A BASIS FOR INVARIANTS IN NON–ABELIAN GAUGE THEORIES[*]

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An algorithm is described to convert Lorentz and gauge invariant expressions in non–Abelian gauge theories with matter into a standard form, consisting of a linear combination of basis invariants. This algorithm is needed for computer calculations of effective actions. The defining properties of the basis invariants are reported. The number of basis invariants up to mass dimension 16 are presented.

1. Introduction

Effective actions of gauge theories are space–time integrals over gauge and Lorentz invariant expressions. From the mathematical point of view, they are, up to some factors, functional traces of heat kernel coefficients, known as Schwinger–DeWitt,[1] Gilkey–Seeley,[2] or Hadamard coefficients.[3] In flat space–time, these coefficients are polynomials constructed from a matrix potential and from the gauge field strength tensor by multiplication, gauge covariant differentiation, and contraction of Lorentz indices. Due to Bianchi identities and the product rule for covariant derivatives, the form of the coefficients is not unique. Furthermore, the physically interesting functional trace of the coefficients allows cyclic exchanges of matrix factors and integration by parts.

New methods of computing effective actions, such as the string–inspired world line path integral formalism,[4,5] but also the implementation of established calculation algorithms on computers[6] enable the extension of known results to higher order in the inverse mass expansion. To manage the corresponding increasing number of terms and to compare results of different methods,[7,8] a standard basis of invariants is needed, in terms of which all results can be expressed. An algorithm should be provided to convert a Lorentz scalar given in a non–standard form into terms of the basis. For gravitational invariants, constructed from the Riemann and the metric tensor, such normal forms were presented up to order eight in the mass dimension by Fulling et al.[9] In the general case with matter, gauge fields, and gravity, basis sets of non–local invariants up to third order in the curvature were constructed. They are used in the expansion of effective actions in terms of Barvinsky–Vilkovisky form factors.[10]

This contribution analyzes the formal structure of invariant monomials in non–Abelian gauge theories with matter in flat space–time. Step by step, the operations applicable to invariants are used to convert them into a fixed form. Thus, a basis of invariants is specified, and simultaneously, a procedure to expand an arbitrary given Lorentz invariant expression in terms of the basis is obtained. The proof of the basis property of the specified set of invariants will be published elsewhere.[11]

2. Notations

Notations are introduced on the basis of a concrete example. Let us consider a gauged scalar field theory described by the massive complex field ϕ^a and the Hermitian matrix valued gauge field A_μ^{ab}. The gauge covariant derivative in the fundamental representation is $\mathcal{D}_\mu^{ab} = \delta^{ab}\partial_\mu - iA_\mu^{ab}$. The coupling constant is contained in the gauge field. Integrating the quantum fluctuations of the field ϕ^a in the given backgrounds φ^a and A_μ^a, we obtain, in a first approximation, the one–loop effective action $\Gamma^{(1)}[\varphi, A]$ which can be expanded in gauge invariant terms[7]

$$\Gamma^{(1)}[\varphi, A] = \operatorname{Tr}\ln\left(-\mathcal{D}^2 + V + m^2\right) = \int d^d x \sum_i \frac{C_i}{m^{\mu_i - d}} \operatorname{tr}\left(I_i\left(F, V\right)\right). \quad (1)$$

V is a matrix potential originating from the matter fields. The C_i are complex numbers and $I_i\left(F, V\right)$ matrix valued Lorentz scalars composed of the potential V, the field strength tensor $F_{\mu\nu}^{ab} = i\left[\mathcal{D}_\mu, \mathcal{D}_\nu\right]^{ab} = \partial_\mu A_\nu^{ab} - \partial_\nu A_\mu^{ab} - i\left[A_\mu, A_\nu\right]^{ab}$, and the gauge covariant derivative in the adjoint representation $D_\mu = [\mathcal{D}_\mu, .] = \partial_\mu - i[A_\mu, .]$. D_μ acts on the matrix potential and on the field strength tensor. d is the dimension of space–time. μ_i is the mass dimension of the scalar $I_i(F, V)$ according to the mass dimensions of its constituents $[V] = 2$, $[F_{\mu\nu}] = 2$, and $[\mathcal{D}_\mu] = [D_\mu] = 1$.

The form (1) is not unique due to several equalities, namely the product rule for covariant derivatives, integration by parts, cyclic permutations, the Bianchi identity, the antisymmetry of the field strength tensor, and the exchange of derivatives:

$$D_\mu(XY) = D_\mu XY + XD_\mu Y, \quad \int dx\, \operatorname{tr}\left(D_\mu X_\mu Y\right) = -\int dx\, \operatorname{tr}\left(X_\mu D_\mu Y\right), \quad (2a,b)$$
$$\operatorname{tr}(XY\ldots Z) = \operatorname{tr}(Y\ldots ZX), \quad D_\mu F_{\kappa\lambda} = D_\kappa F_{\mu\lambda} + D_\lambda F_{\kappa\mu}, \quad (2c,d)$$
$$F_{\mu\nu} = -F_{\nu\mu}, \quad D_\mu D_\nu X = D_\nu D_\mu X - i\left[F_{\mu\nu}, X\right]. \quad (2e,f)$$

Let us call a V, an F, or covariant derivatives of them a *simple factor*, i.e.

$$(\text{simple factor}) \in \{V,\ F_{\kappa\lambda},\ D_{\mu_1} D_{\mu_2} \ldots D_{\mu_n} V,\ D_{\mu_1} D_{\mu_2} \ldots D_{\mu_n} F_{\kappa\lambda}\}. \quad (3)$$

Simple factors containing the matrix potential are called *V–factors*, the others *F–factors*. With the product rule (2a), expression (1) can be converted into a form where the invariants $I_i(F, V)$ are monomials, i.e. products of simple factors. Subsequently, the invariants are supposed to have this form.

If the gauge group representation is unitary, the additional symmetries

$$V^\dagger = V, \quad A_\mu^\dagger = A_\mu, \quad F_{\mu\nu}^\dagger = F_{\mu\nu}, \quad (D_\mu X)^\dagger = D_\mu X \quad \text{if} \quad X^\dagger = X \quad (4)$$

hold. Consequently, simple factors are Hermitian. For simple factors X, Y, and Z this leads to

$$\overline{\operatorname{tr}(XYZ\ldots)} = \operatorname{tr}\left(\ldots Z^\dagger Y^\dagger X^\dagger\right) = \operatorname{tr}\left(\ldots ZYX\right). \quad (5)$$

Thus, an invariant monomial can be expressed by the complex conjugate of its mirror image with identical factors, but in inverted order. Therefore we call eq. (5) a mirror transformation. In general, a monomial and its complex conjugate are independent of each other, so that operation (5) cannot be used to reduce the number of terms in eq. (1). However, Lagrangians are real. Hence, in an appropriate basis, an arbitrary invariant monomial $I(F, V)$ and its mirror image have complex conjugate coefficients C and $\bar{C}$ so that they add to $2\Re\mathfrak{e}\,(C \cdot I(F, V))$. Another exception occurs for real ϕ^a and imaginary A_μ^{ab}.[a] Then V–factors are real and F–factors imaginary. In this case, monomials and their complex conjugates are not independent of each other and eq. (5) reduces the number of terms in eq. (1) indeed.

3. The Basis

3.1. *The reduction algorithm*

We start from an arbitrary Lorentz invariant given in the form (1). The product rule must be used whenever derivatives of products are encountered. This may happen at each stage of the algorithm. The manipulations (2b–f, 5) must be applied in the sequence of the following sub-subsections to obtain a standard result. The rules given there do not entirely fix all details of the algorithm. Therefore, the algorithm can be executed in different ways, but the results will be expressed by the same basis of invariants and, hence, will be identical. The procedure will require exchanges of derivatives by eq. (2f). Since thereby additional invariants with more F–factors and fewer derivatives are produced, the algorithm starts with the invariants with the most F–factors and descends to invariants with fewer and fewer F–factors.

3.1.1. *Integration by parts*

The indices in a Lorentz invariant monomial can be contracted between different factors and within the same factor. We call the latter self–contractions. They always include a covariant derivative. Therefore, *we apply integration by parts to covariant derivatives in self–contractions.* Thereby all self–contractions are eliminated.

3.1.2. *The Bianchi identity*

The Bianchi identity (2d) exchanges the index of one derivative with the indices of $F_{\mu\nu}$ within an F–factor. All other factors remain unchanged. Therefore, we need a prescription that specifies the derivatives which are candidates for applying the Bianchi identity in the F–factor under consideration. Let us consider the example

$$\mathrm{tr}\big(\overbrace{\underbrace{D_\mu}^{L}\underbrace{D_\nu}^{R}\underbrace{D_\rho}^{M} D_\sigma F_{\kappa\lambda}}_{\substack{\text{factor under}\\\text{consideration}}} \underbrace{\ldots X'_\nu \ldots}_{\substack{\text{right}\\\text{sector}}} Y_{\sigma\kappa} \underbrace{\ldots X''_\rho \ldots}_{\substack{\text{middle}\\\text{sector}}} Z_\lambda \underbrace{\ldots X_\mu \ldots}_{\substack{\text{left}\\\text{sector}}}\big) \quad . \tag{6}$$

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[a] This is the case for real orthogonal representations of the gauge group. Then iA_μ^{ab} is real and antisymmetric in a and b.

The indices of $F_{\kappa\lambda}$ are contracted with the factors $Y_{\sigma\kappa}$ and Z_λ, which divide the remaining factors into three, possibly empty, sectors. We call them "right sector", "middle sector", and "left sector", as indicated, because, due to cyclic invariance (2c), the "left sector" is connected with the left–hand side of the factor under consideration.

The derivatives of the factor under consideration are called left ("L"), right ("R"), and middle ("M") corresponding to the sector they are contracted with. Not all derivatives are left, right, or middle (e.g. D_σ). The Bianchi identity (2d) mixes all three kinds of derivatives. Therefore it can be used to eliminate one kind of index in all factors of all monomials. Since the middle sector is invariant under the mirror transformation (left and right sectors are interchanged), *we apply the Bianchi identity to middle derivatives.* Each such application of the Bianchi identity reduces the number of factors in the corresponding middle sector. Thus, after finitely many steps, all middle derivatives are eliminated.

Finally, we convert multiple contractions between factors into a standard form by

$$\ldots F_{\mu\nu} \ldots D_\mu D_\nu X \ldots \quad \Rightarrow \quad -\tfrac{i}{2} \ldots F_{\mu\nu} \ldots [F_{\mu\nu}, X] \ldots \tag{7}$$

$$\ldots F_{\mu\nu} \ldots D_\mu F_{\nu\kappa} \ldots \quad \Rightarrow \quad \tfrac{1}{2} \ldots F_{\mu\nu} \ldots D_\kappa F_{\nu\mu} \ldots \tag{8}$$

$$\ldots D_\mu F_{\nu\kappa} \ldots D_\nu F_{\mu\lambda} \ldots \quad \Rightarrow \quad \ldots D_\mu F_{\nu\kappa} \ldots D_\mu F_{\nu\lambda} \ldots +$$
$$+ \tfrac{1}{2} \ldots D_\kappa F_{\nu\mu} \ldots D_\lambda F_{\mu\nu} \ldots \quad . \tag{9}$$

The first equality uses the antisymmetry (2e) of the field strength tensor and the commutation rule (2f). The second transformation relies on the antisymmetry (2e) together with the Bianchi identity (2d). The third rule results by applying the Bianchi identity (2d) to one of the factors and subsequently using eq. (8).

3.1.3. *The arrangement of factors*

Cyclic factor permutations (2c) and, possibly, mirror transformations (5) can be used to identify invariants. Applying eqs. (2c) and (5) in all possible ways to a given invariant monomial, we obtain a class of equivalent invariants. *We pick a representative of each equivalence class.* This may be done by introducing an ordering relation in the equivalence classes Then we pick the smallest (or greatest) invariant of each equivalence class as the representative.

3.1.4. *The arrangement of indices*

Derivatives and indices of F's can be exchanged by means of eqs. (2f) and (2e) in all factors of all invariant monomials. Let us consider a certain factor within an invariant. It can be shifted completely to the left–hand side by eq. (2c), as a result of which the achieved arrangement of factors is temporarily destroyed[b] (cf. example (6)). *After this operation, we rearrange the derivatives and/or indices*

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[b]The arrangement of the factors has to be restored after reordering the indices and is, in the end, not affected by this procedure.

of the F (if present) in the considered factor according to the contracted counter indices. In example (6) $Y_{\sigma\kappa}$ is located left of Z_λ. Thus the indices of $F_{\kappa\lambda}$ have the correct order. The locations of X'_ν, $Y_{\sigma\kappa}$, X''_ρ, and X_μ define the correct order of the derivatives to be $D_\nu D_\sigma D_\rho D_\mu$. Since the mirror transformation inverts the ordering of the factors, it has to be applied *before* rearranging the indices. Cyclic factor permutations and the arrangement of indices do not interfere with each other.

3.2. *The defining properties of the basis*

Pursuing the above algorithm, we state the following properties of basis invariants:

- The invariants are products of simple factors.
- Indices are contracted only between different factors of an invariant monomial.
- There are no "middle" derivatives.
- In multiple contractions between factors, derivatives are contracted with derivatives and indices of F's with indices of F's (cf. eqs. (7 − 9)) except for contractions of an index of an F with a derivative where the other index of the F is contracted with a third factor.
- The order of derivatives and of indices of the F's is as described in sub-subsection 3.1.4.

These properties allow to count the basis invariants of a certain mass dimension. Up to mass dimension 16, this was performed by a C language program (table 1). Results of higher dimension or divided by the number of F's are available.

Table 1. The number of basis invariants **with** and *without* the mirror transformation. v is the number of occurrences of the matrix potential V in the invariants.

	Mass dim.	Total		$v = 0$		1		2	
1	2	**1**	*1*	**0**	*0*	**1**	*1*		
2	4	**2**	*2*	**1**	*1*	**0**	*0*	**1**	*1*
3	6	**5**	*5*	**2**	*2*	**1**	*1*	**1**	*1*
4	8	**17**	*18*	**7**	*7*	**4**	*5*	**4**	*4*
5	10	**79**	*105*	**29**	*36*	**24**	*36*	**17**	*23*
6	12	**554**	*902*	**196**	*300*	**184**	*329*	**119**	*191*
7	14	**5283**	*9749*	**1788**	*3218*	**1911**	*3655*	**1096**	*2020*
8	16	**65346**	*127072*	**21994**	*42335*	**24252**	*47844*	**13333**	*25861*

Table 1. (Continued)

	Mass dim.	$v = 3$		4		5		6		7		8	
1	2												
2	4												
3	6	**1**	*1*										
4	8	**1**	*1*	**1**	*1*								
5	10	**6**	*7*	**2**	*2*	**1**	*1*						
6	12	**39**	*63*	**13**	*16*	**2**	*2*	**1**	*1*				
7	14	**370**	*670*	**96**	*158*	**18**	*24*	**3**	*3*	**1**	*1*		
8	16	**4452**	*8638*	**1095**	*2020*	**186**	*329*	**30**	*41*	**3**	*3*	**1**	*1*

4. Conclusions and Outlook

A prescription for defining a standard basis set of invariants in non−Abelian gauge theories was obtained. A reduction algorithm was presented to convert a given

Lorentz scalar by partial integration, by the Bianchi identity, and by cyclic invariance of the trace into a linear combination of this basis set of invariants. The proof that this set is a basis indeed, relies on a graphical representation of invariants and is given elsewhere.[11]

For cases where, in addition, the mirror transformation reduces the number of independent invariants, a general proof of the basis property is still lacking. However at least up to mass dimension 16, it can be shown by counting the invariants that the standard set remains a basis.

Another open problem is to take into account additional identities which exist for particular choices of the gauge group representation.

Acknowledgements

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