H. Kröger and N. Scheu

Département de Physique, Université Laval, Québec, Québec G1K 7P4, Canada. E-mail: nscheu@phy.ulaval.ca

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We compute structure functions in the Hamiltonian formalism on a momentum lattice using a physically motivated regularisation that links the maximal parton number to the lattice size. We show for the ϕ^4_{3+1} theory that our method allows to describe continuum physics. The critical line and the renormalised mass spectrum close to the critical line are computed and scaling behaviour is observed in good agreement with Lüscher and Weisz' lattice results. We then compute distribution functions and find a Q^2 behaviour and the typical peak at $x_B \to 0$ like in QCD.

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Hadron structure is probed by deep inelastic scattering (DIS). Over recent years a great deal of experimental data has been gathered from high energy collider experiments. While perturbative quantum chromodynamics (QCD) describes successfully the large Q^2 dependence of DIS structure functions, it fails to predict the correct dependence on the Bjørken variable x_B. Thus much effort has been devoted to compute quark or gluon distribution functions and proton structure functions from QCD with *non-perturbative* methods. E.g., Martinelli et al. [1] have computed the first two moments of the pion structure function via Monte Carlo lattice simulations. These calculations are notoriously difficult (for the present status of lattice calculations of structure functions see Ref. [2]). This situation calls for alternative techniques. In this letter we present such a new approach. Its basic ingredients are: (i) We use a Hamiltonian formulation, based on (ii) a momentum lattice as regulator, and (iii) use a Breit frame (*not* the rest frame) corresponding to the scattering process. We apply our method to the scalar model in $3+1$ dimensions, which has been extensively studied, and compute the distribution function. As a result we find an Altarelli-Parisi like behaviour leading to a sharp forward peak at small x_B at high resolution Q^2, as it typically shows up in high energy DIS hadron scattering experiments. We extract continuum physics: Close to the critical point our results are in perfect agreement with the predicted scaling behaviour as well as with Euclidian lattice results by Lüscher and Weisz [3].

Let us briefly outline the reasons for the choice of our method: (i) Structure functions are computed from wave functions. Wave functions are defined in Minkowsky space where they can be computed directly from a Hamiltonian formulation. The Hamilonian approach offers the advantage of allowing to compute directly Minkowsky space observables. E.g., scattering wave functions for glueball-like states in compact QED_{2+1} have been computed in a Hamiltonian formulation on a momentum lattice [4] (for a review of Hamiltonian lattice methods see [5–7]). (ii) The usefulness of a momentum lattice to compute physics close to a critical point has been demonstrated in Ref. [8,9]. (iii) The reason for our choice of the Breit frame will be explained below. However, Hamiltonian methods are known to lead to numerical problems because of the huge number of degrees of freedom involved. Nobody has succeeded yet in observing scaling behaviour indicating continuum physics in a (3+1)-dimensional Hamiltonian formulation. In this work we shall demonstrate for the scalar theory that those difficulties can be overcome.

The most important experiment in order to probe the structure of hadrons is deep inelastic scattering (DIS)). Its simplest form is inclusive scattering of an unpolarised lepton off a hadronic target. Let us recall some basic notations [10]. The hadron in its ground state with four momentum P interacts with the probing lepton by the exchange of a virtual photon (our neutrino) with space-like four-momentum q. In Feynman's parton model it is assumed that the proton consists of constituents, the partons, which are weakly bound, i.e. its binding energy is small compared to the resolution ability $Q := \sqrt{-q_\mu q^\mu}$ of the probing photon. In this approximation, the so-called Bjørken scaling variable $x_B := \frac{Q^2}{2P_\mu q^\mu}$ can be interpreted as the momentum fraction of the struck parton if we work in the Breit frame. The Breit frame is defined by the requirements that the photon energy q_0 be zero and that the photon momentum $\vec{q}$ be antiparallel to the hadron momentum $\vec{P}$. In this frame the following relation between the parton momentum $\vec{p}$ and the proton momentum holds:

$$(\vec{p} - \vec{P}/2)^2 \leq |\vec{P}/2|^2. \tag{1}$$

The rationale for this particular choice of frame being that QCD structure functions $F(x_B, Q)$ can be interpreted as a linear combination of parton momentum distribution functions $f(x_B, Q)$, which have a more intuitive interpretation.

The latter is defined by th Structure functions are another way of expressing scattering cross sections. The distribution function of a parton counts the number of those partons with a given momentum fraction x_B in the proton. For a precise definition see Ref. [10].

Because the Breit frame introduced above refers to a particular struck parton and we want to describe a many-parton system (proton) we need to extend the definition to a generalised Breit frame: $q_0 = 0$ but $\vec{q}$ needs no longer be antiparallel to $\vec{P}$. Although the parton momentum needs no longer be collinear in general to the proton momentum $\vec{P}$, we nevertheless impose Eq.(1) as kinematical condition. While the generalised Breit frame has been introduced for the purpose of practical calculations, it should be noted that the strict relation between distribution and structure functions, characteristic for original Breit-frame no longer holds in a strict sense. However, this relation is recovered for the generalised Breit frame in the continuum limit.

Because we are working in the Hamiltonian approach we need to define a basis of the Hilbert space. We construct the Hilbert space as a Fock space of free particles and select (parton) momenta $\vec{p}$ from a bounded domain corresponding to DIS as given by Eq.(1). This is an *assumption* based on the physical intuition that the experimentally observable parton momenta are those which dominate the quantum dynamics. This assumption has been tested by computing critical behaviour of renormalised masses and a good agreement with analytical scaling behaviour has been observed (see below).

Now we introduce a momentum lattice regularisation: In order to have a practically convenient lattice we further constrain the parton momenta from Eq.(1), namely by selecting a regular cube centered at $\vec{P}/2$ and located inside the ball given by Eq.(1). I.e., the parton momenta $\vec{p}$ lie in the domain

$$0 \leq p_i \leq \Lambda = \frac{\sqrt{3}}{2}|\vec{P}| \quad \text{for } i = x, y, z. \tag{2}$$

We define lattice momenta $\vec{p} := \vec{n}\Delta p$ where $\vec{n}$ is an integer vector and Δp is the momentum lattice spacing covering the domain given by Eq.(2). One notices that all lattice momenta are *positive* (non negative). Contrary to a regularisation in the rest frame which does *not* limit the particle number, our approach has the following important property: For any given Hilbert state with non-zero total momentum, the Fock space particle numbers are bounded. Consequently the ultraviolet cutoff Λ given by Eq.(2) implies a total particle number cutoff and thus drastically reduces the dimension of the Hilbert space.

Mass spectrum and critical behaviour of the ϕ^4_{3+1} theory

Before discussing structure functions we need to convince ourselves that the method allows to compute correctly physical observables. We have chosen the scalar ϕ^4_{3+1} theory because it is a quite well understood theory and has a second order phase transition, allowing to test our method near a critical point. The Hamiltonian of the ϕ^4 theory is given by

$$H = \int d^3x \; \frac{1}{2}(\frac{\partial\phi}{\partial t})^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{m_0^2}{2}\phi^2 + \frac{g_0}{4!}\phi^4, \tag{3}$$

where m_0 and g_0 are the bare mass and coupling constant, respectively. We express the Hamiltonian in terms of free field creation and annihilation operators corresponding to the lattice momenta. Because the Hamiltonian and the momentum operators commute, we compute the energy spectrum E_n in a Hilbert space sector of given momentum $\vec{P}$. Since we are not in the rest frame we have to use the mass-shell condition $M_n := \sqrt{E_n^2 - \vec{P}^2}$ in order to obtain the physical mass spectrum. It is known [3] that the critical line between the symmetric and the broken phase lies entirely in the region where the bare parton mass squared m_0^2 is negative. Hence we cannot build up the Fock-space in terms of partons with those masses. As a remedy we have split the bare mass squared $m_0^2 = m_{kin}^2 + m_{int}^2$ into a positive kinetic part m_{kin}^2 and an interaction part m_{int}^2. The Fock states are built from positive bare masses m_{kin}. The best choice of m_{kin} seems to be to take the renormalised mass m_R (which however, requires a separate calculation). In numerical calculations close to the critical point shown in Fig.[1] we have chosen for simplicity a small positive value. We found that the lower lying physical mass spectrum is not very sensitive to the value of m_{kin} (this is not the case for higher lying masses).

We diagonalised the Hamiltonian on two lattices: $\Lambda/\Delta p = 3$ and $\Lambda/\Delta p = 4$. This would correspond to symmetric lattices ($-\Lambda$ and $+\Lambda$) of size 7^3 and 9^3 nodes, respectively. This results in a very small Hilbert space of only 6 and 21 states, respectively. In order to compare our results to those of Lüscher and Weisz [3] we express the bare parameters m_0 and g_0 in terms of the parameters λ and κ: $m_0^2 = (1-2\lambda)/\kappa - 8$ and $g_0 = 6\frac{\lambda}{\kappa^2}$. Fig.[1] displays the renormalised mass m_R versus κ. One observes that our results computed on very small lattices are quite close to the results of Lüscher and Weisz [3]. Masses M computed on the lattice must obey $a < 1/M < L$ where L is the length of the lattice and a denotes the lattice spacing of a space-time lattice $\Lambda = \frac{\pi}{a}$. It can be shown from perturbation theory [11,3] that the physical masses close to the critical point obey the following scaling law $M \sim C\tau^{1/2}|ln\tau|^{-1/6}$, where $\tau := 1 - \kappa/\kappa_{crit}$ and C is a constant. Since the results of Ref. [3] are based on the solution of the renormalisation group equations, this scaling law fits their results. One should note, however, that two different regularisations (this work and that of Ref. [3]) in general correspond to two different critical lines. In Tab. [1] we have displayed our results for the critical points κ_{crit} as a function of λ and compared our results with those of Ref. [3]. Again, our results are very close to those of Lüscher and Weisz. These results cover a domain of the bare parameter space extending quite far away from the Gaussian fixed point $\kappa = 1/8$ and $\lambda = 0$.

Another way to test continuum physics is to look at the mass ratios M_n/M_1 from the spectrum on the lattice and check if they become independent of the cutoff Λ or else independent of the coupling constant $g_0(\Lambda)$ (i.e., they scale). Those mass ratios M_n/M_1 are shown in Fig.[2]. As can be seen, for a number of states $M_n/M_1 \rightarrow const$ in a wide range of κ-values, i.e., they scale. However, for some states M_n/M_1 diverges, i.e., there is no scaling. The physical reason behind this is the following: The ϕ^4_{3+1} model describes a gas of partons repelling each other [3]. The spectrum of Fig.[2] shows states dominated by the 1-,2-,3-,4- particle Fock space sectors plus a spectrum of excited (scattering) states. The picture of repulsive two-particle-exchange force is confirmed by observation that the mass of the lowest-lying n-body state is larger than n-times the mass of the one-body state. The states which scale are just those lowest-lying n-body states. The higher-lying part of the spectrum consits of states with more nodes in the wave-function than lattice points, having also a wider range and contributions from higher Fock-state sectors. Because in the calculation corresponding to Fig.[2], the parameters Δp, Λ and the parton number cutoff are all kept fixed, we cannot properly describe those higher-lying states. Consequently, they do not show scaling. When we go to bigger lattices ($\Delta p \rightarrow 0$) then we observe (not displayed here) more states which show scaling.

Distribution functions

The distribution function $f(x_B, Q)$ of finding some parton with momentum fraction x_B inside the hadron is determined by the parton momentum distribution function $\tilde{f}(\vec{p}, \vec{P})$ for finding a parton with momentum $\vec{p}$ inside the hadron with momentum $\vec{P}$. Since Q is a dimensionful quantity, its scale is set by the lattice spacing a, i.e., $Q \sim \dfrac{1}{a(m_0, g_0)}$ and thus depends on the bare parameters if one keeps the renormalised mass fixed. The continuum limit $a \rightarrow 0$ corresponds to the the limit towards arbitrarily high resolution ability. If one keeps the renormalised mass and the renormalised coupling constant fixed, then Q is a function of the bare coupling constant g_0 and vice versa – invertibility of $Q(g_0)$ assumed. Hence $f(x_B, Q)$ is related via the function $Q(g_0)$ to the distribution function $\bar{f}(x_B, g_0)$ which only depends on dimensionless parameters. Consequently, a calculation of the distribution function along a renormalisation group trajectory can be used to compute the Q-dependence of the quark structure functions in QCD.

While QCD possesses bound states of quarks and gluons, the existence of corresponding bound states in the scalar ϕ^4_{3+1} is not evident. According to Ref. [3] they do not exist in the symmetric phase and there is little chance to find them in the broken phase, either. This is confirmed by our numerical findings. In order to compute distribution functions of a bound state of partons in the scalar model we have taken recurrence to the ϕ^3 model. We calculate the distribution function of the ϕ^3 theory, because the ϕ^3-interaction describes forces which are attractive one-particle exchange forces [3]. This allows formation of bound states as in QCD. However, this theory is known to suffer from an unstable vacuum since it is unbounded from below. The unstable vacuum of the ϕ^3 theory prevents to calculate meaningful ground state masses which are needed to specify renormalisation group trajectories and hence the exact relation between the resolution Q and the bare coupling constant g_0. While in QCD one computes $\bar{f}(x_B, g_0$ and $g_0(Q)$ to obtain $f(x_B)$, here we can only compute the distribution function $f(x_B, g_0(Q)$. We have computed the distribution function in 1-,2- and 3 space dimensions. For a given parton number cutoff, these curves look very much alike. In order to analyse the behaviour at small x_B we have chosen to present our result corresponding to a calculation in one space dimension (Fig.[3]). When increasing the coupling g_0 we see that the distibution function develops a peak at momentum fraction $x_B = 0$. This is so, because increasing the coupling means that more partons are produced which share the total momentum fraction. The behaviour of the distribution function seen here is typical for QCD, where $g_0(Q)$ increases with the resolution Q. It is seen in DIS experiments and described by the Altarelli-Parisi equations. If we had applied a parton number cutoff independent of Λ, the small x_B behaviour of Fig.[3] which is a typical many-body effect [12], would not have been seen. This is so because a system of n identical observable particles must have an expectation value of x_B around $1/n$ for symmetry reasons.

In conclusion, we have devised a Hamiltonian method able to compute physical observables in Minkowsky space. We have applied it to the scalar model and obtained the correct scaling behaviour of the mass spectrum at the critical point. Moreover, we have computed distribution functions showing a peak at small x_B as described by the Altarelli-Parisi equations in QCD. Work is in progress to compute structure functions for full QCD.

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Fig.1 The ground state mass m_R in lattice units ($a \equiv 1$) versus κ for $\lambda = 0.00345739$ ($\bar{\lambda} = 0.01$ in Ref. [3]). The dots correspond to results of Ref. [3]. Our results correspond to $\Lambda/\Delta p = 3$ (dashed line) and $\Lambda/\Delta p = 4$ (solid line).

Fig.2 The lowest lying mass spectrum versus κ. The ground state mass is set to one. λ as in Fig.[1].

Fig.3 The distribution function $\bar{f}(x_B, g_0(Q))$ of ϕ^3_{1+1} versus the momentum fraction x_B and the coupling constant $g_0(Q)$. The bare mass m_0 has been to be $m_0 = 3\Delta k$. $\Lambda/\Delta p = 11$.

I. TABLE CAPTION

λ	0.0005	0.001	0.005	0.01	0.05	0.1
κ^{LW}_{crit}	0.125101	0.125202	0.125991	0.126968	0.132368	0.13601
α	0.99997	0.99993	0.99972	0.9993	1.0073	1.0275

The critical points κ_{crit} versus λ. κ^{LW}_{crit} is taken from Ref. [3]. $\alpha := \kappa^{KS}_{crit}/\kappa^{LW}_{crit}$ denotes the ratio between the results of this work and Ref. [3]. In this work, κ_{crit} has been determined by the condition that the renormalised mass m_R becomes imaginary. $\Lambda/\Delta p = 4$.