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Building GUTs from Strings

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Abstract

We study in detail the structure of Grand Unified Theories derived as the low-energy limit of orbifold four-dimensional strings. To this aim, new techniques for building level-two symmetric orbifold theories are presented. New classes of GUTs in the context of symmetric orbifolds are then constructed. The method of permutation modding is further explored and $SO(10)$ GUTs with both 45 or 54-plets are obtained. $SU(5)$ models are also found through this method. It is shown that, in the context of symmetric orbifold $SO(10)$ GUTs, only a single GUT-Higgs, either a 54 or a 45, can be present and it always resides in an order-two untwisted sector. Very restrictive results also hold in the case of $SU(5)$. General properties and selection rules for string GUTs are described. Some of these selection rules forbid the presence of some particular GUT-Higgs couplings which are sometimes used in SUSY-GUT model building. Some semi-realistic string GUT examples are presented and their properties briefly discussed.

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1 Introduction

In a recent paper [1], we discussed the construction of four-dimensional string models whose massless sector constitute grand unified $SO(10)$ and $SU(5)$ theories. In spite of the popularity of both four-dimensional strings and GUT ideas, it is surprising how little effort has been devoted to making compatible both approaches to unification by constructing GUTs from strings. The reason for this is that standard GUTs require the presence of GUT-Higgs chiral fields, e.g. adjoints in $SU(5)$, adjoints or 54s in $SO(10)$, in the massless spectrum. In order to have that type of massless chiral fields in a chiral $N=1$ theory, the affine Lie algebra associated to the GUT group has to be realized at level two or higher. Straightforward compactifications of the supersymmetric heterotic string always have level-one algebras inherited from the $E_8 \times E_8$ or $SO(32)$ ten-dimensional heterotic string. To obtain 4-d strings with higher level it is necessary to go beyond simple compactifications of the heteorotic string. At the beginning it was thought that such higher level models would be very complicated to construct. This explains why, in the early days of string model-building, there were no attempts in this direction. In fact, only a few papers have dealt with the explicit construction of 4-d strings with affine Lie algebras at higher levels [2, 3, 4, 5].

Actually, it turns out that it is not particularly complicated to construct higher level models and hence 4-d strings whose massless sector constitute $SU(5)$ and $SO(10)$ GUTs. In ref. [1], we referred to this type of models as string GUTs and investigated their construction in the framework of symmetric Abelian orbifolds. Abelian toroidal orbifolds with $N=1$ unbroken supersymmetry are probably the simplest non-trivial 4-d strings and hence provide a natural context for a general study of string GUTs. Furthermore, consistency with world-sheet supersymmetry, which is often a problem [4, 5] in the alternative fermionic construction, is built-in in the orbifold formalism.

The general prescription to build higher level models is the following. The starting point is a $(0,2)$ orbifold compactification of the 10-d heterotic string. It turns out convenient to use the $Spin(32)/Z_2$ instead of the $E_8 \times E_8$ lattice. Models in which the gauge group has the structure $G_{GUT} \times G$, where G in turn contains as a subgroup a copy of G_{GUT}, i.e. $G_{GUT} \subset G$, are searched for. At this point we have an usual level-one (0,2) orbifold model with a particular gauge structure. The next step is to perform a modding or projection such that only gauge bosons corresponding to the *diagonal* G^D_{GUT} subgroup of $G_{GUT} \times G_{GUT}$ survive in the massless spectrum. We are thus left with a gauge symmetry $G^D_{GUT} \times G'$ in which G^D_{GUT} is realized at level two.

In the case of orbifolds, the final reduction $G_{GUT} \times G_{GUT} \rightarrow G^D_{GUT}$ can be achieved following three different methods explained in refs. [1, 3]. In method I, the underlying level-one model with $G_{GUT} \times G$ group is obtained by embedding the twist action of the orbifold into the gauge degrees of freedom by means of an automorphism of the gauge lattice instead of a shift. Then, a "continuous Wilson-line" [6] background is added in such a way that the symmetry is broken continuously to the diagonal subgroup G^D_{GUT}. In method II, the final step is a modding of the original model by a Z_2 twist under which the two factors of G_{GUT} are explicitly permuted. Method III is field-theoretical, the original symmetry is broken down to the diagonal subgroup by means of an ordinary Higgs mechanism.

Although the above three methods in principle look different, there are many level-two models that can be built equivalently using more than one of them. In fact, the scheme is quite general and may be implemented in other classes of 4-d string constructions. For example, it may be used within the class of $(0,2)$ models [7, 8] obtained by adding gauge backgrounds and/or discrete torsion to Gepner and Kazama-Suzuki models. The final step leading to a level-two group can be achieved by embedding an internal order-two symmetry into a permutation of the two G_{GUT} factors.

In the present paper we report on a number of new results obtained in the construction of string GUTs from symmetric orbifolds. These new results are discussed in the different chapters as follows.

In chapter two we describe in detail the conditions for modular invariance in the method in which a permutation is used to obtain the level-two GUT. This is done by a careful study of the partition function. We undertake this analysis of the permutation method for two reasons. Modding by permutation of gauge coordinates has not been studied in any detail in the literature and there are several non-trivial technical issues to be fixed before the actual construction of GUT models. Moreover, unlike in the other two methods, the resulting examples are not necessarily continuously connected to an original level-one theory. In chapter three we then apply our analysis to enlarge our class of models and construct examples that we were unable to discover before. In particular, considering the simultaneous action of a shift and a permutation in the gauge lattice, allows us to derive the first symmetric orbifold $SO(10)$ GUTs with an adjoint in the massless spectrum. We also obtain $SU(5)$ models using the permutation plus shift method. In contrast, in [1], the $SO(10)$ GUTs contained only 54s and $SU(5)$ examples were only constructed using the method of flat directions.

In chapter four we further explore the general structure of $SO(10)$ and $SU(5)$ symmetric orbifold string GUTs. Expanding our analysis of ref. [1] we show that this class of $SO(10)$ string GUTs can only have either one 54 or one adjoint 45 and that this GUT-Higgs can only surface in the untwisted sector. Furthermore we show that only orbifolds with one complex plane rotated by just order-two twists can lead to $SO(10)$ string GUTs. Abelian Z_N and $Z_N \times Z_M$ orbifolds with this property are the Z_4, Z_6, Z_8, Z_{12}, $Z_2 \times Z_2$, $Z_2 \times Z_4$ and $Z_2 \times Z_6$ orbifolds. Hence, many symmetric orbifold models are ruled out for the purpose of $SO(10)$ GUT-building. In the case of $SU(5)$ the situation is slightly less tight. The adjoint 24 can belong either in untwisted sectors or in twisted sectors corresponding to certain particular order four or six twists. Many orbifolds such as Z_3, $Z_3 \times Z_3$, Z_7, $Z_2 \times Z_6'$ and Z_8', are thus unsuitable for string GUTs. When GUT Higgses do occur, their structure is very restricted and implies a set of selection rules on couplings that are also presented in chapter four. Other selection rules that apply to general string GUTs, and not exclusively to symmetric orbifolds, are also given.

In chapter five we briefly discuss four-generation models as well as a model with *almost* three generations. As we explained above, only orbifolds of even order may lead to string GUTs. Due to this even order there is the tendency to find an even number of generations and hence four generations is the simplest option. Although a three-generation $SU(5)$ model is presented, it also contains an extra exotic $(15+9\cdot\overline{5})$ chiral family. We believe that obtaining three net generations within the context of symmetric orbifold GUTs is going to be a difficult

task. However, we do not think that this is a general property of string GUTs. It should be possible to obtain three-generation examples using similar techniques but applied to other constructions such as those based on Gepner or Kazama-Suzuki models.

In chapter six we summarize our conclusions and discuss the outlook of string GUTs.

2 The Permutation Method Revisited

In this chapter we upgrade and complete the formalism discussed in [1] for the construction of level-two models from a permutation modding of the gauge coordinates. In ref. [1] we discussed three methods for obtaining level-two symmetric orbifold string-GUTs. Two of them, the method of continuous Wilson lines and the method of field theoretical flat directions, are continuously connected to level-one models. This means that at particular points of the scalar fields moduli space the level-two gauge symmetry is enhanced to a (bigger) level-one symmetry. In this sense the level-two model is more generic than its level-one parent.

In the case of $SO(10)$ string GUTs this continuous connection implies, as argued in [1], that only 54s GUT Higgs fields can be obtained. In order to be able to find adjoint 45s we have to use a method that yields models not necessarily continuously connected to an underlying level-one model. Since the permutation modding involves a discrete order-two twist, it offers this kind of possibility, and this is one of the motivations for the detailed analysis presented in this section. It turns out that to obtain 45 representations, we have to simultaneously act with a discrete shift on the gauge degrees of freedom, as will be shown below. The permutation modding method will also allow us to construct directly $SU(5)$ string GUTs which we were only able to derive in [1] through the flat direction method.

The starting point is the ten-dimensional heterotic string with gauge group $E_8\times E_8$ or $SO(32)$ corresponding to an affine Lie algebra at level $k=1$. We use the bosonic formulation in which the gauge group originates in 16 left-moving coordinates F_J compactified on a torus with $E_8\times E_8$ or $Spin(32)/Z_2$ lattice denoted Λ_{16}. An usual orbifold compactification [9, 10] leads to to a 4-d model with gauge group $G_1\times G_2\times\cdots$ in which each non-Abelian factor realizes an affine algebra still at level $k=1$.

To obtain higher level algebras, a further process of twisting must be performed.

Indeed, the main ingredient in the permutation method is an order-two operation Π that exchanges the F_J coordinates associated to two identical group factors $G \times G$, thus producing the diagonal subgroup G_D at level $k = 2$. In the orbifold construction, the permutation Π must be accompanied by an action on the internal degrees of freedom. In the following we wish to explain how Π is implemented as a quantized Wilson line.

Our strategy will be to construct a modular invariant partition function for the orbifold with the Π action. To do this we carefully deduce the contribution of the simply-twisted sector and then generate the remaining sectores by modular transformations and by requiring an operator interpretation. We will develop explicitly the case of the Z_4 orbifold on an $SU(4) \times SU(4)$ internal lattice. However, many of the statements will apply to other orbifolds. In particular, at the end we will consider the $Z_2 \times Z_2$ orbifold. These two are the simplest even-order Abelian orbifolds.

2.1 Notation and Embedding

We introduce our notation by recalling some basic facts about orbifolds. We denote the internal twist generator by θ and the internal lattice by Γ, its basis vectors are e_i. The eigenvalues of θ are of the form $e^{\pm 2\pi i v_a}$, $a = 1, 2, 3$, with $0 \leq |v_a| < 1$. The v_a can be chosen so that $v_1 + v_2 + v_3 = 0$ and define a twist vector (v_1, v_2, v_3). For convenience we also set $v_0 = 0$. The orbifold action on the internal coordinates is represented by an space-group element $(\theta, n_i e_i)$, meaning boundary conditions

$$x(\sigma + 1, t) = \theta x(\sigma, t) + n_i e_i \tag{1}$$

In turn this implies that in the simply-twisted sector, the string Hilbert space splits into sub-sectors in which the string center of mass sits at fixed points x_f satisfying $(1 - \theta) x_f = n_i e_i \bmod \Gamma$. In each $SU(4)$ sub-lattice there are four distinct choices for $n_i e_i$, namely $n_i e_i = 0, e_1, e_1 + e_2, e_1 + e_2 + e_3$.

The embedding of $(\theta, n_i e_i)$ in the gauge degrees of freedom is represented by an element (Θ, V), meaning boundary conditions

$$F(\sigma + 1, t) = \Theta F(\sigma, t) + V + P \tag{2}$$

where $P \in \Lambda_{16}$ and Θ is an automorphism of Λ_{16}. The full orbifold action is represented by elements $g = (g_{int} | g_{gauge})$ such as $(\theta, n_i e_i | \Theta, V)$. In an orbifold compactification with Wilson line backgrounds [11], the generators of the full orbifold action are $(\theta, 0 | 1, V)$ and $(1, e_i | 1, a_i)$, where the a_i are the Wilson lines.

In the permutation method we assign the operation Π as a Wilson line in the e_1 direction and rather consider the second generator to be $(1, e_1 | \Pi, 0)$. This then implies the following elements in the simply-twisted sector

$$\begin{aligned}
g_0 &= (\theta, 0 | 1, V) \\
g_1 &= (\theta, e_1 | \Pi, \Pi V) \\
g_2 &= (\theta, e_1 + e_2 | 1, \Pi V) \\
g_3 &= (\theta, e_1 + e_2 + e_3 | \Pi, V)
\end{aligned} \tag{3}$$

Since θ is of order four and the e_i are completely rotated by θ, $g_i^4 \equiv 1$ and V must satisfy

$$2(V - \Pi V) \in \Lambda_{16} \tag{4}$$

The interpretation of eqs.(3) is that each θ-subsector feels a different gauge action. This is precisely what happens in an orbifold with quantized Wilson lines [11].

For definiteness, in the following we will deal with the $Spin(32)/Z_2$ lattice. The analysis can be straightforwardly extended to the $E_8 \times E_8$ lattice. We will consider a Π action consisting of the permutation of two sets of L gauge bosonic coordinates. More precisely,

$$\begin{aligned} &\Pi(F_1, \cdots, F_L, F_{L+1}, \cdots, F_{2L}, F_{2L+1}, \cdots, F_{16}) \\ &= (F_{L+1}, \cdots, F_{2L}, F_1, \cdots, F_L, F_{2L+1}, \cdots, F_{16}) \end{aligned} \tag{5}$$

In this F basis, Π acts as a non-diagonal matrix. It proves convenient to work in another basis that diagonalizes Π. Then, for $J = 1, \cdots, L$, we define symmetric, $\overline{F}_J$, and antisymmetric, $\widehat{F}_J$, combinations

$$\begin{aligned} \overline{F_J} &= \frac{F_J + F_{J+L}}{\sqrt{2}} \\ \widehat{F_J} &= \frac{F_J - F_{J+L}}{\sqrt{2}} \end{aligned} \tag{6}$$

Clearly, $\overline{F}_J \to \overline{F}_J$ and $\widehat{F}_J \to -\widehat{F}_J$, under Π.

Given a 16-d vector B, we define its symmetric and antisymmetric parts as

$$\begin{aligned} \overline{B} &= (\frac{B_1 + B_{1+L}}{\sqrt{2}}, \cdots, \frac{B_L + B_{2L}}{\sqrt{2}}; B_{2L+1}, \cdots, B_{16}) \\ \widehat{B} &= (\frac{B_1 - B_{1+L}}{\sqrt{2}}, \cdots, \frac{B_L - B_{2L}}{\sqrt{2}}; 0, \cdots, 0) \end{aligned} \tag{7}$$

Notice that $B^2 = \overline{B}^2 + \widehat{B}^2$.

For future purposes we also need to define $(16 - L)$-dimensional lattices Λ_I and Λ_I^*, related to Λ_{16}. A vector $P \in \Lambda_{16}$ is of the form

$$\begin{aligned} P &= (m_1 + \frac{s}{2}, \cdots, m_L + \frac{s}{2}, m_{L+1} + \frac{s}{2}, \cdots, m_{2L} + \frac{s}{2}, m_{2L+1} + \frac{s}{2}, \cdots, m_{16} + \frac{s}{2}) \\ s &= 0, 1 \quad ; \quad (m_1 + \cdots + m_{16}) = even \end{aligned} \tag{8}$$

The type of $\overline{P}$ constructed from a vector P invariant under Π motivates the definition of the invariant lattice Λ_I. Thus, a vector $P_I \in \Lambda_I$ is of the form

$$\begin{aligned} P_I &= (\sqrt{2}[n_1 + \frac{s}{2}], \cdots, \sqrt{2}[n_L + \frac{s}{2}]; n_{2L+1} + \frac{s}{2}, \cdots, n_{16} + \frac{s}{2}) \\ s &= 0, 1 \quad ; \quad (n_{2L+1} + \cdots + n_{16}) = even \end{aligned} \tag{9}$$

The dual of Λ_I, denoted Λ_I^*, is the set of vectors $\overline{Q}$ such that $P_I \cdot \overline{Q} =$ int. It is easy to see that $\overline{Q} \in \Lambda_I^*$ must have the form

$$\begin{aligned} \overline{Q} &= (\frac{n_1}{\sqrt{2}}, \cdots, \frac{n_L}{\sqrt{2}}; n_{2L+1} + \frac{s}{2}, \cdots, n_{16} + \frac{s}{2}) \\ s &= 0, 1 \quad ; \quad (n_1 + \cdots + n_L + n_{2L+1} + \cdots + n_{16}) + sL = even \end{aligned} \tag{10}$$

2.2 The Partition Function

For each element in the full orbifold group there are twisted sectors in which the boundary conditions in σ are of type (1). Moreover, in each sector there must be a projection on invariant states. This projection is implemented by considering twisted boundary conditions in the t direction. We will denote by $(g;h)$ the boundary conditions in the $(\sigma;t)$ directions. It can be shown [9] that the partition function can be written as

$$Z = \frac{1}{|\mathcal{P}|} \sum_{g} \sum_{h|[h,g]=0} Z(g;h) \tag{11}$$

where $Z(g;h)$ is a path-integral evaluated with boundary conditions $(g;h)$. $|\mathcal{P}|$ is the order of the orbifold point group.

The various pieces $Z(g;h)$ are related one another by modular transformations of the world-sheet parameter τ. More precisely, under $\tau \to \frac{a\tau+b}{c\tau+d}$, the boundary conditions change as $(g;h) \to (g^d h^c; g^b h^a)$. Therefore

$$Z(g;h) \to Z(g^d h^c; g^b h^a) \tag{12}$$

Hence, we can generate pieces of Z by applying modular transformations. Moreover, since in general $Z(g;h)$ picks up phases under these transformations and there are some that leave the boundary conditions unchanged (e.g. $\tau \to \tau + N$ in the Z_N orbifold), there will appear modular invariance restrictions on the possible forms of the full orbifold group.

$Z(g;h)$ is made up of contributions from the different string right- and left-movers. The contribution from the space-time and internal bosonic coordinates is well known [10], the left and right parts being conjugate of each other since we are dealing with a symmetric orbifold. The contribution from the right-handed fermions is given in terms of ϑ-functions and it depends only on the internal twist vector v. For example, if $g_{int} = (\theta, n_i e_i)$, and $h_{int} = (1,0)$, we have up to phases,

$$Z_\psi(\theta, n_i e_i; 1, 0) \sim \sum_{\alpha,\beta=0,\frac{1}{2}} \overline{\eta}_{\alpha\beta} \prod_{a=0}^{3} \frac{\vartheta \begin{bmatrix} \alpha + v_a \\ \beta \end{bmatrix} (\bar{\tau})}{\eta(\bar{\tau})} \tag{13}$$

where the sum is over spin structures. Modular invariance fixes $\overline{\eta}_{0\frac{1}{2}} = \overline{\eta}_{\frac{1}{2}0} = -\overline{\eta}_{00}$. We make the standard choices $\overline{\eta}_{00} = 1$ and $\overline{\eta}_{\frac{1}{2}\frac{1}{2}} = 1$. Using properties of ϑ-functions, (13) can be written as

$$Z_\psi(\theta, n_i e_i; 1, 0) \sim \frac{1}{\eta^4(\bar{\tau})} \sum_{\substack{r_a \in \mathbb{Z} \\ (r_0 + \cdots + r_3) = \mathrm{odd}}} \left[\bar{q}^{\frac{1}{2}(r+v)^2} - \bar{q}^{\frac{1}{2}(r+v+S)^2} \right] \tag{14}$$

where $q = e^{2i\pi\tau}$, $S = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and here $v = (0, v_1, v_2, v_3)$.

The contribution of the gauge bosons in a $(g;h)$ sector can be computed as $Tr(q^{H_F(g)} h)$. $H_F(g)$ is the Hamiltonian of the F fields with σ-boundary conditions twisted by g and the trace is evaluated over the corresponding Hilbert space. We will now review this construction when the g action on the F is given by $(1,V)$. We will then extend the analysis to embeddings of type $(\Pi,\Pi V)$.

We consider the general expansion

$$F(\sigma,t)=F_0+M\sigma_-+\frac{i}{2}\sum_r\frac{\alpha_r}{r}\,e^{-2i\pi r\sigma_-} \tag{15}$$

where $\sigma_-=\sigma-t$. If $g_{gauge}=g_V\equiv(1,V)$, the σ boundary condition (2) implies that $M=P+V$ and that the oscillator level r is an integer. The Hamiltonian for the F bosons is thus given by

$$H_F(g_V)=\frac{1}{2}(P+V)^2+\sum_{n=1}^{\infty}\;:\alpha_n^\dagger\alpha_n:-\frac{16}{24} \tag{16}$$

Now, if the h action on the F is trivial, we have up to phases,

$$Z_F(g_V;1)=Tr(q^{H_F(g_V)})\sim\frac{1}{\eta^{16}(\tau)}\sum_{P\in\Lambda_{16}}q^{\frac{1}{2}(P+V)^2} \tag{17}$$

Combining with (14) we obtain the complex piece of $Z(g;1)$ for $g=(\theta,n_ie_i|1,V)$. For a Z_4 orbifold this piece must transform in itself under $\tau\to\tau+4$. From the transformations of (14) and (17) we then obtain the modular invariance condition

$$4(V^2-v^2)=0\,mod\,2 \tag{18}$$

together with the embedding condition $4V\in\Lambda_{16}$.

For completeness we also consider the effect of a t boundary condition h encoded by the element $(1,V)$. This is

$$F(\sigma,t+1)=F(\sigma,t)+V+P' \tag{19}$$

This tells us that in this case h acts as a translation. Then, up to phases,

$$Z_F(g_V;g_V)=Tr(q^{H_F(g_V)}e^{2i\pi V\cdot M})\sim\frac{1}{\eta^{16}(\tau)}\sum_{P\in\Lambda_{16}}q^{\frac{1}{2}(P+V)^2}e^{2i\pi(P+V)\cdot V} \tag{20}$$

If the phases in (14) and (17) are chosen to be unity, modular invariance implies an overall phase $e^{-i\pi(V^2-v^2)}$ in $Z_{\psi F}(g;g)$, $g=(\theta,n_ie_i|1,V)$ [10].

We now turn to the case of boundary conditions involving permutations. We will consider $g_1=(\theta,e_1|\Pi,\Pi V)$. Since Π does not affect the coordinates F_K, $K=2L+1,\cdots,16$; their expansions and contribution to Z_F are as before. Changes are due to the coordinates $\overline{F}_J$ and $\widehat{F}_J$, $J=1,\cdots,L$, with expansions

$$\begin{aligned}\overline{F}(\sigma,t)&=\overline{F}_0+\overline{M}\sigma_-+\frac{i}{2}\sum_r\frac{\overline{\alpha_r}}{r}\,e^{-2i\pi r\sigma_-}\\ \widehat{F}(\sigma,t)&=\widehat{F}_0+\widehat{M}\sigma_-+\frac{i}{2}\sum_r\frac{\widehat{\alpha_r}}{r}\,e^{-2i\pi r\sigma_-}\end{aligned} \tag{21}$$

To simplify, in the above, the untouched coordinates coordinates F_K, have been included together with the $\overline{F}_J$. Since $g_{gauge} = g_\Pi \equiv (\Pi, \Pi V)$, the boundary conditions are

$$\begin{aligned}
\overline{F}_J(\sigma+1,t) &= \overline{F}_J(\sigma,t) + \frac{P_J + V_{J+L} + P_{J+L} + V_J}{\sqrt{2}} \\
\widehat{F}_J(\sigma+1,t) &= -\widehat{F}_J(\sigma,t) + \frac{P_J + V_{J+L} - P_{J+L} - V_J}{\sqrt{2}} \\
\overline{F}_K(\sigma+1,t) &= \overline{F}_K(\sigma,t) + P_K + V_K
\end{aligned} \tag{22}$$

Imposing these conditions to the expansions (21) implies that the oscillators $\overline{\alpha}_r$ have integer levels $r = l$ whereas the $\widehat{\alpha}_r$ have $r = l + \frac{1}{2}$. Furthermore, the momenta must satisfy

$$\begin{aligned}
\overline{M} &= \overline{Q} + \overline{V} \\
\widehat{M} &= 0
\end{aligned} \tag{23}$$

where $\overline{Q}$ and $\overline{V}$ are respectively the symmetric parts of P and V. In fact, $\overline{Q} \in \Lambda_I^*$, according to (10). Finally, the symmetric zero mode $\overline{F}_{0J}$ is arbitrary while $2\widehat{F}_{0J} = (P_J + V_{J+L} - P_{J+L} - V_J)/\sqrt{2}$.

The Hamiltonian $H_F(g_\Pi)$ splits into a symmetric and antisymmetric part, $H_F = \overline{H}_F + \widehat{H}_F$, where

$$\begin{aligned}
\overline{H}_F(g_\Pi) &= \frac{1}{2}(\overline{Q}+\overline{V})^2 + \sum_{n=1}^{\infty} : \overline{\alpha}_n^\dagger \overline{\alpha}_n : - \frac{L}{24} - \frac{(16-2L)}{24} \\
\widehat{H}_F(g_\Pi) &= \sum_{r\in \mathbb{Z}+\frac{1}{2}} : \widehat{\alpha}_r^\dagger \widehat{\alpha}_r : + \frac{L}{48}
\end{aligned} \tag{24}$$

The σ boundary conditions (22) then define states $|\overline{M}, \overline{N}, \widehat{N}\rangle$, where $\overline{N}$ and $\widehat{N}$ are oscillator numbers. Taking the trace over these states gives

$$Z_F(g_\Pi;1) = Tr(q^{H_F(g_\Pi)}) = \sum_{\bar{Q}\in\Lambda_I^*} \frac{q^{\frac{1}{2}(\overline{Q}+\overline{V})^2}}{\eta(\frac{\tau}{2})^L \eta(\tau)^{16-2L}} \tag{25}$$

The $\eta(\tau)^{16-2L}$ in the denominator arises from the $16-2L$ coordinates untouched by the permutation. The term $\eta(\frac{\tau}{2})^{-L}$ originates as

$$\eta(\frac{\tau}{2})^{-L} = \left[q^{\frac{L}{24}} \prod_{n=1} (1-q^n)^L \right]^{-1} \left[q^{-\frac{L}{48}} \prod_{n=1} (1-q^{n-1/2})^L \right]^{-1} \tag{26}$$

The first (second) factor comes from the trace over the L symmetric (antisymmetric) oscillators. Finally, in (25), we have made a definite choice of phases so as to have an equality.

Combining (25) and (14) gives $Z_{\psi F}(g_1;1)$. Performing a $\tau \to \tau+4$ transformation gives the modular invariance constraint

$$4\left[\overline{V}^2 - v^2 + \frac{L}{8}\right] = 0 \, mod \, 2 \tag{27}$$

together with the embedding condition $4\overline{V} \in \Lambda_I$.

We next wish to generate other contributions to the partition function by applying modular transformations to (25). We begin with $Z_F(1;g_\Pi)$ which is obtained from a $\tau \to -1/\tau$ transformation Using the Poisson resummation formula [12] we obtain

$$Z_F(1;g_\Pi) = \sum_{P_I \in \Lambda_I} \frac{q^{\frac{1}{2}P_I^2} e^{2i\pi P_I \cdot \overline{V}}}{\eta(2\tau)^L \eta(\tau)^{16-2L}} \tag{28}$$

We are also interested in deriving this result directly by imposing g_Π boundary conditions in the t direction and computing $Tr(q^{H_F(1)} g_\Pi)$. In this way we will gain a better understanding about the action of the permutation Π.

The Hamiltonian $H_F(1)$ is essentially given in (16) and the trace must be taken over states $|P,N\rangle$. Now, to derive the effect of inserting the g_Π projection, it is convenient to split P into its $\overline{P}$ and $\widehat{P}$ parts, as well as to consider two set of oscillators $\overline{\alpha}_n$ and $\widehat{\alpha}_n$. For the antisymmetric momenta we have $\langle \widehat{P} | \Pi \widehat{P} \rangle = 0$, unless $\widehat{P} = 0$. Therefore, recalling the definition (7) of $\widehat{P}$, we see that the only momentum states that survive in the trace are those invariant under permutations, namely those with $P_J = P_{J+L}$, for $J = 1, \cdots, L$. For the symmetric momenta, $\Pi|\overline{P}\rangle = |\overline{P}\rangle$ so that $g_\Pi |\overline{P}\rangle = e^{2i\pi \overline{P}\cdot\overline{V}} |\overline{P}\rangle$. Moreover, when $\widehat{P} = 0$, $\overline{P}_J = \sqrt{2} P_J$, for $J = 1, \cdots, L$. Hence, $\overline{P} = P_I \in \Lambda_I$.

The above discussion explains the sum and the numerator in (28). The $\eta(2\tau)$ in the denominator appears because Π multiplies the $\widehat{\alpha}_n$ oscillators by a phase $e^{i\pi}$. Indeed, the permuted oscillator contribution is

$$\left[q^{\frac{L}{24}} \prod_{n=1} (1-q^n)^L \cdot q^{\frac{L}{24}} \prod_{n=1} (1+q^n)^L \right]^{-1} = \left[q^{\frac{2L}{24}} \prod_{n=1} (1-q^{2n})^L \right]^{-1} = \eta(2\tau)^{-L} \tag{29}$$

Basically, we see that Π also projects onto invariant oscillator states.

We have now a neat interpretation of the action of Π in the t direction when inserted into the trace. The factor of two in $\eta(2\tau)$ as well as the factor of $\sqrt{2}$ in P_I can be heuristically explained as follows. Suppose that there is a Hamiltonian $H = H_1 + H_2$, where H_1 and H_2 have the same Hilbert space. Suppose also that a permutation Π exchanges states $|a_1\rangle$ and $|a_2\rangle$. Then,

$$\langle a_1, a_2 | \Pi | a_1, a_2 \rangle = \begin{cases} \langle a_1, a_2 | a_1, a_2 \rangle & \text{if} \quad |a_1\rangle = |a_2\rangle \\ 0 & \text{otherwise} \end{cases} \tag{30}$$

Therefore,

$$\begin{aligned} Tr(q^{H_1+H_2}\Pi) &= \sum_{a_1,a_2} \langle a_1, a_2 | q^{H_1+H_2} \Pi | a_1, a_2 \rangle \\ &= \sum_{a_1} \langle a_1, a_1 | q^{H_1+H_2} | a_1, a_1 \rangle = \sum_{a_1} \langle a_1 | q^{2H_1} | a_1 \rangle \end{aligned} \tag{31}$$

This means that the two identical theories being permuted collapse into one but evaluated at 2τ. This becomes even more explicit if we rewrite eq. (28) in terms of Riemann theta functions.

From $Z_F(g_\Pi;1)$ we can also generate $Z_F(g_\Pi^2;g_\Pi)$ thus developing some insight about the doubly-twisted sector. The first step is to apply a $\tau \to \tau+2$ transformation to (25) to arrive at

$$Z_F(g_\Pi;g_\Pi^2) = e^{-2i\pi(\overline{V}^2-\frac{L}{8})} \sum_{\overline{Q}\in\Lambda_I^*} \frac{q^{\frac{1}{2}(\overline{Q}+\overline{V})^2} e^{2i\pi(\overline{Q}+\overline{V})\cdot 2\overline{V}} e^{2i\pi\overline{Q}^2}}{\eta(\frac{\tau}{2})^L \eta(\tau)^{16-2L}} \tag{32}$$

where we have neglected constant phases that cancel against those in Z_ψ. Since Λ_I^* is not an integer lattice, here we cannot get rid of the phase quadratic in $\overline{Q}$ as it is done in an ordinary level-one $Z_F(g_V;g_V^2)$. This is an indication that in computing (32) directly as a trace, a $\overline{Q}$-dependent phase must be allowed. This fact is not evident from the boundary conditions.

The next step is to derive $Z_F(g_\Pi^2;g_\Pi)$ by applying to (32) a $\tau \to -1/\tau$ transformation followed by a $\tau \to \tau+1$ transformation. To this end it is necessary to shift $\overline{Q}^2$ into a linear term in $\overline{Q}$ so that the Poisson resummation formula can be used. In fact, from (10) we see that

$$\begin{aligned} \overline{Q}^2 &= \frac{n_1}{2} + \cdots + \frac{n_L}{2} + \frac{sL}{2} \quad mod \quad 1 \\ &= 2\overline{Q}\cdot\overline{G} \quad mod \quad 1 \end{aligned} \tag{33}$$

The auxiliary vector $\overline{G}$ is defined as

$$\begin{aligned} 2\overline{G} &\equiv (\frac{1}{\sqrt{2}},\cdots,\frac{1}{\sqrt{2}};0,\cdots,0) + \xi\Upsilon \\ \Upsilon &\equiv (0,\cdots,0;1,0,\cdots,0) \end{aligned} \tag{34}$$

with $\xi=0$ for L even and $\xi=1$ for L odd. Hence, $e^{2i\pi\overline{Q}^2} = e^{2i\pi\overline{Q}\cdot 2\overline{G}}$.

After using the Poisson resummation formula we arrive finally at

$$Z_F(g_\Pi^2;g_\Pi) = e^{-2i\pi(\overline{V}^2-2\overline{G}^2-\frac{L}{8})} \sum_{P_I\in\Lambda_I} \frac{q^{\frac{1}{2}(P_I+2\overline{V}+2\overline{G})^2} e^{2i\pi(P_I+2\overline{V}+2\overline{G})\cdot\overline{V}}}{\eta(2\tau)^L \eta(\tau)^{16-2L}} \tag{35}$$

From the above we see that to a g_Π-twisted sector there corresponds a g_Π^2-twisted sector in which the allowed momenta that survive the g_Π projection are of the form $P_I + 2\overline{V} + 2\overline{G}$. We now turn to deriving this result from boundary conditions in order to better understand the meaning of the vector $2\overline{G}$.

Since Π is of order two, we assume that $g_\Pi^2 = (1;2V) = g_{2V}$. Then, in the g_Π^2-twisted sector, the Hamiltonian is of the form (16) and

$$Z_F(g_\Pi^2;g_\Pi) = Tr(q^{H_F(g_{2V})} g_\Pi) \tag{36}$$

The trace is taken over states $|M,N\rangle$, where $M = P+2V$. Now, as we have seen before, inserting g_Π in the trace has the effect of projecting onto invariant oscillator states and this is reflected in the $\eta(2\tau)^L$ factor. Also, g_Π projects onto invariant momenta, selecting states with $\widehat{M}=0$. This implies

$$P_J + 2V_J = P_{J+L} + 2V_{J+L} \quad ; \quad J=1,\cdots,L \tag{37}$$

The shift V must satisfy $2(V_J - V_{J+L}) =$int, since for any $P\in\Lambda_{16}$, $P_I - P_K =$ int. This is guaranteed by the embedding condition (4). Then, using (37) we can assert that the lattice momenta that survive are such that

$$\begin{aligned} M &= (m_1+\frac{s}{2},\cdots,m_L+\frac{s}{2},m_1+\frac{s}{2},\cdots,m_L+\frac{s}{2};m_{2L+1}+\frac{s}{2},\cdots,m_{16}+\frac{s}{2}) \\ &+ 2(V_1,\cdots,V_L,V_1,\cdots V_L;V_{2L+1},\cdots V_{16}) \end{aligned} \tag{38}$$

Clearly, $\widehat{M}=0$.

To compute $\overline{M}$, we first notice that, since $P=(M-2V)\in\Lambda_{16}$, it must be that

$$\mathcal{S} + (m_{2L+1}+\cdots+m_{16}) = \ even \tag{39}$$

where

$$\mathcal{S} = \sum_{J=1}^{L} 2(V_J - V_{J+L}) \tag{40}$$

Thus, for $\mathcal{S}$ =even, $(m_{2L+1}+\cdots+m_{16})$ =even. To have the same type of constraint when $\mathcal{S}$ =odd, we redefine $m_{2L+1}\to m_{2L+1}+1$. With this convention it then follows that $\overline{M}$ can be written as

$$\overline{M} = P_I + 2\overline{V} + 2\widehat{V} + \zeta\Upsilon \tag{41}$$

where $P_I\in\Lambda_I$ and $\zeta=\mathcal{S}\bmod 2$.

The upshot of the above discussion is that computing directly from boundary conditions leads to $Z_F(g_\Pi^2;g_\Pi)$ of the form (35), provided that

$$2\widehat{V} + \zeta\Upsilon = 2\overline{G} \quad mod \quad \Lambda_I \tag{42}$$

Moreover, in this case

$$2\overline{G}^2 + \frac{L}{8} = \widehat{V}^2 \quad mod \quad 1 \tag{43}$$

Hence, the overall phase in (35) agrees with the expected value $e^{-2i\pi V^2}$.

It is interesting to notice that, formally, the above results look quite similar to those obtained in refs. [13, 14] (in the context of $N=2$ coset model contructions) where characters $\chi(M\tau)$ emerge after modding out by a cyclic permutation symmetry of M internal theories. In fact, the origin of the factor M is the same, the collapse of M internal theories. However, there are two relevant differences. On the one hand, since we are considering permutations in the gauge sector, this operation is intrinsically asymmetric in the sense that it affects only the left sector. As a consequence, stringent constraints such as eqns. (27) and (42) must be satisfied in order to match left and right sectors while level matching is automatic when permuting similar theories in both sectors. On the other hand, let us stress that we are introducing permutations here as Wilson lines, associated to given directions in the internal lattice. The underlying model here is not necessarily invariant under permutations. Of course, if a model invariant under permutation of gauge factors, in the sense that if a representation (R, R') appears so does (R', R), is constructed, then we can mod out by this symmetry. This modding is an external projection unrelated to orbifold operations. The addition of sectors twisted by permutations to recover modular invariance, will lead to even more severe contraints, because now Π will be felt in all θ^n sectors. For example, in the Z_4 orbifold case, only permutations of four or eight gauge bosonic coordinates are allowed.

The previous analysis has focused on the Z_4 orbifold. We will now consider the $Z_2 \times Z_2$ orbifold with $SO(4)^3$ lattice. In this case there are two internal twist generators θ and ω with twist vectors $a = (0, 1/2, 0, -1/2)$ and $b = (0, 0, 1/2, -1/2)$. In the absence of Wilson lines the generators of the full orbifold group are $(\theta, 0|1, A)$ and $(\omega, 0|1, B)$, where $2A \in \Lambda_{16}$ and $2B \in \Lambda_{16}$. The permutation Π is embedded as a Wilson line in the first $SO(4)$ sub-lattice, the corresponding generator being $(1, e_1|\Pi, 0)$.

As reviewed in ref. [1], there are several ways in which the twists ω and θ can act on the six dimensional lattice. They have different fixed points structure and therefore lead to different multiplicities in the spectrum. Moreover, consistency constraints, due to the inclusion of Wilson lines, also depend on the realization.

Let us consider the realization

$$\theta = (-1, 1, -1) \quad ; \quad \omega = (1, -1, -1) \tag{44}$$

Since ω does not rotate the first sub-lattice, the ω sector is not split by Π. On the other hand, the θ sector is split in a way encoded by the elements $(\theta, 0|1, A)$, $(\theta, e_1|\Pi, \Pi A)$, $(\theta, e_2|\Pi, A)$ and $(\theta, e_1 + e_2|1, \Pi A)$. Similarly, the relevant elements in the $\theta\omega$ sector are $(\theta\omega, 0|1, C)$, $(\theta\omega, e_1|\Pi, \Pi C)$, etc., where $C = A - B$.

Defining $g_{\Pi A} = (\Pi, \Pi A)$, we find that the partition function $Z_F(g_{\Pi A}; 1)$ is of the form (25). Combining with the fermionic piece and performing a $\tau \to \tau + 2$ transformation, that must leave the partition function invariant, we find the constraints

$$\begin{aligned} \overline{A}^2 - a^2 + \frac{L}{8} &= 0 \, mod \, 1 \\ 2\overline{A} &= 2\overline{G} \, mod \, \Lambda_I \end{aligned} \tag{45}$$

Similar conditions must hold for $\overline{C}$. It then follows that

$$2\overline{B} \in \Lambda_I \tag{46}$$

The constraints (45) and (46) supplement the regular modular invariance conditions $(A^2 - a^2) = \text{mod } 1$, $(B^2 - b^2) = \text{mod } 1$ and $(A \cdot B - a \cdot b) = \text{mod } 1$.

The partition function $Z_F(g_{\Pi A}; g_B)$ cannot be connected to $Z_F(g_{\Pi A}; 1)$ by modular transformations. We then choose to connect it to $Z_F(g_B; g_{\Pi A})$ which in turn is computed as $Tr(q^{H_F(g_B)} g_{\Pi A})$ using our prescription for the action of $g_{\Pi A}$ when inserted into the trace. With the usual choice of phase we then find

$$Z_F(g_B; g_{\Pi A}) = e^{-i\pi A\cdot B} \sum_{P_I \in \Lambda_I} \frac{q^{\frac{1}{2}(P_I+B_I)^2} e^{2i\pi(P_I+B_I)\cdot\overline{A}}}{\eta(2\tau)^L \eta(\tau)^{16-2L}} \tag{47}$$

The shift B_I appears when projecting into states with $\widehat{M} = 0$, where $M = P + B$ and it is given by

$$B_I = \overline{B} + \widehat{B} + \zeta_B \Upsilon \tag{48}$$

where $\zeta_B = \mathcal{S}_B \bmod 2$ and $\mathcal{S}_B = \sum_{J=1}^{L} (B_J - B_{J+L})$. In fact, the existence of solutions to $\widehat{M} = 0$ requires

$$B_J - B_{J+L} = 0 \, mod \, 1 \tag{49}$$

Equivalently, $(\widehat{B} + \zeta_B \Upsilon) \in \Lambda_I^*$. A simple way to satisfy the constraints on the shift B is to choose $\widehat{B} = 0$. In this case we find

$$Z_F(g_{\Pi A}; g_B) = e^{-i\pi \overline{A}\cdot\overline{B}} \sum_{\overline{Q} \in \Lambda_I^*} \frac{q^{\frac{1}{2}(\overline{Q}+\overline{A})^2} e^{2i\pi(\overline{Q}+\overline{A})\cdot\overline{B}}}{\eta(\frac{\tau}{2})^L \eta(\tau)^{16-2L}} \tag{50}$$

where we have used that $B_I = \overline{B}$.

An alternative to the above realization of the twists, usually more attractive since it leads to lower multiplicities, corresponds to the choice

$$\theta = (-1, \sigma_1, \sigma_1) \quad ; \quad \omega = (-\sigma_1, -1, -\sigma_1) \tag{51}$$

In this case the fixed sets in the θ sector that feel the action of Π are not fixed by either ω or $\theta\omega$. As a consequence, contributions such as $Z_F(g_B; g_{\Pi A})$ or $Z_F(g_{\Pi A}; g_B)$ will not appear. Therefore, no extra constraints apart from eq. (45) will be required.

To end this section let us briefly consider the contribution of the internal bosonic coordinates. For our purposes it is sufficient to know that

$$Z_B(g; h) = \tilde{\chi}(g,h) \left\{ \overline{q}^{E_0(g) - \frac{6}{24}} (1 + \cdots) \right\} \left\{ q^{E_0(g) - \frac{6}{24}} (1 + \cdots) \right\} \tag{52}$$

where the ellipsis stands for higher powers of q corresponding to oscillator states. The twisted vacuum energy E_0 depends on the g_{int}-twist vector (v_1, v_2, v_3) according to

$$E_0(g) = \sum_{a=1}^{3} \frac{1}{2} |v_a| (1 - |v_a|) \tag{53}$$

$\tilde{\chi}(g,h)$ is a numerical factor that basically counts the number of simultaneous fixed points of g_{int} and h_{int}. In the presence of Wilson lines it is convenient to decompose it as a sum of terms $\tilde{\chi}(g, h|x_g)$ where the x_g are the fixed points of g_{int}. For more details we refer the reader to the Appendix of [1].

2.3 Massless Spectrum and Generalized GSO Projectors

In the last section we have seen how modular invariance and the operator interpretation of the partition function impose constraints on the shift V and the number of permuted coordinates. We now want to discuss how the partition function determines the allowed massless states.

The massless spectrum can be divided into twisted sectors acoording to the σ boundary conditions. Let us begin with the simply-twisted sector, which in turn is divided into four sub-sectors according to (3). The partition function in each sub-sector is obtained from $Z(g_i;1)$ by applying $\tau \to \tau+1$ transformations. Moreover, since $\forall P \in \Lambda_{16}$, $\Pi P \in \Lambda_{16}$ and $\overline{(\Pi V)} = \overline{V}$, it can be shown that $Z(g_0;1) = Z_(g_2;1)$ and $Z(g_1;1) = Z_(g_3;1)$. This implies that the simply-twisted partition function actually splits into two pieces that we will denote $Z(\theta)$ and $Z(\theta_\Pi)$. Explicitly,

$$\begin{aligned} Z(\theta) &= \frac{1}{2}\left[Z(g_0;1) + Z(g_0;g_0) + Z(g_0;g_0^2) + Z(g_0;g_0^3)\right] \\ Z(\theta_\Pi) &= \frac{1}{2}\left[Z(g_1;1) + Z(g_1;g_1) + Z(g_1;g_1^2) + Z(g_1;g_1^3)\right] \end{aligned} \tag{54}$$

The $Z(g_i;g_i^n)$ only differ in phase factors, the powers of $q, \overline{q}$ are identical and fix the mass level. Actually, for massless states the phase factors are all unity. The left masslessness conditions are

$$\begin{aligned} Z(\theta) \;&:\; M_L^2 = N_L + \frac{1}{2}(P+V)^2 + E_0(\theta) - 1 = 0 \\ Z(\theta_\Pi) \;&:\; M_L^2 = N_L + \frac{1}{2}(\overline{Q}+\overline{V})^2 + E_0(\theta) + \frac{L}{16} - 1 = 0 \end{aligned} \tag{55}$$

where N_L includes the x- and F-oscillator numbers. The right masslessness condition, common for both sectors, is given by

$$M_R^2 = N_R + \frac{1}{2}(r+v)^2 + E_0(\theta) - \frac{1}{2} = 0 \tag{56}$$

States satisfying (55) and (56) appear with multiplicity $2\tilde{\chi}(g_i, 1|x_{g_i})$. This is multiplicity 8 in the Z_4 orbifold with $SU(4) \times SU(4)$ lattice.

Let us now consider the doubly-twisted sector. Since $g_\Pi^2 = g_V^2 = g_{2V}$, the partition function in this sector cannot be split into sub-sectors. Rather, there is just a partition function $Z(\theta^2)$ and all massless states satisfy

$$\begin{aligned} Z(\theta^2) \;:\; & M_L^2 = N_L + \frac{1}{2}(P+2V)^2 + E_0(\theta^2) - 1 = 0 \\ & M_R^2 = N_R + \frac{1}{2}(r+2v)^2 + E_0(\theta^2) - \frac{1}{2} = 0 \end{aligned} \tag{57}$$

$Z(\theta^2)$ includes terms $Z(g_i^2;1)$ and their $\tau \to \tau+1$ modular transforms $Z(g_i^2;g_i^2)$, which only differ in phases that actually vanish for massless states. There are also terms $Z(g_i^2;g_i)$ and their $\tau \to \tau+1$ modular transforms $Z(g_i^2;g_i^3)$, that also contribute equally to massless states. Moreover, they are obtained from modular transformations applied to $Z(g_i; g_i^2)$ as explained before. In particular, from (35) we see that $Z(g_1^2; g_1)$ can only contribute to invariant gauge momentum and oscillator states. The conclusion is that there are two types of massless states according to whether or not they are invariant under Π. In the Z_4 orbifold with $SU(4) \times SU(4)$ lattice, their respective multiplicities turn out to be

$$\begin{aligned} D_{inv}(\theta^2) &= 2 + 2\, e^{2i\pi\Delta_2} \\ D_{non-inv}(\theta^2) &= 2 + e^{2i\pi\Delta_2} \end{aligned} \tag{58}$$

The phase Δ_n is given by

$$\Delta_n = (P + nV) \cdot V - (r + nv) \cdot v - \frac{n}{2}(V^2 - v^2) - \rho \tag{59}$$

where ρ appears in the case of x-oscillator states and depends on how the oscillator is rotated by θ.

Finally, let us discuss the untwisted sector. All massless states satisfy

$$\begin{aligned} Z(1) \quad : \quad & M_L^2 = N_L + \frac{1}{2}P^2 - 1 = 0 \\ & M_R^2 = N_R + \frac{1}{2}r^2 - \frac{1}{2} = 0 \end{aligned} \tag{60}$$

The surviving states must fulfill the usual projection $\Delta_0 =$int. We must also construct Π-invariant combinations since we are embedding Π as a Wilson line. This is the equivalent of the condition $P \cdot a_i = 0$ on massless states in the presence of quantized Wilson lines.

In the case of $Z_2 \times Z_2$ with twists (44), the partition function in the θ sector also divides into two distinct pieces originating sub-sectors θ and θ_Π. The $\theta\omega$ sector splits in a similar way. The ω sector is no split but invariant and non-invariant states are projected differently. In the alternative case (51), the θ sector also divides into two sub-sectors. The ω and the $\theta\omega$ sectors are unaffected. The relevant mass formulae and multiplicities follow from the form of the partition function discussed in the previous section.

3 Examples of $SO(10)$ and $SU(5)$ level 2 models

In this section we apply the techniques developed in the previous section to the construction of explicit GUT examples. We present a couple of four-generation $SO(10)$ models with an adjoint Higgs in the massless spectrum (we were only able to construct models with a 54 and no adjoint in ref.[1]). We also build a four-generation $SU(5)$ model.

A natural extension of the permutation method just described is to combine the action of Π with a shift S in Λ_{16}. More precisely, we consider a generator $(1, e_1|\Pi, S)$. Since Π is of order two, it must be that $2S \in \Lambda_{16}$. In the Z_4 orbifold, the simply-twisted sector splits according to the elements $(\theta, 0|1, V)$, $(\theta, e_1|\Pi, S+\Pi V)$, $(\theta, e_1+e_2|1, S+\Pi S+\Pi V)$ and $(\theta, e_1+e_2+e_3|\Pi, \Pi S+V)$. The original translation V must satisfy eq. (4). The structure of the partition function is completely analogous to the case $S=0$. In particular, there appear modular invariant conditions of the form of eqs. (27) and (42), with V replaced by $V+S$.

The main effect of the shift S is to change the projection onto invariant states in the untwisted sector. States with invariant momentum, $\Pi P = P$, must satisfy $P.S = \mathrm{int}$. On the other hand, if $\Pi P \neq P$, $P\cdot(S-\Pi S) = \mathrm{int}$ and $P\cdot(V-\Pi V) = \mathrm{int}$, the states survive into combinations invariant under the full action of Π and S. This means, for instance, that when $P\cdot S = \frac{1}{2}$, the combination changes sign under Π. As a consequence, antisymmetric representations such as adjoints in $SO(10)$, can arise in the untwisted sector. This result will be illustrated in our first example.

3.1 $SO(10)_2$ model with an adjoint and 4 generations.

This model is based on the Z_4 orbifold with $SU(4)\times SU(4)$ compactifying lattice. The starting $SO(10)\times SO(18)\times SU(2)\times U(1)$ model is derived by a gauge embedding with shift

$$V = \frac{1}{4}(2,2,2,2,2,\ 0,0,0,0,0,\ 0,0,0,0,1,1) \tag{61}$$

An $SO(10)$ gauge group at level 2 emerges after implementing a permutation Π of the form (5) with $L=5$, accompanied by a shift

$$S = \frac{1}{4}(1,1,1,1,1,\ 1,1,1,1,1,\ 1,1,1,1,3,3) \tag{62}$$

The resulting model has gauge group $SO(10)\times SU(4)\times SU(2)\times U(1)^2$ where $SO(10)$ is realized at level two. The massless spectrum is found basically as described in section 2.3. There are 8 generations. More notably, there is an adjoint 45 in the U_3 untwisted sector. Let us now explain how this 45 does materialize.

In the U_3 sector, states must have $P\cdot V = \pm\frac{1}{2}$. In the original model there are states with

$$P_{10,10} = (\underline{\pm1,0,0,0,0},\ \underline{\pm1,0,0,0,0},\ 0,0,0,0,0,0) \tag{63}$$

where underlining means permutations. These states are part of a $(10,18)$ of $SO(10)\times SO(18)$. Some of the above momenta are invariant under Π and are projected out because they have $P_{10,10}\cdot S = \pm\frac{1}{2}$. There are also non-invariant momenta with $P_{10,10}\cdot S = 0$ that give rise to invariant combinations such as

$$[(+1,0,0,0,0,\ -1,0,0,0,0,\ 0,\cdots,0)\oplus(-1,0,0,0,0,\ +1,0,0,0,0,\ 0,\cdots,0)] \tag{64}$$

Finally, there are non-invariant momenta with $P_{10,10}\cdot S = \pm\frac{1}{2}$ that give rise to invariant combinations of the form

$$\pm[(+1,0,0,0,0,\ 0,+1,0,0,0,\ 0,\cdots,0)\ominus(0,+1,0,0,0,\ +1,0,0,0,0,\ 0,\cdots,0)] \tag{65}$$

$Sector$	$SO(10)\times SU(2)^3\times U(1)^3$	Q_1	Q_2	Q_3
U_1,U_2	2 (1,1,2,2)	0	1/2	1/2
U_3	(45,1,1,1)	0	0	0
(θ,V)	4 ($\overline{16}$,1,1,1)	0	0	1/4
$(\theta,V+W)$	no massless states			
$(\theta,\overline{V}+\overline{W})$	4 (10,1,1,1)	1/4	1/4	0
	4 (1,2,1,1)	$-1/4$	1/4	0
	4 (1,1,2,1)	$+1/4$	$-1/4$	0
	4 (1,1,1,1)	$-1/4$	$-1/4$	1/2
	4 (1,1,1,1)	$-1/4$	$-1/4$	$-1/2$
$(\theta,\overline{V}+\overline{S}+\overline{W})$	4 (10,1,1,1)	$-1/4$	1/4	0
	4 (1,2,1,1)	1/4	1/4	0
	4 (1,1,2,1)	$-1/4$	$-1/4$	0
	4 (1,1,1,1)	1/4	$-1/4$	1/2
	4 (1,1,1,1)	1/4	$-1/4$	$-1/2$
$(\theta^2,2V)$	4 (10,1,1,1)	0	0	$-1/2$
	4 (10,1,1,1)	0	0	1/2
	3 (1,2,1,1)	1/2	0	1/2
	1 (1,2,1,1)	$-1/2$	0	$-1/2$
	3 (1,2,1,1)	$-1/2$	0	1/2
	1 (1,2,1,1)	1/2	0	$-1/2$
	3 (1,1,2,1)	0	1/2	1/2
	1 (1,1,2,1)	0	$-1/2$	$-1/2$
	3 (1,1,2,1)	0	1/2	$-1/2$
	1 (1,1,2,1)	0	$-1/2$	1/2
$Osc.$	2$\times$ 3(1,1,1,2)	0	0	0
	2$\times$ 1 (1,1,1,2)	0	0	0

Table 1: Particle content of the model of section 3.1.

The 45 comprises all 25 states of type (64) plus all 20 states of type (65).

The number of generations may be further reduced by associating a Wilson line W ($W = \Pi W$) to the second $SU(4)$ lattice, for instance

$$W = \frac{1}{4}(0,0,0,0,0,\ 0,0,0,0,0,\ 2,2,0,0,0,0) \tag{66}$$

This second order Wilson line breaks the $SU(4) \times U(1)$ group above into $SU(2) \times SU(2) \times U(1)^2$ and reduces the number of $SO(10)$ generations to 4. The massless spectrum of the model is given in Table 1.

3.2 $SU(5)_2$ model with 4 generations.

This model is straightforwardly obtained by interchanging the roles of S and W in the above $SO(10)_2$ example. In this way, a $SU(5)_2 \times SU(2)^3 \times U(1)^4$ model with four generations finally emerges. The massless spectrum of the model is shown in Table 2. Interestingly enough, the two examples above show two possible uses of the inclusion of a shift accompanying the permutation twist. In the $SO(10)$ case it helps in picking up the adjoint (asymmetric) representation in the decomposition $(10,10) = 54 + 1 + 45$, in the $SU(5)$ case it reduces the number of generations.

3.3 $SO(10)_2$ model with an adjoint and a $(16 + \overline{16})$ pair

This example is based on $Z_2 \times Z_2$ with the twist realization (51). The starting $SO(10)^3 \times U(1)$ model is derived with the gauge embedding

$$\begin{aligned} A &= \frac{1}{2}(1,1,1,1,1,\ 0,0,0,0,0,\ 1,0,0,0,0,0) \\ B &= \frac{1}{2}(0,0,0,0,0,\ 0,0,0,0,0,\ 1,1,1,1,1,-1) \end{aligned} \tag{67}$$

To obtain $SO(10)$ at level two we act with a permutation Π of type (5) with $L = 5$, plus a shift

$$S = \frac{1}{4}(1,1,1,1,1,\ 1,1,1,1,1,\ 1,1,1,1,1,-3) \tag{68}$$

The resulting model has gauge group $SO(10) \times SU(5) \times U(1)^2$, with $SO(10)$ realized at level two. The massless spectrum, shown in Table 3, is found along the usual procedure. There are 4 net generations and a $(16 + \overline{16})$ pair. There is an adjoint 45 in the U_1 untwisted sector that appears exactly as explained in section 3.1. The normalization of the $U(1)$ charges is chosen so as to simplify the values given in the table. However, in order to verify the anomaly consistency conditions such as $Tr\, Q = 8 Tr\, Q^3$, we must rescale the charges appropriately. For instance, all values of Q_1 must be divided by $\sqrt{2}$.

$Sector$	$SU(5)_2 \times SU(2)^3 \times U(1)^4$	Q_0	Q_1	Q_2	Q_3
U_1, U_2	2 (1,1,1,2)	0	0	1/2	1/2
U_3	(24+1,1,1,1)	0	0	0	0
(θ, V)	4 $(\overline{10}$,1,1,1)	1/2	0	0	1/4
	4 (5 ,1,1,1)	$-3/2$	0	0	1/4
	4 (1,1,1,1)	5/2	0	0	1/4
$(\theta, V+W)$	4 (5,1,1,1)	1	1/4	1/4	0
	4 $(\overline{5},1,1,1)$	-1	1/4	1/4	0
	4 (1,2,1,1)	0	$-1/4$	1/4	0
	4 (1,1,2,1)	0	1/4	$-1/4$	0
	4 (1,1,1,1)	0	$-1/4$	$-1/4$	1/2
	4 (1,1,1,1)	0	$-1/4$	$-1/4$	$-1/2$
$(\theta, \overline{V}+\overline{S})$	no massless states				
$(\theta, \overline{V}+\overline{S}+\overline{W})$	4 (5,1,1,1)	1	$-1/4$	1/4	0
	4 $(\overline{5},1,1,1)$	-1	$-1/4$	1/4	0
	4 (1,2,1,1)	0	1/4	1/4	0
	4 (1,1,2,1)	0	$-1/4$	$-1/4$	0
	4 (1,1,1,1)	0	1/4	$-1/4$	1/2
	4 (1,1,1,1)	0	1/4	$-1/4$	$-1/2$
θ^2	4 (5,1,1,1)	1	0	0	1/2
	4$(\overline{5},1,1,1)$	-1	0	0	1/2
	4 (5,1,1,1)	1	0	0	$-1/2$
	4$(\overline{5},1,1,1)$	-1	0	0	$-1/2$
	3 (1,1,2,1)	0	0	1/2	1/2
	1 (1,1,2,1)	0	0	$-1/2$	$-1/2$
	3 (1,1,2,1)	0	0	1/2	$-1/2$
	1 (1,1,2,1)	0	0	$-1/2$	1/2
	3 (1,2,1,1)	0	1/2	0	1/2
	1 (1,2,1,1)	0	$-1/2$	0	$-1/2$
	3 (1,2,1,1)	0	$-1/2$	0	1/2
	1 (1,2,1,1)	0	1/2	0	$-1/2$
$Osc.$	2× 3(1,1,1,2)	0	0	0	0
	2× 1(1,1,1,2)	0	0	0	0

Table 2: Particle spectrum of the model of section 3.2.

$Sector$	$SO(10) \times SU(5)$	Q_1	Q_2
U_1	(45,0)	0	0
	(1,5)	-1	1
	$(1,\overline{5})$	1	-1
(θ, A)	$2(16,1)$	-1/2	0
$(\theta, \overline{A}+\overline{S})$	$(10,1)$	-1/4	5/4
	$(10,1)$	1/4	-5/4
	$(1,\overline{5})$	-1/4	1/4
	$(1,5)$	1/4	-1/4
	$(1,1)$	3/4	5/4
	$(1,1)$	-3/4	-5/4
ω	$3(1,\overline{10})$	-1/2	1/2
	$3(1,5)$	-1/2	-3/2
	$3(1,1)$	-1/2	5/2
	$(1,10)$	1/2	-1/2
	$(1,\overline{5})$	1/2	3/2
	$(1,1)$	1/2	-5/2
$\theta\omega$	$3(16,1)$	-1/2	0
	$(\overline{16},1)$	1/2	0

Table 3: Particle content of the model of section 3.3.

4 General aspects of orbifold GUTs

There are many properties of the string GUTs presently described that are general and apply to any orbifold GUT. In fact some of them are valid even for any level-two string GUT, independently of the string-building method. Some of these general properties were spelled out in ref. [1]. In the present section we further elaborate on the arguments to show that this class of string GUTs is very much restricted both in particle content and couplings. One of the main sources of constraints is the fact that any massless particle must fulfill the condition

$$N_x \ + \ h_F \ + \ E_0 \ = \ 1 \tag{69}$$

N_x is a bosonic oscillation number. h_F is the contribution to the left-handed conformal weight coming from the gauge coordinates. E_0, given in (53), is the contribution of the internal twisted vacuum. There is a finite number of possibilities for E_0, each one associated to a twist vector (v_1, v_2, v_3) (see Table 1 in ref. [1]). On the other hand, in our type of constructions, the gauge coordinates generically realize an affine Lie algebra together with a coset conformal field theory (CFT) with conformal dimensions such that $c_F = c_{KM} + c_{coset} = 16$. Moreover, $h_F = h_{KM} + h_{coset}$, where h_{KM} only depends on the level of the algebra and on the gauge quantum numbers of the particle according to

$$h_{KM}(R_1, R_2, \cdots) \ = \ \sum_a \frac{C(R_a)}{k_a + \rho_a} \tag{70}$$

Here $C(R_a)$ is the quadratic Casimir of the representation R_a, k_a is the level and ρ_a is de dual Coxeter number. The sum runs over the different gauge groups.

4.1 General properties of $SO(10)$ orbifold GUTs

Using the constraint (69) and (70) it follows that fields transforming like a 54 can only be present in the untwisted sector while adjoint 45s can in principle exist either in the untwisted sector or else in twisted sectors corresponding to twist vectors $(0, \frac{1}{4}, -\frac{1}{4})$ of Z_4 type or $(0, \frac{1}{6}, -\frac{1}{6})$ of Z_6 type [1]. The fact that a 54 can only appear in the untwisted sector is easily demonstrated since $h_{KM}(54) = 1$ for $SO(10)$ at level two. Then eq.(69) can only be verified for $E_0 = 0$ which means untwisted sector for symmetric orbifolds. In the case of an adjoint 45, $h_{KM}(45) = 4/5$ and hence in principle any twist vector with $E_0 \leq 1/5$ leaves room enough for massless 45s in the corresponding twisted sector. The unique twists with $E_0 \leq 1/5$ are precisely the Z_4 and Z_6 twists mentioned above which have $E_0 = 3/16$ and $E_0 = 5/36$ respectively. This is why in ref. [1] we left open the possibility of finding models with 45s in these twisted sectors. We will now show that with a little more effort this possibility can be ruled out to reach the important conclusion that GUT-Higgs fields in orbifold $SO(10)$ string GUTs can only appear in untwisted sectors. This in turn has dramatic consequences for the possible couplings of 45s and 54s. To prove the above statement, it is useful to consider in more detail the conformal field theory aspects of the gauge sector. This will also allow us to understand other general properties of these $SO(10)$ theories.

As we recalled in the introduction, $SO(10)$ string models obtained from continuous Wilson lines are continuously connected to level-one models with gauge group factors $SO(2N)\times SO(2M)\times G$, with $N \geq M \geq 5$. The 45s or 54s can only arise from level-one representations $(2N, 2M)$ which have conformal weight one for any N, M. Thus, neither these representations nor the induced 45s or 54s can be present in any twisted sector. The same is true, of course, for any $SO(10)$ string-GUT constructed from flat directions in a level-one model. Thus, 45s in a twisted sector could only occur in a model which is not continuously connected to level one.

Of the three methods considered in ref. [1], only that based on a permutation action does not necessarily produce level-two models continuously connected to level one. In the permutation method, $SO(10)$ at level two is obtained by starting from an orbifold with level one gauge group $SO(2N) \times SO(2M) \times G$, $M \geq N \geq 5$. The permutation Π exchanges L gauge coordinates inside $SO(2N)$ with another L inside $SO(2M)$. As described in the previous section, massless states can originate in permuted twisted sectors like θ_Π or non-permuted like θ. The latter cannot contain 45s because their particle content is a simple GSO-like projection of the level one model that cannot contain $(2N, 2M)$ multiplets in twisted sectors. Thus, 45s can only possibly exist in permuted sectors. Let us then concentrate on this possibility.

From our previous results we know that eq.(69) is equivalent to

$$N_L \ + \ \frac{1}{2}(\overline{Q}+\overline{A})^2 \ + \ \frac{L}{16} \ + \ E_0 \ = \ 1 \tag{71}$$

where A is a generic gauge embedding in Λ_{16}. Here $N_L = N_x + N_F$ contains the oscillator number N_x of the bosonic string coordinates and the oscillator number N_F of the gauge coordinates. The quantum numbers of the state are encoded in the shifted Λ_I^* momenta $(\overline{Q}+\overline{A})$ and in N_F. In particular, choosing to embed $SO(10)$ in the first five entries of the shifted lattice, a 45 must correspond to solutions of eq.(71) of the form

$$N_L = 0 \ \ ; \ \ (\overline{Q}+\overline{A}) \ = \ (\underline{\pm\tfrac{1}{\sqrt{2}}, \pm\tfrac{1}{\sqrt{2}}, 0, 0, 0}, \ Z; Y) \tag{72}$$

$$N_L = N_F = 1/2 \ \ ; \ \ (\overline{Q}+\overline{A}) \ = \ (0,0,0,0,0, Z; Y) \tag{73}$$

where $Z = (Z_6, \cdots, Z_{2L})$ and $Y = (Y_{2L+1}, \cdots, Y_{16})$. The states with $N_F = 1/2$ are the permuted oscillator states needed to complete the 45. The Z and Y vectors must coincide in both types of states to ensure that the full 45 has the same quantum numbers under the rest of the gauge group.

The explicit solutions in (73) are such that

$$N_L \ + \ \frac{1}{2}(\overline{Q}+\overline{A})^2 \ \geq \ \frac{1}{2} \tag{74}$$

Combining this result with eq.(71) implies

$$\frac{L}{16} \ + \ E_0 \ \leq \ \frac{1}{2} \tag{75}$$

As mentioned before, the only twisted sectors in which a 45 has any chance to appear have either $E_0 = 3/16$ or $E_0 = 5/36$. In either case, eq.(75) can only be fulfilled for $L \leq 5$. Thus, only models with permutations of five gauge coordinates can possibly work and necessarily $Z = 0$. In a Z_4 sector ($E_0 = 3/16$), the inequality in (75) is saturated so that $Y = 0$ whereas $Y^2 = 14/144$ in a Z_6 sector. Since in this case $h_{KM} = \frac{4}{5} + \frac{Y^2}{2}$, eq. (69) in both sectors requires that the 45 be accompanied by a coset field with $h_{coset} = 1/80$.

To proceed we need to further investigate the left-handed CFT associated to the gauge degrees of freedom. Since we have $L = 5$ it is now clear that the relevant CFT has the structure

$$SO(10)_2 \;\otimes\; \left(\frac{SO(10)_1 \times SO(10)_1}{SO(10)_2}\right) \otimes\; G_Y\ . \tag{76}$$

The central charges associated to each of the three factors is $c_{SO(10)} = 9$, $c_{coset} = 1$ and $c_Y = rank G_Y = 6$. In terms of these theories we can write

$$h_F = h_{SO(10)} \;+\; h_{coset} \;+\; \frac{Y^2}{2} \tag{77}$$

Thus, a particle will have contributions to its left-handed gauge conformal weight coming from its $SO(10)$ and G_Y quantum numbers and from the $c = 1$ coset.

We thus see that in building level-two $SO(10)$ string GUTs the coset theory $\mathcal{C} = SO(10)_1 \times SO(10)_1/SO(10)_2$ plays an important rôle. Interestingly enough, this coset has a remarkably simple CFT structure (which is not the case for the equivalent coset in $SU(5)$ unification). Indeed, as explained in ref. [15] , $\mathcal{C}$ is an element in the series of $c = 1$ models of type $SO(N)_1 \times SO(N)_1/SO(N)_2$ that correspond to Z_2 orbifolds of a free boson compactified on a torus at radius $r = \sqrt{N}/2$. The conformal weights of winding and momenta states of the circle compactification at this radius are given by the simple expression

$$h_{m,n} \;=\; \frac{1}{2}\left(\frac{m}{2(\sqrt{N}/2)} + n\frac{\sqrt{N}}{2}\right)^2 \;=\; \frac{1}{8N}(2m + nN)^2\ . \tag{78}$$

In our case $N = 10$ and the weights in the circle compatification are of the form $q^2/20$, $q \in \mathbf{Z}^+$. There are also Z_2 twist and excited twist operators giving rise respectively to states of conformal dimensions $\frac{1}{16}$ mod 1 and $\frac{9}{16}$ mod 1. The particles in our $SO(10)$ string GUT will have associated some of the possible coset weights. No weight is as small as $h_{coset} = 1/80$, which is the required contribution for a massless 45 in the Z_4 or Z_6 sectors. Therefore, we conclude that there cannot be 45s in any twisted sector.

The fact that the GUT Higgs fields in $SO(10)$ string GUTs are necessarily in the untwisted sector has important consequences for the structure of the couplings in the theory. To begin, we can immediately show that there will be just one GUT Higgs, either a 54 or a 45, associated to one and only one of the three untwisted sectors. Indeed, in all the level two $SO(10)$ constructions the starting level-one model has generic gauge group $SO(2N) \times SO(2M) \times G$, with $M \geq N \geq 5$.

To produce such a group, the gauge embedding must have the general form $A = (\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}, 0, 0, \cdots, 0, A_{M+N+1}, \cdots, A_{16})$ modulo elements of the Λ_{16} lattice. Representations transforming as 54 or 45 in the untwisted sector correspond to specific subsets of lattice vectors of the form given in (63). Since $P_{10,10} \cdot A = \frac{1}{2}$ modulo integer, for these particles to survive the generalized GSO projection, they must be associated to a complex plane which is twisted only by order-two twists. This means that the internal twist vector must be of the form $(v_1, v_2, \frac{1}{2})$. This is the case of the Z_4, Z_6, Z_8, Z_{12}, $Z_2 \times Z_2$, $Z_2 \times Z_4$ and $Z_2 \times Z_6$ orbifolds. All of these have a complex plane which suffers only order-two twists and hence may have a 45 or a 54 in the corresponding untwisted sector. All other symmetric orbifolds are ruled out for the purpose of $SO(10)$ model-building. Finally, in the level-two $SO(10)$ model the 54 and the 45 cannot both survive at the same time in the mentioned untwisted sector since one of them is necessarily projected out due to the opposite properties that these representations have under permutations of the underlying $SO(10) \times SO(10)$ structure. In the case of a level-one $SO(10)^2$ model continuously connected to the level-two $SO(10)$ GUT, only a 54 may be present in the massless sector since, as discussed in [1], the 45 is swallowed by the Higgs mechanism.

We thus see that the structure of symmetric orbifold string $SO(10)$ GUTs is quite characteristic. The GUT-Higgs lives in one of the three untwisted sectors, say in the third U_3. This very much constraints its couplings. In particular, it is well known that the only Yukawa couplings among purely untwisted particles is of the type $U_1U_2U_3$. Also point-group invariance obviously forbids couplings of type UUT. Thus, necessarily couplings of type 54^3 or $X54^2$, $X45^2$ are absent (45^3 couplings are anyway absent due to its antisymmetric character). In fact, couplings of type 45^n, 54^n are forbidden for arbitrary n due to the three R-symmetries associated to each of the three complex planes in any orbifold.

There are also some conclusions that may be drawn *independently of the string-construction method* and apply to methods like asymmetric orbifolds or the fermionic construction. We already remarked that for the 54, $h_{SO(10)} = 1$. Then, eqns. (69) and (77) necessarily imply that a 54 of $SO(10)$ cannot have any extra gauge quantum numbers, otherwise it would be superheavy. Furthermore, $h_{coset} = 0$, so that its transformation properties under the coset CFT are trivial. This implies the absence of couplings of type $X(54)^2$ or (in theories with more than one 54) $X(54)(54)'$, where X is any $SO(10)$ singlet field. Indeed, if X is charged under some other gauge symmetry, this gauge symmetry will forbid those couplings since the 54s are neutral. If on the other hand X is neutral, it necessarily has a non-trivial coset CFT structure. In this second case the couplings also vanish because the corresponding correlators should vanish given the trivial coset structure of the 54s. Notice however that a 54 might have discrete (R-symmetry) quantum numbers coming from the right-moving factor of its vertex operator.

4.2 General aspects of $SU(5)$ orbifold GUTs

In ref. [1] it was found that the constraint (69) reduced the number of possibilities for the location of adjoint 24s in the different twisted sectors, since $h(24) = 5/7$. In particular, it was found that a 24 could not possibly be located in twisted sectors with twist vectors: $(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$, $(\frac{1}{4}, \frac{1}{4}, -\frac{1}{2})$, $(\frac{1}{6}, \frac{1}{3}, -\frac{1}{2})$ and $(\frac{1}{8}, \frac{3}{8}, -\frac{1}{2})$. Arguing along the lines of the previous section, we now show that actually 24s cannot appear in any twisted sector except in those of Z_4 and Z_6 type with $(0, \frac{1}{4}, -\frac{1}{4})$ and $(0, \frac{1}{6}, -\frac{1}{6})$. Furthermore, we will also show that 24s cannot possibly appear in the untwisted sectors of several Abelian orbifolds.

In the string GUT-building method in which the $SU(5)$ GUT is continuosly connected through a flat direction to a level-one group such as $SU(5) \times SU(5)$, the 24s have their origin in representations of type $(5, \bar{5})$. Now, since such fields have $h_{SU(5)^2} = 4/5$, only twisted sectors with $E_0 \leq 1/5$ can contain a 24. Only the Z_4 and Z_6 twisted sectors mentioned above have this property ($E_0 = 3/16$ and $E_0 = 5/36$ respectively). It is now easy to prove that actually only these twisted sectors may contain 24s even if we consider the permutation method which is not necessarily continuously connected to level one. Indeed, notice that we can still use formulae (71) and (75) in this $SU(5)$ case, since the $SU(5)$ roots are a subset of those in eq. (73). Hence, we are again confined to the Z_4 and Z_6 twisted sectors with $E_0 \leq 3/16$. Unlike the $SO(10)$ case, we cannot further argue about the absence of 24s in these two sectors since now $h(24) = 5/7$ and values of h_{coset} larger than $(1 - E_0 - 5/7)$ are allowed. This leaves room for 24s in these twisted sectors. In fact, a $Z_2 \times Z_4$ orbifold example of a $SU(5)$ model with 24s in a Z_4 sector was presented in ref. [1].

The alternative is having the 24s in the untwisted sector. Let us then study whether all orbifolds may have 24s in the untwisted sector and, if so, how many. The untwisted 24s have their origin in chiral fields of type $(5, \bar{5})$ or $(\bar{5}, 5)$ in the underlying $SU(5) \times SU(5)$ level-one structure (the group may be bigger but must contain $SU(5)^2$). These representations have $P^2 = 2$ weights of the form $P = (\underline{1,0,0,0,0}, \underline{-1,0,0,0,0}, 0,0,0,0,0,0)$ and $-P$ respectively. For a model to have one 24 in the untwisted sector, its level-one ancestor must contain *at least two fields* of this type in its own untwisted sector. The reason is that one of these fields (or a combination of both) dissappears in the process of going from the underlying level-one model to the level-two $SU(5)$ GUT. Indeed, if the transition occurs by giving vevs to one of these fields, one of them is Higgssed away. Thus, two of these fields are needed so that one of them is left over. If level two is reached by permutation modding, only half of the weights $\pm P$ (either symmetric or antisymmetric combinations) will survive the projection, so that the final effect is similar.

We can see that there are just two ways to have two such fields. One possibility is having an orbifold in which there is a degeneracy factor 2 or 3 coming from the right-movers. There are only three such orbifolds:Z_3 (multiplicity 3 for any untwisted chiral field) and Z_4, Z_6' (multiplicity 2 for the untwisted fields in the first two complex directions). The second possibility is having an orbifold with one of the complex planes feeling only order-two twists (as in the $SO(10)$ case). In this case,

GUT	Z_3	Z_4	Z_6	Z_6'	Z_7	Z_8	Z_8'	Z_{12}	Z_{12}'
$SO(10)$	–	U	U	–	–	U	–	U	–
$SU(5)$	–	U	U	U	–	U, T_4	–	U, T_6	T_4

Table 4: Z_N orbifolds allowing for the construction of string GUTs. The table indicates whether the GUT-Higgs are in the untwisted sector (U) or in some order-four or six twisted sector ($T_{4,6}$).

GUT	$Z_2 \times Z_2$	$Z_3 \times Z_3$	$Z_2 \times Z_4$	$Z_2 \times Z_6'$	$Z_2 \times Z_6$	$Z_4 \times Z_4$	$Z_3 \times Z_6$	$Z_6 \times Z_6$
$SO(10)$	U	–	U	–	U	–	–	–
$SU(5)$	U	–	U, T_4	–	U, T_6	T_4	T_6	T_6

Table 5: $Z_M \times Z_N$ orbifolds allowing for the construction of string GUTs. The table indicates whether the GUT-Higgs are in the untwisted sector (U) or in some order-four or six twisted sector ($T_{4,6}$).

P and $-P$ are allowed in the untwisted sector, since a particle with weight P as above has $(\pm P) \cdot A = 1/2 \mod 1$ (A is the shift in the underlying level-one theory). Therefore, in order-two complex planes if there is a $(5, \overline{5})$ there is also a $(\overline{5}, 5)$ and a 24 survives in the level-two $SU(5)$ model. Since for orbifolds other than Z_3 ,Z_4 and Z_6' a given weight in the gauge lattice appears only once in the untwisted sector, we conclude that the possibilities for obtaining at least one 24 in the untwisted sector are exhausted.

The above discussion allows us to rule out a number of orbifolds for the purpose of constructing $SU(5)$ GUTs. Orbifolds without an order-two plane or different from Z_3 and Z_6' cannot lead to 24s (unless they arise from the order 4 or 6 twisted sectors that we mentioned above). This eliminates orbifolds based on Z_7, Z_8', $Z_3 \times Z_3$ and $Z_2 \times Z_6'$. In fact, the Z_3 orbifold, can also be discarded because if it has $(5, \overline{5})$s in its untwisted sector, it automatically has vanishing net number of fermion generations. We will spare the reader the proof of this statement. It can be shown explicitly by considering the most general shifts and Wilson lines compatible with the presence of these representations in the untwisted sector. Thus, we conclude that for any $SU(5)$ symmetric orbifold GUT, if the 24s reside in the untwisted sector (by far the commonest case) there is *only one* such 24 associated to one of the three untwisted sectors. The general situation concerning the possibilities offered by symmetric Z_N and $Z_M \times Z_N$ orbifolds for the construction of $SO(10)$ and $SU(5)$ GUTs is summarized in Tables 4 and 5. Notice that only even-order orbifolds can give rise to string GUTs.

As it happened in $SO(10)$, the fact that 24s must belong either to the untwisted or else to Z_4 or Z_6 twisted sectors has important implications for the couplings of the GUT-Higgs. If the 24 is in the untwisted sector, 24^n couplings will be forbidden by the three R-symmetries of the orbifold, and the same will be true for couplings of

$X(24)^2$ type, X being any singlet particle. If, on the other hand, a 24 is present in a Z_4 (Z_6) sector, couplings of type 24^{4n} (24^{6n}) are in principle allowed by point group invariance. However, if the 24 is continuously connected to a $(5, \overline{5})$ representation of an underlying $SU(5) \times SU(5)$ theory, then this underlying symmetry will force the lowest dimension self-coupling to be of order 20 (30). In any case cubic self-couplings 24^3 as in the model of ref. [16] are forbidden. Notice however that the 24s can couple to 5 and $\overline{5}$s in the theory, it is only self-couplings that are so much restricted.

4.3 Summary of selection rules

The above discussion may be summarized in a series of selection rules which we collect in this subsection for the phenomenologically oriented reader. There are some properties that are *general* and apply to any level-two string GUT, whatever the construction technique used. These are the following:

i. All superpotential terms have $dim \geq 4$ (i.e. no mass terms).

ii. At level two the only representations that can be present in the massless spectrum are: 5, 10, 15, 40 and 24 of $SU(5)$; 10, 16, 45 and 54-plets of $SO(10)$.

iii. A 54 of $SO(10)$ cannot be charged under any other gauge group.

iv. There cannot be couplings of type $X(54)^2$ or, if several 54s exist, $X(54)(54)'$, X being any $SO(10)$-singlet chiral field.

The first two rules were already discussed in refs. [3, 1], whereas the last two are discussed above.

If we further focus on the particular case of string GUTs obtained from *symmetric orbifolds*, we can add the extra rules:

v. $SO(10)$: There is only *either* one 54 or one 45. They do not have any selfcouplings 54^n, 45^n to any order. Couplings bilinear in the GUT-Higgs such as $(54)^2XY\cdots$, $(45)^2XY\cdots$, are forbidden. The 45 or 54 do on the other hand couple in general to Higgs 10-plets.

vi. $SU(5)$: There can be more than one 24 only in a few orbifold models containing particular Z_4 or Z_6 twists. Otherwise, there can be only one 24 with no self-couplings 24^n to any order. Adjoints in twisted sectors can in principle have couplings $(24)^{4n}$ or $(24)^{6n}$.

It is important to remark that properties **v** and **vi** apply only to *symmetric orbifold GUTs*. It cannot be concluded that all string GUTs have this structure. Properties **i** to **iv** are general.

It is compelling to compare the resulting string GUT structure with some $SO(10)$ models found in the recent SUSY-GUT literature. In refs. [17, 18], $SO(10)$ models in which doublet-triplet splitting is effected by a version of the Dimopoulos-Wilczek mechanism are considered. They involve three adjoints, one 54 and a pair $(16 + \overline{16})$ with judiciously chosen couplings. This complicated arrangement is required to guarantee doublet-triplet splitting while avoiding Goldstone bosons and extra massless fields beyond those of the MSSM. Refs. [19, 20] study $SO(10)$ models whose spectrum is claimed to be inspired by explicit fermionic construction of string GUTs (in fact, the particular string $SO(10)$ models referred to turned out to be inconsistent due to lack of world-sheet supersymmetry). What the authors actually do is to dispose of the 54 while trying to acomplish all the phenomenological goals with one single 45, and a pair $(16+\overline{16})$. As they stand, all these models would have difficulties to work in the string context since they violate the first of the above selection rules by including explicit mass terms in the superpotential.

In ref. [21] an attempt is made to get rid of explicit mass terms and to use only dimension four potential terms. The structure required to make this work while maintaining doublet-triplet splitting, no Goldstone bosons and a massless sector identical to that of the MSSM, is quite contrived. Six adjoint 45s, two 54s, a pair $(16+\overline{16})$ and extra singlets, along with particular superpotential couplings, are invoked in this scheme. Such a multiplicity of GUT-Higgs fields is rather unexpected in explicit string models (we just saw how in the case of symmetric orbifolds only one GUT-Higgs is obtained). An extra problem may concern the couplings. For example, a coupling of type $X(54)(54)'$ is used ($S_2S'S$ in the notation of [21]) which is strictly forbidden in any possible level-two string GUT. This is a small detail which could probably be easily cured. What seems difficult is to simultaneously procure doublet-triplet splitting and the MSSM massless content while using only $dim \geq 4$ couplings. We think that this is a generic problem for $SO(10)$ GUTs and not solely for those based on an underlying string theory.

5 Four-generation and three-generation models

In this section we present examples of string GUTs with reduced number of generations (3 or 4). Models with an even number of net generations are the commonest in our orbifold constructions. Qualitatively speaking, what happens is that the number of generations depends on the multiplicity of each twisted (and untwisted) sector. The multiplicity of a twisted sector is in turn given by the number of fixed points which is very often an even number for even orbifolds, leading to even number of generations. This is an oversimplification since there are even-order orbifolds with some twisted sectors having odd multiplicities. Furthermore, the degeneracy factors can be modulated to some extent through the addition of Wilson lines. However, the above remark reflects what actually occurs in explicit models.

Consider in particular the case of $SO(10)$ string GUTs. As we have explained, these can only be realized in orbifolds in which one of the three complex planes has at most an order-two twist. This reduces the list of possibilities just to the even orbifolds shown in Tables 4 and 5. Since we are dealing with strings based on the $Spin(32)/Z_2$ lattice, there are no spinorial vectors with $P^2 = 2$ and hence 16s of $SO(10)$ can never arise in the untwisted sector, all generations must originate in twisted sectors. Thus, the replication of generations results merely from degeneracy factors in different twisted sectors that are even most of the time. We have not carried out a full computer search of three-generation $SO(10)$ string GUTs but just an exploratory scan in the lowest order orbifolds (Z_4, Z_6, $Z_2 \times Z_2$ and $Z_2 \times Z_4$) and we have not found three-generation models. However, we do not think that this is a general feature of string theory and probably three-generation examples can be found. After all, the first Calabi-Yau compactifications historically found all had even number of generations, it took some time to produce three-generation examples!. Notice in this context that the three-generation $SO(10)$ models reported in [4, 5] were later on withdrawn due to lack of world-sheet supersymmetry.

The situation of $SU(5)$ GUTs is slightly different. According to Tables 4 and 5, the list of possible orbifolds is somewhat larger. Furthermore, unlike the $SO(10)$ case, there can be generations in the untwisted sector since $(10+\overline{5})$ representations need not come from Λ_{16} spinorial weights. Thus, in principle the $SU(5)$ GUT construction is more flexible with respect to the generation number. For instance, if there were four generations from the twisted sectors, there could be an antigeneration in the untwisted sector to adjust three net families. An example of this sort will be shown below. However, it turns out that these attempts to look for three generations lead to models considerably more complicated than the four-generation models that we obtain. Therefore, we think that it is interesting to first examine some aspects of four-generation models. In particular, we want to discuss the Yukawa coupling structure of the four-generation $SO(10)$ GUT called "example 1" in ref. [1], due to its relative simplicity and also because there are many similar $SO(10)$ string GUTs that can be obtained with the present techniques. We also briefly discuss a few aspects of the four-generation examples in section 3. A recent study of the phenomenological viability of four-generation SUSY models is carried out in ref. [22].

5.1 A four-generation $SO(10)$ GUT

We consider the first example discussed in chapter 4 in ref. [1]. This model can in fact be built by the three orbifold methods: continuous Wilson lines, permutation modding or flat directions. In ref. [1] it is constructed through continuous Wilson lines on a $Z_2 \times Z_2$ orbifold with cubic $SO(4)^3$ torus lattice. The $Z_2 \times Z_2$ twists are such that the twisted sectors have reduced multiplicities (see [1] for details). The gauge group is $SO(10)_2 \times SO(8) \times U(1)^2$ where the GUT is realized at level two and the rest at level one. The complete chiral field massless spectrum is shown in Table 6.

The $U(1)$ with charge Q_A is anomalous but this anomaly is cancelled by the Green-Schwarz mechanism. The fact that $TrQ_A \neq 0$ induces a one-loop effective Fayet-Illiopoulos term. Upon minimization of the scalar potential this term will force some chiral field to acquire a vev in order to cancel the D-term. Since TrQ_A is possitive, giving a vev to X_{0-1} will be sufficient for this cancellation.

It is useful to recall some properties of the Yukawa couplings. Generically denoting fields by their corresponding sector, the allowed Yukawa couplings in the $Z_2 \times Z_2$ orbifold are of the form

$$U_1U_2U_3 \ , \ U_2\theta\theta \ , \ U_1\omega\omega \ , \ U_3(\theta\omega)(\theta\omega) \ , \ \theta\omega(\theta\omega) \tag{79}$$

These are the only couplings allowed by the point-group and H-momentum discrete symmetries. Among the allowed cubic couplings a small subset could be actually

$Sector$	$SO(10)\times SO(8)$	Q	Q_A	Notation
U_1	(1,8)	1/2	1/2	
	(1,8)	-1/2	-1/2	
U_2	(1,8)	-1/2	1/2	
	(1,8)	1/2	-1/2	
U_3	(54,1)	0	0	Φ_{54}
	(1,1)	0	0	X_0
	(1,1)	0	1	X_{01}
	(1,1)	1	0	X_{10}
	(1,1)	-1	0	X_{-10}
	(1,1)	0	-1	X_{0-1}
θ	$3(16,1)$	1/4	1/4	16^i_θ
	$(\overline{16},1)$	-1/4	-1/4	$\overline{16}_\theta$
ω	$3(16,1)$	-1/4	1/4	16^i_ω
	$(\overline{16},1)$	1/4	-1/4	$\overline{16}_\omega$
$\theta\omega$	$4(10,1)$	0	1/2	10^a_+
	$4(10,1)$	0	-1/2	10^a_-
	$3(1,8)$	0	1/2	
	$(1,8)$	0	-1/2	
	$8(1,1)$	1/2	0	X^r_+
	$8(1,1)$	-1/2	0	X^r_-

Table 6: Particle content and charges of the 4-generation $SO(10)$ model of section 5.1.

forbidden by extra discrete symmetries coming from space-group selection rules that depend on details of the structure of fixed points and space-group conjugacy classes. Since we just want to point out a few qualitative features we will not take into account these further restrictions. It is interesting to remark that, in spite of the many possible couplings forbidden by point-group and H-momentum selection rules, the distribution of the particles in the different untwisted and twisted sectors is such that many of the phenomenologically required couplings are indeed present. The list of interesting Yukawa couplings in the model includes the following:

i) The singlets $X_0, X_{01}, X_{10}, X_{-10}, X_{0-1}$ in the U_3 untwisted sector have couplings with the $SO(8)$ octets in the sectors U_1, U_2 and $\theta\omega$. If some or all of these singlets get vevs (we already saw that the Fayet-Illiopoulos term induces a vev at least for X_{0-1}), most or all of this extra matter will dissapear from the massless spectrum. At the same time the $U(1)^2$ symmetry is spontaneously broken.

ii) The singlets X_+^r, X_-^r in the $\theta\omega$ sector have couplings to some $(16+\overline{16})$ pairs:

$$h_{ri} X_+^r (16_\omega^i)(\overline{16}_\theta) \quad ; \quad h'_{rj} X_-^r (16_\theta^i)(\overline{16}_\omega) \tag{80}$$

where h, h' are Yukawa coupling constants of order unity. These couplings may have several effects. Some of these singlets $X_{+,-}$ could get a vev and give large masses to one or two $(16+\overline{16})$ pairs. A second use of these couplings is perhaps more interesting. We know that the breaking of the $SO(10)$ symmetry down to the standard model requires vevs both for the 54 and a $(16+\overline{16})$ pair. If one of the two $\overline{16}$, say $\overline{16}_\theta$, gets a vev, the right-handed neutrinos of the three 16_ω generations will combine with three of the eight X_+^r singlets and will dissapear from the massless spectrum. Thus, this would solve the neutrino mass problem for these three generations that would be naively identified with the observed three generations. On the other hand, the fourth net generation would be one of the three 16_θ^i fields (or a lineal combination). Since they do not couple to the $\overline{16}_\theta$ field, their right and left-handed neutrinos can only have Dirac type masses that could be large if their standard Yukawa coupling is of order one. In principle this could explain why LEP sees only three and not four massless neutrinos, although of course a more detailed analysis would be needed to reach a definitive conclusion. Such detailed analysis goes beyond the scope of this paper in which a systematic study of the phenomenology of the models is not pursued.

iii) Yukawa couplings of the standard type that can give usual Dirac masses to the fermions are also present. The symmetries allow for the couplings:

$$h_{ija} 16_\theta^i 16_\omega^j 10_-^a \quad ; \quad h'_{ija} \overline{16}_\theta^i \overline{16}_\omega^j 10_+^a \tag{81}$$

iv) The electroweak Higgsses $10_{+,-}$ couple to the fields Φ_{54}, X_0, X_{01} and X_{0-1} of the U_3 sector :

$$(10_+, 10_-) \begin{pmatrix} X_{0-1} & X_0 + \Phi_{54} \\ X_0 + \Phi_{54} & X_{01} \end{pmatrix} \begin{pmatrix} 10_+ \\ 10_- \end{pmatrix} \tag{82}$$

These couplings are similar to those of the singlets in U_3 to the $SO(8)$ octets mentioned above. There is however an important difference in this case signaled by the presence of couplings to the GUT-Higgs Φ_{54}. If there is a vev $\langle\Phi_{54}\rangle = v\,\mathrm{diag}\,(2,2,2,2,2,2,-3,-3,-3,-3)$, the $SO(10)$ symmetry breaks down to $SU(4)\times SU(2)\times SU(2)$. Then, there are regions in the "moduli-space" of the fields Φ_{54}, X_0, X_{01} and X_{0-1} in which some electroweak multiplets $(1,2,2)$ can be light and their $SO(10)$ partners, the $(6,1,1)$s are heavy. Thus, the possibility of doublet-triplet splitting exists but, as already remarked in [1] , it remains to understand why the regions of moduli-space giving some massless doublets should be dynamically preferred. The existence of all the above phenomenologically required couplings is non-trivial. However, a much more detailed analysis of the different couplings and a better understanding of the doublet-triplet splitting would be needed.

Before closing this subsection let us briefly comment on the phenomenologically interesting couplings of the four-generation models constructed in section 3. The model in section 3.1 is an $SO(10)$ GUT with a 45 and no $(16+\overline{16})$ pairs (see Table 1). The four $\overline{16}$ generations in the (θ, V) sector have standard Yukawa couplings with the first set of 10-plets in the $(\theta^2, 2V)$ sector. The adjoint 45 in the U_3 sector can potentially split doublets from triplets due to its coupling to the two sets of Higgs 10-plets in the $(\theta^2, 2V)$ sector. In spite of the existence of these interesting couplings, the absence of $(16+\overline{16})$ pairs makes impossible the breaking of the $B-L$ symmetry down to the standard model and also renders the right-handed neutrinos insufficiently massive. The $SO(10)$ model in section 3.3 (see Table 3) has a $(16+\overline{16})$ pair and the breaking of the symmetry down to the standard model can easily proceed. However, the standard fermion Yukawa couplings and the coupling of the 45 to the Higgs 10-plets are not present at the renormalizable level. Although the required effective Yukawa couplings might appear from nonrenormalizable terms, it is fair to say that the $SO(10)$ model with a 54 discussed above has a simpler structure.

The $SU(5)$ model in section 3.2 (see Table 2) has an interesting Yukawa coupling structure. The $SU(5)$ 10-plets belong to sector (θ, V) and couplings to the last of the $\overline{5}$-plets in the θ^2 sector exist. These give masses to the U-type quarks. Similar couplings exist for the D-type quarks. Couplings of the $24+1$ representation in the U_3 sector to the $(5+\overline{5})$s fields in the θ^2 sector do also exist. These couplings potentially allow for doublet-triplet splitting. Of course, these comments about these four-generation models just reflect the most obvious aspects of their Yukawa coupling structure. A more detailed analysis including the effects of non-renormalizable couplings would be needed to gauge their phenomenological viability.

5.2 A $SU(5)$ string GUT with "almost" three generations

Perhaps the most efficient method for the purpose of model searching is to start with a level-one model with replicated GUT groups and then find flat directions in which the symmetry is broken down to a diagonal level-two GUT. Here we present an $SU(5)$ GUT with adjoint Higgsses obtained by taking flat directions breaking a level-one gauge group $SU(6)\times SU(5)$ to a level-two $SU(5)$. This is a Z_6 orbifold generated by the twist vector $(\frac{1}{6}, \frac{2}{6}, -\frac{3}{6})$ acting on an $SU(3)^3$ compactification lattice.

The embedding in the gauge degrees of freedom is realized by a shift:

$$V = \frac{1}{6}(1,1,1,1,1,1,-2,-2,-2,-2,-2,3,3,3,3,0) \quad (83)$$

This shift fulfills the usual level matching condition $6(V^2 - v^2) = even$. The $SO(32)$ gauge symmetry is broken down to $SU(6) \times SU(5) \times SO(8) \times U(1)^3$. No Wilson lines are needed in this particular model. The massless chiral spectrum shown in Table 7 is found following the usual projection techniques.

In the third untwisted sector, associated to the complex plane with an order-two twist, there appear the multiplets $(6, \overline{5}, 1) + (\overline{6}, 5, 1)$. There is an F-flat and D-flat field direction corresponding to the vevs:

$$\langle (6^i, \overline{5}^a, 1) \rangle = \langle (\overline{6}^i, 5^a, 1) \rangle = \nu \delta^{ia} \quad ; \quad i, a = 1, \cdots, 5 \quad (84)$$

with $i \neq 6$. These vevs break the $SU(6) \times SU(5)$ symmetry down to $SU(5)_2 \times U(1)$, where the $SU(5)_2$ is realized at level two. The model so obtained constitutes an $SU(5)$ GUT. There is an adjoint 24 in the U_3 untwisted sector arising from the chiral fields in $(6, \overline{5}, 1) + (\overline{6}, 5, 1)$ that are not swallowed by the Higgs mechanism.

The net number of $SU(5)$ generations is found by looking for the 10-plets in the spectrum. In the θ^3 twisted sector there are four net $\overline{10}$ multiplets. However, there is an extra massless 10-plet from the untwisted sector so that there are indeed three chiral $(\overline{10} + 5)$ generations plus a number of vector-like $(10 + \overline{10})$ and $(5 + \overline{5})$ multiplets. The origin of this additional 10-plet in the untwisted sector is as follows. Naively we would say that there are in fact two 10-plets in the untwisted sector coming from the $(1, 10, 1)$ and $(15, 1, 1)$ fields (recall that the latter decomposses under $SU(5)$ as $15 = 10 + 5$). However, the following couplings between untwisted particles exist:

$$(6, \overline{5})(1, 10)(\overline{6}, \overline{5}) \quad ; \quad (\overline{6}, 5)(15, 1)(\overline{6}, \overline{5}) \quad (85)$$

Once the $(6, \overline{5})$, $(\overline{6}, 5)$ acquire the vevs in eq. (84), the antisymmetric piece of the $(\overline{6}, \overline{5})$ combines with a linear combination of the two 10-plets inside $(15, 1)$ and $(1, 10)$ and becomes massive. Thus, only one 10-plet from the untwisted sector remains in the massless spectrum and the net number of standard $SU(5)$ generations is indeed three.

Unfortunately, apart from this three generations and the vector-like matter there is an extra "exotic family" in the massless spectrum. As it is well known, the simplest chiral anomaly-free combination in $SU(5)$ is $(10 + \overline{5})$. The next to simplest chiral anomaly-free combination is $(15 + 9 \cdot \overline{5})$, where this 15-plet is the two index symmetric tensor (do not mistake it with the other 15-plet above which is the antisymmetric $SU(6)$ tensor). One of these "exotic families" is present in the massless spectrum of this model. A $\overline{15}$ is indeed obtained from the symmetric components of the $(\overline{6}, \overline{5})$ fields and remains in the massless spectrum. There is also a surplus of nine 5-plets that cancel the $SU(5)$ anomalies. It would be interesting to study in detail to what extent this extra "exotic family" could be made phenomenologically viable. Under the standard model group, $15 = (6, 1, -2/3) + (3, 2, 1/6) + (1, 3, 1)$

$Sector$	$SU(6)\times SU(5)\times SO(8)$	Q_1	Q_2	Q_3
U_1	$(6,1,1)$	1	0	1
	$(6,1,1)$	1	0	-1
	$(\overline{6},\overline{5},1)$	-1	-1	0
	$(1,5,8_V)$	0	1	0
U_2	$(1,\overline{5},1)$	0	-1	1
	$(1,\overline{5},1)$	0	-1	-1
	$(\overline{6},1,8_V)$	-1	0	0
	$(1,10,1)$	0	2	0
	$(15,1,1)$	2	0	0
U_3	$(1,1,8_V)$	0	0	1
	$(1,1,8_V)$	0	0	-1
	$(6,\overline{5},1)$	1	-1	0
	$(\overline{6},5,1)$	-1	1	0
θ	$12(6,1,1)$	-1	5/6	-1/2
	$12(1,1,1)$	-2	5/6	1/2
θ^2	$3(1,1,8_{\bar{s}})$	-1	-5/6	1/2
	$3(1,1,8_{\bar{s}})$	1	5/6	1/2
	$6(1,1,8_s)$	-1	-5/6	-1/2
	$6(1,1,8_s)$	1	5/6	-1/2
	$3(6,1,1)$	-1	-5/3	0
	$6(\overline{6},1,1)$	1	5/3	0
	$3(1,\overline{5},1)$	2	2/3	0
	$6(1,5,1)$	-2	-2/3	0
θ^3	$4(1,1,1)$	0	5/2	-1/2
	$8(1,\overline{10},1)$	0	1/2	-1/2
	$4(1,5,1)$	0	-3/2	-1/2
	$8(1,1,1)$	0	-5/2	1/2
	$4(1,10,1)$	0	-1/2	1/2
	$4(1,\overline{5},1)$	0	3/2	1/2

Table 7: Particle content and charges of the $SU(6)\times SU(5)$ model of section 5.2.

and hence we would have at least an extra colour-sextet quark. Objects of this type have been considered in the past [23] . Apart from this potential problem, the structure of Yukawa couplings is much less appealing than that found for the previous four-generation $SO(10)$ model (see ref. [24] for the general structure of Z_6' Yukawa couplings). For example, the $\overline{10}$-plets corresponding to the three physical generations are in the θ^3 twisted sector. However there are no Yukawa couplings of the type $\overline{10}\,5\,\overline{10}$, so that, for instance, the top-quark would have no Yukawas at this level. It is also not obvious how doublet-triplet splitting could occur.

6 Conclusions and Outlook

In this paper we have further explored the construction and patterns of 4-d symmetric orbifold strings whose massless spectrum constitue $SO(10)$ and $SU(5)$ GUTs. From this study there arises a very characteristic structure for this class of string GUTs. We find that only certain even-order Abelian orbifolds can be used to construct such models. In particular, we have shown on general grounds that $SO(10)$ or $SU(5)$ GUTs cannot be obtained from the Z_3, Z_7, $Z_3 \times Z_3$, Z_8' and $Z_2 \times Z_6'$ orbifolds. String $SO(10)$ models can only be obtained in the subset of even-order orbifolds that have a complex plane rotated only by order-two twists. They only contain either a single 54 or a single 45 that belongs in the untwisted sector corresponding to that order-two twist. Due to their location in an untwisted sector, these GUT-Higgs fields behave very much like string moduli.

In the case of string $SU(5)$ models the situation is slightly less tight. The models usually contain a single adjoint 24 located in an untwisted sector, which again leads to modulus-like behaviour for this field. Multiple 24s can however arise in a restricted class of Abelian orbifolds that contain certain twisted sectors of order four or six. In both $SO(10)$ and $SU(5)$ string symmetric orbifolds the couplings of the GUT-Higgs fields are very constrained. In section 4 we list a number of selection rules for their couplings. Some of these rules have more general validity and apply to any possible level-two string GUT, and not only to symmetric orbifolds.

We have also extended our previous analysis concerning the permutation method to construct level-two models. The models obtained through this method, unlike the other two used in [1] are not necessarily continuously connected to level-one models. This offers new model-building possibilities. In particular this method allows us to obtain $SO(10)$ GUTs with adjoint 45 Higgs fields (in the other two methods only 54s can be obtained). Our study of the permutation method includes a careful study of the partition function in order to obtain the appropriate modular invariance conditions as well as the generalized GSO projectors.

In the context of symmetric orbifold GUTs it is natural to obtain models with an even number of generations. As we said above, only even-order orbifolds can be used to construct GUTs and this typically leads to even degeneracies in the twisted sectors which in turn implies an even number of generations. To illustrate our method we have presented several four-generation $SO(10)$ and $SU(5)$ models. It is intriguing that many of the phenomenologically required couplings, e.g. couplings giving masses to right-handed neutrinos, couplings which could induce doublet-triplet splitting, usual Yukawa couplings, etc., are indeed present. Recently there has been some interest in the study of four-generation versions of the SSM [22] . It could perhaps be interesting to study in detail the possible phenomenological viability of some of these four-generation string GUTs. The closest we get to a three-generation model is an $SU(5)$ example with an extra exotic $(15 + 9 \cdot \overline{5})$ "family", but its phenomenological properties do not seem to be better than those of the four-generation models. Although it indeed seems that even number of generations is the most common situation in symmetric orbifold string GUTs, our search has not been systematic and we cannot rule out at the moment the existence of three-generation examples in this class of string GUTs. A computer search for models could perhaps be appropriate.

The methods presented here can be generalized in several ways. An obvious possibility is the construction of level-two models based on *asymmetric* orbifolds. Indeed, as remarked in [1], the strong constraints coming from eq. (69) are much weaker in the case of asymmetric orbifolds since, for instance, there can be cases in which the right-movers are twisted but the left-movers are not. In this case $E_0 = 0$ in eq. (69) and it becomes easier to find GUT-Higgs fields in twisted sectors, leading to models with multiple GUT-Higgses. Indeed, string GUTs with multiple GUT-Higgs fields based on asymmetric orbifolds can be constructed [25, 26]. The trouble is that very little model-building, if any, has been done using asymmetric orbifolds and it would be necessary to first develop some theoretical tools. In particular, a good control of the generalized GSO projectors for asymmetric orbifolds would be necessary to find the complete massless spectrum of each model. Although in principle this can be extracted from the original literature [27] in practice this is a non-trivial task. This task becomes even more involved in the presence of the extra modding required to get the level-two model. Furthermore, the addition of discrete Wilson lines in asymmetric orbifolds turns out to be rather restricted compared to the symmetric case [10] , and this makes the construction less flexible from the model-building point of view.

A different, perhaps more promising, approach is the derivation of level-two GUTs based on the Gepner or Kazama-Suzuki constructions. The techniques employed for symmetric orbifolds can be easily generalized to start with a coset compactification of the $Spin(32)/Z_2$ heterotic string with an $SO(26)$ gauge group (plus possible additional $U(1)$s). The internal $c = 9$ system in this case is a tensor product of $N = 2$ superconformal coset models. As discussed in detail in refs. [7, 8], different gauge groups and particle contents can be obtained by twisting the $N = 2$ superconformal models by discrete symmetries while embedding these symmetries in the $SO(26)$ degrees of freedom through a shift. In this way, level-one models with replicated groups are found as in the orbifold case. Then, if the given coset compactification has an order-two symmetry, it could be embedded through an order-two permutation of two GUT group factors, taking care that left-right level matching is mantained. The class of string GUTs obtained in this way would be rather large and so will be the chances to find three generation models.

The outstanding challenge in GUT model-building, with or without strings, is the origin of doublet-triplet splitting. The structure of the standard model is quite bizarre. There is a chiral piece, the quark-lepton generations, and a vector-like piece, the Higgs pair H_1, H_2. Even with supersymmetry, the presence of the light Higgs fields is a mistery. In the case of GUTs this mistery comes in the form of the doublet-triplet splitting problem. But in the context of other level-one, non-GUT, string models the mistery still exists of why a set of Higgs fields remains light anyhow (even though no splitting is needed). Thus, we do not think that the situation of string GUTs concerning this issue is conceptually very different from the situation of other string level-one models. On the contrary, as discussed in ref. [1], within the context of string GUTs, doublet-triplet splitting appears as a dynamical problem in which the crucial issue is the quantum moduli space of the GUT-Higgs fields in the model. Perhaps recent developements in the understanding of non-perturbative phenomena in supersymmetric theories could help in solving this problem.

In closing, let us comment on level-two versus level-one 4-d strings. One may have the wrong impression that somehow level-one heterotic theories are more generic than level-two or higher. In fact, one may argue for the contrary. We remarked at the beginning of section two (see also ref.[3]) that the higher level theories are often continuously connected to level-one theories. For *generic* values of some charged moduli fields the gauge group has higher level and it is only at particular points of moduli space that the gauge group is enhanced to a bigger one realized at level-one. Thus, in this sense, reduced rank models realized at higher levels are more generic than level-one models. Independently of its use to construct GUT theories, level-two (and higher) string theories could be relevant in realizing the low-energy physics. In particular, it is conceivable that all or some of the non-Abelian symmetries of the standard model could correspond to higher level affine Lie algebras. We hope that the techniques developed in [1] and in the present article will be useful to the general study of higher level heterotic strings, independently of their use in GUT model-building.

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