu_k = O(1)$ is very different from that in the usually considered case of a narrow resonance ($q \ll 1$), where $2\pi\mu_k \ll 1$. In particular, it proceeds during a tiny part of each oscillation of the field ϕ when $1 - \cos z \sim q^{-1}$ and the induced effective mass of the field χ (which is determined by the condition $m_\chi^2 = g^2\Phi^2/2$) is less than m_ϕ. As a result, the number of particles grows exponentially within just a few oscillations of the field ϕ. This leads to an extremely rapid (explosive) decay of the classical scalar field ϕ. This regime occurs only if $q \gtrsim \pi^{-1}$, i.e. for $g\Phi \gtrsim m_\phi$, so that $m_\phi \ll gM_{\rm P}$ is the necessary condition for it. One can show that a typical energy E of a particle produced at this stage is determined by equation $A - 2q \sim \sqrt{q}$, and is given by $E \sim \sqrt{gm_\phi M_{\rm P}}$ [61].

Creation of χ-particles leads to the two main effects: transfer of the energy from the homogeneous field $\phi(t)$ to these particles and generation of the contribution to the effective mass of the ϕ field: $m_{\phi,eff}^2 = m_\phi^2 + g^2\langle\chi^2\rangle_{ren}$. The last term in the latter expression quickly becomes larger than m_ϕ^2. One should take both these effects into account when calculating backreaction of created particles on the process. As a result, the stage of the broad resonance creation ends up within the short time $t \sim m_\phi^{-1}\ln(m_\phi/g^5M_{\rm P})$, when $\Phi^2 \sim \langle\chi^2\rangle$ and $q = \frac{g^2\Phi^2}{4m_{\phi,eff}^2}$ becomes smaller than 1. At this time the energy density of produced particles $\sim E^2\langle\chi^2\rangle \sim gm_\phi M_{\rm P}\Phi^2$

is of the same order as the original energy density $\sim m_\phi^2 M_{\rm P}^2$ of the scalar field ϕ at the end of inflation. This gives the amplitude of oscillations at the end of the stage of the broad resonance particle creation: $\Phi^2 \sim \langle\chi^2\rangle \sim g^{-1} m_\phi M_{\rm P} \ll M_{\rm P}^2$. Since $E \gg m_\phi$, the effective equation of state of the whole system becomes $p \approx \rho/3$. Thus, explosive creation practically eliminates a prolonged intermediate matter-dominated stage after the end of inflation which was thought to be characteristic to many inflationary models. However, this does not mean that the process of reheating has been completed. Instead of χ-particles in the thermal equilibrium with a typical energy $E \sim T \sim (m M_{\rm P})^{1/2}$, one has particles with a much smaller energy $\sim (g m_\phi M_{\rm P})^{1/2}$, but with extremely large mean occupation numbers $n_k \sim g^{-2} \gg 1$.

After that the Universe expands as $a(t) \propto \sqrt{t}$, and the scalar field ϕ continues its decay in the regime of the narrow resonance creation $q \approx \frac{\Phi^2}{4\langle\chi^2\rangle} \ll 1$. As a result, ϕ decreases rather slowly, $\phi \propto t^{-3/4}$. This regime is very important because it makes the energy of the ϕ field much smaller than that of the χ-particles. One can show that the decay finally stops when the amplitude of oscillations Φ becomes smaller than $g^{-1} m_\phi$ [61]. This happens at the moment $t \sim m_\phi^{-1}(g M_{\rm P}/m_\phi)^{1/3}$ (in the case $m < g^7 M_{\rm P}$ decay ends somewhat later, in the perturbative regime). The physical reason why the decay stops is rather general: decay of the particles ϕ in our model occurs due to its interaction with another ϕ-particle (interaction term is quadratic in ϕ and in χ). When the field ϕ (or the number of ϕ-particles) becomes small, this process is inefficient. The scalar field can decay completely only if a single scalar ϕ-particle can decay into other particles, due to the processes $\phi \to \chi\chi$ or $\phi \to \psi\psi$, see eq. (43). If there is no spontaneous symmetry breaking and no interactions with fermions in our model, such processes are impossible.

At later stages the energy of oscillations of the inflaton field decreases as $a^{-3}(t)$, i.e. more slowly than the decrease of energy of hot ultrarelativistic matter $\propto a^{-4}(t)$. Therefore, the relative contribution of the field $\phi(t)$ to the total energy density of the Universe rapidly grows. This gives rise to an unexpected possibility that the inflaton field by itself, or other scalar fields can be cold dark matter candidates, *even if they strongly interact with each other.* However, this possibility requires a certain degree of fine tuning; a more immediate application of our result is that it allows one to rule out a wide class of inflationary models which do not contain interaction terms of the type of $g^2\sigma\phi\chi^2$ or $h\phi\bar{\psi}\psi$.

So far we have not considered the term $\frac{\lambda}{4}\phi^4$ in the effective potential. Meanwhile this term leads to production of ϕ-particles, which in some cases appears to be the leading effect. Let us study the ϕ-particle production in the theory with $V(\phi) = \frac{m_\phi^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$ with $m_\phi^2 \ll \lambda M_{\rm P}^2$. In this case the effective potential of the field ϕ soon after the end of inflation at $\phi \sim M_{\rm P}$ is dominated by the term $\frac{\lambda}{4}\phi^4$. Oscillations of the field ϕ in this theory are not sinusoidal, they are given by elliptic functions, but with a good accuracy one can write $\phi(t) \sim \Phi \sin(c\sqrt{\lambda}\int \Phi dt)$, where $c = \frac{\Gamma^2(3/4)}{\sqrt{\pi}} \approx 0.85$. the Universe at that time expands as at the radiation-dominated stage: $a(t) \propto \sqrt{t}$. If one neglects the feedback of created ϕ-particles on the homogeneous field $\phi(t)$, then its amplitude $\Phi(t) \propto a^{-1}(t)$, so that $a\Phi = const$. Using a conformal time η, exact equation for quantum fluctuations $\delta\phi$ of the field ϕ can be reduced to the Lame equation. The results remain essentially the same if we use an approximate equation

$$\frac{d^2(\delta\phi_k)}{d\eta^2}+\left[k^2+3\lambda a^2\Phi^2\,\sin^2(c\sqrt{\lambda}a\Phi\eta)\right]\delta\phi_k=0,\quad \eta=\int\frac{dt}{a(t)}=\frac{2t}{a(t)}\,, \tag{46}$$

which leads to the Mathieu equation with $A=\frac{k^2}{c^2\lambda a^2\Phi^2}+\frac{3}{2c^2}\approx\frac{k^2}{c^2\lambda a^2\Phi^2}+2.08$, and $q=\frac{3}{4c^2}\approx 1.04$. Looking at the instability chart, we see that the resonance occurs in the second band, for $k^2\sim 3\lambda a^2\Phi^2$. The maximal value of the coefficient μ_k in this band for $q\sim 1$ approximately equals to 0.07. As long as the backreaction of created particles is small, expansion of the Universe does not shift fluctuations away from the resonance band, and the number of produced particles grows as $\exp(2c\mu_k\sqrt{\lambda}a\Phi\eta)\sim\exp(\frac{\sqrt{\lambda}\Phi t}{5})$.

After the time interval $\sim M_{\rm P}^{-1}\lambda^{-1/2}|\ln\lambda|$, backreaction of created particles becomes significant. The growth of the fluctuations $\langle\phi^2\rangle$ gives rise to a contribution $3\lambda\langle\phi^2\rangle$ to the effective mass squared of the field ϕ, both in the equation for $\phi(t)$ and in Eq. (46) for inhomogeneous modes. The stage of explosive reheating ends when $\langle\phi^2\rangle$ becomes greater than Φ^2. After that, $\Phi^2\ll\langle\phi^2\rangle$ and the effective frequency of oscillations is determined by the term $\sqrt{3\lambda\langle\phi^2\rangle}$. The corresponding process is described by Eq. (45) with $A(k)=1+2q+\frac{k^2}{3\lambda a^2\langle\phi^2\rangle}$, $q=\frac{\Phi^2}{4\langle\phi^2\rangle}$. In this regime $q\ll 1$, and particle creation occurs in the narrow resonance regime in the second band with $A\approx 4$. Decay of the field in this regime is extremely slow: one can show [61] that the amplitude Φ decreases only by a factor $t^{1/12}$ faster that it would decrease without any decay, due to the expansion of the Universe only, i.e., $\Phi\propto t^{-7/12}$. Reheating stops altogether when the presence of non-zero mass m_ϕ though still small as compared to $\sqrt{3\lambda\langle\phi^2\rangle}$ appears enough for the expansion of the Universe to drive a mode away from the narrow resonance. It happens when the amplitude Φ drops up to a value $\sim m_\phi/\sqrt{\lambda}$.

In addition to this process, the field ϕ may decay to χ-particles. This is the leading process for $g^2\gg\lambda$. The equation for χ_k quanta has the same form as eq. (46) with the obvious change $\lambda\to g^2/3$. Initially parametric resonance is broad. The values of the parameter μ_k along the line $A=2q$ do not change monotonically, but typically for $q\gg 1$ they are 3 to 4 times greater than the parameter μ_k for the decay of the field ϕ into its own quanta. Therefore, this pre-heating process is very efficient. It ends at the moment $t\sim M_{\rm P}^{-1}\lambda^{-1/2}\ln(\lambda/g^{10})$ when $\Phi^2\sim\langle\chi^2\rangle\sim g^{-1}\sqrt{\lambda}M_{\rm P}^2$. The typical energy of created χ-particles is $E\sim(g^2\lambda)^{1/4}M_{\rm P}$. The following evolution is essentially the same as that described above for the case of a massive scalar field decaying into χ-particles.

Finally, let us consider the case with symmetry breaking. In the beginning, when the amplitude of oscillations is much greater than σ, the theory of decay of the inflaton field is the same as in the case considered above. The most important part of pre-heating occurs at this stage. When the amplitude of the oscillations becomes smaller than $m_\phi/\sqrt{\lambda}$ and the field begins oscillating near the minimum of the effective potential at $\phi=\sigma$, particle production due to the narrow parametric resonance typically becomes very weak. The main reason for this is related to the backreaction of particles created at the preceding stage of pre-heating on the rate of expansion of the Universe and on the shape of the effective potential [61]. A rather interesting effect which makes investigation of this regime especially complicated is a temporary (non-thermal) symmetry restoration which occurs because of the interaction of the field ϕ with its fluctuations $<\phi^2>$.

Importance of spontaneous symmetry breaking for the theory of reheating should not be underestimated, since it gives rise to the interaction term $g^2\sigma\phi\chi^2$ which is linear in ϕ. Such terms are necessary for a complete decay of the inflaton field in accordance with the perturbation theory (43).

In this section we presented the new theory of reheating developed in [60], where we performed an investigation of reheating with an account of expansion of the Universe and of the backreaction of created particles, both in the broad resonance regime and in the narrow resonance case. As a result of this investigation, we obtained equations for the power-law decrease of the amplitude of an oscillating scalar field with an account taken of all of these effects. During the last year there appeared many other papers on the theory of reheating [64]–[68], which made the physical picture of reheating even more clear. Unfortunately, it is not easy to compare the results obtained in [64]–[68] with the results of our work [60]. For example, a very thorough investigation of reheating in the narrow resonance regime without a complete account of backreaction was performed in [64, 67, 68], and their results in this approximation agree with the corresponding results of [60]. However, as we have seen, at the first, most efficient stages of reheating the resonance is broad, and when it becomes narrow a complete account of backreaction becomes necessary [60]. Backreaction was studied in a very detailed way in ref. [65], but their investigation was performed neglecting expansion of the Universe, which was an important part of our work. That is why in this review we concentrated on the results obtained in [60]. However, to obtain a complete theory of reheating a much more detailed investigation will be necessary, and in this respect many of the results obtained in [64]–[68] should be very useful.

We should emphasize that the stage of parametric resonance is just the first stage of the process. If one naively takes the energy density at the end of explosive reheating and assumes that this energy density instantaneously transfers to heat, one may overestimate the reheating temperature by many orders of magnitude. Indeed, after the stage of explosive reheating the bose-particles created at this stage have enormously large occupation numbers, and they should further decay into the usual elementary particles. This may take a lot of time, during which the energy density of the Universe may decrease dramatically. To find the reheating temperature one should investigate the subsequent decay of the particles created at the stage of explosive reheating. This decay can be described by the old perturbative methods developed in [58]. Note, however, that now this theory should be applied not to the decay of the original large and homogeneous oscillating inflaton field, but to the decay of particles produced at the stage of pre-heating, as well as to the decay of small remnants of the classical inflaton field. This makes a lot of difference, since typically coupling constants of interaction of the inflaton field with matter are extremely small, whereas coupling constants involved in the decay of other bosons can be much greater. As a result, the reheating temperature can be much higher than the typical temperature $T_r \lesssim 10^9$ GeV which could be obtained neglecting the stage of parametric resonance [61]. In addition, one should make a careful study of the process of establishing of thermal equilibrium [66]. On the other hand, such processes as baryon creation after inflation occur best of all outside the state of thermal equilibrium. Therefore, the stage of explosive reheating (pre-heating), which produces fields and particles outside of the state of thermal equilibrium, may play an extremely important role in the cosmological theory. Another consequence of the resonance effects is an almost instantaneous change of equation of state from the vacuum-like one to the equation of state of relativistic matter $p = \rho/3$. This may be important for investigation of the primordial black hole formation, which could appear from growing density perturbations if equation of state after inflation for a long time was $p = 0$.

A rather nontrivial example of reheating appears in inflationary models based on supergravity, see e.g. [69]–[72]. The leading mode of the single-inflaton decay in such models often involves creation of a gravitino, which is a fermion. This does not necessarily mean that the first explosive stage cannot be realized in such models. Indeed, just as in the theory $\frac{\lambda}{4}\phi^4$, at the first stage the homogeneous classical oscillating inflaton field ϕ may decay into decoherent waves or particles of the same field ϕ. However, this will be just a first stage of reheating, after which one should consider decay of the inflaton particles by the usual perturbative methods. In such a situation one does not expect any deviations of the reheating temperature from its value obtained by perturbative methods [58], [69]–[72].

One should note also that in certain models the oscillations of the scalar field from the very beginning occur in the region where the conditions for the explosive reheating formulated in [60] are not satisfied. Such a situation occurs, e.g., in “natural inflation” [73], where the change of the effective mass of the inflaton field during its oscillations is relatively small, and the conditions of existence of narrow resonance in expanding Universe derived in [60] are violated.

Let us briefly summarize our results:

1. In many models where decay of the inflaton field can occur in the purely bosonic sector the first stages of reheating occur due to parametric resonance. This process (pre-heating) is extremely efficient even if the corresponding coupling constants are very small. However, there is no explosive reheating in the models where decay of the inflaton field is necessarily accompanied by fermion production.

2. The stage of explosive reheating due to a broad resonance typically is very short. Later the resonance becomes narrow, and finally the stage of pre-heating finishes altogether. Interactions of particles produced at this stage, their decay into other particles and subsequent thermalization typically require much more time that the stage of pre-heating, since these processes are suppressed by the small values of coupling constants.

3. The last stages of reheating typically can be described by the elementary theory of reheating [58]. However, this theory should be applied not to the original inflaton field, but to the products of its decay formed at the stage of explosive reheating. In some models it changes the final value of the reheating temperature.

4. Existence of the intermediate stage between the end of explosive reheating and the beginning of thermal equilibrium may have important implications for the theory of baryogenesis.

5. Reheating never completes in the theories where a single ϕ-particle cannot decay into other particles. This implies that reheating completes only if the theory contains interaction terms like $\phi\sigma\chi^2$ of $\phi\bar{\psi}\psi$. In most cases the theories where reheating never completes contradict observational data. On the other hand, this result suggests an interesting possibility that the classical scalar

fields (maybe even the inflaton field itself) may be responsible for the dark matter of the Universe even if they strongly interact with other matter fields.

6 Conclusions

Inflationary theory is already more than 15 years old, and its main principles seem to be well understood. Nevertheless, it is young enough to bring us many new surprises. Originally we expected that inflation was a short intermediate stage after the hot big bang. Now it seems that the standard big bang theory is only a part of inflationary cosmology which describes local (but not global) properties of the self-reproducing inflationary universe. Even though each part of the Universe expands (or collapses), the Universe as a whole may be stationary. One of the main purposes of inflationary cosmology was to solve the primordial monopole problem by expanding the distance between the monopoles. Recently we learned that the monopoles themselves may expand exponentially and become as large as a universe [74]. On the other hand, we learned that an infinitely large open inflationary universe may fit into an interior of a single bubble of a finite size produced during the false vacuum decay. This demonstrated that even though $\Omega = 1$ remains one of the rather robust predictions of inflationary cosmology, it will be impossible to kill inflation by proving that our universe is open. The process of creation of matter after inflation also happened to be extremely interesting and complicated, involving investigation of nonperturbative resonance effects in an expanding universe. Rapid development of the inflationary theory is a very good sign indicating that we are moving fast towards a complete cosmological theory – assuming, as we all hope, that we have chosen the right direction.

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